DISCRETE EQUATIONS: THEIR LOCAL INFORMATION, MELLIN TRANSFORM AND ELLIPTIC EQUATIONS

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Abstract

Our overarching goal is to study various types of algebraic difference equations using techniques analogous to the ones used to study D-modules. Concretely, we study actions of discrete groups on algebraic curves and equivariant sheaves for these actions. We focus on understanding the local behavior of equivariant sheaves.

We start by recalling basic notions about equivariant sheaves, the analogous notion to a D-module in the discrete situation. We give a definition for the restriction of an equivariant sheaf to the formal neighborhood of a point, in the case of a reduced curve acted on by a group which is virtually the integers. This definition should play the role of the usual notion of restriction to the disk of a D-module.

We give two reasons why this definition behaves in the desired way. The first one is that equivariant sheaves can be glued from their restriction to the complement of a point together with the restriction to the formal disk around the point (using our new definition), in a manner analogous to the Beauville-Laszlo Theorem.

The second reason is that in the case of the integers acting by translation on the affine line, we are able to recover familiar notions from the theory of D-modules. Namely, we are able to define vanishing cycles for an equivariant sheaf, and we show that the Mellin transform identifies the vanishing cycles of an equivariant sheaf with some nearby cycles of its Mellin transform, which is a D-module on the punctured line. We also show that vanishing cycles capture the intuitive notion of a singularity of a difference equation.

We also study symmetric elliptic difference equations from this point of view. We discuss the precise relation between such elliptic equations and equivariant sheaves, showing that while they are not equivalent, elliptic equations can be compared to equivariant sheaves. Further, we study the relation between elliptic equations defined on a reduced singular curve and those defined on its normalization. In particular we show that we can apply the theory for equivariant sheaves to study the local type of elliptic equations as well, and that these too can be recovered from glueing data.

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Notation and Symbols

$\mathcal{A}_g$	An equivariant structure
C	A reduced quasiprojective curve
C^*	$C \setminus Gp$
$\mathcal{D}_{\mathbb{G}_m}$	The ring of differential operators of $\mathbb{A}^1 \setminus 0$, $k[x^{\pm 1}]\langle \partial_x \rangle$
$\Delta_{\mathbb{A}^1}$	The ring of difference operators of $\mathbb{A}^1$, $k[z]\langle \tau^{\pm 1} \rangle$
E	A curve in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(2,2)$
G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$	The category of G-equivariant sheaves on C whose generic stalks are finitely generated
G-$\mathbf{Mod}(U_p)$	The category of equivariant sheaves on the formal neighborhood of p, see Definition 3.2
$\overline{g}$	$(g^{-1})^* \circ \mathcal{A}_g$
G	A discrete group, sometimes the infinite dihedral group
$\mathcal{H}ol(U_p)$	Objects in $\mathbb{Z} - \mathbf{Mod}(U_p)$ which are finitely generated over $k[z]_p$
$\mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$	Definition 4.31
$\widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$	Definition 4.33
$\widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$	Modules in $\widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$ on which τ acts as a unit
$\iota_p^{\rightarrow}$	$M \mapsto k((\tau)) \otimes_k M$
j^*	The localization that removes the point p
K_p	The ring of fractions of R_p
$\mathcal{M}$	The Mellin transform
$\mathcal{M}^{(p,q)}$	The local Mellin transform from p to q
$\mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$	Definition 4.30
$\widehat{\mathbf{Mod}}(\Delta^l_{\mathbb{A}^1})_p$	Definition 4.32

$M^\star$	Any of M^l, M^r, M^{lr}
M_p	$R_p \otimes M$ tensored over the stalk at p
R_p	The completed local ring of C at p
St_p	The stabilizer of p
St_p^*	$\{h \in \mathrm{St}_p : h\tau = \tau h\}$

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Chapter 1

Introduction

This thesis concerns the results of [13] and [14] about equivariant sheaves on curves and their local study. Equivariant sheaves can be interpreted as an algebraic counterpart to discrete equations: these include difference equations, which are linear recurrence relations of the form $y(t+1) = A(t)y(t)$ for $A(t) \in GL_n(\mathbb{C}(t))$ and y is a column vector; and q-difference equations, which take the form $y(qt) = A(t)y(t)$ for a given $q \in \mathbb{C}^\times$. The relation between equivariant sheaves and discrete equations is analogous to the relation between D-modules and differential equations.

1.1 The local type of discrete equations

Our main goal is to provide a notion for the local data of an equivariant sheaf around the formal neighborhood of a point p on a curve C (this is Definition 3.5), for the action of certain groups. We show that this definition is reasonable in that a sheaf can be recovered from its restriction to the formal neighborhood around p, its restriction to the open set $C \setminus p$ and an isomorphism between these two modules on the intersection, i.e. the punctured formal neighborhood around p.

Let us state it precisely: We are given a group G that has a finite index subgroup isomorphic to $\mathbb{Z}$, acting on a reduced curve C over a field k. We will focus on equivariant sheaves whose stalks at every generic point of the curve are finitely generated, and call the category they form G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$. The restriction $|_{U_p}$ we define lands in a category of modules on the formal neighborhood U_p with extra structure, which we will call G-$\mathbf{Mod}(U_p)$. We may also consider modules on the open set $C^* = C \setminus p$, and similarly restrict them to the (punctured) neighborhood U_p^* (see Definition 3.1). The usual pullback of quasicoherent sheaves can be enhanced in a natural way for equivariant sheaves on (the formal neighborhood in) a curve. When we include the localization functors we obtain a commutative (up to natural isomorphism) square of restrictions:

$$\begin{array}{ccc} \text{G-}\mathbf{Mod}^{\mathrm{gfg}}(C) & \xrightarrow{j^*} & \text{G-}\mathbf{Mod}^{\mathrm{gfg}}(C^*) \\ \downarrow{\scriptstyle |_{U_p}} & & \downarrow{\scriptstyle |_{U_p}} \\ \text{G-}\mathbf{Mod}(U_p) & \xrightarrow{j^*} & \text{G-}\mathbf{Mod}(U_p^*). \end{array} \tag{1.1}$$

Theorem 1.1. *The Diagram (1.1) is a cartesian square of categories.*

More explicitly, it induces an equivalence between G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$ *and the category* G-$\mathbf{Mod}(U_p) \times_{\text{G-}\mathbf{Mod}(U_p^*)}$ G-$\mathbf{Mod}^{\mathrm{gfg}}(C^*)$*. This is the category of triples* $(M_{U_p}, M_{C^*}, \cong)$*, consisting of objects* $M_{U_p} \in$ G-$\mathbf{Mod}(U_p)$*,* $M_{C^*} \in$ G-$\mathbf{Mod}^{\mathrm{gfg}}(C^*)$ *and a fixed isomorphism* $j^* M_{U_p} \cong M_{C^*}|_{U_p}$*.*

This theorem validates the definition of $|_{U_p}$ in that it ensures that at the very least no information is lost. It could be also interpreted as saying that $|_{U_p}$ provides a classification of singularities of discrete equations. It can also be thought of as analogous to the Beauville-Laszlo Theorem from [4], in the equivariant situation.

All the relevant definitions and the proof of the Theorem can be found in Chapter 3.

1.2 The local Mellin transform

If we focus on the situation where $C = \mathbb{A}^1$ and $G = \mathbb{Z}$ acting by $z \mapsto z + 1$, then equivariant sheaves are the same as modules over the noncommutative ring $\Delta_{\mathbb{A}^1} = k[z]\langle \tau^{\pm 1} \rangle$, which is defined by the relation $\tau z = (z-1)\tau$. In this case, $\mathbb{Z} \to \mathbb{G}_a$ is a generalized 1-motive in the sense of [15], and its Cartier dual is $\mathrm{Spf}\, k[[t]] \to \mathbb{G}_m$. The Fourier transform of Loc. cit. states that there is an equivalence between modules on a 1-motive and its dual: in this case, modules on $\mathbb{Z} \to \mathbb{G}_a$ are $\mathbb{Z}$-equivariant sheaves on $\mathbb{A}^1$ and modules on $\mathrm{Spf}\, k[[t]] \to \mathbb{G}_m$ are D-modules on $\mathbb{G}_m = \mathbb{A}^1 \setminus 0$. Since these groups are affine, the Fourier transform is exact, and in this case it corresponds to the ring isomorphism $\Delta_{\mathbb{A}^1} \cong k[x, x^{-1}]\langle \partial_x \rangle$ mapping $z \leftrightarrow x\partial_x$ and $\tau^{\pm 1} \leftrightarrow x^{\pm 1}$. This is called the Mellin transform, and we denote it $\mathcal{M} : D - \mathbf{Mod}(\mathbb{G}_m) \to \mathbf{Mod}(\Delta_{\mathbb{A}^1})$.

Our goal is to relate the local information of a module with that of its Mellin transform, by introducing local Mellin transforms, following [5] and [1], where the analogous results are proven for the Fourier transform. For any pair of points $p \in \overline{\mathbb{G}_m} = \mathbb{G}_m \cup \{0, \infty\}$ and $q \in \overline{\mathbb{A}^1} = \mathbb{A}^1 \cup \{\infty\}$, there should be a local Mellin transform $\mathcal{M}^{(p,q)}$ relating the the local data of a D-module N at p to the local data of $\mathcal{M}(N)$ at q. Further, one should be able to recover the "local data" of $\mathcal{M}(N)$ at q from the collection $\{\mathcal{M}^{(p,q)}(N_p)\}_{p \in \overline{\mathbb{G}_m}}$. Here we are deliberately loose with the meaning of "local data" and the notation N_p: depending on the choices of p and q, one needs to take different notions of local data to make the statements work. In the case of the Fourier transform, these are vanishing cycles for finite points and nearby cycles for ∞, see [1]. Going back, for a difference module M one shold be able to recover the local data of $\mathcal{M}^{-1}(N)$ at p from the collection $\{(\mathcal{M}^{(p,q)})^{-1}(N_q)\}_{q \in \overline{\mathbb{A}^1}}$. We will abbreviate $\mathcal{M}^{-(p,q)} = (\mathcal{M}^{(p,q)})^{-1}$.

In the case of the Mellin transform, if $p \in \mathbb{A}^1$ and $q \in \mathbb{G}_m$, then $\mathcal{M}^{(p,q)} = 0$. The case of $\mathcal{M}^{(p,\infty)}$ was shown in [10] for any $p \in \overline{\mathbb{G}_m}$ to have all the desirable properties, so the remaining case is showing the existence of $\mathcal{M}^{(0,p)}$ and $\mathcal{M}^{(\infty,p)}$ for $p \in \mathbb{A}^1$, which we do in this thesis.

In order to relate local types of D-modules at $q \in \{0, \infty\}$ with difference modules at $p \in \mathbb{A}^1$, we need to make sense of what the local type is. We define vanishing cycles for difference modules in terms of the restriction $|_{U_p}$, in simple terms starting from a module in G-$\mathbf{Mod}(U_p)$ (Definition 4.21). For every point p, there are two different vanishing cycles, which we denote Φ_p^l and Φ_p^r, and their target is the category of finite length $k[z]$-modules supported at p, which we will denote $\mathbf{Mod}(k[z]_p)_{\mathrm{fin}}$. Vanishing cycles are the right ingredients for the local Mellin transform.

These functors approximately compute a familiar notion: if a difference equation is given as a matrix difference equation $y(t+1) = A(t)y(t)$, where $A(t) \in GL_n(\mathbb{C}(t))$, then Φ_p^r computes the poles of the matrix A (i.e. the points where it is not defined), and Φ_p^l computes the zeroes (i.e. the points where A^{-1} is not defined). This is not completely true, since taking a rational isomorphism might change the set of zeroes and poles, where a rational isomorphism can be thought of as a matrix $B \in GL_n(\mathbb{C}(t))$, and performing the gauge transformation $A(t) \mapsto B(t+1)^{-1}A(t)B(t)$. Further, there is the added difficulty that one may mean different things by "determining a difference module by a matrix equation". These difficulties can be partially worked around: for example if a given $\mathbb{Z}$-orbit doesn't contain both zeroes and poles of the matrix, then zeroes and poles are computed by Φ_p applied to the intermediate extension (Construction 4.16).

On the D-module side, we use the classification of D-modules over the formal disk (originally proved by Turrittin [25] and Levelt [16], but the proof can also be found in [26]). Given a holonomic module $N \in \mathbf{Mod}(k[x^{\pm 1}]\langle\partial_x\rangle)$, its nearby cycles are defined as $\Psi_0 M = k((x)) \otimes_{k[x]} M$ and $\Psi_\infty N = k((x^{-1})) \otimes_{k[x]} N$. Now, a holonomic $k((x))\langle\partial_x\rangle$-module can be split as the direct sum of its regular part and its irregular part, and furthermore, the regular part can be split according to the leading term, which is well-defined up to adding an integer. We denote $\mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg},(p)}$ the category of those $k((x))\langle\partial_x\rangle$-modules whose leading term is in $p+\mathbb{Z}$, and we will denote $\cdot^{\mathrm{reg},(p)}$ the functor that picks out the regular part of leading term p. These theorems only hold for holonomic modules, since the notions are not defined for non-holonomic ones.

Theorem 1.2. *Suppose the base field k has characteristic* 0*.*

1) *For any $p \in \mathbb{A}^1$, there is an equivalence*

$$\mathcal{M}^{(0,p)} : \mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg},(p)} \longrightarrow \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}.$$

Further, there is a natural isomorphism

$$\Phi_p^l \circ \mathcal{M} \cong \mathcal{M}^{(0,p)} \circ \cdot^{\mathrm{reg},(p)} \circ \Psi_0 : \mathcal{H}ol(k[x^{\pm 1}]\langle\partial_x\rangle) \to \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}.$$

2) *For any $p \in \mathbb{A}^1$, there is an equivalence*

$$\mathcal{M}^{(\infty,p)} : \mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg},(-p)} \longrightarrow \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}.$$

Further, there is a natural isomorphism

$$\Phi_p^r \circ \mathcal{M} \cong \mathcal{M}^{(\infty,p)} \circ \cdot^{\mathrm{reg},(-p)} \circ \Psi_\infty : \mathcal{H}ol(k[x^{\pm 1}]\langle\partial_x\rangle) \to \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}.$$

For a D-module $N \in \mathcal{H}ol(\mathcal{D}_{K_0})$, the classification tells us that $N = \bigoplus_{p\in\mathbb{A}^1/\mathbb{Z}} N^{\mathrm{reg},(p)} \oplus N^{\mathrm{irreg}}$. The results of [10] ensure that for a difference module $N \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$, $\mathcal{M}^{-1}(N)^{\mathrm{irreg}}$ is determined by $\mathcal{M}^{-(p,\infty)}(N)$ for varying $p \in \overline{\mathbb{G}_m}$. Therefore, we conclude the following corollary.

Corollary 1.3. *Let $N \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$. Then for $p \in \mathbb{G}_m$, we have the following (*[10]*)*

$$\Phi_p(\mathcal{M}^{-1}N) = \mathcal{M}^{-(p,\infty)}\Psi_\infty N.$$

For $p = 0$ or ∞, we have ([10, 13]*):*

$$\Psi_p(\mathcal{M}^{-1}N) = \bigoplus_{q\in\mathbb{A}^1} \mathcal{M}^{-(p,q)}(\Phi_q N) \oplus \mathcal{M}^{-(p,\infty)}(\Psi_\infty N).$$

In all cases, the local type of $\mathcal{M}^{-1}(N)$ at p is determined by the local types of N at $q \in \overline{\mathbb{A}^1}$.

1.3 Elliptic equations

Symmetric elliptic difference equations are one of our main motivations to study discrete equations. They were introduced in [20] in order to give an interpretation to the elliptic Painlevé equation arising in Sakai's classification of surfaces associated to the Painlevé equations [23]. It was first shown that the classical Painlevé equations correspond to isomonodromy deformations of moduli spaces of differential equations [17], which are some of the surfaces in the classification. However, not all the surfaces in Sakai's classification arise this way. Others arise as moduli spaces of discrete equations, such as difference equations [2]. Symmetric elliptic difference equations complete this picture by providing a moduli interpretation for the elliptic Painlevé equation.

Symmetric elliptic difference equations, which we will call **elliptic equations** from here on, arise as follows: discrete equations on the line take the form $y(\tau(x)) = A(x)y(x)$ for some automorphism τ of $\mathbb{P}^1$. For an elliptic equation, the role of τ is played by a correspondence in $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. a curve $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ which we require to have degree 2 over each component and to be symmetric when the coordinates are interchanged. An elliptic equation is given by a matrix meromorphic function $A : E \to GL_n(\mathbb{C})$, and it takes the form $y(t) = A(s,t)y(s)$ whenever $(s,t) \in E$. The matrix A is required to satisfy the relation $A(s,t) = A(t,s)^{-1}$. In Chapter 5 we ellaborate on elliptic modules, the counterpart to D-modules for this setting.

In the case where E is the union of the graphs of τ and τ^{-1} for $\tau \in \mathrm{Aut}(\mathbb{P}^1), \tau^2 \neq \mathrm{Id}$, elliptic equations are τ-difference equations on $\mathbb{P}^1$. Further, if E is the nonreduced double diagonal, certain elliptic modules are equivalent to D-modules on $\mathbb{P}^1$ (Proposition 5.13). Part of the interest on elliptic equations resides on the fact that they can degenerate to all these situations.

Elliptic equations can be interpreted as equations on E rather than $\mathbb{P}^1$: the pullback $\widetilde{y}(s,t) = y(s) : E \to \mathbb{C}^n$ of a solution satisfies the equations $\widetilde{y}(s,t) = \widetilde{y}(s,t')$ and $\widetilde{y}(t,s) = A(s,t)\widetilde{y}(s,t)$. The involutions $(s,t) \mapsto (s,t')$ and $(s,t) \mapsto (t,s)$ generate a dihedral group G acting on E, and the equations satisfied by y can be thought of as describing the G-equivariance of $\widetilde{y}$. At the level of modules, we show in Proposition 5.4 that elliptic modules embed fully faithfully into the category of equivariant sheaves on E. Further, under some flatness assumptions at the singularities, we can extend the result to the normalization of $\widetilde{E}$, in Proposition 5.5. Using this comparison, we can rephrase Theorem 1.1 in the situation of elliptic equations. This is the content of Theorem 5.11.

1.4 Structure of this thesis

Chapter 2 is a summary of background material, and contains no original work. Section 2.1 introduces the definitions for equivariant sheaves, and their relation to traditional discrete equations is discussed in Section 2.2. Section 2.3 deals with the original motivation to study discrete equations from this point of view, which comes from the study of Painlevé equations and their deep connections to algebraic geometry. Next, in Section 2.4 we introduce the local type in the particular case of the affine line, and we work sout some examples that will hopefully give the reader some intuition on the topic. Finally, in Section 2.5 we introduce elliptic equations and a possible algebraic counterpart.

Chapter 3 contains the definition of the local type in the biggest generality, and the proof of Theorem 1.1.

Chapter 4 concerns equivariant sheaves for the translation on the affine line, and the local Mellin transform. Section 4.2 contains some introductory facts about equivariant sheaves on the line. Section 4.3 outlines the proof of Theorem 1.2 and the longer proofs in this section can be found in Section 4.4.

Finally, Chapter 5 deals with elliptic equations and elliptic modules. Section 5.1 details the relation between elliptic equations and equivariant sheaves on various spaces, and these results are applied in Section 5.2 to give a version of Theorem 1.1 in the setting of elliptic modules. In Section 5.3 we discuss the most degenerate elliptic equations, for curves which are not integral, and how these are related to equivariant sheaves on $\mathbb{P}^1$ and D-modules on $\mathbb{P}^1$.

Chapter 2

Background: Equivariant Sheaves

The contents of this chapter are meant as review of equivariant sheaves and their relation with linear recurrences, i.e. discrete equations. We also discuss how to understand the local type in terms of difference equations and provide illustrative examples. None of the contents of this chapter are original work.

2.1 Difference equations and equivariant sheaves

Algebraic difference equations are linear recurrences of the form $f(z+1) = A(z)f(z)$, where f is a vector function $\mathbb{C} \to \mathbb{C}^n$ and $A \in GL_n(\mathbb{C}(z))$. More generally, one could replace $\mathbb{C}$ by a Riemann surface or an algebraic curve C, and replace the automorphism $z \mapsto z+1$ by any other automorphism $\tau \in \mathrm{Aut}(C)$. One could go a small step further and consider actions by groups other than $\mathbb{Z}$. One could take any group G acting on C, and in this case a difference equation consists of a collection of matrices $A_g(z)$ for $g \in G$ whose entries are rational functions on C, satisfying the following conditions:

1. $A_1(z) = \mathrm{Id}$ for any $z \in C$
2. $A_{g_1}(g_2 z)A_{g_2}(z) = A_{g_1 g_2}(z)$ for any $g_1, g_2 \in G, z \in C$.

Then the difference equation takes the form $y(gz) = A_g(z)y(z)$ for all $g \in G$. Our goal is to study such difference equations from the point of view of geometry. For the reader familiar with the theory of D-modules, we follow the same reasoning. With this goal in mind, we would like our objects of study to be local in nature: there should be a notion of a difference equation on an open set, and difference equations defined on open covers should be susceptible to be glued. We allow difference equations to be defined on sections of any vector bundle (or quasicoherent sheaf), instead of on functions.

Definition 2.1. *Let G be a (possibly formal) group scheme acting on a scheme C by a map $\alpha : G \times C \to C$. An* ***equivariant sheaf*** *is a sheaf $M \in \mathbf{QCoh}(C)$ together with an isomorphism $\mathcal{A} : \pi_C^* M \xrightarrow{\sim} \alpha^* M$, satisfying the following two conditions:*

1. *On $G \times G \times C$, $(m \times \mathrm{Id}_C)^* \mathcal{A} = (\mathrm{Id}_G \times \alpha)^* \mathcal{A} \circ \pi_{23}^* \mathcal{A}$, where m is the multiplication on G and π_{23} is the projection onto $G \times C$ that forgets the first factor.*
2. *$(i \times \mathrm{Id})^* \mathcal{A} = \mathrm{Id}_M$, where $i \times \mathrm{Id} : C \to G \times C$ is the inclusion of the identity of G.*

Morphisms of equivariant sheaves $\mathrm{Hom}^G((M, \mathcal{A}_M), (N, \mathcal{A}_N))$ *are morphisms of sheaves* $\phi : M \to N$ *such that* $\alpha^*\phi \circ \mathcal{A}_M = \mathcal{A}_N \circ \pi_C^*\phi$.

Example 2.2. *If* C *and* G *are varieties over an algebraically closed field* k, *an equivariant structure on the trivial bundle* $\mathcal{O}^n$ *is equivalent to a difference equation such that the matrices* A_g *glue to an algebraic map* $G \times C \to M_n(k)$ *given by* $(g, z) \mapsto A_g(z)$. *An equivariant structure on* $k(C)^n$, *where* $k(C)$ *is the sheaf of rational functions of* C, *is equivalent to a difference equation* $\{A_g\}$ *that glues to a rational map* $G \times C \dashrightarrow M_n(k)$, *such that for any* $g \in G(k)$, $A_g : C \dashrightarrow M_n(k)$ *is defined on a dense open set of* C.

The same statement holds if C *is a variety over* $k = \overline{k}$ *and* G *is a discrete group.*

Proof. Suppose we have an equivariant structure on $\mathcal{O}^n$. The map $\mathcal{A}$ is the same as an algebraic map of vector bundles:

$$\widetilde{\mathcal{A}} : G \times C \times k^n \to G \times C \times k^n.$$

This map is the identity on $G \times C$ and it is linear on the fibers. By restricting to the standard basis $\{e_j\}$ of k^n, we obtain a map $A_g(z) : G \times C \to M_n(k)$ with the property that $A_g(z) = (\mathcal{A}_{(g,z)}(e_1) \cdots \mathcal{A}_{(g,z)}(e_n))$. Now, we have the following identities:

$$\begin{aligned} (m \times \mathrm{Id}_C)^*\mathcal{A} = (\mathrm{Id}_G \times \alpha)^*\mathcal{A} \circ \pi_{23}^*\mathcal{A} &\Rightarrow \mathcal{A}_{(g_1 g_2, z)} = \mathcal{A}_{(g_1, g_2 z)} \circ \mathcal{A}_{(g_2, z)} \\ (i \times \mathrm{Id})^*\mathcal{A} = \mathrm{Id}_{\mathcal{O}^n} &\Rightarrow \mathcal{A}_{(1,z)} = \mathrm{Id} \end{aligned}$$

Plugging in the basis vectors we obtain the analogous relations $A_{g_1 g_2}(z) = A_{g_1}(g_2 z) A_{g_2}(z)$ and $A_1(z) = \mathrm{Id}$. If we start with a difference equation, we can reverse this process to turn the matrix map $A_g(z) : G \times C \to M_n(k)$ into a map of vector bundles.

Suppose now that we have an equivariant structure on $k(C)^n$, we carry out the same proof replacing C by $\eta = \mathrm{Spec}\, k(C)$. We have the flat map $i : \eta \to C$ and flat base change ensures the map $\mathcal{A} : \pi_C^* i_* \mathcal{O}^n \to \alpha^* i_* \mathcal{O}^n$ is equivalent to a map $\mathcal{A}' : (\mathrm{Id}_G \times i)_* \pi_\eta^* \mathcal{O}^n \to (\mathrm{Id}_G \times i)_* \alpha|_\eta^* \mathcal{O}^n$. Since $(\mathrm{Id}_G \times i)_*$ is fully faithful, this is the same as a map $\mathcal{A}' : \pi_\eta^* \mathcal{O}^n \to \alpha|_\eta^* \mathcal{O}^n$, i.e. an equivariant structure on the trivial vector bundle on η. As before, we obtain a map $A_g(z) : G \times C \dashrightarrow M_n(k)$, and we obtain matrices with rational entries for $g \in G$ by restricting this map to $\{g\} \times C$. The rest of the reasoning is analogous.

□

We will only consider discrete groups G (as a formal scheme, $G \cong \bigsqcup_{g \in G} \mathrm{Spec}\, k$). In this case, $G \times C = \bigsqcup_{g \in G} C$, $\pi_C = \bigsqcup_g \mathrm{Id}$ and $\alpha = \bigsqcup_g g : \bigsqcup C \to C$.

Remark 2.3. *Let* G *be a discrete group acting on a scheme* C. *An equivariant sheaf consists of the data of* $M \in \mathbf{QCoh}\, M$, *together with* $\mathcal{A}_g : M \xrightarrow{\sim} g^* M$ *for every* $g \in G$. *The cocycle condition becomes the relation* $\mathcal{A}_{g_1 g_2} = g_2^* \mathcal{A}_{g_1} \circ \mathcal{A}_{g_2}$, *and the condition at the identity becomes* $\mathcal{A}_1 = \mathrm{Id}$. *A morphism of sheaves* ϕ *in this situation is a morphism of equivariant sheaves if and only if for every* $g \in G$, $\mathcal{A}_g \circ \phi = g^*\phi \circ \mathcal{A}_g$.

Given an equivariant sheaf M, we can consider for $g \in G$ the map $(g^{-1})^* \circ \mathcal{A}_g : M \to M$, which we will simply denote by $\overline{g}$. This is not a map of sheaves: rather, for every open set $U \subset C$, $\mathcal{A}_g$ maps $M(U)$ to $g^*M(U)$, and g^* identifies $g^*M(U)$ with $M(gU)$, so $\overline{g}$ maps sections on U to sections on gU. It is also not $\mathcal{O}$-linear, like $\mathcal{A}_g$ is, but rather if for a local function $f \in \mathcal{O}(U)$ we denote $f^g = f \circ g \in \mathcal{O}(g^{-1}U)$ (this is the right action of G on $\mathcal{O}$), we have the relation

$$\overline{g}(fs) = (f \circ g^{-1}) \cdot \overline{g}s = f^{g^{-1}}\overline{g}s \in M(gU).$$

We can interpret this as the relation $\overline{g}f = f^{g^{-1}}\overline{g}$, or $f\overline{g} = \overline{g}f^g$. Using this notation, the relation $\mathcal{A}_{g_1g_2} = g_2^*\mathcal{A}_{g_1} \circ \mathcal{A}_{g_2}$ becomes $\overline{(g_1g_2)} = \overline{g}_1 \circ \overline{g}_2$: note that for a morphism of sheaves ϕ, $g^*\phi = g^* \circ \phi \circ (g^{-1})^*$. Therefore,

$$\overline{g_1g_2} = (g_1^{-1})^* \circ (g_2^{-1})^* \circ \mathcal{A}_{g_2g_1} = (g_1^{-1})^* \circ (g_2^{-1})^* \circ g_2^*\mathcal{A}_{g_1} \circ \mathcal{A}_{g_2} =$$

$$= (g_1^{-1})^* \circ \mathcal{A}_{g_1} \circ (g_2^{-1})^* \circ \mathcal{A}_{g_2} = \overline{g}_1 \circ \overline{g}_2.$$

And the same reasoning shows that if $\overline{g_1g_2} = \overline{g}_1 \circ \overline{g}_2$, then the maps $\mathcal{A}_g = g^* \circ \overline{g}$ indeed define an equivariant structure on the sheaf M. Using this notation, a morphism of sheaves ϕ is a morphism of equivariant sheaves if and only if for every $g \in G$, $\overline{g}\circ\phi = \phi\circ\overline{g}$.

In particular, if G is given by generators and relations, the equivariant structure is determined by a collection of isomorphisms $\{\mathcal{A}_g : M \to g^*M\}$ for g in a generating set of G, and a collection of isomorphisms $\{\mathcal{A}_g\}$ for g in a generating set will determine an equivariant structure if and only if for every relation $g_1 \cdots g_m = 1$, the corresponding map $\overline{g}_1 \circ \cdots \circ \overline{g}_m : M \to M$ is the identity (note that since $g_1 \cdots g_m = 1$, in this case the map will be an $\mathcal{O}$-linear isomorphism of sheaves).

If the group action is not faithful, we must take care to note which group the equivariant structure is for. For instance, given an automorphism g of C such that $g^2 = \mathrm{Id}$, any isomorphism $\mathcal{A}_g : M \to g^*M$ will give rise to a $\mathbb{Z}$-equivariant structure, where $\mathbb{Z}$ is generated by g. However, to obtain a $\mathbb{Z}/2\mathbb{Z}$-equivariant structure, we must also have the relation $\mathrm{Id} = \mathcal{A}_{g^2} = g^*\mathcal{A}_g \circ \mathcal{A}_g$.

2.2 Relation to discrete equations

Linear recurrences give rise to equivariant sheaves: a linear recurrence for a group action takes the form $y(gx) = A_g(x)y(x)$ for all $g \in G$, where y is a column vector and A_g is an invertible matrix, of size r. We must have the conditions that $A_{g_1g_2}(x) = A_{g_1}(g_2x)A_{g_2}(x)$, and $A_1(x) = \mathrm{Id}$. We may construct an equivariant sheaf by interpreting the recurrence as generators and relations: start with a free $\mathcal{O}$-module F with basis $\{s_{i,g}\}$ parametrized by $1 \leq i \leq r$ and $g \in G$. Let F have the equivariant structure given by $\overline{g}s_{i,h} = s_{i,hg^{-1}}$. Let $A_g = (a_g^{ij})$. We consider the subsheaf K of F generated by the elements $\{s_{i,gh} - \sum_j a_g^{ij}s_{j,h}\}$ for all $g, h \in G$ and i. Then $M = F/K$ is the desired equivariant sheaf (notice that the equivariant structure preserves K). In the category of equivariant sheaves it correpresents the functor of solutions to the recurrence. In other words, maps from M to another equivariant sheaf N are in bijection with solutions to the system of equations inside N. This is analogous to the situation for D-modules.

For example, maps from M to any sheaf of functions (e.g. meromorphic functions) are the set of solutions to the recurrence in said sheaf. Indeed, if $s_{i,1}$ map to certain functions $f_i(x)$, then $\overline{g^{-1}}s_{i,1} = s_{i,g}$ must map to $\overline{g^{-1}}f_i(x) = f_i(gx)$. Therefore the relation $s_{i,gh} = \sum_j a_g^{ij} s_{j,h}$ implies that $f_i(ghx) = \sum_j a_g^{ij} f_j(hx)$. Conversely, any solution to the recurrence will yield a morphism of sheaves using these formulas.

2.2.1 Difference equations on the affine line

Our main example of equivariant sheaves will be for the translation action of $\mathbb{Z}$ on $\mathbb{A}^1$. Let us call this translation $\tau : z \mapsto z + 1$.

Remark 2.4. *$\mathbb{Z}$-equivariant sheaves on $\mathbb{A}^1$ are equivalent to modules over the noncommutative ring $\Delta_{\mathbb{A}^1} = \mathbb{C}[z]\langle \tau, \tau^{-1} \rangle$, where $\tau z = (z-1)\tau$.*

Proof. Since $\mathbb{A}^1$ is affine, a quasicoherent sheaf on $\mathbb{A}^1$ is the same as a module M over $k[z]$. The action of τ is given by $\overline{\tau} = (\tau^{-1})^* \circ \mathcal{A}_\tau : M \to M$. The relation $\overline{\tau}(f(z)m) = f^{\tau^{-1}}(z)\overline{\tau}m = f(z-1)\overline{\tau}m$ implies that indeed $\tau z = (z-1)\tau$. Conversely, any $\Delta_{\mathbb{A}^1}$-module M gives rise to a quasicoherent sheaf with equivariant structure $\mathcal{A}_\tau = \tau^* \circ \tau : M \to \tau^* M$. □

2.2.2 Geometric interpretations

Another geometric way to interpret equivariant sheaves on a scheme C acted on by G is as those sheaves that come from the quotient C/G. Any sheaf that is pulled back from $\pi : C \to C/G$ comes with an equivariant structure: the relation $\pi \circ g = \pi$ induces an isomorphism $\mathcal{A}_g : \pi^* \cong g^* \circ \pi^*$. Further, since the action of G is associative, the condition $\mathcal{A}_{g_1 g_2} = g_2^* \mathcal{A}_{g_1} \circ \mathcal{A}_{g_2}$ is satisfied.

When the quotient has nice properties, this construction is an equivalence between quasicoherent sheaves on the quotient and equivariant sheaves on C. Since we are not using this result for anything other than motivation, we will prove it in the topological setting. In the algebraic situation, our main example of $\mathbb{A}^1/\mathbb{Z}$ is not a scheme or even an algebraic stack, although in the analytic setting it is $\mathbb{C}^*$.

Remark 2.5. *Let X be a topological space, and let $G \to \mathrm{Aut}(X)$ be a group with a free and properly discontinuous action. Then the category of sheaves of sets on X/G is equivalent to the category of G-equivariant sheaves of sets on X, i.e. sheaves F with a isomorphisms $\mathcal{A}_g : F \to g^* F$ satisfying the conditions from Remark 2.3. The equivalence is induced by the pullback $\pi^* : \mathrm{Sh}(X/G) \to \mathrm{Sh}(X)$.*

Proof. The discussion above shows how π^* of a sheaf on X/G becomes equivariant. Let us show how to go back. Let F be an equivariant sheaf on X. Since the action is free and properly discontinous, π is a covering map, and X/G is covered by open sets U such that $\pi^{-1}(U) = \bigsqcup_i U_i$, where $\pi : U_i \cong U$. We define a sheaf F' on X/G by letting $F'|_U = \pi_* F|_{U_{i_0}}$ for any choice of i_0. The isomorphisms $\mathcal{A}_g$ identify different choices of i, so this is well-defined independently of the preimage chosen for every open set U. The glueing isomorphisms $\phi'_{UV} : F'|_U|_{U\cap V} \cong F'|_V|_{U\cap V}$ can be chosen by picking a base point p on $\pi^{-1}(U \cap V)$, and then choosing $U_0 \overset{\pi}{\cong} U$ and $V_0 \overset{\pi}{\cong} V$ such that $p \in U_0 \cap V_0$. Then, we define

$$\phi'_{UV} : F'|_U|_{U\cap V} \overset{\pi|^*_{U_0}}{\to} F|_{U_0}|_{U_0\cap V_0} \overset{\phi_{U_0V_0}}{\to} F|_{V_0}|_{U_0\cap V_0} \overset{\pi|_{V_0 *}}{\to} F'|_V|_{U\cap V}.$$

Again the equivariant structure ensures that these morphisms are well-defined. We leave the rest of the details to the reader. □

Another geometric interpretation for an equivariant sheaf is as an extension of the action of G to the total space of the quasicoherent sheaf.

Remark 2.6. *Let C be a scheme with an action α of a group G, and let V be a vector bundle on C. Then there is a bijection between equivariant structures on (the sheaf of sections of) V and actions of G on the total space $|V| = \operatorname{Spec}\operatorname{Sym}^\bullet V^\vee$, extending the action of G on C and such that they are linear on the fibers, i.e. such that G acts by maps of vector bundles.*

Proof. A map of coherent sheaves $\mathcal{A} : \pi_C^* V \to \alpha^* V$ induces a map of total spaces $|\mathcal{A}| : G \times |V| \to |\alpha^* V|$ which is linear on the fibers. For any map of schemes $f : C_1 \to C_2$ and a vector bundle V on C_2, let us denote $f^\diamondsuit : |f^* V| \to |V|$ the corresponding projection. Let $\widetilde{\alpha} = \alpha^\diamondsuit \circ |\mathcal{A}|$. We have the following commutative diagram, which ensures that $\widetilde{\alpha}$ is compatible with the projection: the left square is the bundle map $|\mathcal{A}|$, and the right square is the pullback of V by α. The composition of the top row is $\widetilde{\alpha}$.

$$\begin{array}{ccccc} G \times |V| & \xrightarrow{|\mathcal{A}|} & |\alpha^* V| & \xrightarrow{\alpha^\diamond} & |V| \\ \downarrow{\scriptstyle \pi_{G\times C}} & & \downarrow{\scriptstyle \pi_{G\times C}} & & \downarrow{\scriptstyle \pi_C} \\ G \times C & = & G \times C & \xrightarrow{\alpha} & C. \end{array}$$

To show that this is indeed an action, we need the relation $\widetilde{\alpha} \circ (m \times \mathrm{Id}_{|V|}) = \widetilde{\alpha} \circ (\mathrm{Id}_G \times \widetilde{\alpha}) : G \times G \times |V| \to |V|$. We have:

$$\begin{aligned} \mathrm{Id}_G \times \widetilde{\alpha} &= (\mathrm{Id}_G \times \alpha^\diamondsuit) \circ (\mathrm{Id}_G \times |\mathcal{A}|) = (\mathrm{Id}_G \times \alpha)^\diamondsuit \circ |\pi_{23}^* \mathcal{A}|; \\ |\mathcal{A}| \circ (\mathrm{Id}_G \times \alpha)^\diamondsuit &= (\mathrm{Id}_G \times \alpha)^\diamondsuit \circ |(\mathrm{Id}_G \times \alpha)^* \mathcal{A}|; \\ m \times \mathrm{Id}_{|V|} &= (m \times \mathrm{Id}_C)^\diamondsuit; \\ |\mathcal{A}| \circ (m \times \mathrm{Id}_{|V|}) &= |\mathcal{A}| \circ (m \times \mathrm{Id}_C)^\diamondsuit = (m \times \mathrm{Id}_C)^\diamondsuit \circ |(m \times \mathrm{Id}_C)^* \mathcal{A}|. \end{aligned}$$

The second and fourth equalities hold because for any morphism of vector bundles ϕ and any map of schemes g, $|\phi| \circ g^\diamond = g^\diamond \circ |g^*\phi|$. Putting everything together, we have the desired relation:

$$\begin{aligned}\widetilde{\alpha} \circ (m \times \mathrm{Id}_{|V|}) &= \alpha^\diamondsuit \circ |\mathcal{A}| \circ (m \times \mathrm{Id}_{|V|}) = (\alpha \circ (m \times \mathrm{Id}_C))^\diamondsuit \circ |(m \times \mathrm{Id}_C)^* \mathcal{A}| = \\ &= \alpha^\diamondsuit \circ (\mathrm{Id}_G \times \alpha)^\diamondsuit \circ |(\mathrm{Id}_G \times \alpha)^* \mathcal{A}| \circ |\pi_{23}^* \mathcal{A}| = \\ &= \alpha^\diamondsuit \circ |\mathcal{A}| \circ (\mathrm{Id}_G \times \alpha)^\diamondsuit \circ |\pi_{23}^* \mathcal{A}| = \widetilde{\alpha} \circ (\mathrm{Id}_G \times \widetilde{\alpha}).\end{aligned}$$

Further, we have that $1 \in G$ acts as $\mathrm{Id}_{|V|}$ because $\mathcal{A}_1 = \mathrm{Id}$, so $\widetilde{\alpha}$ is an action as desired. Following the proof backwards we obtain an equivariant structure $\mathcal{A}$ from an action $\widetilde{\alpha}$. □

2.3 Motivation: Painlevé equations

This section is independent from everything else. Painlevé equations are an original source of motivation to study discrete equations from the point of view of algebraic geometry, but in this thesis we don't have any results directly related to Painlevé equations. Classical Painlevé equations can be studied from three different perspectives:

1. As non-linear second order ODEs whose singularities have nice (i.e. Painlevé) properties.

2. As differential equations that arise naturally in the study of moduli spaces of connections.

3. As deformations of a specific kind of surfaces, namely Okamoto-Painlevé pairs.

We will briefly discuss all three points of view and then move on to the discrete analogues of the Painlevé equations.

The Painlevé property. The original problem studied by Painlevé was the classification of ODEs whose only **movable** singularities are poles. Let's unravel this definition: The general solution to a (let's say second order) differential equation $y''(t) = g(y', y, t)$ is a family of functions of the form $y(t) = f_c(t)$ depending on some (2-dimensional family of) parameters c. A singularity of the differential equation is then defined to be a value t such that its solution $f_c(t)$ is not holomorphic for some c.

A singularity is called **movable** if the value of t for which f_c is singular depends on c: for example, the families $f_c(t) = 1/(t-c)$ and $f_c(t) = \sqrt{t-c}$ have movable singularities at $t = c$, and the family $f_c(t) = c^3/t$ has an unmovable singularity at $t = 0$.

A differential equation is said to have the Painlevé property if the only movable singularities that appear are poles. In the 1900s Painlevé and others [9, 18] classified the second order differential equations $y'' = g(y', y, t)$ with the Painlevé property, and such that g is a rational function. There are only 6 families, which are called Painlevé equations and are usually denoted $P_{\mathrm{I}}, P_{\mathrm{II}}, P_{\mathrm{III}}, P_{\mathrm{IV}}, P_{\mathrm{V}}, P_{\mathrm{VI}}$.

A geometric approach to the Painlevé property. Consider one of the second order Painlevé equations, written as $(x, y)'(t) = f(x, y, t)$, and let $B \subseteq \mathbb{C}$ be the complement of the unmovable singularities ($\mathbb{C} \setminus B$ is a finite set). In principle, we view the solutions as meromorphic maps $(x, y) : B \dashrightarrow \mathbb{C}^2$.

In other words, solutions are (meromorphic, non-algebraic) sections of the trivial bundle $\mathbb{C}^2 \times B \to B$, and the image of these sections are the trajectories for these equations. If they didn't have poles, they would form a foliation of $\mathbb{C}^2 \times B$.

In the 1970s, Okamoto [17] made the following construction: to account for the poles, we compactify the image of (x, y), so that they become functions $B \to \mathbb{CP}^2$, which are well-defined because the singularities of (x, y) are only poles. After doing this, a new issue arises: on $\mathbb{C}^2$, there's a unique trajectory through each point, but on the points at infinity many trajectories come together. This is solved by replacing $\mathbb{P}^2$ by a surface Σ which is an open set in a blowup of $\mathbb{P}^2$.

Okamoto proved the following: for each of the Painlevé equations one can construct a (smooth, rational, non-compact) surface $\Sigma \supseteq \mathbb{C}^2$ and on the fiber bundle $\Sigma \times B \to B$ there's a foliation $\mathcal{F}$. By a foliation we mean that $\Sigma \times B$ is the disjoint union of curves (called leaves of the foliation), in such a way that locally it's isomorphic to the trivial foliation $\mathbb{C}^3 \cong \mathbb{C}^2 \times \mathbb{C}$. It has the following properties:

- When restricted to $\mathbb{C}^2 \times \mathbb{C} \subseteq \Sigma \times \mathbb{C}$, the leaves of the foliation are the trajectories of the solutions to the Painlevé equation.

- The projection restricted to any leaf $\mathcal{F}_0 \subseteq \Sigma \times B \to B$ is a covering space of B. In particular, solutions can be extended uniquely along any path.

These surfaces are called the spaces of initial conditions for Painlevé equations.

Painlevé equations as moduli spaces of connections. There is a natural way in which Painlevé equations appear in algebraic geometry, which is as isomonodromy deformations for moduli spaces of connections. We will take the example of P_{VI}, the sixth Painlevé equation.

We look at the following connections. Consider four points $x_1, \dots, x_4 \in \mathbb{P}^1$, and four generic parameters $\lambda_1, \dots, \lambda_4$. We consider connections on a rank 2 bundle L on $\mathbb{P}^1$, with simple poles at $x_1, \dots, x_4$, i.e. maps $L \to L \otimes_{\mathcal{O}} \Omega(x_1 + \cdots + x_4)$ satisfying the Leibniz rule. Further, we require that the residue at x_i, which is an endomorphism of L_{x_i}, have eingenvalues $\{\lambda_i, -\lambda_i\}$. Finally, we add the additional data of an isomorphism $(\wedge^2 L, \nabla) \cong (\mathcal{O}, d)$.

The (coarse) moduli space of these connections is a rational surface, namely an open set in a blowup of $\mathbb{P}^2$. If we make $(x_1, x_2, x_3, x_4) = (0, 1, \infty, t)$, we get (for every choice of λ_i's) a family of surfaces $\{\Sigma_t\}$ parametrized by $\mathbb{P}^1_t \setminus \{0, 1, \infty\}$. This is the family of surfaces $\Sigma \times B$ from before. The surfaces Σ_t are not algebraically isomorphic to each other, though they are biholomorphic. Let us call this family $\widetilde{\Sigma} \to \mathbb{P}^1 \setminus \{0, 1, \infty\}$.

When we see $\widetilde{\Sigma}$ as an algebraic variety, this differential equation, which is defined using transcendental tools (since the Riemann-Hilbert correspondence essentially consists on taking solutions to differential equations) is in fact algebraic, and it has a very explicit formula, namely Schlesinger's equations.

Okamoto-Painlevé pairs: Painlevé property as an intrinsic property of a surface. In [22] it was shown how Painlevé equations have an origin purely from the geometry of surfaces. They consider what they call **Okamoto-Painlevé pairs**. An Okamoto-Painlevé pair is a pair (S, Y) consisting of a complex smooth projective surface S together with an effective divisor Y on S satisfying the following conditions. If we let $Y = \sum a_i Y_i$, then

1. There exists a nowhere vanishing 2-form ω on S whose divisor of poles is Y.
2. For all i, $Y \cdot Y_i = 0$.
3. The surface $S \setminus Y$ contains $\mathbb{C}^2$ as a Zariski open set. Further, the complement of this $\mathbb{C}^2$ is a divisor with normal crossings.

All the spaces Σ of initial conditions for Painlevé pairs give rise to Okamoto-Painlevé pairs: every Σ has a unique nowhere vanishing 2-form ω, and Y is taken to be (negative) the divisor of ω on S, the smallest compactification of Σ.

Appart from classifying such pairs ([21]), it was shown that they can be used to obtain Painlevé equations. In broad strokes, the process is as follows: one studies deformations of the compact surface S, together with the divisor Y. Among these, they focus on the deformations which are trivial when restricted to $\Sigma = S \setminus Y$, and they show that a canonical choice of such a connection can be made. The result is a 1-dimensional (algebraic) family of surfaces $\widetilde{S}$ over a base B, whose fibers are Okamoto-Painlevé pairs $(S_t, Y_t)_{t \in B}$. The open subfamily $(\Sigma_t)_{t \in B}$ is locally trivial, so it carries a connection that coincides with one of the Painlevé equations. Since $\widetilde{\Sigma}$ is locally biholomorphic to $\Sigma \times B \to B$, the connection can be seen as the canonical lift of a vector field $\partial/\partial t$ to $\Sigma \times B$.

2.3.1 Discrete Painlevé equations

Painlevé differential equations have many desirable properties from the point of view of dynamical systems, so in this field it was a natural question to seek an analogous property to the Painlevé property for discrete equations, such as second order difference equations (i.e. of the form $(x, y)(t+1) = g(x(t), y(t), t)$ for a rational function g) and q-equations, which take the form $(x, y)(qt) = g(x(t), y(t), t)$, where q is a fixed parameter.

Many examples of equations with nice behavior have been found (see [11] for a summary), and they were found to degenerate to differential Painlevé equations when one takes the appropriate limit. Also, there are some possible definitions for what the Painlevé property should be for discrete dynamical systems [12].

Instead of tackling the dynamical systems story, we will focus on how discrete Painlevé equations arise from geometry.

Discrete Painlevé equations arising from isomonodromic transformations of connections. We will focus on the same example of a moduli space of connections from before. Consider the set of parameters $\theta = (x_1, x_2, x_3, x_4, \lambda_1, \lambda_2, \lambda_3, \lambda_4)$, and the corresponding moduli space Σ_θ. In [3], one can find a list of all isomorphisms between Σ_θ's for varying θ's. One example comes about by letting $\mathbb{Z}^4$ act on $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$.

The essential construction is the elementary modification of a vector bundle at a point p on a smooth curve C. By a modification of a vector bundle V at p we mean a vector bundle V' which is isomorphic to V away from p. Due to the ampleness of the line bundle $\mathcal{O}(p)$, such a vector bundle will always satisfy $V(-Np) \subseteq V' \subseteq V(Np)$ for big enough N. An elementary upper (resp. lower) modification is one such that $V \subset V'$ (resp. $V' \subset V$) and $V' \setminus V$ (resp. $V' \setminus V$) is a length one skyscraper sheaf.

This procedure can be done on a formal germ of the curve. On $\operatorname{Spec} \mathbb{C}[[t]]$, the connections we are studying split as a direct sum of line bundles, so we will look at those for computations. We are studying the line bundle $\mathbb{C}[[t]] \cdot s$, with the connection $\nabla s = t^{-1}\lambda s dt$ for some eigenvalue λ. After a modification, for example considering the bundle $\mathbb{C}[[t]]ts \subset \mathbb{C}[[t]]s$, the connection becomes

$$\nabla(ts) = sdt + t\nabla s = sdt + \lambda sdt = t^{-1}(\lambda + 1)(ts)dt$$

So the eigenvalue shifts by 1 after a modification. This phenomenon applied to a connection as above yields isomorphisms $M_\theta \to M_{\theta'}$, whenever $(\lambda_1, \ldots, \lambda_4) - (\lambda'_1, \ldots, \lambda'_4)$ lies in $\frac{1}{2}\mathbb{Z}^4$.

Recall that M_θ are all rational surfaces, so choosing some rational coordinates (x, y) the isomorphisms can be seen as birational automorphisms of $\mathbb{P}^2$. These automorphisms are an example of a a discrete Painlevé equation.

Discrete Painlevé equations from surface theory. Viewing a discrete dynamical system as a birational map $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, one may ask if these maps can be studied from the point of view of surfaces alone [23]. The surfaces to be studied are slight generalizations of Okamoto-Painlevé pairs: out of the definition above we remove condition 3, but the surface is still required to be rational and Y has to be unique. All the surfaces that appear are blowups of $\mathbb{P}^2$ at 9 points (which may be infinitesimally close). The choice of points can be thought of as roughly a parameter space for the surfaces.

In [23], the groups of automorphisms of $\mathrm{Pic}(S)$ for these surfaces S are studied, and more concretely, the subgroup fixing the intersection pairing, the canonical class and the effective cone. This subgroup is called the Cremona group $\mathrm{Cr}(S)$. The group $\mathrm{Cr}(S)$ is shown to act on the family of surfaces, by mapping a surface to an isomorphic one for a new set of parameters in $(\mathbb{P}^2)^9$. These actions on the surfaces are birational automorphisms of $\mathbb{P}^2$, and they are the discrete Painlevé equations.

In the end, one obtains all these discrete equations as birational transformations $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, which become isomorphisms after blowing up both $\mathbb{P}^2$'s at a set of 9 points (which could be different sets of points for the domain and the image).

Discrete Painlevé equations as isomonodromy deformations of discrete connections. Sakai's classification includes the surfaces coming from Painlevé equations, and the example of isomonodromy deformations of moduli of connections. However, there are more types of surfaces in the classification which do not come from moduli spaces of connections. However, they do have a moduli interpretation, and it's as isomonodromy[1] deformations of discrete equations, such as difference equations, q-equations and elliptic equations. In Sakai's classification, some surfaces denegerate into others, and the most generic surface was shown to be a moduli space of elliptic equations, see [20] and the reference therein by Arinkin, Borodin and Rains.

An example of this behavior is dPV of [2], which concerns difference equations. We start with some parameters $\theta = (a_1, a_2, a_3, a_4; \rho_1, \rho_2, d_1, d_2)$. We are looking for difference equations $f(z+1) = A(z)f(z)$, where A is a 2-by-2 matrix, with the following properties:

- The matrix $A(z)$ has simple zeroes at each a_i, and it has no zeroes or poles anywhere else. A zero (resp. pole) is understood to be a point a where $A(a)^{-1}$ (resp. $A(a)$) is not defined, and it's simple if $A(a)$ (resp. $A(a)^{-1}$) is defined and has rank 1.
- Around ∞, on $\mathrm{Spec}\,\mathbb{C}((z^{-1}))$, there is a basis in which $A(z)$ has the form

$$A(z) = \begin{pmatrix} \rho_1(z^2 + d_1 z) & 0 \\ 0 & \rho_2(z^2 + d_2 z) \end{pmatrix}$$

The moduli space of such objects (called d-connections) is a rational surface M_θ. As in the case of moduli spaces of connections, there are discrete changes in the parameters that give isomorphic moduli spaces. Also as before, the isomorphisms are given by elementary modifications of the vector bundles involved. This time, the change in the parameters amounts to shifting the a_i's by integers.

[1]The use of the word "monodromy" here seems to imply we have a well-established notion of what monodromy of a difference equation is, but we do not know of a universally accepted one.

The isomorphisms between the surfaces are again the Painlevé equations of Sakai. The example dPV here happens to be the same family of surfaces as the moduli spaces of connections that give rise to the Painlevé equation P_{VI}, which we have described above. The moduli problems can be shown to be equal via the Mellin transform. Note that this requires understanding how singularities of a connection result in singularities of its Mellin transform. In [2], there is another similar moduli space of connections, which was known before as a discrete Painlevé equation, but does not appear as a moduli of connections.

2.4 Singularities of discrete equations

In this section we will discuss what singularities of discrete equations are and how Definition 3.5 makes this notion rigorous. We will focus on the case of $\mathbb{Z}$ acting on $\mathbb{A}^1_{\mathbb{C}}$, since it is illustrative of the most important behaviors. Suppose we are given a matrix difference equation with $A(z) \in GL_n(\mathbb{C}(z))$:

$$y(z+1) = A(z)y(z).$$

The intuitive notion of singularities comes from looking at A: we might call **poles** the complex points z for which $A(z)$ is not defined, and similarly call **zeroes** the complex points z for which $A(z)^{-1}$ is not defined.

First of all, we notice that zeroes and poles can shift by an integer after a change of basis (or a gauge transformation, as it is called in the context of differential equations). Note that a change of coordinates $y(z) = B(z)\widetilde{y}(z)$ amounts to making the gauge transformation $A(z) \mapsto B(z+1)^{-1}A(z)B(z)$. For example, we can start with the first order equation $y(z+1) = zy(z)$, which has a zero at $z = 0$. After making the change of variables $\widetilde{y}(z) = zy(z)$, we end with the equation $\widetilde{y}(z+1) = (z+1)\widetilde{y}(z)$, which has a zero at $z = -1$.

This notion of zeroes and poles has some problems. The first is that shifting poles and zeroes may cancel each other, resulting in apparent singularities. For example: consider the equation $y(z+1) = \frac{z}{z-1}y(z)$. According to our definition, it has a zero at $z = 0$ and a pole at $z = 1$. However, the change of variables $\widetilde{y}(z) = \frac{1}{z-1}y(z)$ turns it into the equation $\widetilde{y}(z+1) = \widetilde{y}(z)$. Our construction for the local type explains precisely in what nontrivial ways zeroes and poles can interact.

We will present here the construction in this case of the affine line. This is a particular case of Definition 3.5.

Definition 2.7. *Let $M \in \mathbf{Mod}(\Delta_{\mathbb{A}^1})$ and $p \in \mathbb{C}$, such that $\mathbb{C}(z) \otimes_{\mathbb{C}[z]} M$ is finite dimensional. The local type of M at p is defined as follows. Consider inside of M a finitely generated $\mathbb{C}[z]$-module L such that M/L is torsion. Then the local type $M|_{U_p}$ is defined to be the following triple of $\mathbb{C}[[z]]$-modules:*

$$M|^l_{U_p} := \mathbb{C}[[z-p]] \otimes (\tau^n L) \subseteq \mathbb{C}[[z-p]] \otimes_{\mathbb{C}[z]} M \supseteq \mathbb{C}[[z-p]] \otimes (\tau^{-n} L) =: M|^r_{U_p}.$$

Where n is taken to be a very large natural number. We denote this triple $M|_{U_p} = (M|^l_{U_p}, M_p, M|^r_{U_p})$.

This definition has no ambiguity and it is indeed a functor, as we will show in Proposition 3.10. We will now show in several examples that this captures the desired behavior. Also, this definition enables us to define vanishing cycles.

Definition 2.8. *Let $M \in \mathbf{Mod}(\Delta_{\mathbb{A}^1})$ and $p \in \mathbb{C}$, such that $\mathbb{C}(z) \otimes_{\mathbb{C}[z]} M$ is finite dimensional. The (left and right) vanishing cycles of M at p are defined as*

$$\Phi^l_p M = \frac{\mathbb{C}[[z-p]] \otimes M}{M|^l_{U_p}}; \quad \Phi^r_p M = \frac{\mathbb{C}[[z-p]] \otimes M}{M|^r_{U_p}}.$$

It is true that $\Phi^l_p M = 0$ whenever M has no zeroes in any reasonable sense of the word: see Proposition 4.28.

Example 2.9. *Consider the difference module given by generators and relations as $M = \langle s : (z-1)\tau s = s \rangle$. The relation implies that $\tau^{-1}s = zs$, so we have that*

$$M = \mathbb{C}[z] \cdot \langle \tau^n s : n \in \mathbb{Z} \rangle = \mathbb{C}[z] \cdot \langle \tau^n s : n \in \mathbb{Z}_{\geq 0} \rangle.$$

Further, M embeds into $\mathbb{C}(z) \cdot s$, by mapping

$$\tau^n s \mapsto \frac{1}{(z-1)\cdots(z-n)} s; \quad \tau^{-n} s \mapsto z(z+1)\cdots(z+n-1)s.$$

Let us compute the local type at $p = 0$. We can compute $M|_{U_0}$ by taking $L = \mathbb{C}[z] \cdot s$: we see that $(\tau^n L)_0$ is the same for all $n \geq 0$ and for all $n < 0$:

$$M_0 = \mathbb{C}[[z]] \cdot s; \quad M|^l_{U_0} = \mathbb{C}[[z]] \cdot s; \quad M|^r_{U_0} = \mathbb{C}[[z]] \cdot zs \subsetneq M_0.$$

Therefore, we see that $\Phi^r_p M \cong \mathbb{C}[[z]]/z\mathbb{C}[[z]] \neq 0$. This is indicative of the pole present in the original equation.

Example 2.10. *Take $M = \langle s : (z-1)\tau s = z\tau s \rangle$. This module embeds into $\mathbb{C}(z) \cdot s$ by*

$$\tau^n s \mapsto \frac{z}{z-n} s; \quad \tau^{-n} s \mapsto \frac{z}{z+n} s.$$

. Carrying out the computation yields the following:

$$M_0 = \mathbb{C}[[z]] \cdot s; \quad M|^l_{U_0} = \mathbb{C}[[z]] \cdot zs; \quad M|^r_{U_0} = \mathbb{C}[[z]] \cdot zs.$$

In this case both vanishing cycles are nonzero, reflecting the fact that this equation has a zero and a pole. However, this module contains a module with no singularities, namely the module generated by zs, and we can see that the triple $M|_{U_0}$ contains the smaller triple $(\mathbb{C}[[z]] \cdot zs, \mathbb{C}[[z]] \cdot zs, \mathbb{C}[[z]] \cdot zs)$, which has no singularities.

Example 2.11. *Torsion modules have singularities. For example,* $M = \langle s : zs = 0 \rangle$ *has local data given by* $M|_{U_0} = (0, \mathbb{C}[[z]]s/z\mathbb{C}[[z]]s, 0)$.

Example 2.12. *In this module, the zeroes and poles interact nontrivially. Let*

$$M = \langle s_1, s_2 : \tau s_1 = s_1 + s_2, z s_2 = 0 \rangle.$$

As a $\mathbb{C}[z]$*-module,* $M \cong \mathbb{C}[z] \cdot s_1 \oplus \mathbb{C}[z]/(z) \cdot s_2$. *This module is a nonsplit extension of two simple modules:*

$$0 \longrightarrow \langle s : zs = 0 \rangle \xrightarrow{s \mapsto s_2} M \xrightarrow[s_2 \mapsto 0]{s_1 \mapsto s'} \langle s' : \tau s' = s' \rangle \longrightarrow 0$$

We have that for $n > 0$

$$\tau^n s_1 = s_1 + s_2 + \tau s_2 + \cdots \tau^{n-1} s_2; \quad \tau^{-n} s_1 = s_1 + \tau^{-1} s_2 + \cdots \tau^{-n} s_2.$$

Since $\tau^n s_2$ *is supported on* $z = n$, *when we take the stalk at* 0 *all these terms vanish unless* $n = 0$. *Therefore the local type is*

$$M_0 = \mathbb{C}[[z]] \cdot s_1 \oplus \frac{\mathbb{C}[[z]]}{(z)} \cdot s_2; \quad M|^l_{U_0} = \mathbb{C}[[z]] \cdot (s_1 + s_2); \quad M|^r_{U_0} = \mathbb{C}[[z]] \cdot s_1.$$

Now we can see that there are both zeroes and poles. However, $M|^l_{U_0} \neq M|^r_{U_0}$ *in this case, and this local type is a non-split extension of* $(\mathbb{C}[[z]], \mathbb{C}[[z]], \mathbb{C}[[z]])$ *by* $(0, \frac{\mathbb{C}[[z]]}{(z)}, 0)$, *so the local type is able to remember these special behaviors.*

2.5 Elliptic equations

Symmetric elliptic difference equations are our main motivation to study discrete equations. They were introduced in [20] in order to give an interpretation to the elliptic Painlevé equation arising in Sakai's classification of surfaces associated to the Painlevé equations [23].

Symmetric elliptic difference equations, which we will call **elliptic equations** for the remainder of this thesis, arise as follows: discrete equations on the line take the form $y(\tau(x)) = A(x)y(x)$ for some automorphism τ of $\mathbb{P}^1$. For an elliptic equation, the role of τ is played by a correspondence in $\mathbb{P}^1 \times \mathbb{P}^1$, i.e. a curve $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ which we require to have degree 2 over each component and to be symmetric when the coordinates are interchanged. An elliptic equation is given by a matrix meromorphic function $A : E \to GL_n(\mathbb{C})$, and it takes the form $y(t) = A(s,t)y(s)$ whenever $(s,t) \in E$. The matrix A is required to satisfy the relation $A(s,t) = A(t,s)^{-1}$.

We will define elliptic modules as an algebraic counterpart to elliptic equations. There seem to be more than one natural definition for them, and they are not equivalent but we can describe their relation. See Remark 5.10.

In the case where E is the union of the graphs of τ and τ^{-1} for $\tau \in \mathrm{Aut}(\mathbb{P}^1), \tau^2 \neq \mathrm{Id}$, elliptic equations are τ-difference equations on $\mathbb{P}^1$. Further, if E is the nonreduced double diagonal, certain elliptic modules become equivalent to D-modules on $\mathbb{P}^1$ (Proposition 5.13). Part of the interest on elliptic equations resides on the fact that they can degenerate to all these situations.

Elliptic equations can be interpreted as equations on E rather than $\mathbb{P}^1$: the pullback $\widetilde{y}(s,t) = y(s) : E \to \mathbb{C}^n$ of a solution satisfies the equations $\widetilde{y}(s,t) = \widetilde{y}(s,t')$ and $\widetilde{y}(t,s) = A(s,t)\widetilde{y}(s,t)$. The involutions $(s,t) \mapsto (s,t')$ and $(s,t) \mapsto (t,s)$ generate a dihedral group G acting on E, and the equations satisfied by y can be thought of as describing the G-equivariance of $\widetilde{y}$. At the level of modules, we show in Proposition 5.4 that elliptic modules embed fully faithfully into the category of equivariant sheaves on E. Further, under some flatness assumptions at the singularities, we can extend the result to the normalization of $\widetilde{E}$, in Proposition 5.5. Using this comparison, we can rephrase Theorem 1.1 in the situation of elliptic equations. This is the content of Theorem 5.11.

We define elliptic modules to capture the equations on $\mathbb{P}^1$ of the form $f(y) = A(x,y)f(x)$ for $(x,y) \in E$. For discrete equations, the relations induced by τ and τ^{-1} must be the same. Similarly, for elliptic equations we must have $A(x,y) = A(y,x)^{-1}$.

Definition 2.13. *Let $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a degree $(2,2)$ symmetric curve. Let $\pi_1, \pi_2 : E \to \mathbb{P}^1$ be the projections and let $\sigma : E \to E$ be the automorphisms interchanging the factors. We assume that the projections π_i are finite, i.e. E has no vertical components.*

*An **elliptic module**, or **E-module** for short, is a quasicoherent sheaf M on $\mathbb{P}^1$, together with an isomorphism $\mathcal{A} : \pi_1^*M \to \pi_2^*M$, subject to the condition that $\sigma^*\mathcal{A} = \mathcal{A}^{-1}$.*

We denote the category of E-modules as E-**Mod**. *A morphism $\phi \in \mathrm{Hom}_{\text{E-}\mathbf{Mod}}(M,N)$ of E-modules is a morphism ϕ of sheaves on $\mathbb{P}^1$ such that $\mathcal{A} \circ \pi_1^*\phi = \pi_2^*\phi \circ \mathcal{A}$.*

At the level of stalks, $\mathcal{A}$ is an isomorphism $\mathcal{A}_{x,y} : M_x \to M_y$ whenever $(x,y) \in E$, and $\mathcal{A}_{x,y} = \mathcal{A}_{y,x}^{-1}$. These should properly be called **symmetric elliptic difference modules**. Elliptic difference modules are sheaves on an elliptic curve E equivariant under the translation by a specified point on E. In our situation, if we choose the origin of E to be a ramified point of π_1, then π_1 identifies every point on E with its opposite according to the group law of E. Since (symmetric) elliptic modules come from $\mathbb{P}^1$, their stalks at both points on the fibers of π_1 are identified, hence the name symmetric. A precise statement is provided by Proposition 5.4.

Chapter 3

The local type

We will define the local type of an equivariant sheaf at a point and we will show Theorem 1.1.

3.1 Notation

In this chapter we work over an arbitrary field k. Unless specified otherwise, when we talk about a point in a scheme, we will always mean a schematic point, i.e. a prime ideal, and not a field-valued point. We consider the action of a group G that is an extension of a finite group by $\mathbb{Z}$. Note that this includes all groups G containing subgroups $H_1 \triangleleft H_2 \triangleleft G$ such that H_1 and G/H_2 are finite and $H_2/H_1 \cong \mathbb{Z}$: the projection $H_2 \to \mathbb{Z}$ necessarily has a section, so $\mathbb{Z}$ is a finite index subgroup of G, and some finite index subgroup of this $\mathbb{Z}$ is a normal subgroup of G. Throughout, we will let τ be a chosen generator of a normal finite index subgroup isomorphic to $\mathbb{Z}$.

Throughout we will let C be a (possibly singular, possibly reducible) reduced quasi-projective curve over k with an action of G, with the notation of Section 2.1. We will study G-equivariant quasicoherent sheaves on C. We will say a sheaf M is generically finitely generated if the stalks at every generic point of C are finitely generated, or equivalently if it contains a coherent sheaf L such that M/L is supported on closed points. We denote the category of equivariant sheaves by G-$\mathbf{Mod}(C)$, and the full subcategory of generically finitely generated elliptic modules by G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$.

3.2 Definitions

We will let $p \in C$ be a closed point, and $\mathrm{St}_p < G$ be its stabilizer (the stabilizer of the closed point, i.e. of the corresponding ideal). Depending on whether St_p contains an infinite order element, St_p is either finite or it has finite index. We let $\mathrm{St}_p^* = \{h \in \mathrm{St}_p : h\tau h^{-1} = \tau\}$. Note that either $\mathrm{St}_p^* = \mathrm{St}_p$ or it is a subgroup of index 2. Throughout, we distinguish further into three cases:

(i) St_p is finite and $\mathrm{St}_p^* = \mathrm{St}_p$.

(ii) St_p is finite and $\mathrm{St}_p^* \neq \mathrm{St}_p$.

(iii) St_p is finite index.

Note that in situations (i) and (ii), p must be a smooth point, as it has an infinite orbit.

Definition 3.1. *We let $C^* = C \setminus Gp$. $\text{G-}\mathbf{Mod}(C^*)$ is defined as the full subcategory of $\text{G-}\mathbf{Mod}(C)$ on which functions vanishing only on Gp act as units, or equivalently as the category of G-equivariant quasicoherent sheaves on $C \setminus Gp$. The full subcategory of generically finitely generated modules is denoted $\text{G-}\mathbf{Mod}^{\text{gfg}}(C^*)$. We denote the forgetful functor $j_* : \text{G-}\mathbf{Mod}(C^*) \to \text{G-}\mathbf{Mod}(C)$, and we use the same notation for its restriction $\text{G-}\mathbf{Mod}^{\text{gfg}}(C^*) \to \text{G-}\mathbf{Mod}^{\text{gfg}}(C)$.*

The pushforward j_* has a right adjoint j^*, which is given by pullback of quasicoherent sheaves to C^*, endowed with the natural G-action.

In what follows, we will let R_p be the complete local ring at p, a local ring of dimension 1. We will let K_p be its total ring of fractions, i.e. the direct sum of the function fields of its minimal primes. If R_p is a domain, for example if p is smooth, then K_p is the fraction field of R_p. Note that St_p acts on $\text{Spec}\, R_p$ by restricting the action on C, so we may talk of St_p-equivariant modules on R_p.

Definition 3.2. *The category of local types of equivariant sheaves is defined as follows, in cases (i), (ii) and (iii) above:*

(i) $\text{G-}\mathbf{Mod}(U_p)$ *is the category of R_p-modules M, together with the additional information of two finite rank free submodules $M^l, M^r \subseteq M$, such that $M/M^l, M/M^r$ are supported at p. Additionally, M is St_p-equivariant, and the action of St_p preserves M^l and M^r. Morphisms in $\text{G-}\mathbf{Mod}(U_p)$ are morphisms of equivariant R_p-modules which preserve the chosen submodules.*

(ii) $\text{G-}\mathbf{Mod}(U_p)$ *is the category of R_p-modules M, together with a single finite rank free submodule M^{lr}, as above, such that M/M^{lr} is supported at p. Additionally, M is St_p-equivariant, and the action of St_p preserves M^{lr}. Morphisms are defined analogously.*

(iii) $\text{G-}\mathbf{Mod}(U_p)$ *is the category of St_p-equivariant R_p-modules.*

We will often write $M^\star$ to denote either one of M^l, M^r or M^{lr} to avoid repetition.

Remark 3.3. *In cases (i) and (ii), $\text{G-}\mathbf{Mod}(U_p)$ is not an abelian category, because not all morphisms have cokernels: a map $\phi : M \to N$ could have the property that $N^l/(\phi M^l)$ is not free, or the map into $N/\phi(M)$ might not be injective. However, it is an exact category, because it is a full subcategory of an abelian category which is closed under extensions, namely the category of diagrams of St_p-equivariant R_p modules with no restrictions about the arrows being injective or the modules being free. In particular, short exact sequences in $\text{G-}\mathbf{Mod}(U_p)$ are short exact sequences of R_p-modules $M_1 \to M_2 \to M_3$ for which all the sequences of the form $M_1^\star \to M_2^\star \to M_3^\star$ are exact; and whenever kernels or cokernels exist, they can be computed in the larger category of diagrams.*

Definition 3.4. *We define the categories of punctured local types of equivariant sheaves as the full subcategory* $\text{G-}\mathbf{Mod}(U_p^*) \subset \text{G-}\mathbf{Mod}(U_p)$ *consisting of modules* M *such that* $K_p \otimes_{R_p} M \cong M$. *The forgetful functor will be denoted* j_*.

The left adjoint to j_* *is denoted by* j^*, *and it is given by* $j^*M = K_p \otimes_{R_p} M$, *with* $(j^*M)^\star$ *defined to be the image of* $M^\star$ *inside of* j^*M.

We now define the restriction to the formal disk. From now on, we will denote $M_p = R_p \otimes M$ (where the tensor is over the stalk of $\mathcal{O}$ at p).

Definition 3.5. *The restriction to the formal disk is defined in the following ways:*

(i),(ii) *Let* $M \in \text{G-}\mathbf{Mod}^{\text{gfg}}(C)$. *Choose (arbitrarily) some coherent sheaf* $L \subseteq M$ *such that* M/L *is supported on closed points. We define* $M|_{U_p} \in \text{G-}\mathbf{Mod}(U_p)$ *by* $M|_{U_p} = M_p$. *In case (i) we make* $M|^l_{U_p} = (\overline{\tau}^n L)_p$ *for* $n \gg 0$, *and* $M|^r_{U_p} = (\overline{\tau}^{-n} L)_p$ *for* $n \gg 0$, *and in case (ii) we let* $M|^{lr}_{U_p} = (\overline{\tau}^n L)_p$ *for* $n \gg 0$. *Then* St_p *acts on* $M|_{U_p}$ *via the restriction.*

(iii) *The restriction* $|_{U_p} : \text{G-}\mathbf{Mod}(C) \to \text{G-}\mathbf{Mod}(U_p)$ *consists of making* $M|_{U_p} = M_p$ *and restricting the action of* St_p.

In Proposition 3.10 we show that the restriction is well-defined independently of choices. Note that in cases (i) and (ii) $|_{U_p}$ is defined on generically finitely generated modules, while in case (iii) we can extend the definition to all modules. From now on, we will refer to $\text{G-}\mathbf{Mod}^{\text{gfg}}(C)$ in all three cases. All our proofs will extend to arbitrary modules in case (iii).

Observation 3.6. *Note that* $|_{U_p}$ *maps* $\text{G-}\mathbf{Mod}^{\text{gfg}}(C^*)$ *into* $\text{G-}\mathbf{Mod}(U_p^*)$. *Further, the following square commutes (up to natural isomorphism).*

$$\begin{array}{ccc} \text{G-}\mathbf{Mod}^{\text{gfg}}(C) & \xrightarrow{j^*} & \text{G-}\mathbf{Mod}^{\text{gfg}}(C^*) \\ \big\downarrow |_{U_p} & & \big\downarrow |_{U_p} \\ \text{G-}\mathbf{Mod}(U_p) & \xrightarrow{j^*} & \text{G-}\mathbf{Mod}(U_p^*) \end{array} \tag{3.1}$$

Theorem 3.7. *[Theorem 1.1] The diagram (3.1) is a cartesian square of categories.*

More explicitly, it induces an equivalence between $\text{G-}\mathbf{Mod}^{\text{gfg}}(C)$ *and the category* $\text{G-}\mathbf{Mod}(U_p) \times_{\text{G-}\mathbf{Mod}(U_p^*)} \text{G-}\mathbf{Mod}^{\text{gfg}}(C^*)$. *This is the category of triples* $(M_{U_p}, M_{C^*}, \cong)$, *consisting of objects* $M_{U_p} \in \text{G-}\mathbf{Mod}(U_p)$, $M_{C^*} \in \text{G-}\mathbf{Mod}^{\text{gfg}}(C^*)$ *and a fixed isomorphism* $j^*M_{U_p} \cong M_{C^*}|_{U_p}$. *A morphism between two triples* $f : (M_{U_p}, M_{C^*}, \cong) \to (N_{U_p}, N_{C^*}, \cong)$ *is a pair of morphisms* $f_{U_p} : M_{U_p} \to N_{U_p}$ *and* $f_{C^*} : M_{C^*} \to N_{C^*}$ *that commutes with the isomorphims.*

Let us denote $\mathcal{C} = \text{G-}\mathbf{Mod}(U_p) \times_{\text{G-}\mathbf{Mod}(U_p^*)} \text{G-}\mathbf{Mod}^{\text{gfg}}(C^*)$, and let Φ be the induced functor from $\text{G-}\mathbf{Mod}^{\text{gfg}}(C)$ to $\mathcal{C}$. In Section 3.3 we will build the necessary tools to construct an inverse to Φ.

Remark 3.8. *In case (iii), Theorem 3.7 still holds after replacing* $\text{G-}\mathbf{Mod}^{\text{gfg}}(C)$ *by* $\text{G-}\mathbf{Mod}(C)$*. We do not use the generically finitely generated assumption anywhere, except to be able to define* $|_{U_p}$ *in cases (i) and (ii). We will keep referring to* $\text{G-}\mathbf{Mod}^{\text{gfg}}(C)$ *everywhere to simplify notation.*

Finally, we note that the same theorem holds when we replace "formal neighborhoods" by "stalks" everywhere. The complete local ring R_p will be replaced by the stalk $\mathcal{O}_p$ and K_p is replaced by the ring of rational functions of C, $k(C)$. We will denote $V_p = \operatorname{Spec} \mathcal{O}_p$ and likewise $V_p^* = \operatorname{Spec} k(C)$.

Proposition 3.9. *Assuming Theorem 1.1, the following diagram is a fiber product of categories:*

$$\begin{array}{ccc} \text{G-}\mathbf{Mod}^{\text{gfg}}(C) & \xrightarrow{j^*} & \text{G-}\mathbf{Mod}^{\text{gfg}}(C^*) \\ \big\downarrow{\scriptstyle |_{V_p}} & & \big\downarrow{\scriptstyle |_{V_p}} \\ \text{G-}\mathbf{Mod}(V_p) & \xrightarrow{j^*} & \text{G-}\mathbf{Mod}(V_p^*) \end{array}$$

Where all the definitions are analogous to the ones in this section, replacing $R_p = \widehat{\mathcal{O}}_p$ *by the stalk* $\mathcal{O}_p$ *and* K_p *by* $k(C)$*, the ring of rational functions of* C*. We are denoting* $V_p = \operatorname{Spec} \mathcal{O}_p$*.*

Proof. Consider the diagram

$$\begin{array}{ccccc} \text{G-}\mathbf{Mod}^{\text{gfg}}(C) & \xrightarrow{|_{V_p}} & \text{G-}\mathbf{Mod}(V_p) & \xrightarrow{R_p \otimes_{\mathcal{O}_p}} & \text{G-}\mathbf{Mod}(U_p) \\ \big\downarrow{\scriptstyle j^*} & & \big\downarrow{\scriptstyle j^*} & & \big\downarrow{\scriptstyle j^*} \\ \text{G-}\mathbf{Mod}^{\text{gfg}}(C^*) & \xrightarrow{|_{V_p}} & \text{G-}\mathbf{Mod}(V_p^*) & \xrightarrow{R_p \otimes_{\mathcal{O}_p}} & \text{G-}\mathbf{Mod}(U_p^*) \end{array}$$

The horizontal arrows on the right are given by mapping a module M to $R_p \otimes_{\mathcal{O}_p} M$, with $(R_p \otimes M)^\star = R_p \otimes (M^\star)$, and extending the St_p action in the expected way: for $g \in \mathrm{St}_p$, we make $\overline{g}(f \otimes m) = f^{g^{-1}} \otimes \overline{g}m$. We claim that the right square is Cartesian, from which it follows that Theorem 1.1 implies the statement by the following abstract fact: if the outer rectangle is cartesian and the right square is cartesian as well, then the left square is cartesian.

Let us denote $K = k(C)$ for short. First of all, we check that $\mathbf{Mod}(\mathcal{O}_p) \cong \mathbf{Mod}(R_p) \times_{\mathbf{Mod}(K_p)} \mathbf{Mod}(K)$. The functor F from left to right is given by tensoring. The inverse functor G is given by mapping a triple $(M_{R_p}, M_K, \phi : K_p \otimes_{R_p} M_{R_p} \cong K_p \otimes_K M_K)$ to the kernel of the difference $M_{R_p} \oplus M_K \to K_p \otimes M_{R_p} \cong K_p \otimes M_K$.

Let us start by showing that the following is a short exact sequence of $\mathcal{O}_p$-modules: $\mathcal{O}_p \to R_p \oplus K \to K_p$. The map $i : \mathcal{O}_p \to R_p$ is injective, since its kernel $\ker i = \bigcap_{n \geq 0} p^n$ has the property that $p \ker i = \ker i$, so $\ker i = 0$ by Nakayama's Lemma. Now, in the middle, suppose $a/b \in R_p \cap K \subset K_p$, where $a, b \in R$. Since $a/b \in R_p$, we must have that for every n, $a/b \cong c_n \mod p^n$, where $c_n \in R/p^n$, i.e. $a \in (b) + p^n$ for every n. Since b is a nonzerodivisor and R has dimension 1, b is not contained in any minimal prime, so $p^n \subset (b)$ for some b. Therefore, $a \in (b)$, so $a/b \in R$ as desired. Finally, let us see that R_p and K together generate K_p as $\mathcal{O}_p$-modules. Consider an element of K_p, which we can write as a/b, with $a, b \in R_p$ and b a nonzerodivisor. Consider $a', b' \in R$ such that $a - a', b - b' \in p^n R_p$ for some $n \gg 0$ which we will fix later. This condition forces b' to be a nonzerodivisor in R_p as well, since for any minimal prime ideal I, we can find an n big enough such that $b \notin p^n + I$, so $b' \notin p^n + I \supseteq I$. Let n_0 be big enough so that $(b) \supseteq p^{n_0}$. Then if $n \geq n_0$, we have that

$$\frac{b'}{b} - 1 = \frac{b' - b}{b} \in p^{n-n_0} R_p.$$

Therefore we can write $b' = b(1+c)$, where $c \in p^n R_p$, and in particular b'/b is a unit of R_p. Now, $\frac{a}{b} = \frac{a'}{b'} + \frac{ab' - a'b}{bb'}$. Choosing n bigger than $2n_0$, we have that

$$ab' - a'b \equiv (a - a')b' - a'(b - b') \equiv 0 \mod p^n \overset{n \geq 2n_0}{\Rightarrow} ab' - a'b \in p^{2n_0} \subseteq (b^2) = (bb').$$

So $\frac{a'}{b'} \in K$ and $\frac{ab'-a'b}{bb'} \in R_p$, and the desired sequence is exact on the right.

Let us show that $GF \cong \mathrm{Id}$: given $M \in \mathbf{Mod}(\mathcal{O}_p)$, we tensor it with the short exact sequence $\mathcal{O}_p \to R_p \oplus K \to K_p$. Since the completion is flat, we obtain the short exact sequence $M \to R_p \otimes M \oplus K \otimes M \to K_p \otimes M$, which shows that M is the desired equalizer.

To see that $FG \cong \mathrm{Id}$, let us take $M = (M_{R_p}, M_K)$ in the fiber product. Consider the following exact sequence of $\mathcal{O}_p$-modules:

$$S = \left(0 \to G(M) \to M_{R_p} \oplus M_K \to K \otimes M_{R_p}\right).$$

Localizing S yields $K \otimes G(M) \cong M_K$, and we can apply the five lemma to the following natural map of short exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G(M) & \longrightarrow & R_p \otimes G(M) \oplus K \otimes G(M) & \longrightarrow & K_p \otimes G(M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G(M) & \longrightarrow & M_{R_p} \oplus M_K & \longrightarrow & K_p \otimes M_K & & \end{array}$$

This yields the isomorphism $FG \cong \mathrm{Id}$.

Given that $\mathbf{Mod}(\mathcal{O}_p) \cong \mathbf{Mod}(R_p) \times_{\mathbf{Mod}(K_p)} \mathbf{Mod}(K)$, we can now prove that G-$\mathbf{Mod}(V_p) \cong$ G-$\mathbf{Mod}(U_p) \times_{\text{G-}\mathbf{Mod}(V_p^*)}$ G-$\mathbf{Mod}(V_p)$. We can extend the functors F and G from above to this situation obtaining functors F' and G': F' is induced by the commutative square at the start of this proof. G' is defined by mapping a pair $(M_1, M_2) \in$ G-$\mathbf{Mod}(U_p) \times$ G-$\mathbf{Mod}(V_p^*)$ (where $j^* M_1$ is identified with $R_p \otimes M_2$) to $G'(M_1, M_2) = G(M_1, M_2)$, where $G'(M_1, M_2)^\star = M_2^\star = G(M_1^\star, K \otimes M_2^\star)$. Notice that $M_2^\star \subset G(M_1, M_2)$. We have the map in $\mathbf{Mod}(R_p) \times_{\mathbf{Mod}(K_p)} \mathbf{Mod}(K)$ from $(M_1^\star, K \otimes M_2^\star)$

to (M_1, M_2), so applying G to this map we obtain the required inclusion $G'(M_1, M_2)^\star \to G'(M_1, M_2)$. The equivariant structure is extended by applying G to the maps $\mathcal{A}_g$, and noticing that F and G commute with g^*.

Let us show that F' and G' defined in this way are inverse equivalences. Starting with $(M, M^\star) \in$ G-$\mathbf{Mod}(V_p)$, we have that the arrow $G'F'(M^\star) \to GF(M)$ is isomorphic to the arrow $M^\star \to M^\star$, by applying the discussion above and the functoriality of F' and G'. Further, the equivariant structure is the same: the maps $\mathcal{A}_g$ for $g \in \mathrm{St}_p$ are preserved, and we only need to verify that g^* commutes with the four restriction functors. Similarly, we show that $F'G' \cong \mathrm{Id}$.

□

3.3 Proof of the main theorem

Proposition 3.10. *The functor $|_{U_p}$ has the following properties. Note that in case (iii) all of them are clear or vacuous. We use $\star$ to denote any of l, r or lr.*

1. *Its definition has no ambiguity, i.e. $|_{U_p}$ doesn't depend on the coherent sheaf $L \subseteq M$ as long as M/L is supported on closed points and n is chosen to be big enough (depending on the choice of L). Further, St_p preserves $M|^\star_{U_p}$.*

2. *$M^\star$ is indeed a finite rank free module and $M|_{U_p}/M|^\star_{U_p}$ is supported on p.*

3. *The functor $|_{U_p}$ maps morphisms in G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$ to morphisms in G-$\mathbf{Mod}(U_p)$, i.e. for a morphism $f : M \to N$, $f(M|^\star_{U_p}) \subseteq M|^\star_{U_p}$. Further, $f|_{U_p}$ is St_p-equivariant.*

4. *It is an exact functor, in the sense of exact categories: it maps short exact sequences to short exact sequences.*

5. *Let $f : M \to N$ be a morphism in G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$. Then $f|^\star_{U_p} : M^\star \to N^\star$ is a morphism of free R_p-modules that has constant rank, i.e. its cokernel is a free module. Further, $N^\star/f|_{U_p}M^\star$ embeds into $N/f|_{U_p}M$, so $\operatorname{coker} f$ is an object of G-$\mathbf{Mod}(U_p)$.*

Proof. 1. Let L_1, L_2 be two coherent subsheaves of M. Then $(L_1+L_2)/L_i$ is a coherent sheaf supported on closed points, and hence a finite length sheaf. This implies that the stalks of L_1 and L_2 can only differ at a finite set of points. Also notice that $\overline{g}$ identifies $L_{g^{-1}p}$ and $(\overline{g}L)_p$ as $\overline{g}$ identifies $M_{g^{-1}p}$ with M_p. Applying this to $L_1 = L$ and $L_2 = \overline{\tau}L$, $(\overline{\tau}^{\pm n}L)_p = (\overline{\tau}^{\pm n+1}L)_p$ for $n \gg 0$, which shows that the definition does not depend on n as long as $|n|$ is big enough. Similarly, if two different coherent subsheaves are chosen then their stalks will be equal at $\tau^n p$ as long as $|n| \gg 0$.

Let us show the invariance. Start by assuming that $h \in \mathrm{St}_p^*$. Since we are in situations (i) or (ii), St_p is finite, so we can replace L by the $\overline{h}$-invariant $L_2 = L + \overline{h}L + \cdots + \overline{h}^m L$, where m is the order of h. By the previous paragraph, L and L_2 will yield the same $M|_{U_p}^{\star}$, so we may assume that L is $\overline{h}$-invariant. Then we can see that

$$\overline{h}M|_{U_p}^{\star} = \overline{h}(\overline{\tau}^{\pm n}L)_p = (\overline{h}\overline{\tau}^{\pm n}L)_p \overset{h\in\mathrm{St}_p^*}{=} (\overline{\tau}^{\pm n}\overline{h}L)_p = (\overline{\tau}^{\pm n}L)_p = M|_{U_p}^{\star}.$$

Lastly, if $h \in \mathrm{St}_p^* \setminus \mathrm{St}_p$, as before we may assume L is $\overline{h}$-invariant. Then

$$\overline{h}M|_{U_p}^{lr} = \overline{h}(\overline{\tau}^{n}L)_p = (\overline{\tau}^{-n}\overline{h}L)_p = (\overline{\tau}^{-n}L)_p =$$

$$= (\overline{\tau}^{-n}L)_{hp} = (\overline{h}^{-1}\overline{\tau}^{-n}L)_p = (\overline{\tau}^{n}L)_p = M|_{U_p}^{lr}.$$

2. Since L is a coherent sheaf, its torsion has finite support, so $M|_{U_p}^{\star} = (\overline{\tau}^{\pm n}L)_p$ is torsion-free for $n \gg 0$, so it is free, because in cases (i) and (ii) p must be a smooth point. Further, $(\overline{\tau}^{\pm n}L)_p$ is finitely generated since L is coherent. Further, since $M/\overline{\tau}^{\pm n}L$ is torsion, $M|_{U_p}/M|_{U_p}^{\star} = (M/\overline{\tau}^{\pm n}L)_p$ is torsion as well.

3. Let $L \subseteq M$ be a coherent sheaf such that M/L is supported on closed points. Then $f(L) \subseteq N$ is coherent, so we may choose some $L' \supseteq f(L)$ such that N/L' is supported on closed points. Then for $n \gg 0$, $f|_{U_p}(M|_{U_p}^{\star}) = (\overline{\tau}^{\pm n}f(L))_p \subseteq (\overline{\tau}^{\pm n}L')_p = N|_{U_p}$. The equivariance of the map is straightforward.

4. Given a short exact sequence $M \to N \to P$ in G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$, take a coherent $L_N \subseteq N$ such that N/L_N is supported on closed points. Then $L_M = L \cap M$ is coherent and M/L_M is supported on closed points, and similarly $L_P = L_N/L_M$ is a coherent subsheaf of P and P/L_P is supported on closed points. The short exact sequence $L_N \to L_M \to L_P$ yields the desired statement after applying $\overline{\tau}^{\pm n}$ and taking formal fibers.

5. Decompose f as an epimorphism followed by a monomorphism, so that we have two short exact sequences: $\ker f \to M \to f(M)$ and $f(M) \to N \to \operatorname{coker} f$. Then the exactness of $|_{U_p}$ implies that the cokernel of $f|_{U_p}$ is $(\operatorname{coker} f)|_{U_p}$, which is an object of G-$\mathbf{Mod}(U_p)$, so in particular $N^{\star}/f(M^{\star}) = (\operatorname{coker} f)|_{U_p}^{\star}$ is free and it embeds into $N/f(M) = (\operatorname{coker} f)|_{U_p}$.

□

Proposition 3.11. *$\mathcal{C}$ is an abelian category.*

Proof. Consider a morphism $f = (f_{U_p}, f_{C^*}) : (M_{U_p}, M_{C^*}, \cong_M) \to (N_{U_p}, N_{C^*}, \cong_N)$ in $\mathcal{C}$. We will often ommit the reference to the isomorphism $j^*M_{U_p} \cong_M M_{C^*}|_{U_p}$, and simply understand that these modules are identified. Similarly we will say that $j^*f_{U_p} = f_{C^*}|_{U_p}$

for simplicity. Note that kernels in G-$\mathbf{Mod}(U_p)$ always exist, and G-$\mathbf{Mod}(C^*)$ is an abelian category. However, a priori it is not clear that coker f_{U_p} exists: it would require $f^\star_{U_p}$ to have constant rank (for the relevant choices of $\star = l, r, lr$). By definition of j^*, $M^\star_{U_p} = j^* M^\star_{U_p}$ and $f^\star_{U_p} = j^* f^\star_{U_p}$. Further, $j^* f_{U_p} = f_{C^*}|_{U_p}$, so applying Proposition 3.10 we see that indeed $f^\star_{U_p}$ has constant rank as desired.

The kernel (resp. the cokernel) of the morphism f is the pair $(\ker f_{U_p}, \ker f_{C^*})$ (resp. $(\operatorname{coker} f_{U_p}, \operatorname{coker} f_{C^*})$). These are indeed objects of $\mathcal{C}$, i.e. they agree on G-$\mathbf{Mod}(U_p^*)$ by the isomorphism induced from M (resp. N):

$$j^*((\mathrm{co})\ker f_{U_p}) = (\mathrm{co})\ker j^* f_{U_p} \cong_M (\mathrm{co})\ker f_{C^*}|_{U_p} = ((\mathrm{co})\ker f_{C^*})|_{U_p}.$$

The remaining properties are straightforward to check based on the analogous properties in G-$\mathbf{Mod}(C^*)$ and G-$\mathbf{Mod}(U_p)$: the kernel and cokernel satisfy the required universal property, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel. □

Remark 3.12. *Take $M \in$ G-$\mathbf{Mod}(C)$. Then the kernel and cokernel of the adjunction map $M \to j_* j^* M$ are supported on Gp, since they vanish after applying j^*. Thus, we may write any module M in an exact sequence as follows:*

$$0 \to i_! i^! M \to M \to j_* j^* M \to i_! R^1 i^! M \to 0.$$

Here $i^!$ is the left adjoint to pushforward from Gp, and $R^1 i^!$ is the first derived functor of $i^!$. However, we don't require these facts so we will not prove them here, and we can take the above sequence as the definition of $i!i^!$ and $i_! R^1 i^!$. We will let $\overline{M}$ be the image of $M \to j_* j^* M$. It can be characterized as the largest quotient of M with no sections supported on p.

Remark 3.13. *The category $\mathcal{C}$ has the same structure as* G-$\mathbf{Mod}(C)$ *from Remark 3.12 above. The role of $j_* j^*$ is played by the functor $(j_* j^*, \mathrm{Id})$:* G-$\mathbf{Mod}(U_p) \times_{\text{G-}\mathbf{Mod}(U_p^*)}$ G-$\mathbf{Mod}^{\mathrm{gfg}}(C^*) \to$ G-$\mathbf{Mod}(U_p) \times_{\text{G-}\mathbf{Mod}(U_p^*)}$ G-$\mathbf{Mod}^{\mathrm{gfg}}(C^*)$*, and modules "supported on p" are pairs $(M_{U_p}, 0) \in \mathcal{C}$. The long exact sequence for $M = (M_{U_p}, M_{C^*})$ takes the form*

$$0 \to (i_! i^! M_{U_p}, 0) \to (M_{U_p}, M_{C^*}) \to (j_* j^* M_{U_p}, M_{C^*}) \to (i_! R^1 i^! M_{U_p}, 0) \to 0$$

Where again $i_! i^!$ and $i_! R^1 i^!$ can be characterized as the kernel and cokernel of the map $\mathrm{Id} \to j_* j^*$*. In this case, $\overline{M}$ is again defined as the image of M in $j_* j^* M$, and it is the largest quotient of M such that $\overline{M}_{U_p}$ has no sections supported on p. We will use the notation j^* and $i^!$ for $\mathcal{C}$ as well from now on.*

We will now construct an inverse to Φ : G-$\mathbf{Mod}(C) \to \mathcal{C}$. First, let us construct a functor ι_* : G-$\mathbf{Mod}(U_p) \to$ G-$\mathbf{Mod}(C)$, which will we will prove to be the right adjoint to $|_{U_p}$. Let $M \in$ G-$\mathbf{Mod}(U_p)$ and an open set $V \subseteq C$. Also, from now on fix $\Xi \subset G$ a (finite) set of representatives of $G/\langle\tau\rangle$. We will distinguish our three cases: in case (i), we let

$$\iota_*M(V) = \left\{ (m_g)_{g\in G} : \begin{array}{c} m_g \in \left\{\begin{smallmatrix} M \text{ if } gp\in V \\ K_p\otimes M \text{ if } gp\notin V\end{smallmatrix}\right. ; \begin{smallmatrix} m_{\gamma\tau^i} \in M^l_{U_p} \text{ for } i \ll 0, \forall \gamma\in\Xi; \\ m_{\gamma\tau^i} \in M^r_{U_p} \text{ for } i \gg 0, \forall\gamma\in\Xi \end{smallmatrix} \\ m_{gh^{-1}} = \overline{h} m_g \forall g\in G, h\in \mathrm{St}_p \end{array} \right\}$$

In case (ii), we let

$$\iota_*M(V) = \left\{ (m_g)_{g\in G} : \begin{array}{c} m_g \in \left\{\begin{smallmatrix} M \text{ if } gp\in V \\ K_p\otimes M \text{ if } gp\notin V\end{smallmatrix}\right. ; m_{\gamma\tau^i} \in M^{lr}_{U_p} \text{ for } |i| \gg 0, \forall\gamma\in\Xi; \\ m_{gh^{-1}} = \overline{h} m_g \forall g\in G, h\in \mathrm{St}_p \end{array} \right\}$$

In case (iii):

$$\iota_*M(V) = \left\{ (m_g)_{g\in G} : m_g \in \left\{\begin{smallmatrix} M \text{ if } gp\in V \\ K_p\otimes M \text{ if } gp\notin V\end{smallmatrix}\right. ; m_{gh^{-1}} = \overline{h} m_g \forall g\in G, h\in \mathrm{St}_p \right\}$$

We give $\iota_*M(V)$ the structure of an $\mathcal{O}(V)$-module by letting $f\in\mathcal{O}(V)$ act by $f(m_g)_g = (f^g m_g)_g$. One checks that this definition indeed makes ι_*M into a quasicoherent sheaf, where the restriction maps are induced by the map $M|_{U_p}\to K_p\otimes M|_{U_p}$ (notice in particular that if f is regular at gp, then f^g is regular at p). Further, the condition $\overline{h}m_g = m_{gh^{-1}}$ is preserved by multiplication by $f\in\mathcal{O}$:

$$m_{gh^{-1}} = \overline{h}m_g \Rightarrow f^{gh^{-1}} m_{gh^{-1}} = f^{gh^{-1}}\overline{h}m_g = \overline{h}(f^g m_g)$$

We endow ι_*M with the following G-equivariant structure: for $g_0\in G$, we make $\overline{g}_0(m_g)_g := (m_{g_0^{-1}g})_g = (m_g)_{g_0g}$. Recall that $\overline{g}_0 = (g_0^{-1})^*\circ\mathcal{A}_{g_0}$. As before, we can easily check that the condition $\overline{h}m_g = m_{gh^{-1}}$ is preserved. One checks that $\overline{g}_0(\iota_*M(V)) = \iota_*M(g_0V)$, and further let us verify that the G-action is compatible with the $\mathcal{O}$-action: for $f\in\mathcal{O}$ and $g_0\in G$,

$$f\overline{g}_0(m_g)_g = f(m_{g_0^{-1}g})_g = (f^g m_{g_0^{-1}g})_g = (f^{g_0g}m_g)_{g_0g} = \overline{g}_0(f^{g_0g}m_g)_g = \overline{g}_0 f^{g_0}(m_g)_g.$$

Lemma 3.14. *The functor ι_*, defined as above, is the right adjoint to $|_{U_p}$.*

Note that $|_{U_p}$ is only partially defined, since its domain is G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$ rather than G-$\mathbf{Mod}(C)$. However, the notion of an adjoint makes sense since G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$ is a full subcategory: we mean that for $M\in$ G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$ and $N\in$ G-$\mathbf{Mod}(U_p)$, there is a natural isomorphism

$$\mathrm{Hom}_{\text{G-}\mathbf{Mod}(U_p)}(M|_{U_p}, N) \cong \mathrm{Hom}_{\text{G-}\mathbf{Mod}(C)}(M, \iota_*N).$$

Proof. Let $M \in$ G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$ and $N \in$ G-$\mathbf{Mod}(U_p)$, and let $\phi : M \to \iota_* N$ be a map in G-$\mathbf{Mod}(C)$. A local section $m \in M$ is mapped to a sequence $(\phi_g(m))_g$. Consider ϕ_1, which maps the stalk of M at p to N, and is $\mathcal{O}_C$-linear. We must check that ϕ_1 is St_p-equivariant provided that ϕ is G-equivariant. Indeed, if $\phi(m) = (\phi_g(m))$,

$$\forall h \in G, (\phi_g(\overline{h}m))_g = \phi(\overline{h}m) = \overline{h}\phi(m) = \overline{h}(\phi_g(m))_g = (\phi_{h^{-1}g}(m))_g \Rightarrow \tag{3.2}$$

$$\Rightarrow \forall h \in \mathrm{St}_p, \phi_1(\overline{h}m) = \phi_{h^{-1}}(m) = \overline{h}\phi_1(m)$$

Finally, we must check that ϕ_1 maps $M|^\star_{U_p}$ into $N^\star$. Let us show this in the case where $\star = l$, and the other situations will be analogous. There exists some coherent sheaf $L \subseteq M$ such that $(\overline{\tau}^n L)_p = M^l_p$ for every $n \geq 0$. Then $\phi_1(L_p) = \phi_1((\overline{\tau}^n L)_p) = \phi_{\tau^{-n}}(L_p)$, which for $n \gg 0$ is contained in N^l. This is the case because the stalk at p of L is finitely generated, so we only need to use that $\phi_{\tau^{-n}}(m) \in N^l$ for $n \gg 0$ and m in a finite generating set of L.

In the other direction, let $\psi : M_{U_p} \to N$ be a map in G-$\mathbf{Mod}(U_p)$, i.e. a St_p-equivariant map such that $\psi M^\star_{U_p} \subseteq N^\star$. We define the map $\phi : M \to \iota_* N$ as follows: on a local section m,

$$\phi(m) = (\psi((\overline{g}^{-1}m)_p))_g.$$

If m is regular at gp, then $\overline{g}^{-1}m$ is regular at p, i.e. $\psi((\overline{g}^{-1}m)_p)$ is contained in N rather than $K_p \otimes N$, so the map is well-defined as a map of sheaves. Further, we check that it is $\mathcal{O}$-linear and G-equivariant: if $f \in \mathcal{O}$ and $g_0 \in G$,

$$f\phi(m) = (f^g\psi(\overline{g}^{-1}m)_p)_g = (\psi(\overline{g}^{-1}fm)_p)_g = \phi(fm);$$

$$\overline{g}_0\phi(m) = (\psi(\overline{g}^{-1}\overline{g}_0 m)_p)_g = \phi(\overline{g}_0 m).$$

We must check that the image of ϕ is contained in $\iota_* N$: the condition $\overline{h}m_g = m_{gh^{-1}}$ amounts to $\overline{h}\psi((\overline{g}^{-1}m)_p) = \psi((\overline{h}\overline{g}^{-1}m)_p)$, which follows from the St_p-equivariance of ψ.

Lastly, we must see that for $n \ll 0$ and $\gamma^{-1} \in \Xi$, $\psi((\overline{\tau}^{-n}\overline{\gamma}m)_p) \in N^l$, and similarly for N^r. This is indeed the case, since m is contained in some coherent sheaf L, such that M/L is supported on closed points. For $n \ll 0$ and $\gamma^{-1} \in \Xi$, $(\overline{\tau}^{-n}\overline{\gamma}L)_p = M|^l_{U_p}$ (recall that Ξ is a finite set), and therefore $(\overline{\tau}^{-n}\overline{\gamma}m)_p \in M|^l_{U_p}$, which ψ maps into N^l by assumption.

It is straightforward to check that these two maps are natural. Let us see that they are inverse bijections between $\mathrm{Hom}_{\text{G-}\mathbf{Mod}(C)}(M, \iota_* N)$ and $\mathrm{Hom}_{\text{G-}\mathbf{Mod}(U_p)}(M|_{U_p}, N)$.

Starting with $\phi : M \to \iota_* N$ given by $\phi(m) = (\phi_g(m))_g$, we obtain $\psi = (\phi_1)_p$, and applying the other map, we obtain $\widetilde{\phi} : m \mapsto ((\phi_1)_p(\overline{g}^{-1}m))_g$. So we must check that $\phi_g = \phi_1 \circ g^{-1} = \widetilde{\phi}_g$. This is the content of equation (3.2) above. Going back, if we start with $\psi : M|_{U_p} \to N$, we construct $\phi : m \mapsto (\psi(\overline{g}^{-1}m))_g$, and from here we take $\phi_1 = (\psi|_M)_p = \psi$. This concludes the proof.

□

We now define Ψ, which we will prove is the inverse of Φ. Let $M = (M_{U_p}, M_{C^*}) \in \mathcal{C}$. The adjunction $j^* \vdash j_*$ yields a natural map $f_1 : M_{U_p} \to j_*j^*M_{U_p} \cong M_{C^*}|_{U_p}$ (recall that this isomorphism is part of the data of M). The adjunction $|_{U_p} \vdash \iota_*$ yields a map $f_2 : M_{C^*} \to \iota_* M_{C^*}|_{U_p}$. We define ΨM as the equalizer of $\iota_* f_1$ and f_2, i.e.

$$\Psi(M_{U_p}, M_{C^*}) = \ker(\iota_* M_{U_p} \oplus M_{C^*} \longrightarrow \iota_*(M_{C^*}|_{U_p})) = \ker(\iota_* M_{U_p} \oplus M_{C^*} \longrightarrow \iota_*(j^* M_{U_p}))$$

Lemma 3.15. *Ψ is right adjoint to Φ.*

Proof. This follows formally from previous discussion. Let $N = (N_{U_p}, N_{C^*}) \in \mathcal{C}$, and let $M \in \text{G-}\mathbf{Mod}^{\text{gfg}}(C)$. Then,

$$\begin{aligned}\operatorname{Hom}(M, \Psi(N)) &\cong \ker(\operatorname{Hom}(M, \iota_* N_{U_p}) \oplus \operatorname{Hom}(M, N_{C^*}) \to \operatorname{Hom}(M, \iota_*(N_{C^*}|_{U_p})) \cong \\ &\cong \ker\left(\operatorname{Hom}(M|_{U_p}, N_{U_p}) \oplus \operatorname{Hom}(j^*M, N_{C^*}) \to \operatorname{Hom}(M|_{U_p}, N_{C^*}|_{U_p})\right) \cong \\ &\cong \operatorname{Hom}_{\mathcal{C}}((M|_{U_p}, j^*M), (N_{U_p}, N_{C^*})) = \\ &= \operatorname{Hom}_{\mathcal{C}}(\Phi(M), N).\end{aligned}$$

The second isomorphism follows from the adjunction $|_{U_p} \vdash \iota_*$, and the third is the definition of morphisms in $\mathcal{C}$. $\square$

Lemma 3.16. *Φ is exact and Ψ is left exact. Further, the following short exact sequence remains exact on the right after applying Ψ:*

$$0 \to i_! i^! M \to M \to \overline{M} \to 0$$

Proof. Since Φ is a left adjoint, it is right exact, so we only need to show that it preserves injections. This follows from the fact that both $|_{U_p}$ and j^* are exact. Likewise, Ψ is left exact due to being a right adjoint.

Let $M = (M_{U_p}, M_{C^*}) \in \mathcal{C}$. We have the short exact sequence $i_! i^! M \to M \to \overline{M}$ from Remark 3.13. Let us check that after applying Ψ it remains exact on the right. A local section $m \in \Psi\overline{M}$ is a pair consisting of $(\overline{m}_g)_g \in \iota_* \overline{M}_{U_p}$ and $m_{C^*} \in M_{C^*}$, agreeing on $\iota_* j^* M_{U_p}$. A preimage of m must be a pair $((\widetilde{m}_g)_g, m_{C^*}) \in \Psi M \subseteq \iota_* M_{U_p} \oplus M_{C^*}$, where $\widetilde{m}_g$ map to $\overline{m}_g$.

Let us construct such a preimage in case (i). Note that the induced map $M^\star_{U_p} \to \overline{M}^\star_{U_p}$ is an isomorphism, since it is the quotient of a finite rank free module by its torsion. Therefore, for $n \ll 0$ and $\gamma \in \Xi$, $\overline{m}_{\gamma\tau^n}, \in \overline{M}^l_{U_p}$ has a unique preimage in $M^l_{U_p}$, and similarly for $M^r_{U_p}$ (taking $n \gg 0$). Let $\Theta^l \subset G$ be the subset of g's such that $\overline{m}_g \in \overline{M}^l_{U_p}$, and similarly for Θ^r. Since St_p preserves $M^\star_{U_p}$, it follows that $\Theta^\star \operatorname{St}_p = \Theta^\star$, and $G \setminus (\Theta^l \cup \Theta^r)$ is finite.

We choose $\widetilde{m}_g$ for all $g \in \Theta^l$ as the only preimage of $\overline{m}_g$ contained in $M^l_{U_p}$, and we make the analogous choice for $g \in \Theta^r \setminus \Theta^l$. By the uniqueness of the choice and the fact that St_p preserves $M^\star_{U_p}$ and $\Theta^\star$, it must follow that for $g \in \Theta^l \cup \Theta^r$ and any $h \in \operatorname{St}_p$,

$\widetilde{m}_{gh^{-1}} = \overline{h}\widetilde{m}_g$. For $G \setminus (\Theta^r \cup \Theta^l)$ (a finite set), we choose a set of representatives of $(G \setminus \Theta^r \cup \Theta^l)/\operatorname{St}_p$, and for these representatives g we let $\widetilde{m}_g$ be an arbitrary preimage of $\overline{m}_g$ in M_{U_p}. For the remaining g's, the elements $\widetilde{m}_g$ are chosen in the unique way that ensures the condition that $\widetilde{m}_{gh^{-1}} = \overline{h}\widetilde{m}_g$.

This provides an element $(\widetilde{m}_g) \in \iota_* M_{U_p}$ mapping to $(\overline{m}_g) \in \iota_* \overline{M}_{U_p}$. We must check that the element $((\widetilde{m}_g), m_{C^*})$ is in ΨM, i.e. that this pair agrees on $\iota_* j^* M|_{U_p}$. This is the case because the map $M_{U_p} \to j^* M_{U_p}$ factors through $M_{U_p} \to \overline{M}_{U_p}$, and it is given that $(\overline{m}_g)$ and m_{C^*} agree. We have thus produced a preimage of m as we desired.

In case (ii), we proceed as in case (i), replacing $M^{lr}_{U_p}$ by $M^l_{U_p}$, and noting that defining Θ^{lr} analogously ensures that $G \setminus \Theta^{lr}$ is finite.

For case (iii), we choose a (necessarily finite) set of representatives of G/St_p, and for these we arbitrarily choose a preimage $\widetilde{m}_g$ of $\overline{m}_g$. For the remaining g's, we ensure that $\widetilde{m}_{gh^{-1}} = \overline{h}\widetilde{m}_g$, which implies that $\widetilde{m}_{gh^{-1}}$ maps to $\overline{m}_{gh^{-1}}$. Then as before it will follow that $((\widetilde{m}_g), m_{C^*}) \in \Psi M$, because the map to $j^* M_{U_p}$ factors through $\overline{M}_{U_p}$.

□

We can finally show that Φ and Ψ are mutual inverses.

Proof of Theorem 3.7. The adjunction yields natural transformations $\eta : \Phi\Psi \to \mathrm{Id}$ and $\epsilon : \mathrm{Id} \to \Psi\Phi$. Let us start by proving that ϵ is an isomorphism: let $M \in$ G-$\mathbf{Mod}^{\mathrm{gfg}}(C)$. The identity of $\Phi M = (M|_{U_p}, M|_{C^*})$ induces by the adjunction the map $M \to \Psi\Phi M$, which chasing the proofs above is given by

$$\epsilon : M(U) \ni m \longmapsto \big(((\overline{g}^{-1} m)_p)_g, m|_{C^*}\big) \in \Psi\Phi M \subset \iota_*(M|_{U_p}) \oplus M|_{C^*} \subset \prod_g M|_{U_p} \oplus M|_{C^*}.$$

We will use the following exact sequences:

$$0 \to i_! i^! M \to M \to \overline{M} \to 0; \quad 0 \to \overline{M} \to j_* j^* M \to i_! R^1 i^! M \to 0.$$

Applying $\Psi\Phi$, which is left-exact, the adjunction yields the following diagrams with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \overline{M} & \longrightarrow & j_* j^* M & \longrightarrow & i_! R^1 i^! M \\
\big\downarrow & & \big\downarrow{\scriptstyle \epsilon_{\overline{M}}} & & \big\downarrow{\scriptstyle (1)} & & \big\downarrow{\scriptstyle (2)} \\
0 & \longrightarrow & \Psi\Phi\overline{M} & \longrightarrow & \Psi\Phi j_* j^* M & \longrightarrow & \Psi\Phi i_! R^1 i^! M
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & i_! i^! M & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\
\big\downarrow & & \big\downarrow{\scriptstyle (3)} & & \big\downarrow{\scriptstyle \epsilon_M} & & \big\downarrow{\scriptstyle \epsilon_{\overline{M}}} & & \\
0 & \longrightarrow & \Psi\Phi i_! i^! M & \longrightarrow & \Psi\Phi M & \longrightarrow & \Psi\Phi\overline{M}. & &
\end{array}$$

If arrows (1) and (2) are isomorphisms it will follow that $\epsilon_{\overline{M}}$ is an isomorphism as well. Further, if arrow (3) is an isomorphism, the five-lemma implies that ϵ_M is an isomorphism as well. Putting everything together, to show that ϵ is an isomorphism it suffices to prove that ϵ is an isomorphism when restricted to the images of $i_!$ and j_*, i.e. to sheaves supported on Gp and sheaves in G-$\mathbf{Mod}(C^*)$.

In the first case, $M|_{C^*} = 0$, so $M|_{U_p}$ is supported on p, and we want ot prove that $M \cong \iota_*(M|_{U_p})$. It's a matter of writing out the definitions and using the fact that in cases (i) and (ii), $\iota_* M|_{U_p}$ is contained in the direct sum $\bigoplus_g M|_{U_p} \subset \prod_g M|_{U_p}$, as $M|^\star_{U_p} = 0$.

If $M \cong j_*j^*M$, then ϵ is injective, since $m|_{C^*} = m$. Now, consider an element $n = ((m_g)_g, m_{C^*}) \in \Psi\Phi M$. We have that $n = \epsilon(m_{C^*})$, so ϵ is surjective.

It remains to prove that $\eta : \Phi\Psi \to \mathrm{Id}$ is an isomorphism. Starting with $M = (M_{U_p}, M_{C^*}) \in \mathcal{C}$, η is given by η_{U_p} and η_{C^*} as follows:

$$\eta_{U_p} : (\Psi M)|_{U_p} \ni ((m_g)_g, m_{C^*})_p \mapsto m_1 \in M_{U_p}$$

$$\eta_{C^*} : j^*(\Psi M) \ni j^*((m_g)_g, m_{C^*}) \mapsto m_{C^*} \in M_{C^*}$$

We must check that they are both isomorphisms (as Lemma 3.15 guarantees that they are well-defined and that they agree on $j^*(\Psi M)|_{U_p}$). We try the same strategy, with the analogous exact sequences as before (from Remark 3.13). Applying $\Phi\Psi$ we obtain the following diagrams.

$$\begin{array}{ccccccc}
0 & \longrightarrow & \overline{M} & \longrightarrow & j_*j^*M & \longrightarrow & i_!R^1i^!M \\
\uparrow & & \uparrow{\scriptstyle \eta_{\overline{M}}} & & \uparrow{\scriptstyle (1)} & & \uparrow{\scriptstyle (2)} \\
0 & \longrightarrow & \Phi\Psi\overline{M} & \longrightarrow & \Phi\Psi j_*j^*M & \longrightarrow & \Phi\Psi i_!R^1i^!M
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & i_!i^!M & \longrightarrow & M & \longrightarrow & \overline{M} & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle (3)} & & \uparrow & & \uparrow{\scriptstyle \eta_{\overline{M}}} & & \\
0 & \longrightarrow & \Phi\Psi i_!i^!M & \longrightarrow & \Phi\Psi M & \longrightarrow & \Phi\Psi\overline{M} & \longrightarrow & 0.
\end{array}$$

The rows of these diagrams are exact. This time, for the second diagram we need Lemma 3.16.

As before, if arrows (1) and (2) are isomorphisms it will follow that $\eta_{\overline{M}}$ is an isomorphism as well, and if (3) is an isomorphism as well, the five-lemma will imply that η_M is an isomorphism as desired. So we need to check it for modules M with $M_{C^*} = 0$ and for modules with $M_{U_p} \cong j_*j^*M_{U_p}$.

In the case of a module M with $M_{C^*} = 0$, as before it suffices to write the map η: $\Psi M \cong \iota_* M_{U_p}$, so $\eta_{C^*} = 0$ and η_{U_p} is the map $(\iota_* M_{U_p})|_{U_p} \to M_{U_p}$, which as before we can check by hand that it is an isomorphism. For a module of the form j_*M, we have that $\Psi j_* M \cong M_{C^*}$, from the definition of Ψ (since $(j_* M)_{U_p} \cong (j^* j_* M)_{U_p}$), and therefore indeed η is an isomorphism.

□

Chapter 4

Difference equations and Local Mellin transform

In this chapter we prove some basic facts about the particular case of equivariant sheaves on the affine line, under the translation action. Then we move on to show Theorem 1.2 about the existence of the local Mellin transform.

4.1 Introduction

In this chapter we work over a field k of characteristic 0. Let $\mathcal{D}_{\mathbb{G}_m} = k[x^{\pm 1}]\langle \partial_x \rangle$ be the ring of differential operators on $\mathbb{A}^1 \setminus 0$, given by the relation $\partial_x x - x\partial_x = 1$. The Mellin transform is an equivalence $\mathcal{M}$ between $\mathbf{Mod}(\mathcal{D}_{\mathbb{G}_m})$ and $\mathbf{Mod}(\Delta_{\mathbb{A}^1})$, where $\mathcal{D}_{\mathbb{G}_m} = k[x, x^{-1}]\langle \partial_x \rangle$ is the ring of differential operators on the punctured affine line $\mathbb{G}_m$. It can be seen as induced by the ring isomorphism $\mathcal{D}_{\mathbb{G}_m} \cong \Delta_{\mathbb{A}^1}$ mapping x to τ and $x\partial_x$ to z. It is defined in [2], and it is a particular case of the Fourier transform defined in [15].

Our goal is to show that the vanishing cycles of a holonomic $\Delta_{\mathbb{A}^1}$-module coincide with the nearby cycles of its inverse Mellin transform. We will define holonomic in the next section. For now, recall the definition of vanishing cycles, which makes use of Definition 3.5.

Definition 4.1. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$ and $p \in \mathbb{A}^1$. The* ***left (resp. right) vanishing cycles*** *of M at p are defined as*

$$\Phi_p^l M := \frac{M|_{U_p}}{M|_{U_p}^l} \qquad \textit{(resp.)} \quad \Phi_p^r M := \frac{M|_{U_p}}{M|_{U_p}^r}.$$

Note that Φ_p^l and Φ_{-p}^r can be interchanged by the automorphism $z \leftrightarrow -z$ and $\tau \leftrightarrow \tau^{-1}$ on the difference side, and $x \leftrightarrow x^{-1}$ on the D-module side, thus we may just focus on Φ^l from now on. Proposition 3.10 implies that the images of holonomic modules by Φ_p^l land in the category $\mathbf{Mod}(k[z]_p)_{\mathrm{fin}}$ of finite length $k[z]$-modules supported at p. We construct a right adjoint to Φ_p^l, which we denote $\iota_{p!}^{\rightarrow}$ (we should denote it $\iota_{p!}^l$, but we choose to avoid confusion between ! and l). It is simply the functor $N \mapsto k((\tau)) \otimes_k N$, which gives as image a $\Delta_{\mathbb{A}^1}$-module: $\tau^{\pm 1}$ can act on $k((\tau))$, and the action of $k[z]$ can be given as $f(z)(\tau^n \otimes m) = \tau^n \otimes f(z+n)m$.

These are all the necessary ingredients to construct the local Mellin transform. On the D-module side, we should just focus on regular D-modules, and this can be seen just from the fact that irregular D-modules are the input of $\mathcal{M}^{(0,\infty)}$. Thus, we will let $\mathcal{H}ol(\mathcal{D}_{K_0})^{\text{reg}}$ denote the category of holonomic regular modules over $\mathcal{D}_{K_0} = k((x))\langle\partial_x\rangle$. Following [1], we denote by $\mathring{j}_{0*}$ the forgetful functor from $k((x))\langle\partial_x\rangle$-modules to $k[x,x^{-1}]\langle\partial_x\rangle$-modules.

Further, using the classification of D-modules over the formal disk (originally proved by Turrittin [25] and Levelt [16], but the proof can also be found in [26]), we may split them according to the leading term, which is well defined up to adding an integer. We denote $\mathcal{H}ol(\mathcal{D}_{K_0})^{\text{reg},(p)}$ the category of those D-modules whose leading term is in $p+\mathbb{Z}$. We will show the following.

Theorem 4.2.

1) *For any $p+\mathbb{Z}\in\mathbb{A}^1/\mathbb{Z}$, there is an equivalence*

$$\mathcal{M}^{(0,p)} : \mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg},(p)} \longrightarrow \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}.$$

and for any $F\in\mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg},(p)}$, there is a functorial isomorphism

$$\mathcal{M}(\mathring{j}_{0*}(F)) \xrightarrow{\sim} \iota_{p!}^{\rightarrow}(\mathcal{M}^{(0,p)}(F)).$$

The isomorphism is a homeomorphism in the natural topology, i.e. the τ-adic topology. This determines $\mathcal{M}^{(0,p)}$ up to natural isomorphism.

2) *For any $p+\mathbb{Z}\in\mathbb{A}^1/\mathbb{Z}$, there is an equivalence*

$$\mathcal{M}^{(\infty,p)} : \mathcal{H}ol(\mathcal{D}_{K_\infty})^{\mathrm{reg},(-p)} \longrightarrow \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}.$$

and for any $F\in\mathcal{H}ol(\mathcal{D}_{K_\infty})^{\mathrm{reg},(-p)}$, there is a functorial isomorphism

$$\mathcal{M}(\mathring{j}_{\infty*}(F)) \xrightarrow{\sim} \iota_{p!}^{\leftarrow}(\mathcal{M}^{(\infty,p)}(F)).$$

The isomorphism is a homeomorphism in the natural topology, i.e. the τ-adic topology. This determines $\mathcal{M}^{(\infty,p)}$ up to natural isomorphism.

Analogously to the result in [1], the adjunctions $\Phi_p^l \vdash \iota_{p!}^{\rightarrow}$, $\Psi_0 \vdash \mathring{j}_{0*}$ (where Ψ denotes nearby cycles) immediately yield Theorem 1.2 as a corollary, which as desired gives the relation between the local information of a D-module and that of its Mellin transform.

Corollary 4.3. *Let $F\in\mathcal{H}ol(\mathcal{D}_{\mathbb{G}_m})$. For any $p\in\mathbb{A}^1/\mathbb{Z}$, there are natural isomorphisms*

$$\Phi_p^l(\mathcal{M}(F)) \cong \mathcal{M}^{(0,p)}(\Psi_0(F)^{\mathrm{reg},(p)})$$
$$\Phi_p^r(\mathcal{M}(F)) \cong \mathcal{M}^{(\infty,p)}(\Psi_\infty(F)^{\mathrm{reg},(-p)})$$

Here we write $M^{\mathrm{reg},(p)}$ to denote the functor that picks from a $\mathcal{D}_{K_0}$-module its regular singular summand with leading coefficient p (i.e. the left adjoint to the inclusion of these submodules into general $\mathcal{D}_{K_0}$-modules).

4.2 Basic facts

Throughout we will let p be a closed point of $\mathbb{A}^1$, and we will also call $p\subset k[z]$ the corresponding maximal ideal. We will let $\pi=\pi(z)$ be a generator of p. We will use $k[z]_{(p)}$ to denote the local ring at p, and $k[z]_p$ to denote the completed local ring. Recall that we are using the notation $\pi^{\tau^n}=\pi\circ\tau^n=(z\mapsto\pi(z+n))$.

4.2.1 Holonomic difference modules

We will restrict our attention to holonomic difference modules. Over the affine line, they have an analogous definition to the one for D-modules.

Definition 4.4. *A $\Delta_{\mathbb{A}^1}$-module is* ***holonomic*** *if it is finitely generated over $\Delta_{\mathbb{A}^1}$ and every element is annihilated by a nonzero element of $\Delta_{\mathbb{A}^1}$.*

The same definition holds for a D-module (see for example [6, Chapter 10]), so a difference module is holonomic if and only if its Mellin transform is holonomic. It is less clear how to define holonomic equivariant sheaves in general, but there is work on the way on the topic, see [19]. Note that the Mellin transform shows that holonomic difference modules have finite length, since holonomic D-modules do.

Holonomic difference modules satisfy many desirable properties. However, they are not vector bundles over a dense open set, as in the case of D-modules. To see this behavior, see the examples in Section 2.4.

The following property does not hold for holonomic D-modules.

Proposition 4.5. *Any holonomic $\Delta_{\mathbb{A}^1}$-module has finite stalks, i.e. if $M \in Hol(\Delta_{\mathbb{A}^1})$, then for any maximal ideal $p = (\pi) \subset k[z]$, $k[z]_{(p)} \otimes_{k[z]} M$ is a finitely generated $k[z]_{(p)}$-module.*

Proof. Let us first prove it for a cyclic $\Delta_{\mathbb{A}^1}$-module (actually all holonomic modules are cyclic, but we won't need this fact here). Let M be generated by an element s. If M is holonomic, then s is annihilated by a nonzero element of $Q \in \Delta_{\mathbb{A}^1}$, which after multiplying by a suitable power of τ can be written as $Q = \sum_{i=0}^{n} P_i(z)\tau^i$, where $P_n, P_0 \neq 0$. Choose some $N \gg 0$ such that $P_0(z-m)P_n(z-m) \notin p$ whenever $|m| \geq N$. Then $P_0(z-m)$ and $P_n(z-m)$ are invertible in $k[z]_{(p)}$ and the following identities hold:

$$\tau^{m+n}s = \frac{-1}{P_n(z-m)} \sum_{i=0}^{n-1} P_i(z-m)\tau^{i+m}s \qquad \tau^{-m}s = \frac{-1}{P_0(z+m)} \sum_{i=1}^{n} P_i(z+m)\tau^{i-m}s$$

This implies that the finite set $\{\tau^{1-N}s, \ldots, \tau^{N+n-1}s\}$ generates the stalk $k[z]_{(p)} \otimes_{k[z]} M$ over $k[z]_{(p)}$. Note that the proof holds if $n = 0$, in which case the sums above become empty.

If M is not cyclic, then the statement follows by induction on the length. □

Corollary 4.6. *A holonomic $\Delta_{\mathbb{A}^1}$-module has finite generic rank, i.e. if $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$, $k(z) \otimes_{k[z]} M$ is finite dimensional over $k(z)$. Thus, $\mathcal{H}ol(\Delta_{\mathbb{A}^1}) \subset \mathbb{Z} - \mathbf{Mod}^{\mathrm{gfg}}(\mathbb{A}^1)$.*

Remark 4.7. *If a module M is finitely generated over $\Delta_{\mathbb{A}^1}$, then the converse to the corollary above is also true. If M has finite generic rank, then it cannot contain $\Delta_{\mathbb{A}^1} \cong \bigoplus_{i\in\mathbb{Z}} \tau^i k[z]$ as a submodule, because it has infinite rank, and therefore every element is torsion. In our previous notation, $\mathbb{Z} - \mathbf{Mod}^{\mathrm{gfg}}(\mathbb{A}^1) \cap \mathbf{Mod}^{\textit{fin. gen.}}(\Delta_{\mathbb{A}^1}) = \mathcal{H}ol(\Delta_{\mathbb{A}^1})$.*

Definition 4.8. *We will write $\mathcal{H}ol(U_p)$ to denote the ful subcategory of $\mathbb{Z} - \mathbf{Mod}(U_p)$ consisting of triples (M^l, M, M^r) with M a finitely generated R_p-module. By the discussion above, the restriction $|_{U_p}$ maps $\mathcal{H}ol(\Delta_{\mathbb{A}^1})$ into $\mathcal{H}ol(U_p)$.*

4.2.2 Difference modules on the punctured affine line

One of our goals is to relate difference modules on the affine line to difference modules on the punctured affine line. Due to the action of $\mathbb{Z}$, instead of removing a single point from the line, one must remove a whole $\mathbb{Z}$-orbit $p+\mathbb{Z}$. In the sequel, we will let $p \in \mathbb{A}^1$ be fixed, and we will let $\mathbb{A}^{1*} = \mathbb{A}^1 \setminus (p+\mathbb{Z})$. If the maximal ideal corresponding to p has generator $\pi(z)$, then $\mathbb{A}^{1*} = \operatorname{Spec} k\left[z, \frac{1}{\pi}, \frac{1}{\pi^{\tau^{\pm 1}}}, \dots\right]$.

Definition 4.9. *Difference modules on the punctured line $\mathbb{A}^{1*}$ are defined to be left modules over the ring*

$$\Delta_{\mathbb{A}^{1*}} = k\left[z, \left\{\frac{1}{\pi^{\tau^i}}\right\}_{i\in\mathbb{Z}}\right]\langle \tau, \tau^{-1}\rangle.$$

A difference module is said to be ***holonomic*** *if it is finitely generated and every element is annihilated by a nonzero $f \in \Delta_{\mathbb{A}^{1*}}$. We denote the categories of differences modules and holonomic modules on the punctured line by $\mathbf{Mod}(\Delta_{\mathbb{A}^{1*}})$ and $\mathcal{H}ol(\Delta_{\mathbb{A}^{1*}})$, respectively.*

The pullback functor from $\mathbb{A}^1$ to $\mathbb{A}^{1*}$ naturally extends to difference modules.

Definition 4.10. *We define the* ***restriction to the punctured line*** *as $|_{\mathbb{A}^{1*}}$: $\mathbf{Mod}(\Delta_{\mathbb{A}^1}) \to \mathbf{Mod}(\Delta_{\mathbb{A}^{1*}})$ as follows: for $M \in \mathbf{Mod}(\Delta_{\mathbb{A}^1})$,*

$$M|_{\mathbb{A}^{1*}} = k\left[z, \frac{1}{\pi}\right] \otimes_{k[z]} M.$$

The action of τ on $M|_{\mathbb{A}^{1*}}$ is defined by $\tau(P(z) \otimes m) = P(z-1) \otimes \tau m$. It makes $M|_{\mathbb{A}^{1*}}$ into a left $\Delta_{\mathbb{A}^{1*}}$-module. Note that π^{τ^i} acts as a unit on $M|_{\mathbb{A}^{1*}}$ for any $i \in \mathbb{Z}$, since $\pi^{\tau^i} = \tau^{-i}\pi\tau^i$.

The restriction functor can be thought of as analogous to nearby cycles, thought it is not a restriction to a formal disk, but rather a bigger open set $\mathbb{A}^{1*}$. Therefore, it retains much more information than the nearby cycles functor for D-modules.

Observation 4.11. *If M is holonomic, then $M|_{\mathbb{A}^{1*}}$ is holonomic, so $|_{\mathbb{A}^{1*}}$ gives rise to a functor $|_{\mathbb{A}^{1*}} : \mathcal{H}ol(\Delta_{\mathbb{A}^1}) \longrightarrow \mathcal{H}ol(\Delta_{\mathbb{A}^{1*}})$.*

4.2.3 The intermediate extension

One of the first questions one can ask is whether any d-connection, or more generally any holonomic $\Delta_{\mathbb{A}^{1*}}$-module can be extended over the puncture to a $\Delta_{\mathbb{A}^1}$-module in some canonical way. For a D-module, there are three answers, namely j_*, $j_!$ and $j_{!*}$, whose definitions can be found in [1], for example.

For difference modules, we have j_*, the forgetful functor, which has the disadvantage that it doesn't preserve holonomic modules. However, the intermediate extension $j_{!*}$ does have a difference analogue, which preserves holonomicity. We will see it can be constructed, as the smallest possible $\Delta_{\mathbb{A}^1}$-module contained in a given $\Delta_{\mathbb{A}^{1*}}$-module that only differs from it at $p+\mathbb{Z}$. Before we construct it, we will recall the coherent subsheaves used to define $|_{U_p}$.

Definition 4.12. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$. We define the set $\mathcal{S}(M)$ to be the set of $k[z]$-submodules $L \subseteq M$ such that L is finitely generated over $k[z]$ and M/L is a torsion $k[z]$-module.*

Similarly, if $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^{1*}})$, $\mathcal{S}(M)$ is defined to be the set $k[z]$-submodules $L \subseteq M$ that are finitely generated over $k[z]$, such that M/L is a torsion $k[z]$-module.

Definition 4.13. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$ and let $L \in \mathcal{S}(M)$. We define the **zeroes** of L as the finite set $Z_L = \operatorname{supp} \dfrac{L}{L \cap \tau L} = \operatorname{supp} \dfrac{L+\tau L}{\tau L}$, and the **poles** of L as the finite set $P_L = \operatorname{supp} \dfrac{L+\tau L}{L} = \operatorname{supp} \dfrac{\tau L}{L \cap \tau L}$.*

Remark 4.14. *If L is a (necessarily trivial) vector bundle and M is torsion-free, then we can think of τ as inducing a matrix difference equation on L with rational coefficients. Then P_L is the set of points where the matrix for τ is not defined and Z_L is the set where the inverse matrix is not defined.*

Lemma 4.15. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$ and $L \in \mathcal{S}(M)$. Then M/L is supported on finitely many orbits.*

Proof. Two finitely generated modules that agree over $k(z)$ are equal away from a finite set. Therefore, it is enough to prove the statement for any $L \in \mathcal{S}(M)$.

Let $L \in \mathcal{S}(M)$ be chosen such that it contains a finite generating set of M over $\Delta_{\mathbb{A}^1}$. We will prove by induction that $L_n = (L + \tau L + \cdots + \tau^n L)/L$ is supported on $Z_L + P_L + \mathbb{Z}$ (actually on $P_L + \mathbb{Z}_{\geq 0}$). We use the following short exact sequence, together with the fact that the support of a module is contained in the union of the supports of a submodule and quotient:

$$0 \longrightarrow \frac{L+\cdots+\tau^{n-1}L}{L} \longrightarrow \frac{L+\cdots+\tau^n L}{L} \longrightarrow \frac{L+\cdots+\tau^n L}{L+\cdots\tau^{n-1}L} \longrightarrow 0.$$

We note that $\frac{\tau^{n-1}L+\tau^n L}{\tau^{n-1}L}$ surjects onto $\frac{L+\cdots+\tau^n L}{L+\cdots+\tau^{n-1}L}$, so the support on the latter is contained in the support of the former. Further, we note that $\frac{\tau^{n-1}L+\tau^n L}{\tau^{n-1}L} = \tau^{n-1}\frac{L+\tau L}{L}$, which implies that $\operatorname{supp} \frac{\tau^{n-1}L+\tau^n L}{\tau^{n-1}L} \subset P_L + n - 1$. The induction hypothesis then shows that $\operatorname{supp} \frac{L+\cdots+\cdots+\tau^n L}{L} \subset P_L + \mathbb{Z}_{\geq 0}$.

The analogous reasoning yields the same result for negative n's, and these together show the desired result, since $M = \sum_{n\in\mathbb{Z}} \tau^n L$. □

Construction 4.16. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^{1*}})$. The intermediate extension of M, denoted $j_{!*}M$, is constructed as follows: Consider some $L \in \mathcal{S}(M)$ such that $P_L \cap (Z_L + \mathbb{Z}_{>0}) = \emptyset$. We will call such submodules* ***austere***[1].

Then $j_{!*}$ is defined as the $\Delta_{\mathbb{A}^1}$-submodule of M generated by all the subspaces $P(z)^{-1}L$ for all polynomials $P(z)$ with no roots in $p+\mathbb{Z}$, and $L \in \mathcal{S}(M)$ can be arbitrary provided it is austere. In other words, $j_{!*}M$ can be determined by its stalks in the following way: on $p+\mathbb{Z}$, its stalks equal those of L, and away from $p+\mathbb{Z}$, they equal M's stalks.

Proposition 4.17.

1. *Any holonomic module M contains austere submodules $L \in \mathcal{S}(M)$. Furthermore, any $W \in \mathcal{S}(M)$ contains a submodule $L \in \mathcal{S}(M)$ that is austere.*

2. *Any two austere submodules $V \in \mathcal{S}(M)$ generate the same module $j_{!*}M$ by Construction 4.16.*

Proof. 1. Consider any submodule $W \in \mathcal{S}(M)$. We claim that for the submodule $W' = \tau^{-1}W \cap W \in \mathcal{S}(M)$, its poles satisfy $P_{W'} \subseteq P_W - 1$: Indeed

$$\frac{\tau W'}{W' \cap \tau W'} = \frac{W \cap \tau W}{\tau^{-1}W \cap W \cap \tau W} \subseteq \frac{W}{\tau^{-1}W \cap W}.$$

And further $\operatorname{supp} \frac{W}{\tau^{-1}W\cap W} = -1 + \operatorname{supp} \tau \left(\frac{W}{\tau^{-1}W\cap W}\right) = P_W - 1$. Turning to the set of zeroes, we see that $Z'_W \subseteq Z_W$, since

$$\frac{W'}{W' \cap \tau W'} = \frac{\tau^{-1}W \cap W}{\tau^{-1}W \cap W \cap \tau W} \subseteq \frac{W}{W \cap \tau W}.$$

Thus iterating this process shifts the poles of the submodule to the left, while the zeroes don't move at all. Analogously, considering the submodule $W'' = \tau W \cap W$, one can check that $P_{W''} \subseteq P_W$ and $Z_{W''} \subseteq Z_W + 1$, which allows to move the zeroes to the right while keeping the poles in place. This process of shifting the zeroes and poles must indeed reach a submodule V which is austere.

2. Let $L, L' \in \mathcal{S}(M)$ be austere. By virtue of the first part of this proposition, without loss of generality we may assume that $L \subset L'$ (by choosing a third submodule in $\mathcal{S}(M)$ that is contained in $L \cap L'$). We may also assume that L'/L is supported on $p+\mathbb{Z}$, since modifying L away from $p+\mathbb{Z}$ doesn't affect the construction of

[1] *Because they dispense with unnecessary elements.*

$j_{!*}$. For the time being, we will let $j_{!*}M$ be the module generated from L by the procedure above. We will show that $L' \subseteq j_{!*}M$.

We may take the quotient by the $\Delta_{\mathbb{A}^1}$-module $j_{!*}M$. Then, $(L' + j_{!*}M)/j_{!*}M$ is a finite dimensional module supported on $p + \mathbb{Z}$. If $(L' + j_{!*}M)/j_{!*}M$ is nonzero, it is easy to see that it has some pole to the right of some zero, contradicting the assumption that L', and therefore $(L' + j_{!*}M)/j_{!*}M$, is austere.

□

Proposition 4.18. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^{1*}})$. Then the following hold.*

1. $(j_{!*}M)|_{\mathbb{A}^{1*}} = M$.
2. *The intermediate extension has no nonzero submodules or quotient modules with support contained in $p + \mathbb{Z}$.*
3. *Out of the modules contained in M, $j_{!*}M$ is the smallest $\Delta_{\mathbb{A}^1}$-module N such that $N|_{\mathbb{A}^{1*}} = M$.*
4. *The functor $j_{!*}$ is fully faithful.*

Proof.

1. This follows from the construction, since the stalks of M and $j_{!*}M$ are equal away from $p + \mathbb{Z}$.
2. Since $j_{!*}M \subset M$, $j_{!*}M$ has no elements supported on $p + \mathbb{Z}$. Now suppose that $N \subset j_{!*}M$ is a $k[z]\langle \tau, \tau^{-1} \rangle$-submodule such that $j_{!*}M/N$ is supported on $p + \mathbb{Z}$. By Proposition 4.17, there is an austere $L \in \mathcal{S}(N)$, and by the second part of said proposition, L generates all the stalks of $j_{!*}M$ at the points of $p+\mathbb{Z}$, and therefore $N = j_{!*}M$.
3. Let N be such a module. Then $j_{!*}M$ and N coincide outside of $p + \mathbb{Z}$. Therefore, $j_{!*}M/(N \cap j_{!*}M)$ is supported on $p+\mathbb{Z}$, and by the previous part of this proposition, this implies that $j_{!*}M/(N \cap j_{!*}M) = 0$, so $j_{!*}M \subseteq N$.
4. Since $|_{\mathbb{A}^{1*}} \circ j_{!*} \cong \mathrm{Id}$, it is enough to show that $|_{\mathbb{A}^{1*}}$ is faithful on the image of $j_{!*}$. Suppose a map $f : j_{!*}M \to j_{!*}N$ is such that $f|_{\mathbb{A}^{1*}} = 0$. Then the image of f is a torsion submodule of $j_{!*}N$, which implies that its image is 0. Therefore, $|_{\mathbb{A}^{1*}}$ is faithful.

□

4.2.4 Extending a difference module over a puncture

One application of Theorem 1.1 is to compute all the possible ways that a difference module on the punctured line can be extended to a module on the whole affine line.

Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^{1*}})$. First of all, we claim that $j_{!*}M$ must be the module given by the following information: $(j_{!*}M)|_{\mathbb{A}^{1*}} = M$ and

$$(j_{!*}M)|_{U_p} = \left(M|^l_{U_p^*}, M|^l_{U_p^*} + M|^r_{U_p^*}, M|^r_{U_p^*}\right) \subset M|_{U_p^*} = \left(M|^l_{U_p^*}, M|_{U_p^*}, M|^r_{U_p^*}\right).$$

It can be checked that the module above is the intermediate extension, for example by observing that it is the smallest $\Delta_{\mathbb{A}^1}$-module contained in M that restricts back to M (which determines $j_{!*}$ by Proposition 4.18).

Now let N be any other module in $\mathcal{H}ol(\Delta_{\mathbb{A}^1})$ such that $N|_{\mathbb{A}^{1*}} = M$. There is a natural map $N \to M$, whose image we will denote $\overline{N}$, and an injection $j_{!*}M \to M$, whose image is contained inside of $\overline{N}$, by the minimality. Therefore, we have a diagram

$$N \twoheadrightarrow \overline{N} \hookleftarrow j_{!*}M.$$

The kernel of the first arrow and the cokernel of the second are torsion modules supported on $p + \mathbb{Z}$, so they are successive extensions of $\delta = \Delta_{\mathbb{A}^1}/\Delta_{\mathbb{A}^1}p$ (here we think of p as a maximal ideal). Therefore, to understand the collection of possible N's it is sufficient to understand extensions of modules by torsion modules. The following proposition computes all extensions in $\mathcal{H}ol(U_p)$. Notice that even though $\mathcal{H}ol(U_p)$ is just an exact category, since it is closed under extensions in the category of diagrams of $k[z]_p$-modules, Ext groups can be computed in this larger category.

Proposition 4.19. *Let $M = (M^l, M, M^r)$ and $N = (N^l, N, N^r)$ be two modules in $\mathcal{H}ol(U_p)$. There is a short exact sequence*

$$0 \to \frac{\mathrm{Hom}_{k[z]_p}(M^l, N) \oplus \mathrm{Hom}_{k[z]_p}(M^r, N)}{\mathrm{Hom}_{k[z]_p}(M, N)} \longrightarrow \mathrm{Ext}^1_{\mathcal{H}ol(U_p)}(M, N) \xrightarrow{\Theta} \mathrm{Ext}^1_{k[z]_p}(M, N) \to 0.$$

Where Θ is the forgetful functor from $\mathcal{H}ol(U_p)$ to $\mathbf{Mod}(k[z]_p)$, and $\mathrm{Hom}_{k[z]}(M, N)$ is mapped into the group $\mathrm{Hom}(M^l, N) \oplus \mathrm{Hom}(M^r, N)$ by taking the restrictions.

Corollary 4.20. *1. If M is torsion, then*

$$\mathrm{Ext}^1_{\mathcal{H}ol(U_p)}(M, N) \cong \mathrm{Ext}^1_{k[z]_p}(M, N).$$

2. If M is torsion-free, then

$$\mathrm{Ext}^1_{\mathcal{H}ol(U_p)}(M, N) \cong \frac{\mathrm{Hom}_{k[z]_p}(M^l, N) \oplus \mathrm{Hom}_{k[z]_p}(M^r, N)}{\mathrm{Hom}_{k[z]_p}(M, N)}.$$

This corollary is enough to compute all the possible extensions of a module $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^{1*}})$ to some $N \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$. Going back to the diagram $N \twoheadrightarrow \overline{N} \hookleftarrow j_{!*}M$, the first part can be applied to compute the possible $\overline{N}$'s from $j_{!*}M$, which amounts to taking a finitely generated submodule $\overline{N}$ such that $j_{!*}M|_{U_p} \subseteq \overline{N} \subseteq M|_{U_p^*}$. The second part of the corollary can then be applied to obtain all possible extensions of $\overline{N}$ by a torsion module.

Proof of Proposition 4.19. The functor Θ produces a homomorphism $\mathrm{Ext}^1_{\mathcal{H}ol(U_p)}(M,N)$ $\xrightarrow{\Theta} \mathrm{Ext}^1_{k[z]_p}(M,N)$. Let us show that it is surjective. Consider an extension of $k[z]_p$-modules $N \xrightarrow{f} P \xrightarrow{g} M$. We need to find two submodules $P^\star \subset P$ such that there are induced short exact sequences $0 \to N^\star \to P^\star \to M^\star \to 0$. Since $M^\star$ is a projective $k[z]_p$-module, there's a lift $\widetilde{i^\star} : M^\star \to P$ such that $g \circ \widetilde{i^\star} = i^\star : M^\star \to M$. Then, one can take $P^\star = fN^\star + \widetilde{i^\star}M^\star$. Since $fN^\star \subset \ker g$ and $g|_{\widetilde{i^\star}M^\star}$ is injective, this sum is actually a direct sum, so we indeed obtain the desired short exact sequences, and thus some $P = (P^l, P, P^r) \in \mathcal{H}ol(U_p)$ fitting into a short exact sequence $N \to P \to M$.

The kernel of Θ is the group of extensions that take the form

$$0 \longrightarrow N \longrightarrow (P^l, N \oplus M, P^r) \longrightarrow M \longrightarrow 0.$$

These extensions are all given by choosing submodules $P^\star \subset N \oplus M$, such that they contain $N^\star$ and the quotients map isomorphically into $M^\star$. The short exact sequence $N^\star \to P^\star \to M^\star$ splits because $M^\star$ is projective. Let $\widetilde{i^\star} : M^\star \to P^\star \to N \oplus M$ be one such splitting. The M component of $\widetilde{i^\star}$ must be the identity, and therefore, it is determined by a map $M^\star \to N$. Therefore we have a map $\mathrm{Hom}_{k[z]_p}(M^l, N) \oplus \mathrm{Hom}_{k[z]_p}(M^l, N) \to \mathrm{Ext}^1_{\mathcal{H}ol(U_p)}(M,N)$ whose image is the kernel of Θ.

Let us show that this map is a $k[z]_p$-module homomorphism: multiplication by $k[z]_p$ is induced on both sides by multiplication on N, so it is clear that it commute with $k[z]_p$. For the sum, we can use the Baer sum. For two maps $(j^l, j^r) : M^l \oplus M^r \to N$, the corresponding extension is the class of the following module, with the obvious structure of an extension of M by N:

$$P = M \oplus N; P^\star = (\mathrm{Id}_M, j^\star)M^\star + N^\star.$$

Suppose we have two pairs of maps, $j_i^\star$, for $i = 1, 2$, giving rise to two extensions P_i. Their Baer sum is by definition:

$$P_3 = \frac{P_1 \times_M P_2}{\{(a,0) - (0,a) : a \in N\}} \cong \frac{N \oplus N \oplus M}{N}.$$

In the last term, N is embedded in $N \oplus N$ diagonally, so P_3 is isomorphic to $N \oplus M$, via the map $(n_1, n_2, m) \mapsto (n_1 + n_2, m)$. Looking at this map, we see that the image of $M^\star$ by $j^\star$ in P_3 is given by $(j_1^\star + j_2^\star, \mathrm{Id}_M)$, as desired.

We have an exact sequence

$$\mathrm{Hom}_{k[z]_p}(M^l, N) \oplus \mathrm{Hom}_{k[z]_p}(M^r, N) \longrightarrow \mathrm{Ext}^1_{\mathcal{H}ol(U_p)}(M,N) \xrightarrow{\Theta} \mathrm{Ext}^1_{k[z]_p}(M,N) \longrightarrow 0.$$

Let us compute the kernel of the leftmost map. It is made of the pairs of maps $(j^l, j^r) \in \mathrm{Hom}_{k[z]_p}(M^l, N) \oplus \mathrm{Hom}_{k[z]_p}(M^r, N)$ such that the following short exact sequence is split:

$$0 \longrightarrow N \longrightarrow (N^l \oplus (j^l, \mathrm{Id}_M)M^l, N \oplus M, N^r \oplus (j^r, \mathrm{Id}_M)M^r) \xrightarrow{p} M \longrightarrow 0.$$

These short exact sequences split exactly when there's a section of the second arrow, i.e. a map $s : M \to N \oplus M$ such that $s \circ p = 1_M$. This means that s is of the form $s = (j, \mathrm{Id}_M)$, where $j|_{M^\star} = j^\star$. The existence of j is the last piece of the desired statement. □

4.3 Local Mellin transform and vanishing cycles

In this section we discuss vanishing cycles along with some of their properties, and the local Mellin transform. The longer proofs are postponed until Section 4.4 to make the exposition easier to follow.

Definition 4.21. *We define the* ***functor of (left) vanishing cycles*** $\Phi_p^l : \mathcal{H}ol(\Delta_{\mathbb{A}^1}) \to \mathbf{Mod}(k[z]_p)$ *by*

$$\Phi_p^l(M) = \frac{M|_{U_p}}{M|_{U_p}^l}.$$

Which can be made into a functor in the obvious way.

Throughout this section we may abbreviate $\Phi = \Phi_p^l$.

Remark 4.22. *We can make the following observations.*

1. *By Proposition 3.10, $\Phi_p^l M$ is a finite length $k[z]_p$-module.*

2. *The analogous definition yields a second functor $\Phi_p^r(M) = \frac{M|_{U_p}}{M|_{U_p}^r}$. Every statement in this section has an analogous statement for Φ_p^r after interchanging the roles of τ and τ^{-1}.*

3. *Φ_p^l is exact: it is a composition of two exact functors between exact categories $\mathcal{H}ol(\Delta_{\mathbb{A}^1}) \to \mathcal{H}ol(U_p) \to \mathbf{Mod}(k[z]_p)$. Since its source and target are abelian, it is indeed an exact functor in the sense of abelian categories.*

One reason why Φ_p^l is a good replacement for vanishing cycles is that it vanishes exactly for modules with no zeroes (and Φ^r vanishes for modules with no poles). More precisely, in Section 4.3.1 we show the following theorem.

Theorem 4.23. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$. We have the following equivalences.*

- *M is finitely generated over $k[z]\langle\tau\rangle$ if and only if $\Phi_p^l M = 0$ for every $p \in \mathbb{A}^1$.*

- *M is finitely generated over $k[z]\langle\tau^{-1}\rangle$ if and only if $\Phi_p^r M = 0$ for every $p \in \mathbb{A}^1$.*

- *M is finitely generated over $k[z]$ if and only if $\Phi_p^l M = \Phi_p^r M = 0$ for every $p \in \mathbb{A}^1$. Further, in this case M is a vector bundle.*

There is a further reason why Φ works as vanishing cycles, namely that it is compatible with the Mellin transform, in a way analogous to previous work. There is a local Fourier transform relating local information of a D-module to that of its Fourier transform (see [5]). When the point in consideration is a point of $\mathbb{A}^1$ (as opposed to $\infty \in \mathbb{P}^1$), the local information to consider is the vanishing cycles of the D-module (see [1]). There is also a local Mellin transform ([10]) relating the nearby cycles at ∞ of a difference equation with the restriction to the formal disk around ∞ of its Mellin transform, which we can interpret as nearby cycles. The remaining Mellin transform should relate nearby cycles of a D-module at 0 and ∞ with vanishing cycles of its Mellin transform at points in $\mathbb{A}^1$, and this is precisely what we obtain.

The image of Φ is the category of finitely generated modules set-theoretically supported at p, which we will denote $\mathbf{Mod}(k[z]_p)_{\text{fin}}$. We will define an equivalence, which we will call the local Mellin transform:

$$\mathcal{M}^{(0,p)} : \mathcal{H}ol(\mathcal{D}_{K_0})^{\text{reg},(p)} \longrightarrow \mathbf{Mod}(k[z]_p)_{\text{fin}}.$$

This equivalence will fit into a commutative diagram as follows, where $\mathring{j}_{0*}$ is the forgetful functor (we are using [1]'s notation):

$$\begin{array}{ccc}
\mathbf{Mod}(\mathcal{D}_{\mathbb{G}_m}) & \xrightarrow[\sim]{\mathcal{M}} & \mathbf{Mod}(\Delta_{\mathbb{A}^1}) \\
\mathring{j}_{0*}\uparrow & & \uparrow\iota_{p!}^{\to} \\
\mathcal{H}ol(\mathcal{D}_{K_0})^{\text{reg},(p)} & \xrightarrow[\sim]{\mathcal{M}^{(0,p)}} & \mathbf{Mod}(k[z]_p)_{\text{fin}}.
\end{array}$$

Where $\iota_{p!}^{\to}$ is given by the following formula. To avoid confusions between ! and l, we denote it $\iota_{p!}^{\to}$ instead of $\iota_{p!}^{l}$.

$$\begin{array}{rccc}
\iota_{p!}^{\to} : & \mathbf{Mod}(k[z]_p)_{\text{fin}} & \longrightarrow & \mathbf{Mod}(\Delta_{\mathbb{A}^1}) \\
& M & \longmapsto & k((\tau)) \otimes_k M.
\end{array}$$

We will show the following proposition in Section 4.4.2.

Proposition 4.24. *The functors Φ_p^l and $\iota_{p!}^{\to}$ are adjoints in the following sense: if $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$ and $N \in \mathbf{Mod}(k[z]_p)_{\text{fin}}$, there is a natural isomorphism*

$$\operatorname{Hom}_{k[z]_p}(\Phi_p^l M, N) \cong \operatorname{Hom}_{k[z]\langle\tau,\tau^{-1}\rangle}(M, \iota_{p!}N).$$

Remark 4.25. *Technically, it is not true that $\Phi_p^l \vdash \iota_{p!}$ because the image of $\iota_{p!}$ is not made of holonomic modules. However, the statement above is enough for our purposes. Notice that it implies that Φ_p^l is determined by this adjunction.*

From the above commutative diagram and the adjunctions $\Psi_0 \vdash \mathring{j}_{0*}$ and $\Phi_p^l \vdash \iota_{p!}^{\to}$ for D-modules and difference modules respectively, we obtain the following commutative diagram:

$$\begin{array}{ccc} \mathcal{H}ol(\mathcal{D}_{\mathbb{G}_m}) & \xrightarrow[\sim]{\mathcal{M}} & \mathcal{H}ol(\Delta_{\mathbb{A}^1}) \\ \big\downarrow {\scriptstyle \mathrm{reg},(p)\circ\Psi_0} & & \big\downarrow {\scriptstyle \Phi^l_p} \\ \mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg},(p)} & \xrightarrow[\sim]{\mathcal{M}^{(0,p)}} & \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}. \end{array}$$

Remark 4.26. *It is possible to consider vanishing cycles on all $\mathbb{Z}$-orbits at once, by simply making $\Phi^l_{\mathrm{fin}} = \bigoplus_{p'\in S} \Phi^l_{p'}$, where Φ^l_{fin} becomes a $k[z]$-module. The set S can be chosen to be any class of representatives of $\mathbb{A}^1/\mathbb{Z}$, for example the complex numbers with real part in $[0,1)$.*

In this case the local Mellin transform will give an equivalence between $\mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg}}$ and the category of finite length modules supported on S.

Remark 4.27. *Considering Φ^r, the Mellin transform gives a relation with the behavior of a D-module around ∞. It satisfies $\mathcal{M} \circ \mathring{j}_{\infty *} \cong \iota^r_{p!} \circ \mathcal{M}^{(\infty,p)}$, and therefore $\Phi^r_p \circ \mathcal{M} \cong {}^{\mathrm{reg},(-p)} \circ \Psi_\infty \circ \mathcal{M}^{(\infty,p)}$ (note that in this case the leading term of the derivation must equal $-p$).*

4.3.1 When does a difference module have a singularity?

We show that many reasonable notions of zeroes and poles are equivalent, including in terms of the existence of subvector bundles (i.e. gauge transformations) of a module for which one can find a matrix with no zeroes (resp. no poles). This amounts to finding a subvector bundle preserved by τ^{-1} (resp. by τ). In particular, we can describe when a difference module $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$ has a "zero" or a "pole" in terms of the underlying $k[z]$-module. We prove the following proposition in Section 4.4.1.

Proposition 4.28. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$. The following are equivalent:*

1. $\Phi^l_p M = 0$ *for every* $p \in \mathbb{A}^1$.
2. *For some* $L \in \mathcal{S}(M)$*, there is some finite subset* $F \subseteq \mathbb{A}^1$ *such that* M/L *is supported on* $F + \mathbb{Z}_{\geq 0}$.
3. *For every* $L \in \mathcal{S}(M)$*, there is some finite subset* $F \subseteq \mathbb{A}^1$ *such that* M/L *is supported on* $F + \mathbb{Z}_{\geq 0}$.
4. *Any finite subset of* M *is contained in some* $L \in \mathcal{S}(M)$ *with no zeroes, i.e. such that* $\tau^{-1}L \subseteq L$.
5. M *is finitely generated over* $k[z]\langle\tau\rangle$.

Analogously, the following are equivalent:

1. $\Phi^r_p M = 0$ *for every* p.
2. *For some* $L \in \mathcal{S}(M)$*, there is some finite subset* $F \subseteq \mathbb{A}^1$ *such that* M/L *is supported on* $F + \mathbb{Z}_{\leq 0}$.
3. *For every* $L \in \mathcal{S}(M)$*, there is some finite subset* $F \subseteq \mathbb{A}^1$ *such that* M/L *is supported on* $F + \mathbb{Z}_{\leq 0}$.
4. *Any finite set is contained in some* $L \in \mathcal{S}(M)$ *with no poles, i.e. such that* $\tau L \subseteq L$.
5. M *is finitely generated over* $k[z]\langle\tau^{-1}\rangle$.

These can be put together to characterize difference modules with no singularities at finite points.

Corollary 4.29. *Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$. The following are equivalent:*

1. $\Phi^l_{p'} M = \Phi^r_{p'} M = 0$ *for every* p'.
2. M *is finitely generated over* $k[z]$.
3. M *is a vector bundle.*

Proof. 1⇒2 Let $L \in \mathcal{S}(M)$. By Proposition 4.28, M/L is supported on a set of the form $(F_1 + \mathbb{Z}_{\geq 0}) \cap (F_2 + \mathbb{Z}_{\leq 0})$, where F_1, F_2 are finite. Lemma 4.15 implies that M/L has finite support. Finally, Proposition 4.5 implies that M/L is finitely generated, so M is indeed finitely generated over $k[z]$.

2⇒3 If M is finitely generated over $k[z]$ and it is not a vector bundle, it must have a torsion element. Let $s \in M \setminus \{0\}$ be such that its support is a single point a. Then $\operatorname{supp} \tau^n s = a + n$, which implies that the torsion submodule of M is not finitely generated, and therefore M itself is not finitely generated.

3⇒1 This follows directly from Proposition 4.28.

□

4.3.2 A different approach to vanishing cycles

In order to show the compatibility of vanishing cycles with the Mellin transform, we will try to gain a better understanding of the functor $\iota_{p!}^{\rightarrow} : \mathbf{Mod}(k[z]_p)_{\text{fin}} \longrightarrow \mathbf{Mod}(\Delta_{\mathbb{A}^1})$. This functor is not surjective, but it is faithful, which will allow us to describe its image, which we will denote $\widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$, and obtain an equivalence $\iota_{p!}^{\rightarrow} : \mathbf{Mod}(k[z]_p)_{\text{fin}} \longrightarrow \widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$. This will show that the functor is full if one considers $k[[\tau]]\langle z\rangle$-linear morphisms on the target. Using said equivalence we could define "the local Mellin transform" to be the functor $\iota_{p!}^{\rightarrow} \circ \mathcal{M}^{(0,p)} : \mathcal{H}ol(\mathcal{D}_{K_0}) \longrightarrow \widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$. Since $\iota_{p!}^{\rightarrow}$ is an equivalence, both approaches are interchangeable.

In order to describe $\widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$, we take several steps. First we describe the category of $k[z]\langle\tau\rangle$-modules which are supported on an orbit and are finite in the appropriate way, as per the following definition.

Definition 4.30. *The category $\mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$ is the full subcategory of left modules V over $k[z]\langle\tau\rangle$ satisfying the following properties:*

1. *Any $m \in V$ is supported on $p + \mathbb{Z}$, i.e. there exists a $P(z) \in k[z]$ such that $P(z)m = 0$ and the roots of P are contained in $p + \mathbb{Z}$.*

2. *$\tau : V \to V$ is a locally nilpotent map, i.e. for every $m \in V$ there is a natural number n such that $\tau^n m = 0$.*

Definition 4.31. *A module in $\mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$ is **holonomic** if $\tau^{-1}(0)$ is finite dimensional. We denote the category of holonomic modules by $\mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$.*

In Section 4.3.2 we describe the relevant properties for these modules. We consider the collection of $k[[\tau]]\langle z\rangle$-modules which are limits of these modules.

Definition 4.32. *The category $\widehat{\mathbf{Mod}}(\Delta^l_{\mathbb{A}^1})_p$ is defined to be the category of $k[[\tau]]\langle z\rangle$-modules M such that the following natural map is an isomorphism*

$$M \longrightarrow \varprojlim_{L\in\mathcal{L}(M)} M/L.$$

Where $\mathcal{L}(M)$ is the set of $k[[\tau]]\langle z\rangle$-submodules $L \subset M$ such that $M/L \in \mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$.

Definition 4.33. *We say a module $M \in \widehat{\mathbf{Mod}}(\Delta^l_{\mathbb{A}^1})_p$ is **holonomic** if it is of Tate type, i.e. if it has an open finitely generated $k[[\tau]]$-submodule U such that $\tau^{-1}(U)/U$ is a finite dimensional vector space. The category of holonomic modules will be denoted $\widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$.*

Proposition 4.43 gives several equivalent definitions to the definition above. Finally, the image of $\iota_{p!}^{\to}$ is the subcategory $\widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p \subset \widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$ consisting of modules on which τ acts as a unit.

We may define the following functor.

Definition 4.34. *We define the functor* ***"sections with support at*** p***"*** *as follows:* $\iota_p^! : \widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p \to \mathbf{Mod}(k[z]_p)_{\text{fin}}$ *is given by*

$$\iota_p^! : M \longmapsto \iota_p^! M = \{m \in M : \exists n, p^n m = 0\}.$$

Here we think of $p = (\pi)$ *as an ideal in* $k[z]$*.*

The rest of this section will be building up to the proof of the following proposition. The proof can be found in Section 4.4.4.

Proposition 4.35. *The functors* $\iota_p^!$ *and* $\iota_{p!}^{\to}$ *are inverse equivalences* $\widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p \longleftrightarrow \mathbf{Mod}(k[z]_p)_{\text{fin}}$.

Remark 4.36. *The above Proposition can be proven by taking the Mellin transform of modules in* $\widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$*, which turns the difference modules into D-modules on the punctured formal disk. Then the classification of said differential operators can be used to obtain the result. However, we have chosen to take an alternative approach to the proof, which involves only difference modules.*

Difference modules with support on an orbit

Let us show some useful properties about $\mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$.

Remark 4.37. *All modules* V *in* $\mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$ *have an increasing filtration* $V^i = \tau^{-i}(0) = \tau^{-1}(V^{i-1})$*. Note that* τ *induces a map* $V^i \to V^{i-1}$ *with kernel* V^1*, which implies that* $\dim V^i \leq \dim V^{i-1} + \dim V^1$*. We will use this notation in what follows.*

Proposition 4.38. *Both* $\mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$ *and* $\mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$ *are closed under submodules, quotients and extensions. Further, if* W *is a submodule or a quotient of* $V \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$*, then* $\dim W^1 \leq \dim V^1$*.*

Proof. $\mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$ is clearly closed under submodules, quotients and extensions.

Being holonomic is clearly preserved under submodules. For quotients, suppose $V \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$ and W is a submodule of V. Then the only nontrivial condition is that in V/W, $\tau^{-1}(0)$ is finite dimensional, i.e. that $\tau^{-1}(W)/W$ is finite dimensional. Note that $\tau^{-1}(W^i) \cap W = W^{i+1}$. Therefore, $\frac{\tau^{-1}(W)}{W} = \frac{\bigcup_i \tau^{-1}(W^i)}{W} = \bigcup_i \frac{\tau^{-1}(W^i)}{W^{i+1}}$. Since τ induces a map $\tau^{-1}(W^i) \to W^i$ with kernel contained in V^1,

$$\dim \frac{\tau^{-1}(W^i)}{W^{i+1}} = \dim \tau^{-1}(W^i) - \dim W^{i+1} \leq \dim \tau^{-1}(W^i) - \dim W^i \leq \dim V^1.$$

Therefore, $\tau^{-1}(W)/W$ is a union of subspaces of dimension at most $\dim V^1$, and therefore it has dimension at most $\dim V^1$, which implies that $V/W \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$.

Finally, for extensions, observe that a short exact sequence $0 \to U \to V \to W \to 0$ yields an exact sequence of vector spaces $0 \to U^1 \to V^1 \to W^1$. □

We will use the following lemma in the sequel.

Lemma 4.39. *Let $V \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$. The following inequalities hold:*

1. $\dim \iota_p^! V \leq \dim V^1$

2. $\dim V/\tau V \leq \dim V^1$*, with equality if and only if V is finite dimensional.*

Proof.

1. Since V^1 is a torsion $k[z]$-module, we may decompose it based on supports, as $V^1 = \bigoplus_{i\in\mathbb{Z}}(V^1)_{p+i}$. It is finite dimensional, so only a finite number of the components are nonzero. On $\iota_p^! V$, we may consider the filtration by subvector spaces $(\iota_p^! V)^i = V^i \cap \iota_p^! V$. Then there are injections $\tau^i : (\iota_p^! V)^{i+1}/(\iota_p^! V)^i \hookrightarrow (V^1)_{p+i}$, which show that $\dim \iota_p^! V = \sum_i \dim(\iota_p^! V)^{i+1}/(\iota_p^! V)^i \leq \sum_i \dim(V^1)_{p+i} \leq \dim V^1$, as desired.

2. Consider the exact sequence $0 \to V^i \to V^{i+1} \xrightarrow{\tau^i} V^1 \to V^1/(\tau^i V^{i+1}) \to 0$. The dimension of the last term is at most $\dim V_1$ and it is nondecreasing with i. This dimension equals $d_i = \dim V^1 + \dim V^i - \dim V^{i+1}$. If it is ever the case that $d_i = \dim V^1$, this implies that $V^i = V^{i+1} = V$, so V is finite dimensional. It remains to observe that $\dim V^1 + \dim V^i - \dim V^{i+1} = \dim V^i/(\tau V^{i+1})$. In this identity, the limit of the right hand side equals the dimension of $V/\tau V$, and it is at most $\dim V_1$, as desired.

□

Corollary 4.40. *Every module $V \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$ is Artinian.*

Proof. Given a decreasing sequence $V_j \subset V$, the proof of the previous lemma shows that the sequences $(a^i_j)_{i\in\mathbb{N}} = (\dim(V_j)^i)_{i\in\mathbb{N}}$ are nondecreasing (with i and j) and concave (as i varies with fixed j). It is easy to see that a such a sequence of sequences eventually stabilizes. □

Limits of difference modules with support on an orbit

We are particularly interested in the category $\widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$. A way to state some of the properties of these modules is by using the natural topology on them. Recall the definition of the τ-adic topology.

Definition 4.41. *Let M be a $k[\tau]$-module. The **τ-adic topology** on M is defined as follows: a subspace $U \subset M$ is open if for every finitely generated submodule $N \subseteq M$, there is a k such that $\tau^k N \subseteq U$. Open subspaces form a basis of neighborhoods of 0.*

Remark 4.42. *A $k[[\tau]]\langle z\rangle$-module M is in $\widehat{\mathbf{Mod}}(\Delta^l_{\mathbb{A}^1})_p$ if and only if there is a basis $\{U\}$ of open $k[[\tau]]\langle z\rangle$-modules such that for every U and every $s \in M/U$, there is a polynomial $P(z)$ such that $P(z)s = 0$ and the roots of P are contained in $p + \mathbb{Z}$. This is due to the fact that on a $k[[\tau]]$-module the τ-adic topology is always Hausdorff and complete, so $M \to \lim\limits_{L\ open} M/L$ is always an isomorphism.*

Proposition 4.43. *For a module $M \in k[[\tau]]\langle z\rangle$, the following are equivalent:*

1. *$M \in \widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$.*
2. *M contains an open $k[[\tau]]\langle z\rangle$-submodule N that is finitely generated over $k[[\tau]]$ and such that $M/N \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$. Moreover, if $f \in k[z]$ is not contained in the ideal p^{τ^n} for any n, then f acts as a unit on M.*
3. *The identity in M has a basis of open neighborhoods N_i which are $k[[\tau]]\langle z\rangle$-submodules such that $M/N_i \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$ and $\dim \tau^{-1}N_i/N_i$ is bounded.*
4. *The identity in M has a basis of open neighborhoods N_i which are $k[[\tau]]\langle z\rangle$-submodules that are finitely generated over $k[[\tau]]$ and such that $M/N_i \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$.*

Proof. See Section 4.4.3. □

Corollary 4.44. *Both $\widehat{\mathbf{Mod}}(\Delta^l_{\mathbb{A}^1})_p$ and $\widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$ are abelian categories.*

With all these tools we can go on to prove Proposition 4.35, which can be found in Section 4.4.4 so that we can apply the result to the Mellin transform.

4.3.3 Local Mellin transform

Let $\mathcal{D}_{K_0} = k((x))\langle\partial\rangle$ be the ring of differential operators on the punctured formal disk, i.e. the ring given by the relation $\partial x - x\partial = 1$. A (left) $\mathcal{D}_{K_0}$-module is said to be **holonomic** if it is a finite dimensional vector space over $k((x))$, and the category of holonomic $\mathcal{D}_{K_0}$-modules will be denoted $\mathcal{H}ol(\mathcal{D}_{K_0})$.

The local Mellin transform $\mathcal{M}^{(0,\infty)}$ of [10] takes as input a holonomic $\mathcal{D}_{K_0}$ module that has irregular singularities, since these are the modules that yield singularities at ∞ after applying the Mellin transform. The Mellin transform $\mathcal{M}^{(0,p)}$ will take as input $\mathcal{D}_{K_0}$-modules with regular singularities.

Definition 4.45. *Let $M \in \mathcal{H}ol(\mathcal{D}_{K_0})$. We say that M **has regular singularities** if M contains a lattice that is fixed by $x\partial$.*

It follows from the classification of differential operators over K_0, which can be found in [26], that $\mathcal{H}ol(\mathcal{D}_{K_0})$ decomposes as a direct sum $\mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg}} \oplus \mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{irreg}}$. Further, D-modules can be decomposed according to the leading term of the derivation:

$$\mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg}} = \bigoplus_{p \in k/\mathbb{Z}} \mathcal{H}ol(\mathcal{D}_{K_0})^{\mathrm{reg},(p)}.$$

Definition 4.46. *Let $F \in \mathcal{H}ol(K_0)^{\mathrm{reg},(p)}$. The* ***local Mellin transform*** *of F is defined as*

$$\mathcal{M}^{(0,p)}F = \iota_p^!(\mathcal{M}(\mathring{j}_{0*}F)).$$

Where $\mathring{j}_{0*}$ is the forgetful functor $\mathcal{H}ol(\mathcal{D}_{K_0}) \to \mathcal{H}ol(\mathcal{D}_{\mathbb{G}_m})$ (we are using [1]'s notation here). The vector space $\mathcal{M}\mathring{j}_{0*}F$ equals F, together with an action of $k((\tau))\langle z\rangle$ given by $\tau^{\pm 1} \mapsto x^{\pm 1}$ and $z \mapsto x\partial$.

Remark 4.47. *By definition, the x-adic topology on $\mathring{j}_{0*}F$ coincides with the τ-adic topology on $\iota_{p!}^{\rightarrow}\mathcal{M}^{(0,p)}(F)$, and this together with the condition*

$$\mathcal{M}(\mathring{j}_{0*}(F)) \xrightarrow{\sim} \iota_{p!}^{\rightarrow}(\mathcal{M}^{(0,p)}(F)).$$

determines $\mathcal{M}^{(0,p)}$. This follows from the fact that $\iota_p^!$ and $\iota_{p!}^{\rightarrow}$ are mutual inverses, Proposition 4.35.

Proposition 4.48. *The functor $\mathcal{M}^{(0,p)}$ induces an equivalence*

$$\mathcal{M}^{(0,p)} : \mathcal{H}ol(\Delta_{K_0})^{\mathrm{reg},(p)} \xrightarrow{\sim} \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}.$$

Proof. Using Proposition 4.35, this will follow from showing that the following functor is an equivalence, since it remains to compose with $\iota_p^!$

$$\mathcal{M} \circ \mathring{j}_{0*} : \mathcal{H}ol(\Delta_{K_0})^{\mathrm{reg},(p)} \xrightarrow{\sim} \widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p.$$

First of all, we must check that the image of $\mathcal{H}ol(\Delta_{K_0})^{\mathrm{reg},(p)}$ is indeed contained in $\widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$. Let $V \in \mathcal{H}ol(\Delta_{K_0})^{\mathrm{reg},(p)}$. By definition of the leading coefficient and having regular singularities, if $p = (\pi(z))$, we can find a lattice $L \subset V$ such that $\pi(x\partial)^n L \subseteq xL$ for some big enough n. This implies that $\mathcal{M}\mathring{j}_{0*}V/L \in \mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$: if we let $s \in V$, then for some m, $x^m s \in L$. Denoting $\mathcal{M}\mathring{j}_{0*}s = \widehat{s}$ and $\mathcal{M}\mathring{j}_{0*}L = \widehat{L}$, we have that

$$\tau^m \pi(z+m)^n \widehat{s} = \pi(z)^n \tau^m \widehat{s} \in \tau \widehat{L} \Rightarrow \tau^{m-1}\pi(z+m)^n \widehat{s} \in \widehat{L} \Rightarrow (\cdots) \Rightarrow$$

$$\Rightarrow \pi(z+m)^n \cdots \pi(z+1)^n \widehat{s} \in \widehat{L}.$$

A similar computation for $x^i L$ shows condition (4) in Proposition 4.43, so it follows that $\mathcal{M}\mathring{j}_{0*}V \in \widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$.

It is clear that $\mathcal{M}\circ\mathring{j}_{0*}$ is fully faithful, since morphisms on both sides are $k((x))\langle\partial\rangle \cong k((\tau))\langle z\rangle$-linear maps. To show that it is essentially surjective, we just need to produce a preimage for every isomorphism class in $\widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$, or equivalently, for every module of the form $\iota_{p!}^{\rightarrow}M = k((\tau))\otimes_k M$, where $M \in \mathbf{Mod}(k[z]_p)_{\text{fin}}$. We may view $\iota_{p!}^{\rightarrow}M$ as a module over $k((x))\langle\partial\rangle$ via the Mellin transform. It is finite dimensional, since its dimension over $k((x))$ equals $\dim_k M$, and further it is regular and its leading coefficient is p, since it contains the lattice $L = k[[x]]\otimes_k M$ which is a witness to both these facts. □

The corollary below follows directly from the adjunctions $\Psi_0 \vdash \mathring{j}_{0*}$ and $\Phi^l_p \vdash \iota_{p!}^{\rightarrow}$.

Corollary 4.49. *The following square is commutative up to a natural isomorphism:*

$$\begin{array}{ccc}
\mathcal{H}ol(\mathcal{D}_{\mathbb{G}_m}) & \xrightarrow[\sim]{\mathcal{M}} & \mathcal{H}ol(\Delta_{\mathbb{A}^1}) \\
\Big\downarrow{\scriptstyle \text{reg},(p)\circ\Psi_0} & & \Big\downarrow{\scriptstyle \Phi^l_p} \\
\mathcal{H}ol(\mathcal{D}_{K_0})^{\text{reg},(p)} & \xrightarrow[\sim]{\mathcal{M}^{(0,p+\mathbb{Z})}} & \mathbf{Mod}(k[z]_p)_{\text{fin}}.
\end{array}$$

4.4 Proofs for Section 4.3

4.4.1 Proof of Proposition 4.28

$1 \Rightarrow 2$ Suppose that $\Phi^l M = 0$, and let $L \in \mathcal{S}(M)$. The fact that all the vanishing cycles are 0 implies that for every $p \in \mathbb{A}^1$ and $n \gg 0$, $M_p = \tau^n(L_{p-n})$. Since τ^n induces an isomorphism $M_{p-n} \cong M_p$, we deduce that $M_{p-n} = L_{p-n}$ for some big enough n (which can be chosen uniformly for all p's, since only a finite set affect the count by Lemma 4.15). The conclusion follows.

$2 \Rightarrow 3$ Two modules $L_1, L_2 \in \mathcal{S}(M)$ can only differ at a finite set, since $L_i/(L_1 \cap L_2)$ are finite length modules.

$3 \Rightarrow 4$ Choose any $L \in \mathcal{S}(M)$ containing a given finite set. Note that $\operatorname{supp}\frac{\tau^{-n}L+\tau^{-n+1}L}{\tau^{-n+1}L} = Z_L - n$. Therefore, if n is big enough, we take $L' = \tau^{-n}L + \cdots + L \in \mathcal{S}(M)$, and we have that

$$Z_{L'} = \operatorname{supp}\frac{L'+\tau L'}{\tau L'} \subseteq \operatorname{supp}\frac{\tau^{-n}L+\tau^{-n+1}L}{\tau^{-n+1}L} \cap \operatorname{supp}\frac{M}{L} = (Z_L - n)\cap(F+\mathbb{Z}_{\geq 0}) = \emptyset.$$

$4 \Rightarrow 5$ Let $L \in \mathcal{S}(M)$ have no zeroes, chosen to contain a (finite) generating set of M over $\Delta_{\mathbb{A}^1}$. Then, $\tau^{-1}L \subseteq L$, which implies that $M = \sum_{n\in\mathbb{Z}}\tau^n L = \sum_{n\geq 0}\tau^n L$, so a finite set generating L over $k[z]$ also generates M over $k[z]\langle\tau\rangle$, as desired.

$5 \Rightarrow 1$ Suppose that M is finitely generated as a $k[z]\langle\tau\rangle$-module, and let $S = \{s_i\}$ be a finite generating set over this ring. Let $L \in \mathcal{S}(M)$ containing S. By assumption, $\sum_{i\geq 0} \tau^i L = M$. In particular, $\tau^{-1}L \subset L + \tau L + \cdots + \tau^m L$ for some m, since $\tau^{-1}L \in \mathcal{S}(M)$ and therefore it is a Noetherian module. Let $L' = \tau + \cdots + \tau^m L$. Then $\tau^{-1}L' \subseteq L'$, so the sequence $\tau^i L'$ is increasing with i, and we still have that $\sum_{i\geq 0} \tau^i L' = \bigcup_{i\geq 0} \tau^i L' = M$. Fixing a fiber p, we have that $\bigcup_{i\geq 0} \tau^i(L'_{p-i}) = \bigcup_{i\geq 0}(\tau^i L')_p = M_p$. We have that M_p/L'_p is a finite length module, by definition of being in $\mathcal{S}(M)$ and Proposition 4.5. Therefore there is an $N \gg 0$ such that $\tau^n(L'_{p-n}) = M_p$ for any $n \geq N$. By definition of $|_{U_p}$, this implies that $M|_{U_p} = M|^l_{U_p}$, so $\Phi^l_p M = 0$.

4.4.2 Proof of Proposition 4.24

Let $M \in \mathcal{H}ol(\Delta_{\mathbb{A}^1})$, and let $N \in \mathbf{Mod}(k[z]_p)_{\text{fin}}$. We must find a natural isomorphism

$$\mathrm{Hom}_{k[\pi]}(\Phi M, N) \cong \mathrm{Hom}_{k[z]\langle\tau,\tau^{-1}\rangle}(M, \iota_! N).$$

First of all, note that since τ acts as a unit on both M and $\iota_! N$, the forgetful functor gives an isomorphism $\mathrm{Hom}_{k[z]\langle\tau,\tau^{-1}\rangle}(M, \iota_! N) \cong \mathrm{Hom}_{k[z]\langle\tau\rangle}(M, \iota_! N)$.

Throughout this proof, we will denote $\frac{k[\tau,\tau^{-1}]}{\tau^{n+1}k[\tau]} = \tau^n k[\tau^{-1}]$ for short. In other words, we have that

$$k((\tau)) = \lim_{\substack{n\to\infty \\ \leftarrow}} \frac{k[\tau,\tau^{-1}]}{\tau^{n+1}k[\tau]} = \lim_{\leftarrow} \tau^n k[\tau^{-1}].$$

Where the projection maps $\tau^n k[\tau^{-1}] \to \tau^{n-1}k[\tau^{-1}]$ are implied. Using this notation, we will also abbreviate $\tau^n k[\tau^{-1}] \otimes_k N$ to $\tau^n k[\tau^{-1}]N$. By the definition of a limit, we have that

$$\mathrm{Hom}_{k[z]\langle\tau\rangle}(M, \iota_! N) \cong \mathrm{Hom}_{k[z]\langle\tau\rangle}\left(M, \lim_{\leftarrow} \tau^n k[\tau^{-1}]N\right) \cong \lim_{\leftarrow} \mathrm{Hom}_{k[z]\langle\tau\rangle}\left(M, \tau^n k[\tau^{-1}]N\right).$$

Consider now one of the arrows in the right hand side limit:

$$\begin{array}{rccc} \pi: & \mathrm{Hom}\left(M, \tau^{n+1}k[\tau^{-1}]N\right) & \longrightarrow & \mathrm{Hom}\left(M, \tau^n k[\tau^{-1}]N\right) \\ & f & \longmapsto & \pi(f) = f \mod \tau^n N. \end{array}$$

The homomorphism π has an inverse: $\pi^{-1}(f) = \tau \circ f \circ \tau^{-1}$, where τ is seen as the k-linear isomorphism $\tau^n k[\tau^{-1}]N \to \tau^{n+1}k[\tau^{-1}]N$. One verifies that $\pi^{-1}(f)$ is indeed $k[z]\langle\tau\rangle$-linear, and that π and π^{-1} are inverses. Therefore, $\lim_{\leftarrow} \mathrm{Hom}\left(M, \tau^n k[\tau^{-1}]N\right)$ is a limit of a system of isomorphisms, so it is isomorphic to any one of its terms:

$$\mathrm{Hom}_{k[z]\langle\tau,\tau^{-1}\rangle}(M, \iota_! N) \cong \lim_{\leftarrow} \mathrm{Hom}_{k[z]\langle\tau\rangle}\left(M, \tau^n k[\tau^{-1}]N\right) \cong \mathrm{Hom}_{k[z]\langle\tau\rangle}\left(M, k[\tau^{-1}]N\right).$$

We can write a map $f : M \to k[\tau^{-1}]N$ as $f(s) = \sum_{i\geq 0} \tau^{-i}\phi_i(s)$, where ϕ_i is a collection of maps $M \to N$. The conditions of f being $k[z]\langle\tau\rangle$-linear and the image of f landing in $k[\tau^{-1}]N$ rather than in $k[[\tau^{-1}]]N$ boil down to the following three conditions:

$$\begin{array}{rrcl} \forall m \in M & \phi_i(m) & = & \phi_0(\tau^n m); \\ \forall m \in M & \phi_0(zm) & = & z\phi_0(m); \\ \forall m \in M \exists n \in \mathbb{Z}_{\geq 0} & \phi_0(\tau^n m) & = & 0. \end{array}$$

The first two conditions imply that f is determined by a $k[z]$-linear map $\phi_0 : M \to N$, i.e.

$$\mathrm{Hom}_{k[z]\langle\tau\rangle}\left(M, k[\tau^{-1}]N\right) \cong \left\{\phi \in \mathrm{Hom}_{k[z]}(M,N) : \forall m \exists n, \phi(\tau^n m) = 0\right\} =: \mathrm{Hom}^{(\tau)}_{k[z]}(M,N).$$

To finish the proof, we will show that the maps in $\mathrm{Hom}^{(\tau)}_{k[z]}(M,N)$ are precisely the maps that factor through the map $M \to \Phi^l_p M$. Let $\phi \in \mathrm{Hom}^{(\tau)}_{k[z]}(M,N)$, and let $L \in \mathcal{S}(M)$. Since L is finitely generated, it follows that for some big enough n, $\phi(\tau^n L) = 0$. Therefore, the map $\Phi_p : M_p \to (k[\tau^{-1}]N)_p \cong N$ sends $(\tau^n L)_p$ to 0, and therefore it factors (uniquely) through a map $\widetilde{\phi} : \Phi^l_p M = M_p/(\tau^n L)_p \longrightarrow N$.

To go in the opposite direction, let $g : \Phi^l_p M \to N$ and consider the composition $\widetilde{g} = g \circ \pi$, where π is the projection $M \to \Phi^l_p M$. Taking $L \in \mathcal{S}(M)$ containing m, $(\tau^n m)_p \in (\tau^n L)_p$, so if $n \gg 0$, $\pi m = 0$. Therefore, $\widetilde{g} \in \mathrm{Hom}^{(\tau)}_{k[z]}(M,N)$. Putting all the steps together, we have concluded the proof.

4.4.3 Proof of Proposition 4.43

$1 \Rightarrow 2$ Let $U \subseteq M$ be a witness to M being a Tate space, i.e. U is a finitely generated $k[[\tau]]$-module and $\tau^1(U)/U$ is finite dimensional. By the assumption that $M \in \widehat{\mathbf{Mod}}(\Delta^l_{\mathbb{A}^1})_p$, M has a particular basis of open submodules. We may choose an $N \subseteq U$ that is an open $k[[\tau]]\langle z\rangle$-submodule, and $M/N \in \mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$. In fact, $M/N \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$: as a $k[[\tau]]$-module, M/N is an extension of M/U by U/N, and there is an exact sequence

$$0 \longrightarrow \frac{\tau^{-1}(N)\cap U}{N} \longrightarrow \frac{\tau^{-1}(N)}{N} \longrightarrow \frac{\tau^{-1}(U)}{U}.$$

The last term in the sequence is finite dimensional by assumption. The first one is contained in the torsion finitely generated $k[[\tau]]$-module U/N, and therefore it is also finite dimensional. This shows that $M/N \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$. Since $N \subseteq U$, N is finitely generated over $k[[\tau]]$.

Lastly, if $f \notin p^{\tau^n}$ for any n, f acts as a unit because M is an inverse limit of modules in $\mathbf{Mod}(\Delta^l_{\mathbb{A}^1})_p$, on each of which f acts as a unit.

$2 \Rightarrow 3$ Let $N \subseteq M$ be the submodule in the assumption. Then we claim that $\{\tau^i N\}_{i\geq 0}$ is the required basis. It is easy to check that they are $k[[\tau]]\langle z\rangle$-submodules, given that N is one. Let us see that $M/\tau^i N \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$. For each i, consider the exact sequence

$$0 \longrightarrow \frac{N \cap \tau^{-1}\tau^i N}{\tau^i N} \longrightarrow \frac{\tau^{-1}\tau^i N}{\tau^i N} \longrightarrow \frac{\tau^{-1}N}{N}.$$

The dimension of $\frac{N\cap\tau^{-1}\tau^i N}{\tau^i N}$ is bounded above. This is true because N is a finitely generated $k[[\tau]]$-module and this claim can be checked by writing N as a direct sum of a finite module and a free module. Therefore,

$$\dim \frac{\tau^{-1}\tau^i N}{\tau^i N} \leq \dim \frac{N \cap \tau^{-1}\tau^i N}{\tau^i N} + \dim \frac{\tau^{-1}N}{N}.$$

Which shows that $\dim \tau^{-1}\tau^i N/\tau^i N$ is a bounded number. Let us show now that the $\tau^i N$ form a basis of open sets. They are indeed open since N is. If U is any other open subspace, then the fact that N is finitely generated combined with the definition of an open set shows that $\tau^i N \subseteq U$ for a big enough i.

Lastly, let us show that $M/\tau^i N \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$ for every i. It only remains to show that all its elements are supported on $p+\mathbb{Z}$. We consider the short exact sequence

$$0 \longrightarrow \frac{N}{\tau^i N} \longrightarrow \frac{M}{\tau^i N} \longrightarrow \frac{M}{N} \longrightarrow 0.$$

The first term in the sequence is finite dimensional, so the support of its elements is in $p+\mathbb{Z}$, since f acts as a unit on it whenever $f \notin p^{\tau^n}$ for all n. Therefore, $\operatorname{supp} M/\tau^i N \subseteq \operatorname{supp} N/\tau^i N \cup \operatorname{supp} M/N \subseteq p+\mathbb{Z}$.

$3 \Rightarrow 4$ Let $B = \{N_i\}$ be the basis in the statement. We must find a basis of finitely generated $k[[\tau]]$-modules with the required properties. It will be a subset of B, namely we will choose a fixed $N \in B$, and the new basis will consist of the elements of B contained in N. By Proposition 4.38, the quantity $\dim \tau^{-1}(N_i)/N_i$ is nondecreasing as $N_i \in B$ gets smaller. Let $N \in B$ be such that $\dim \tau^{-1}(N)/N = d$ is the maximum among elements $N_i \in B$. We will show that N is finitely generated over $k[[\tau]]$. From here, it follows that $B_N = \{N_i \in B : N_i \subseteq B\}$ is the required basis.

Consider some $N_i \in B_N$, and the following short exact sequence:

$$0 \longrightarrow \frac{N}{N_i} \longrightarrow \frac{M}{N_i} \longrightarrow \frac{M}{N} \longrightarrow 0.$$

Let $\pi_i : M \to M/N_i$ be the projection. Since $V^1 = \operatorname{Tor}_1^{k[[\tau]]}(k[[\tau]]/\tau k[[\tau]], V)$, the sequence above induces the following Tor exact sequence of finite dimensional vector spaces

$$0 \longrightarrow (\pi_i N)^1 \longrightarrow (\pi_i M)^1 \longrightarrow \left(\frac{M}{N}\right)^1 \longrightarrow \frac{\pi_i N}{\tau \pi_i N}.$$

The assumption that $\dim \tau^{-1}N/N$ is maximal implies that the dimension of the two spaces in the middle is equal, which in turn implies that $\dim \frac{\pi_i N}{\tau\pi_i N} \geq \dim(\pi_i N)^1$. By Lemma 4.39 this implies that $\pi_i N$ is finite dimensional, and by Nakayama's lemma it is generated by any system of generators for $\pi_i N/\tau\pi_i N$. Further, $\dim \pi_i N/\tau\pi_i N \leq \dim(\pi_i N)^1 \leq \dim(\pi_i M)^1 = d$.

Consider now the short exact sequences

$$0 \longrightarrow \frac{\tau^{-1}(N_i) \cap N}{N_i} \longrightarrow \pi_i N \xrightarrow{\tau} \pi_i N \longrightarrow \frac{\pi_i N}{\tau \pi_i N} \longrightarrow 0.$$

We claim that the inverse limit of these sequences as $i \to \infty$ is also exact. We can check the Mittag-Leffler conditions and then apply the results on exactness of inverse limits ([24, Tag 0598]). Splitting the exact sequence into two short exact sequences, we have that the Mittag-Leffler conditions hold because the spaces $\tau^{-1}(N_i)/N_i$ are finite dimensional, and because the maps $\tau\pi_i N \to \tau\pi_{i'} N$ are surjections, respectively. Therefore, the limit of the sequences is the exact sequence

$$0 \longrightarrow \frac{\tau^{-1}N}{N} \longrightarrow N \xrightarrow{\tau} N \longrightarrow \frac{N}{\tau N} \longrightarrow 0.$$

In particular, $N/\tau N = \lim_{\leftarrow} \pi_i N/\tau\pi_i N$. On the right hand side we have an inverse limit of surjections of finite dimensional vector spaces of dimension at most d, so all the maps are eventually isomorphisms. Lifting any given basis for $N/\tau N$ will generate all the modules $\pi_i N/\tau\pi_i N$, so by Nakayama's lemma it will generate all the $\pi_i N$'s (which applies since these are finite dimensional), and therefore it will generate N.

This shows that there is an element N in the basis B which is finitely generated over $k[[\tau]]$. Therefore, the basis $B_N = \{N_i \in B : N_i \subseteq N\}$ satisfies the required properties.

$4 \Rightarrow 1$ This is clear.

4.4.4 Proof of Proposition 4.35

Lemma 4.50. *The functor* $\iota^! : \widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p \to \mathbf{Mod}(k[z]_p)_{\mathrm{fin}}$ *is exact.*

Using this lemma, we will now prove the proposition. First of all, it is straightforward to check that $\iota^!\iota_! \cong \mathrm{Id}$, so it remains to show that $\iota_!\iota^! \cong \mathrm{Id}$. There is a natural map

$$\iota_!\iota^! M \longrightarrow M.$$

One can check that it is injective. Let us now show that the map $\iota_! \iota^! \to \mathrm{Id}$ is surjective. Let $P = M/\iota_!\iota^! M$. Let us show that $\iota^! P = 0$: applying $\iota^!$ to the short exact sequence $0 \to \iota_!\iota^! M \to M \to P \to 0$, we obtain

$$0 \longrightarrow \iota^!\iota_!\iota^! M \xrightarrow{\sim} \iota^! M \longrightarrow \iota^! P \longrightarrow 0.$$

The first arrow is an isomorphism because $\iota^!\iota_! \cong \mathrm{Id}$. Therefore, $\iota^! P = 0$.

Let us show that $\iota^! P = 0$ for $P \in \widehat{\mathcal{H}ol}^*(\Delta^l_{\mathbb{A}^1})_p$ implies that $P = 0$. Applying Proposition 4.43, let L be an open finitely generated $k[[\tau]]$ submodule $L \subset P$, such that $zL \subset L$ and $P/L \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$. Then we have a short exact sequence in $\widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$

$$0 \longrightarrow \iota^! L \longrightarrow \iota^! P \longrightarrow \iota^!(P/L) \longrightarrow 0.$$

If $\iota^! P = 0$, then it must be the case that $\iota^!(P/L) = 0$. If $P/L \neq 0$, then it has some nonzero element m supported at some point $\tau^{-j}(p)$ for some j. Then we may replace L by $\tau^j L$, and $\tau^j m$ is a nonzero element of $\iota^!(P/\tau^j L)$. Therefore, $P/\tau^j L = 0$. This implies that $P = \tau^j L$ is a $k((\tau))$-vector space which is finitely generated over $k[[\tau]]$, so indeed $P = 0$.

Proof of Lemma 4.50. In general, $\iota^!$ is left exact. In order to show that it is right exact as well, it will be enough to show that it maps surjections to surjections. First of all, consider a surjection in $\mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$. $f : M \longrightarrow N$. Let us show that $\iota^! f$ is also surjective. Since $M, N \in \mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$, we may write $M = \bigoplus_{i \in \mathbb{Z}} M_{p+i}$ and similarly for N. The morphism f sends each component M_{p+i} to N_{p+i}, so $\iota^! f$ becomes the map $\iota^! f : M_p \longrightarrow N_p$, which must necessarily be surjective.

Now let us consider the case where $M, N \in \widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$. First we need to prove that if $M = \lim\limits_{\leftarrow} M/L$, the natural map

$$\iota^! M = \iota^! \lim_{\leftarrow} \frac{M}{L} \longrightarrow \lim_{\leftarrow} \iota^! \frac{M}{L}.$$

is an isomorphism. It can be checked that it is injective. To see that it is surjective, a system of compatible elements $\{s_L\}$ on the right hand side corresponds to an element s of M, and we must show that this element is torsion. By Proposition 4.43 together with Lemma 4.39, there is a d such that $\dim \iota^! M/L \leq d$ for every L. In particular, if π generates p, $\pi^d \iota^! M/L = 0$, so s_L is annihilated by π^d for every L. This implies that $\pi^d s = 0$, so $s \in \iota^! M$ as we wished.

Suppose now that we have a surjection $f : M \longrightarrow N$ in $\widehat{\mathcal{H}ol}(\Delta^l_{\mathbb{A}^1})_p$. We must show that the corresponding map $\iota^! f : \iota^! M \to \iota^! N$ is an isomorphism. We can write $M = \lim M/L$ for L a basis of neighborhoods of 0. Further, since f is a surjection, it can be checked that $f(L)$ forms a basis of neighborhoods of 0 in N. Thus $\iota^! f$ can be seen as a map

$$\iota^! f : \iota^! \lim_{\leftarrow} \frac{M}{L} \longrightarrow \iota^! \lim_{\leftarrow} \frac{N}{f(L)}.$$

By the discussion above, the map is (naturally) isomorphic to

$$\lim_{\leftarrow} \iota^! f_L : \lim_{\leftarrow} \iota^! \frac{M}{L} \longrightarrow \lim_{\leftarrow} \iota^! \frac{N}{f(L)}.$$

Each of the maps in the limit is surjective. A sufficient condition for an inverse limit of surjective maps to be surjective is the arrows forming the limit being surjections themselves [24, Tag 0598]. This is the case, because we've already shown that $\iota^!$ is right exact when restricted to $\mathcal{H}ol(\Delta^l_{\mathbb{A}^1})_p$. This shows that $\iota^!$ is exact.

□

Chapter 5

Elliptic equations

The main goal of this chapter is to understand the relation between elliptic modules and various kinds of equivariant sheaves. This is based on [14]. We compare elliptic equations to equivariant sheaves on the elliptic curve and equivariant sheaves on its normalization. Also, we study how elliptic modules become equivariant sheaves on $\mathbb{P}^1$ as the elliptic curve becomes reducible, and finally we show what happens in the most degenerate cases, for non-reduced curves.

This comparison enables us to extend the notions of the local type, namely the contents of Definition 3.5 and Theorem 1.1, to the setting of elliptic equations. This is the content of Theorem 5.11.

Throughout, we will work over a field k of characteristic different from 2. Sometimes, we will require k to be perfect. We will let $E \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a curve of degree 2 over each component. If σ interchanges the components, we will require E to be preserved by σ. We will let $\pi_i : E \to \mathbb{P}^1$ be the projections, which we require to be finite. We will also let $\sigma_i : E \to E$ be the deck transformation of π_i, i.e. the unique involution such that $\pi_i \circ \sigma_i = \pi_i$. σ_i is shown to exist in Lemma 5.3. If E is reduced, we will let $\pi : \widetilde{E} \to E$ be its normalization, and we will abuse notation by using $\sigma, \sigma_1, \sigma_2$ to denote the lifts to $\widetilde{E}$.

The group G will always denote the infinite dihedral group generated by σ and σ_1. For a scheme C with an automorphism g and a local function f, we will use the notation $f^g = f \circ g$.

5.1 Elliptic equations as equivariant sheaves

Recall the definition of an elliptic module.

Definition 5.1. *A* ***symmetric elliptic difference module****, or* E***-module*** *for short, is a quasicoherent sheaf M on $\mathbb{P}^1$, together with an isomorphism $\mathcal{A} : \pi_1^* M \to \pi_2^* M$, subject to the condition that $\sigma^* \mathcal{A} = \mathcal{A}^{-1}$.*

We denote the category of E-modules as E-**Mod**. *A morphism $\phi \in \mathrm{Hom}_{\text{E-}\mathbf{Mod}}(M, N)$ of E-modules is a morphism ϕ of sheaves on $\mathbb{P}^1$ such that $\mathcal{A} \circ \pi_1^* \phi = \pi_2^* \phi \circ \mathcal{A}$.*

Remark 5.2. *The words "elliptic module" and "E-module" have been used to refer to other notions. However, these will not appear anywhere in this thesis, so from now on we will refer to these as elliptic modules or E-modules.*

Elliptic modules come with a $\mathbb{Z}/2\mathbb{Z}$-equivariant structure (for the action of σ), and the fact that they are pulled back from $\mathbb{P}^1$ means they have another $\mathbb{Z}/2\mathbb{Z}$-equivariant structure, for the action of σ_1 (Lemma 5.3). Together, they form an equivariant structure for the infinite dihedral group $G = \langle \sigma, \sigma_1 : \sigma^2 = \sigma_1^2 = 1\rangle$.

Elliptic modules as we have defined them are not equivalent to G-equivariant sheaves on E, but they do embed into these. The reason for the difference lies in the fixed points of σ_1: sheaves equivariant under the $\mathbb{Z}/2$-action of σ_1 are sheaves on the stack quotient of E by $\mathbb{Z}/2$, but $\mathbb{P}^1$ is just the coarse moduli space for this stack, and they differ exactly at the branch locus of π_1. The relation between these two is simple: sheaves that descend to $\mathbb{P}^1$ are the ones for which σ_1 acts as the identity on the (derived) fibers at ramified points. This is the content of Lemma 5.3. We now present the main three results of this section, followed by their proofs. In the next section we apply these

For the results of this section it is essential that the characteristic of k is not 2, as well as for Theorem 5.11, since it depends on these statements.

Lemma 5.3. *Let C be a smooth connected curve, and let $\pi : C' \to C$ be a finite flat map of degree 2. In this situation, there is a unique deck involution $\sigma : C' \to C'$ such that $\pi \circ \sigma = \pi$. Let $i : Y \hookrightarrow C'$ be the fixed scheme of σ, i.e. Y is cut out by the ideal sheaf $I_Y = \langle g - g^\sigma : g \in \mathcal{O}_{C'}\rangle$.*

Then π^* induces an equivalence between quasicoherent sheaves on C and $\mathbb{Z}/2\mathbb{Z}$-equivariant sheaves M on C' such that $Li_Y^*\mathcal{A}_\sigma = \mathrm{Id}$. Here $\mathcal{A}_\sigma : M \to \sigma^*M$ denotes the equivariant structure, and by $Li_Y^*\mathcal{A}_\sigma = \mathrm{Id}$ we mean that it agrees with the isomorphism $Li_Y^* \cong Li_Y^* \circ \sigma^*$ induced from $i_Y = \sigma \circ i_Y$.

Proposition 5.4. *Let $E \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a degree $(2,2)$ symmetric curve with no horizontal or vertical components. Let $\sigma : E \to E$ be the automorphism interchanging the factors, and let σ_1 be the deck transformation of $\pi_1 : E \to \mathbb{P}^1$. Let G be the infinite dihedral group generated by σ and σ_1. Let $i : Y \hookrightarrow E$ be the subscheme fixed by σ_1, i.e. the scheme cut out by the ideal sheaf $I_Y = \langle f - f^{\sigma_1} : f \in \mathcal{O}_E\rangle$. Then there is an equivalence between the following categories:*

- *E-modules.*
- *The full subcategory of G-equivariant sheaves on E such that $Li_Y^*\mathcal{A}_{\sigma_1} = \mathrm{Id}$, where Li_Y^* denotes the derived restriction to Y.*

The equivalence of categories maps an E-module M to π_1^*M with the equivariant structure such that $\mathcal{A}_\sigma = \mathcal{A}$ from the E-module structure, and $\mathcal{A}_{\sigma_1}$ is provided by Lemma 5.3.

Proposition 5.5. *Let $E \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ be a* ***reduced*** *degree $(2,2)$ symmetric curve with no horizontal or vertical components. Let the field k be perfect. Let $\pi : \widetilde{E} \to E$ be the normalization of E, let $\sigma : \widetilde{E} \to \widetilde{E}$ be the automorphism interchanging the factors, and let σ_i be the deck transformation of $\pi_i\pi : \widetilde{E} \to \mathbb{P}^1$ (note that $\sigma_2 = \sigma\sigma_1\sigma$). Let G be the* *infinite dihedral group generated by σ and σ_1. Finally, let Z be the singular set of E and let $i_Y : Y \hookrightarrow E$, resp. $i_{\widetilde{Y}} : \widetilde{Y} \hookrightarrow \widetilde{E}$ be the fixed scheme of σ_1.*

The pullback π^* induces an equivalence between the following categories:

- *E-modules which are flat at $\pi_1(Z) \subset \mathbb{P}^1$.*
- *The full subcategory of G-equivariant sheaves on $\widetilde{E}$ satisfying two conditions:*
  1. *At the points of $\pi^{-1}(Z)$, the sheaves are flat.*
  2. *$Li^*_{\widetilde{Y}}\mathcal{A}_{\sigma_1} = \mathrm{Id}$.*

  The equivalence of categories maps an E-module M to π_1^*M with the equivariant structure such that $\mathcal{A}_\sigma = \mathcal{A}$ coming from the E-module structure, and $\mathcal{A}_{\sigma_1}$ is provided by Lemma 5.3.

Proof of Lemma 5.3. Note that in the unramified case this boils down to étale descent for quasicoherent sheaves, [24, Tag 023T].

Let us start by explicitly showing the existence of σ. Since π is finite flat of degree 2, $\pi_*\mathcal{O}_{C'}$ is a locally free $\mathcal{O}_C$-module of rank 2. We will ommit π_* from the notation and just denote $\mathcal{O}_{C'} = \pi_*\mathcal{O}_{C'}$, since we will only talk of sheaves on C. Let us start by showing that $\mathcal{O}_{C'}/\mathcal{O}_C$ is locally free (the flatness implies that $\mathcal{O}_C \subset \mathcal{O}_{C'}$). Since $\mathcal{O}_C$ is (locally) a Dedekind domain, it suffices to show that it is torsion-free. Suppose it had torsion: let $y \in \mathcal{O}_{C'}, a, b \in \mathcal{O}_C$ be such that $ay = b$. The ideal $(a, b) \subset \mathcal{O}_C$ is locally free, so passing to a smaller open cover, we can assume that it is principal: thus we may assume that $a = ca'$, $b = cb'$ and $(a', b') = \mathcal{O}_C$. The flatness of $\mathcal{O}_{C'}$ implies that $c \in \mathcal{O}_C$ is not a zero divisor, so we have that $a'y = b'$. Since $\mathcal{O}_{C'}$ is finite over $\mathcal{O}_C$, y is integral over $\mathcal{O}_C$, i.e. there is a monic polynomial annihilating it: $\sum_{i=0}^n a_i y^i = 0$, where $a_n = 1$. Multiplying by a'^n, we have $\sum_{i=0}^n a'^{n-i} a_i b'^i = 0$, which implies that a' divides b'^n. The conditions that $(a', b') = \mathcal{O}_C$ together with $a'|b'^n$ imply that a' is a unit in $\mathcal{O}_C$. Therefore, $y = b'a'^{-1} \in \mathcal{O}_C$. This shows that $\mathcal{O}_{C'}/\mathcal{O}_C$ is locally free.

We have that both $\mathcal{O}_{C'}$ and $\mathcal{O}_{C'}/\mathcal{O}_C$ are locally free (of ranks 2 and 1, respectively). Consider an open cover over which they are both free, and for each open set choose a lifting $y' \in \mathcal{O}_{C'}$ of a generator of $\mathcal{O}_{C'}/\mathcal{O}_C$. Then (on a fixed open set), $\{1, y'\}$ is a basis of $\mathcal{O}_C$. Therefore, $y'^2 = ay' + b$ for some $a, b \in \mathcal{O}_C$. We replace y' by $y = y' - a/2$, so that $y^2 := x \in \mathcal{O}_C$. Thus we have shown that $\mathcal{O}_{C'}$ is locally of the form $\mathcal{O}_C[y]/(y^2 - x)$, and as an $\mathcal{O}_C$-module it is $\mathcal{O}_C \oplus y\mathcal{O}_C$. The action of σ is $\mathcal{O}_C$-linear and generated by $y \mapsto -y$. This action is independent of the choice of y: one checks directly that any other $\widetilde{y} \in \mathcal{O}_{C'}$ whose square is in $\mathcal{O}_C$ is an element of $\mathcal{O}_C \cdot y$, and therefore the σ-action is unique. Since this canonical action is preserved by localization, it can be glued over the open cover to yield the desired deck transformation. Notice that $\mathcal{O}_C = \mathcal{O}_{C'}^\sigma = \{\alpha \in \mathcal{O}_{C'} : \alpha^\sigma = \alpha\}$.

Now that we know the global existence of σ, we can see that the pullback π^* is a local construction on C. Therefore, it is enough to prove the statement on an open cover.

From now on, we will assume $C = \operatorname{Spec} R$ is affine, and $S := \mathcal{O}_{C'} = R[y]/(y^2 - x)$ for some $x \in R$.

For an R-module M, $\pi^* M = M \oplus yM$, and the natural isomorphism $\sigma^*\pi^* \cong (\pi\sigma)^* = \pi^*$ is the equivariant structure given by $\mathcal{A}_\sigma(m_1 + ym_2) = \sigma^*(m_1 - ym_2) = \sigma^* m_1 + y\sigma^* m_2$. Therefore, on $\pi^*M/y\pi^*M$ we see that $\mathcal{A}_\sigma$ induces the map $m \mapsto \sigma^* m$, while on $y^{-1}(0) \subseteq \pi^* M$, it induces the map $m \mapsto -\sigma^* m$, since $y^{-1}(0) \subseteq yM \subset \pi^* M$. Conversely, suppose we start with an S-module N with an equivariant structure $\mathcal{A}_\sigma$ such that $\mathcal{A}_\sigma m = \sigma^* m$ on N/yN, and $\mathcal{A}_\sigma m = -\sigma^* m$ for $m \in N$ such that $ym = 0$. In this case, we may split N into eigenspaces for $\overline{\sigma} = \sigma^* \circ \mathcal{A}_\sigma$: the $\mathbb{Z}/2\mathbb{Z}$-equivariance exactly imposes the condition that $\overline{\sigma}^2 = 1$, hence the eigenvalues are contained in $\{\pm 1\}$. Let $N = N_+ \oplus N_-$, where $N_\pm$ is the sub-R-module on which $\overline{\sigma}$ acts as $\pm \operatorname{Id}$. The above assumption on $\mathcal{A}_\sigma$ implies that $\ker y \subset N_-$ and that $N_- \subset \operatorname{im} y$, since y interchanges the eigenspaces. Therefore, $N = N_+ \oplus yN_+ = \pi^* N_+$, so choosing the eigenspace N_+ is the inverse to the pullback functor with the equivariant structure. It is straightforward to check that morphisms of R-modules are in bijection (via the pullback) with equivariant morphisms of S-modules.

It only remains to show that for an equivariant module N, the condition $Li_Y^* \mathcal{A}_\sigma = \operatorname{Id}$ is equivalent to the condition that $\overline{\sigma}$ acts as 1 on N/yN and as -1 on $y^{-1}(0) \subseteq N$. Using the presentation $S = R[y]/(y^2 - x)$, we see that $I_Y = \langle g^\sigma - g \rangle = yS$. A direct computation using the resolution $S \xrightarrow{y} S$ shows that $N/yN \cong L^0 i_Y^* N$ and $y^{-1}(0) \cong L^1 i_Y^* N$, yet these isomorphisms do not necessarily commute with $\mathcal{A}_\sigma$, as we will show.

We begin by constructing a free resolution of N that carries a compatible equivariant structure. First, split N into eigenspaces $N = N_+ \oplus N_-$ as before. Take generating sets of N_+ and N_- as R-modules and consider the free S-module generated by the union, which we will write $F_0 = F_0^+ \oplus F_0^-$ ($F_0^\pm$ is generated by a generating set of $N_\pm$). We have the surjection $d_0 : F_0^+ \oplus F_0^- \to N$, and its pullback $\sigma^* F_0^+ \oplus \sigma^* F_0^- \to \sigma^* N$. Next we extend the equivariant structure to F_0: For e a basis element of $F_0^\pm$, we let $\mathcal{A}_\sigma(e) = \pm \sigma^* e$. This ensures that we have the rightmost commutative square in the following diagram:

$$\begin{array}{ccccccccc}
F_2 & \xrightarrow{d_2} & F_1 & \xrightarrow{d_1} & F_0 & \xrightarrow{d_0} & N & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathcal{A}_\sigma} & & \downarrow{\scriptstyle \mathcal{A}_\sigma} & & \downarrow{\scriptstyle \mathcal{A}_\sigma} & & \downarrow{\scriptstyle \mathcal{A}_\sigma} & & \\
\sigma^* F_2 & \xrightarrow{\sigma^* d_2} & \sigma^* F_1 & \xrightarrow{\sigma^* d_1} & \sigma^* F_0 & \xrightarrow{\sigma^* d_0} & \sigma^* N & \longrightarrow & 0.
\end{array}$$

Now, let $K_0 = \ker d_0$. Notice that $\mathcal{A}_\sigma K_0 = \sigma^* K_0$, so K_0 inherits the equivariant structure. Thus, we can iterate the process to obtain the beginning of a free resolution $F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} N \to 0$ where every term is equivariant and the above diagram is commutative.

Let us write $i = i_Y$. Now, $Li^* N$ is represented by the complex $i^* F_\bullet = \cdots \to i^* F_2 \xrightarrow{i^* d_2} i^* F_1 \xrightarrow{i^* d_1} i^* F_0$ (the quasiisomorphism $Li^* F_\bullet \cong Li^* N$ is induced by the map $d_0 : F_0 \to N$), and $i^* M = M/yM$. We note that

$$L^0 i_Y^* N \cong H^0(i^* F_\bullet) = \operatorname{coker}(i^* F_1 \to i^* F_0) = \frac{F_0}{yF_0 + d_1 F_1} \xrightarrow[\cong]{d_0} \frac{N}{yN}.$$

The map d_0 commutes with $\overline{\sigma}$: therefore if $\overline{\sigma}$ acts as the identity on one side, it will do so in the other, as desired. For $L^1 i^* N$, we note the following:

$$L^1 i_Y^* N \cong H^{-1}(i^* F_\bullet) = \frac{\ker(i^* F_1 \to i^* F_0)}{\operatorname{im}(i^* F_2 \to i^* F_1)} = \frac{F_1 \cap d_1^{-1}(yF_0)}{yF_1 + d_2 F_2} =$$

$$= \frac{F_1 \cap d_1^{-1}(yF_0)}{yF_1 + d_1^{-1}(0)} \xrightarrow[\cong]{d_1} \frac{d_1 F_1 \cap yF_0}{yd_1 F_1}.$$

It is straightforward to check that d_1 induces an isomorphism. As before, d_1 commutes with $\overline{\sigma}$. Next, notice that y is not a zero divisor in S, because $y^2 \in R$ cannot be a zero divisor, as S is flat over R. Therefore, for any submodule $F' \subseteq F_0$, $y^{-1}(yF') = F'$, since F_0 is free. Therefore, y induces an isomorphism:

$$\frac{d_1 F_1 \cap yF_0}{yd_1 F_1} \xleftarrow[\cong]{y} \frac{y^{-1} d_1 F_1}{d_1 F_1}.$$

Notice that this map does not commute with $\overline{\sigma}$, but rather $y \circ \overline{\sigma} = -\overline{\sigma} \circ y$. Therefore, if the action of $\overline{\sigma}$ on $L^1 i^* N$ is 1, the action on the right hand side is given by -1. Finally, notice that d_0 maps $y^{-1} d_1 F_1 / d_1 F_1$ isomorphically into $y^{-1}(0) \subseteq N$, and that d_0 commutes with $\overline{\sigma}$. Notice that since $\sigma = \operatorname{Id}$ on Y, $\overline{\sigma} = \sigma^* \circ \mathcal{A}_\sigma = \mathcal{A}_\sigma$. This shows what we wished: if $\overline{\sigma}$ acts as 1 on N/yN and as -1 on $y^{-1}(0) \subset N$, then $\mathcal{A}_\sigma$ acts as the identity on $Li^* N$.

□

Proof of Proposition 5.4. Consider an E-module M, with $\mathcal{A} : \pi_1^* M \to \pi_2^* M$. Lemma 5.3 yields an equivariant sheaf structure on $\pi_1^* M$, $\mathcal{A}_{\sigma_1} : \pi_1^* M \to \sigma_1^* \pi_1^* M$, and $\mathcal{A}_{\sigma_1}$ is the identity at the ramification points. Making $\mathcal{A}_\sigma = \mathcal{A}$, we obtain a G-equivariant structure: the relations on G are generated by $\sigma^2 = \sigma_1^2 = \operatorname{Id}$, and indeed $\sigma^* \mathcal{A} \circ \mathcal{A}_\sigma = \operatorname{Id}$.

Let us now go in the opposite direction. Let $\widetilde{M}$ be an equivariant sheaf on E as in the statement. Lemma 5.3 shows that there's a unique $M \in \mathbf{QCoh}(\mathbb{P}^1)$ such that $\widetilde{M} = \pi_1^* M$ with the induced σ_1-equivariant structure. Further, $\mathcal{A}_\sigma$ induces an elliptic module structure on M.

It is straightforward to check that the constructions are functorial using the fact that Lemma 5.3 provides a functor, and that they are mutually inverse.

□

Proposition 5.5 requires some background. If E is singular and reduced, then the results of [8] allow us to relate quasicoherent sheaves on E with sheaves on its normalization $\widetilde{E}$. These results require flatness at the singular points, so we cannot have an equivalence (see Remark 5.8 for an example). However, we do have an equivalence between the full subcategories of flat sheaves in the equivariant setting, analogously to the theorem in loc. cit. We will recall it here for convenience.

This theorem describes the relation between modules over a fiber product of rings $B\times_{B'}A'$ and modules over B, B' and A'. We reproduce the statement and the constructions here for convenience. Start with a Cartesian square of rings, and the corresponding commutative square of pullbacks (i.e. tensors):

$$\begin{array}{ccc} B\times_{B'}A' & \longrightarrow & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & B' \end{array} \qquad \begin{array}{ccc} \mathbf{Mod}(B\times_{B'}A') & \longrightarrow & \mathbf{Mod}(A') \\ \downarrow & & \downarrow \\ \mathbf{Mod}(B) & \longrightarrow & \mathbf{Mod}(B') \end{array} \quad .$$

The diagram on the right hand side induces a functor T from $\mathbf{Mod}(B\times_{B'}A')$ to $\mathbf{Mod}(B)\times_{\mathbf{Mod}(B')}\mathbf{Mod}(A')$, which concretely is given by

$$T(M)=(B\otimes M, A'\otimes M, \cong)\,.$$

Recall that $\mathbf{Mod}(B)\times_{\mathbf{Mod}(B')}\mathbf{Mod}(A')$ is the category of triples consisting of a B-module N_B, an A'-module $M_{A'}$ and an isomorphism $\phi : B'\otimes N_B \cong B'\otimes N_{A'}$. In the definition of $T(M)$, this isomorphism is the canonical one. Ferrand constructs a right adjoint S to T, defined as follows: an object $N=(N_B, N_{A'}, \phi)$ is mapped to

$$S(N)=\{(n_B, n_{A'})\in N_B\times N_{A'} : \phi(1\otimes n_B)=1\otimes n_{A'}\}.$$

S is defined on morphisms in the obvious way. Théorème 2.2 in [8] includes the following statement.

Theorem 5.6 (Ferrand)**.** *For A', B', B, S, T as above, assume that $A'\to B'$ is surjective. Then S and T are inverse equivalences between the full subcategories of consisting of flat modules.*

Proof of Proposition 5.5. Let us start by showing that we are in the right situation to apply Theorem 5.6. Rings will be replaced by schemes affine over E, and analogous statements hold simply because modules and pullbacks are preserved by localization.

Let $\sigma_2=\sigma\sigma_1\sigma$ be the deck involution for π_2. Let $\widetilde{X}$ be the subscheme of $\widetilde{E}$ given as the fixed subscheme of $\sigma_1\sigma_2$. This is the subscheme cut out by the ideal sheaf $I_{\widetilde{X}}=\langle f-f^{\sigma_1\sigma_2} : f\in\mathcal{O}_{\widetilde{E}}\rangle=\langle f^{\sigma_1}-f^{\sigma_2} : f\in\mathcal{O}_{\widetilde{E}}\rangle$. Letting $X=\pi(\widetilde{X})$, we have a commutative square:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{i_{\widetilde{X}}} & \widetilde{E} \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{i_X} & E. \end{array} \tag{5.1}$$

Lemma 5.7. *With the notation above, $\mathcal{O}_E = \pi_*\mathcal{O}_{\widetilde{E}} \times_{i_{X*}\pi_*\mathcal{O}_{\widetilde{X}}} i_{X*}\mathcal{O}_X$. Further, X is the (affine scheme) quotient of $\widetilde{X}$ by the action of σ_1, so π_1 induces an isomorphism between X and its image. The support of X is exactly Z, the singular set of E. Here we assume that the field k is perfect and not of characteristic* 2.

Proof. Each of the two maps $\pi_i \circ \pi : \widetilde{E} \to \mathbb{P}^1$ is a Galois ramified cover with Galois group $\langle \sigma_i \rangle = \mathbb{Z}/2\mathbb{Z}$, so it identifies $\mathcal{O}_{\mathbb{P}^1}$ with $(\pi_{i*}\pi_*\mathcal{O}_{\widetilde{E}})^{\sigma_i}$, where the notation R^{σ_i} denotes $\{f \in R : f^{\sigma_i} = f\}$. Since E is the image of $\widetilde{E}$ in $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{O}_E$ is generated by functions on each of the $\mathbb{P}^1$ factors. Our first claim is that $\pi_*\mathcal{O}_{\widetilde{E}}^{\sigma_1} \boxtimes \pi_*\mathcal{O}_{\widetilde{E}}^{\sigma_2}$ generates $\mathcal{O}_E$. This statement must be understood in the following sense: there is a basis of open sets U of E such that $\mathcal{O}_E(U)$ is generated by σ_1-invariant functions in $\pi_*\mathcal{O}_{\widetilde{E}}(\pi^{-1}(U))$, together with σ_2-invariant functions in $\pi_*\mathcal{O}_{\widetilde{E}}(\pi^{-1}(U))$. In particular, if we say $f \in \mathcal{O}_{\widetilde{E}}(V)$ is σ_i-invariant, we mean that f^{σ_i} is regular on V as well. Further, we make the same claim about $\pi_{1*}\mathcal{O}_E$: we will show that there is a basis of open sets U of $\mathbb{P}^1$ such that $(\pi_{1*}\mathcal{O}_E)(U)$ is generated by σ_1-invariant functions and σ_2-invariant functions in $(\pi_1\pi)_*\mathcal{O}_{\widetilde{E}}(U)$. Note that all the rings of regular functions we mention can be thought of as contained in the ring of rational functions of $\widetilde{E}$ (recall that $\widetilde{E}$ might be disconnected, in which case its ring of rational functions is a sum of fields), so we can talk about containments and generation.

First, choose a basis of open sets of E of the form $V = E \cap (U_1 \times U_2)$, where $U_i \subseteq \mathbb{P}^1$ are affine open subschemes. The ring $\mathcal{O}_E(V)$ is generated by $\pi_i^*\mathcal{O}_{\mathbb{P}^1}(U_i) = (\pi_*\mathcal{O}_{\widetilde{E}}((\pi_i\pi)^{-1}(U_i)))^{\sigma_i}$, for $i = 1, 2$, where we can think of all the rings as contained in the ring of rational functions of $\widetilde{E}$. Since $\pi_i^{-1}(U_i) \supseteq V$, $\mathcal{O}_{\widetilde{E}}((\pi_i\pi)^{-1}(U_i)) \subseteq \mathcal{O}_{\widetilde{E}}(\pi^{-1}(V))$, so we have the desired statement on E: $\mathcal{O}_E(V) = \pi_*\mathcal{O}_{\widetilde{E}}(\pi^{-1}(V))^{\sigma_1} \cdot \pi_*\mathcal{O}_{\widetilde{E}}(\pi^{-1}(V))^{\sigma_2}$. Let us see what happens on $\mathbb{P}^1$: suppose we have an open set as above, $E \cap (U_1 \times U_2)$, and consider any open $U \subseteq \mathbb{P}^1$ such that $\pi_1^{-1}(U) \subseteq U_1 \times U_2$. In this case, we have the simple observation that $\pi_1^{-1}(U) = E \cap (U \times U_2)$, so the reasoning above applies, and therefore $\mathcal{O}_E(\pi_1^{-1}U)$ is generated by $(\pi_*\mathcal{O}_{\widetilde{E}}((\pi_1\pi)^{-1}(U)))^{\sigma_1}$ and $(\pi_*\mathcal{O}_{\widetilde{E}}((\pi_2\pi)^{-1}(U_2)))^{\sigma_2}$. By assumption, $\pi_1^{-1}(U) \subseteq \pi_2^{-1}(U_2)$, so

$$(\pi_*\mathcal{O}_{\widetilde{E}}((\pi_2\pi)^{-1}(U_2)))^{\sigma_2} \subseteq (\pi_*\mathcal{O}_{\widetilde{E}}((\pi_1\pi)^{-1}(U)))^{\sigma_2} \subseteq \mathcal{O}_E(\pi_1^{-1}U).$$

In particular, $\mathcal{O}_E(U)$ is generated by $(\pi_*\mathcal{O}_{\widetilde{E}}((\pi_1\pi)^{-1}(U)))^{\sigma_i}$ for $i = 1$ and $i = 2$. In particular, there is a basis for the topology on $\mathbb{P}^1$ over which the equation $\pi_{1*}\mathcal{O}_E = \pi_{1*}\pi_*\mathcal{O}_{\widetilde{E}}^{\sigma_1} \cdot \pi_{1*}\pi_*\mathcal{O}_{\widetilde{E}}^{\sigma_2}$ holds.

All four maps in the Diagram (5.1) are affine, as is the map $\pi_1 : E \to \mathbb{P}^1$. We will slightly abuse notation and use $\mathcal{O}_{\widetilde{X}}, \mathcal{O}_X, \mathcal{O}_{\widetilde{E}}, \mathcal{O}_E$ to refer to their pushforwards to $\mathbb{P}^1$ by the map π_1, taking advantage of the fact that schemes affine over $\mathbb{P}^1$ are equivalent to quasicoherent sheaves of $\mathcal{O}_{\mathbb{P}^1}$-algebras. Then, the statement we are tring to prove can be written $\mathcal{O}_E = \mathcal{O}_{\widetilde{E}} \times_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_X$. We will think of quasicoherent sheaves on a scheme Ξ affine over $\mathbb{P}^1$ as sheaves of modules over $\mathcal{O}_\Xi$. Further, the discussion above shows that we may think of $\pi_* \mathcal{O}_E$ as $\pi_{1*}\pi_* \mathcal{O}_{\widetilde{E}}^{\sigma_1} \cdot \pi_{2*}\pi_* \mathcal{O}_{\widetilde{E}}^{\sigma_2}$, which we will just abbreviate as $\mathcal{O}_{\widetilde{E}}^{\sigma_1}\mathcal{O}_{\widetilde{E}}^{\sigma_2}$. We have the following diagram:

$$\begin{array}{ccc} \mathcal{O}_E = \mathcal{O}_{\widetilde{E}}^{\sigma_1}\mathcal{O}_{\widetilde{E}}^{\sigma_2} & \hookrightarrow & \mathcal{O}_{\widetilde{E}} \\ \downarrow & & \downarrow \\ \frac{\mathcal{O}_E}{I_{\widetilde{X}} \cap \mathcal{O}_E} & \hookrightarrow & \mathcal{O}_{\widetilde{X}}. \end{array}$$

We claim it is Cartesian, by first showing that $I_{\widetilde{X}} \subset \mathcal{O}_E$: this is due to the fact that generators of $I_{\widetilde{X}}$ (on some small enough open set) can be written as $f - f^{\sigma_1\sigma_2} = (f + f^{\sigma_1}) - (f^{\sigma_1} + f^{\sigma_1\sigma_2}) \in \mathcal{O}_{\widetilde{E}}^{\sigma_1} + \mathcal{O}_{\widetilde{E}}^{\sigma_2} \subset \mathcal{O}_E$. Now, $\mathcal{O}_E$ is contained in the fiber product $\frac{\mathcal{O}_E}{I_{\widetilde{X}} \cap \mathcal{O}_E} \times_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\widetilde{E}}$, so we need to show the other containment: a local section in the fiber product is $s \in \mathcal{O}_{\widetilde{E}}$ such that $s + I_{\widetilde{X}} \in \mathcal{O}_E + I_{\widetilde{X}}$. Since $I_{\widetilde{X}} \subset \mathcal{O}_E$, it follows that $s \in \mathcal{O}_E$.

We have the desired cartesian square of sheaves of rings. Notice that $X = \pi(\widetilde{X}) = \operatorname{Spec} \mathcal{O}_E / I_{\widetilde{X}}$. Finally, to show that $X = \widetilde{X}/\langle \sigma_1 \rangle$, we need to show that $\left(\mathcal{O}_{\widetilde{E}}/I_{\widetilde{X}}\right)^{\sigma_1} = \mathcal{O}_E/I_{\widetilde{X}}$. First, $\mathcal{O}_E/I_{\widetilde{X}}$ is generated as a sheaf of rings by $\mathcal{O}_{\widetilde{E}}^{\sigma_1}$ and $\mathcal{O}_{\widetilde{E}}^{\sigma_2}$, so in order to show that $\left(\mathcal{O}_{\widetilde{E}}/I_{\widetilde{X}}\right)^{\sigma_1} \supseteq \mathcal{O}_E/I_{\widetilde{X}}$ we only need to check that $\mathcal{O}_{\widetilde{E}}^{\sigma_2} \subseteq \mathcal{O}_{\widetilde{E}}^{\sigma_1} + I_{\widetilde{X}}$. An element $f \in \mathcal{O}_{\widetilde{E}}^{\sigma_2}$ can be written as

$$f = \frac{f + f^{\sigma_1}}{2} + \frac{f - f^{\sigma_1}}{2} = \frac{f + f^{\sigma_1}}{2} + \frac{f - f^{\sigma_2\sigma_1}}{2} \in \mathcal{O}_{\widetilde{E}}^{\sigma_1} + I_{\widetilde{X}}.$$

For the other contaiment, let $f + I_{\widetilde{X}} \in \left(\mathcal{O}_{\widetilde{E}}/I_{\widetilde{X}}\right)^{\sigma_1}$, i.e. suppose $f^{\sigma_1} - f = g \in I_{\widetilde{X}}$. Then $g^{\sigma_1} = -g$, $f + g/2 \in \mathcal{O}_{\widetilde{E}}^{\sigma_1} \subset \mathcal{O}_E$ and $f + I_{\widetilde{X}} = f + g/2 + I_{\widetilde{X}}$, showing the desired containment.

Finally, let us show that the points of X are those where E is singular. First note that $I_{\widetilde{X}}$ is contained in the conductor of $\mathcal{O}_E \subseteq \mathcal{O}_{\widetilde{E}}$, since $I_{\widetilde{X}}\mathcal{O}_{\widetilde{E}} = I_{\widetilde{X}} \subseteq \mathcal{O}_E$, just because $I_{\widetilde{X}}$ is an ideal of $\mathcal{O}_{\widetilde{E}}$. Since the conductor is supported on the singular locus of E (i.e. the points where π is not an isomorphism), it follows that $\widetilde{X}$ contains $\pi^{-1}(Z)$.

For the other containment, suppose $p \in \widetilde{X}$, i.e. $\sigma_1 p = \sigma_2 p$. There are two possible situations, depending on whether $\sigma_1 p = p$. Start by assuming that $\sigma_1 p \neq p$. In this case, for $i = 1, 2$, $\pi_i \pi(p) = \pi_i \pi(\sigma_i p) = \pi_i \pi(\sigma_{3-i} p)$, which implies that the map $(\pi_1 \pi, \pi_2 \pi) : \widetilde{E} \to E \subset \mathbb{P}^1 \times \mathbb{P}^1$ identifies p and $\sigma_1 p$. Therefore, π is not an isomorphism around p, so the stalk of E at $\pi(p)$ is not normal, hence $\pi(p)$ is singular.

Assume now that $\sigma_1 p = p$, and let $m \subset \mathcal{O}_{\widetilde{E}}$ be the coresponding maximal ideal. Suppose σ_1 does not act as the identity on $\mathcal{O}_{\widetilde{E}}/m$. Then $(\mathcal{O}_{\widetilde{E}}/m)^{\sigma_1} \subsetneq \mathcal{O}_{\widetilde{E}}/m$. We have already seen that $\mathcal{O}_E/I_{\widetilde{X}} = (\mathcal{O}_{\widetilde{E}}/I_{\widetilde{X}})^{\sigma_1}$, and therefore

$$\frac{\mathcal{O}_E}{m \cap \mathcal{O}_E} \overset{I_{\widetilde{X}} \subset m \cap \mathcal{O}_E}{=} \frac{\mathcal{O}_E/I_{\widetilde{X}}}{(m \cap \mathcal{O}_E)/I_{\widetilde{X}}} = \frac{(\mathcal{O}_{\widetilde{E}}/I_{\widetilde{X}})^{\sigma_1}}{(m \cap \mathcal{O}_E)/I_{\widetilde{X}}} \subseteq \left(\frac{\mathcal{O}_{\widetilde{E}}}{m}\right)^{\sigma_1} \subsetneq \frac{\mathcal{O}_{\widetilde{E}}}{m}.$$

So, as before, π is not an isomorphism around p, so $\pi(p)$ is singular.

Lastly, suppose σ_1 acts as the identity on $\mathcal{O}_{\widetilde{E}}/m$. Then σ_1 acts linearly on m/m^2, which is a one dimensional $\mathcal{O}_{\widetilde{E}}/m$-vector space. Since σ_1 is an involution, it acts as -1 or as 1. Suppose it acts as 1: then for a generator f of m, we have that $f^{\sigma_1} \in f + m^2$. Therefore, for any n, $(f^n)^{\sigma_1} \in f^n + f^{n-1}m^2 = f^n + m^{n+1}$, so σ_1 acts as the identity on the completion of $\mathcal{O}_{\widetilde{E}}/m$, so it acts as the identity on the connected component of $\widetilde{E}$ containing m. Since we are assuming that E is reduced, this cannot happen. Therefore, σ_1 acts as (-1) on m/m^2.

We are left with the situation where σ_1 acts as the identity on $\mathcal{O}_{\widetilde{E}}/m$ and as (-1) on m/m^2, and so does σ_2, since $p \in \widetilde{X}$, the subscheme where $\sigma_1 = \sigma_2$. Let us prove that $I_{\widetilde{X}} \subset m^2$, i.e. that $\sigma_1 = \sigma_2 \mod m^2$. The map $\sigma_1 - \sigma_2$ is a k-linear derivation of $\mathcal{O}_{\widetilde{E}}/m$ with values in m/m^2: first of all, if $a \in m$, then $a^{\sigma_1} \equiv a^{\sigma_2} \equiv -a \mod m^2$, and $\sigma_1 = \sigma_2 = 1$ when they act on $\mathcal{O}_{\widetilde{E}}/m$, so it is a k-linear map as desired. Notice further that $m/m^2 \cong \mathcal{O}_{\widetilde{E}}/m$ as $\mathcal{O}_{\widetilde{E}}/m$-vector spaces. Finally, we can check it is indeed a derivation: for any $a, b \in \mathcal{O}_{\widetilde{E}}/m$ and any lifts to $\mathcal{O}_{\widetilde{E}}/m^2$, we have that

$$(ab)^{\sigma_1-\sigma_2} - a(b^{\sigma_1-\sigma_2}) - b(a^{\sigma_1-\sigma_2}) = (a-a^{\sigma_1})(b-b^{\sigma_1}) - (a-a^{\sigma_2})(b-b^{\sigma_2}) \in m^2 + m^2 = m^2.$$

Finally, since k is perfect, the finite field extension $k \subseteq \mathcal{O}_{\widetilde{E}}/m$ is separable, and therefore the only k-linear derivation of $\mathcal{O}_{\widetilde{E}}/m$ is 0, so $\sigma_1 = \sigma_2 \mod m^2$ as desired.

Therefore, $I_{\widetilde{X}} \subseteq m^2$, and $(\mathcal{O}_{\widetilde{E}}/m^2)^{\sigma_1} \subsetneq \mathcal{O}_{\widetilde{E}}/m^2$. As before, we have that

$$\frac{\mathcal{O}_E}{m^2 \cap \mathcal{O}_E} \overset{I_{\widetilde{X}} \subset m^2 \cap \mathcal{O}_E}{=} \frac{\mathcal{O}_E/I_{\widetilde{X}}}{(m^2 \cap \mathcal{O}_E)/I_{\widetilde{X}}} = \frac{(\mathcal{O}_{\widetilde{E}}/I_{\widetilde{X}})^{\sigma_1}}{(m^2 \cap \mathcal{O}_E)/I_{\widetilde{X}}} \subseteq \left(\frac{\mathcal{O}_{\widetilde{E}}}{m^2}\right)^{\sigma_1} \subsetneq \frac{\mathcal{O}_{\widetilde{E}}}{m^2}.$$

Therefore, π is not an isomorphism around p, so p is a singular point.

□

Proposition 5.4 shows that the category of E-modules is equivalent to the category of G-equivariant sheaves on E with the condition that $Li_Y^*\mathcal{A}_{\sigma_1} = \mathrm{Id}$. The pullback functor maps sheaves on $\mathbb{P}^1$ flat at $\pi_1(Z)$ to sheaves flat at Z, so the restriction is an equivalence between the desired subcategory of E-modules and the category of G-equivariant sheaves on E which are flat at Z and such that $Li_Y^*\mathcal{A}_{\sigma_1} = \mathrm{Id}$.

Let us start by showing how π^* maps equivariant modules to equivariant modules. Let $M \in$ G-$\mathbf{Mod}(E)$: we have the maps $\pi^*\mathcal{A}_\sigma : \pi^*M \to \pi^*\sigma^*M = \sigma^*\pi^*M$, and $\sigma^*(\pi^*\mathcal{A}) = \pi^*\sigma^*\mathcal{A} = (\pi^*\mathcal{A})^{-1}$. Similarly, we have $\pi^*\mathcal{A}_{\sigma_1}$ and both maps together make π^*M G-equivariant. If M is flat at Z, π^*M is flat at $\pi^{-1}(Z)$. Further, suppose $Li_Y^*\mathcal{A}_{\sigma_1} = \mathrm{Id}$. Then, considering the restriction $\pi : \widetilde{Y} \to Y$, we have that $Li_{\widetilde{Y}}^*L\pi^*\mathcal{A}_{\sigma_1} = L\pi^*Li_Y^*\mathcal{A}_{\sigma_1} = \mathrm{Id}$. Now, note that on a neighborhood of the points of $Y \setminus Z$, π is an isomorphism, and therefore $L\pi^* = \pi^*$. On the other hand, on a neighborhood of the points of $Y \cap Z$, we are assuming that π_1^*M is flat, and therefore $L\pi^*\mathcal{A}_{\sigma_1} = \pi^*\mathcal{A}_{\sigma_1}$. Therefore, $Li_{\widetilde{Y}}^*\pi^*\mathcal{A}_{\sigma_1} = \mathrm{Id}$ as desired. This provides a functor going one way.

Let us now construct the inverse to π^*. Given Lemma 5.7, we are in the situation where Theorem 5.6 applies. We have the adjoint pair of the descent functor $S : \mathbf{QCoh}(\widetilde{E}) \times_{\mathbf{QCoh}(\widetilde{X})} \mathbf{QCoh}(X) \to \mathbf{QCoh}(E)$ and its right adjoint T given by pullbacks to $\widetilde{E}$ and X. S is given on objects by mapping a triple $N_{\widetilde{E}} \in \mathbf{QCoh}(\widetilde{E})$, $N_X \in \mathbf{QCoh}(X)$ and $\phi : i^*_{\widetilde{X}} N_{\widetilde{E}} \cong \pi^* N_X$ to

$$S(N_{\widetilde{E}}, \phi, N_X) = \{(s_{\widetilde{E}}, s_X) \in \pi_* N_{\widetilde{E}} \times i_{X*} N_X : \phi(s_{\widetilde{E}}) = s_X\}.$$

Consider $\widetilde{M} \in$ G-$\mathbf{Mod}(\widetilde{E})$ satisfying the hypotheses in the statement. From $\widetilde{M}$ we construct an object in $\mathbf{QCoh}(\widetilde{E}) \times_{\mathbf{QCoh}(\widetilde{X})} \mathbf{QCoh}(X)$: The $\langle \sigma_1 \rangle$-equivariant structure $\mathcal{A}_{\sigma_1}$ satisfies the hypothesis of Lemma 5.3, so there is a sheaf $M \in \mathbf{QCoh}(\mathbb{P}^1)$ such that $\pi^* \pi_1^* M = \widetilde{M}$ with this equivariant structure. Take $T(\pi_1^* M) = (\widetilde{M}, i_X^* \pi_1^* M)$ to be the desired object. Since $\widetilde{M}$ is flat at $\pi^{-1}(Z)$, M is flat (i.e. torsion-free) at $\pi(Z)$, and $\pi_1^* M$ is flat at Z. Equivalently, by the local criterion for flatness, $\pi_1^* M$ is flat at X: Lemma 5.7 shows Z and X have the same support. Theorem 5.6 then implies that $\pi_1^* M$ and $T(\pi_1^* M)$ are in the categories on which T and S are inverse equivalences, so in particular we have the natural isomorphism $\pi_1^* M \to S(T(\pi_1^* M))$.

To give M the structure of an E-module, we need to construct a σ-equivariant structure. The σ-equivariant structure of $\widetilde{M}$ can be enhanced to one on $T(\pi_1^* M)$, by simply restricting $\mathcal{A}_\sigma$ to $\widetilde{X}$ (note that $\widetilde{X}$ is G-invariant). Now, we simply take $S(\mathcal{A}_\sigma) : S(\widetilde{M}) \to S(\sigma^* M)$. From the definition of S above, it is clear that $S(\sigma^* M)$ is naturally isomorphic to $\sigma^* S(M)$, providing the desired equivariant structure, since indeed $\sigma^* S(\mathcal{A}_\sigma) \circ S(\mathcal{A}_\sigma) = \mathrm{Id}$.

This construction is a functor: a morphism of G-equivariant sheaves $\widetilde{f} : \widetilde{M} \to \widetilde{N}$ is mapped to a morphism $f : M \to N$ of sheaves on $\mathbb{P}^1$ since Lemma 5.3 provides a functor (and σ_1-equivariant maps descend to $\mathbb{P}^1$). Further, $\pi_1^* f$ will be σ_1-equivariant.

It remains to show that if we start with a morphism of G-equivariant sheaves, then $\pi_1^* f$ will be σ-equivariant. Suppose that $\sigma^* \widetilde{f} \circ \mathcal{A}_\sigma = \mathcal{A}_\sigma \circ \widetilde{f}$. We have the morphism $T(\pi_1^* f) : T(\pi_1^* M) \to T(\pi_1^* N)$, and we want to show that $\sigma^* T(\pi_1^* f) \circ \mathcal{A}_\sigma = \mathcal{A}_\sigma \circ T(\pi_1^* f)$. For this, we need the identity to hold on $\widetilde{E}$ and on X. It holds on $\widetilde{E}$ by hypothesis, and in order to hold on X, must have that

$$i_X^* \pi_2^* f \circ i_X^* \mathcal{A}_\sigma = i_X^* \sigma^* \mathcal{A}_\sigma \circ i_X^* \pi_1^* f.$$

It is true that π^* applied to the above equation holds, since $\pi^* i_X^* \pi_1^* f = i^*_{\widetilde{X}} \widetilde{f}$. Now, $\pi : \widetilde{X} \to X$ is the restriction of $\pi_1 \pi : \widetilde{E} \to \mathbb{P}^1$ to the preimage of $\pi_1(X) \cong X$, so it is a faithfully flat map. Therefore, π^* from X to $\widetilde{X}$ is a faithful functor, and the above equation holds since it holds after taking π^*. Now we have that $\sigma^* T(\pi_1^* f) \circ \mathcal{A}_\sigma = \mathcal{A}_\sigma \circ T(\pi_1^* f)$. Applying S to this equation we have the desired equivariance of f.

Given that S and T are mutually inverse, it is straightforward to check that the functors we have constructed are mutually inverse.

□

Remark 5.8. *Notice that the condition of flatness at Z is indeed necessary. Consider the following example: Let an affine open set of E be cut out by the equation $(y-qx)(y-q^{-1}x)=0$, for some $q\in k^\times$ with $q^2\neq 1$. Then $\widetilde{E}$ is the disjoint union of two lines: $\widetilde{E}=\operatorname{Spec} k[t_1]\times k[t_2]$, where π is given by $x\mapsto(t_1,t_2)$ and $y\mapsto(qt_1,q^{-1}t_2)$. The dihedral group G acts as follows:*

$$\begin{array}{lrcl|lrcl} \sigma_1: & x & \leftrightarrow & x & \sigma: & x & \leftrightarrow & y \\ & y & \leftrightarrow & (q+q^{-1})x-y & & y & \leftrightarrow & x \\ & (t_1,0) & \leftrightarrow & (0,t_2) & & (t_1,0) & \leftrightarrow & (0,q^{-1}t_2) \end{array}$$

We can consider the following G-equivariant sheaf on $\widetilde{E}$: let $\widetilde{M}=k[t_1]/(t_1)\times k[t_2]/(t_2)$, and let s_i be a basis element for $k[t_i]/(t_i)$. Consider the following equivariant structure:

$$\mathcal{A}_{\sigma_1}(s_i)=s_i;\mathcal{A}_\sigma(s_i)=s_{2-i};\quad i=1,2$$

This equivariant structure satisfies the condition that $Li^*_{\widetilde{Y}}\mathcal{A}_{\sigma_1}=\mathrm{Id}$, since $\widetilde{Y}$ is empty. Indeed it descends to $M=k[x]/(x)$ on $\operatorname{Spec} k[x]$.

There is no E-module whose pullback is $\widetilde{M}$: it would have to be supported on $M=k[x]/(x)$. However, $\pi_1^*M\cong k[x,y]/((y-qx)(y-q^{-1}x),x)\cong k[x,y]/(y^2,x)$ is not isomorphic to $\pi_2^*M\cong k[x,y]/(y,x^2)$. Therefore, M supports no elliptic modules.

Remark 5.9. *The functor from E-modules to G-equivariant sheaves on $\widetilde{E}$ constructed above from $\pi^*\pi_1^*$ is defined for any E-module, without the flatness assumption. The functor defined this way on the whole category of E-modules is faithful, but not full in general. Consider two E-modules M and N, with their corresponding elliptic structures which we will denote by $\mathcal{A}$ in both cases.*

Lemma 5.3 ensures that $\pi^*\pi_1^*$ is a bijection between morphisms of sheaves from M to N and morphisms of $\mathbb{Z}/2\mathbb{Z}\langle\sigma_1\rangle$-equivariant sheaves from $\pi^*\pi_1^*M$ to $\pi^*\pi_1^*N$. Therefore, the map $\pi^*\pi_1^*:\mathrm{Hom}_{\mathbf{E\text{-}Mod}}(M,N)\to\mathrm{Hom}_G(\pi^*\pi_1^*M,\pi^*\pi_1^*N)$ is injective, since it is the restriction of the bijection

$$\pi^*\pi_1^*:\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(M,N)\xrightarrow{\sim}\mathrm{Hom}_{\mathbb{Z}/2\mathbb{Z}}(\pi^*\pi_1^*M,\pi^*\pi_1^*N).$$

Let us now show by example that the functor is not full. Consider the curve E from Remark 5.8, with q a primitive cubic root of unity, so an affine open set of E is cut out by the equation $y^2+xy+x^2=0$. We will construct two nonisomorphic elliptic modules M_1,M_2 whose pullbacks to $\widetilde{E}$ are isomorphic. For both modules, the underlying sheaf is the module $k[x]/(x^3)$. Let s_i be the generator for M_i. We define the elliptic module structures by

$$\mathcal{A}_1\pi_1^*s_1=\pi_2^*s_1;\mathcal{A}_2\pi_1^*s_2=(1+x^2y)\pi_2^*s_2.$$

When pulled back to $\widetilde{E}$, they both take the form $\mathcal{A}:\pi^*\pi_1^*s_i\mapsto\pi^*\pi_2^*s_i$, so they become isomorphic by mapping s_1 to s_2. However, there are no nonzero maps from the elliptic *module M_1 to M_2. Such a map would take the form $f(s_1) = (a_0 + a_1 x + a_2 x^2)s_2$. The relation $\mathcal{A}_2 \circ \pi_1^* f = \pi_2^* f \circ \mathcal{A}_1$ amounts to*

$$(a_0 + a_1 x + a_2 x^2 + a_0 x^2 y)\pi_2^* s_2 = (a_0 + a_1 x + a_2 x^2)(1 + x^2 y)\pi_2^* s_2 =$$

$$= \mathcal{A}_2 \left((a_0 + a_1 x + a_2 x^2)\pi_1^* s_2\right) = \mathcal{A}_2 \left(\pi_1^*(f(s_1))\right) = \mathcal{A}_2 \left(\pi_1^* f(\pi_1^* s_1)\right) =$$
$$= \pi_2^* f \left(\mathcal{A}_1(\pi_1^* s_1)\right) = \pi_2^* f \left(\pi_2^* s_1\right) = (a_0 + a_1 y + a_2 y^2)\pi_2^* s_2.$$

The only solution to the equation $a_0 + a_1 x + a_2 x^2 + a_0 x^2 y \equiv a_0 + a_1 y + a_2 y^2 \mod (y^2 + xy + x^2, x^3)$ corresponds to the zero morphism.

Remark 5.10. *With Proposition 5.4 in mind, it seems that there are several reasonable definitions for elliptic modules. One could consider the whole category of G-equivariant sheaves on E, which as explained in said Proposition contains* E-**Mod** *as a full subcategory. Alternatively, one could force σ and σ_1 to play symmetric roles by requiring that $\mathcal{A}_\sigma$ act as the identity on the fixed locus of σ, and considering this full subcategory of the one we are calling* E-**Mod** *in this paper.*

Also notice that there are two very different behaviors depending on whether $\sigma_1\sigma$ has finite order. If $(\sigma_1\sigma)^n = \mathrm{Id}_E$, then the composition $(\overline{\sigma_1\sigma})^n$ is an automorphism of $\pi_1^* M$. An interesting full subcategory of elliptic modules is the full subcategory of modules for which this automorphism is the identity. In other words, one might consider sheaves equivariant for a finite dihedral group, rather than the infinite dihedral group.

5.2 The local type of elliptic equations

In light of Proposition 5.4, we can apply Theorem 1.1 to elliptic modules.

Theorem 5.11. *Let E and G be as in Proposition 5.4. Let $p \in E$ be a closed point and let $E^* = E \setminus Gp$. For any scheme on which σ_1 acts, we will denote without ambiguity Y as the fixed scheme of σ_1. Let $p \in E$. Let* G-$\mathbf{Mod}^{\mathrm{gfg}}(E)^\circ$ *(resp.* G-$\mathbf{Mod}^{\mathrm{gfg}}(E^*)^\circ$*) be the full subcategory of* G-$\mathbf{Mod}^{\mathrm{gfg}}(E)$ *(resp.* G-$\mathbf{Mod}^{\mathrm{gfg}}(E^*)$*) consisting of modules for which $Li_Y^* \mathcal{A}_{\sigma_1} = \mathrm{Id}$.*

To define G-$\mathbf{Mod}(U_p)^\circ$*, for every $g \in G$ we will let $Y_g = g^{-1}Y$ be the fixed scheme of $g^{-1}\sigma_1 g$ intersected with the formal neighborhood of p, in particular Y_g is empty unless $\sigma_1 gp = gp$. Then, we let* G-$\mathbf{Mod}(U_p)^\circ$ *be the full subcategory of* G-$\mathbf{Mod}(U_p)$ *consisting of modules for which $Li_{Y_g}^* \mathcal{A}_{g^{-1}\sigma_1 g} = \mathrm{Id}$, for every $g \in G$. We are denoting the embedding of Y_g into U_p by i_{Y_g}.*

Then the restriction of the functors $|_{U_p}$ and $|_{C^*}$ induces an equivalence between G-$\mathbf{Mod}^{\mathrm{gfg}}(E)^\circ$ *and the fiber product* G-$\mathbf{Mod}(U_p)^\circ \times_{\text{G-}\mathbf{Mod}(U_p^*)}$ G-$\mathbf{Mod}^{\mathrm{gfg}}(E^*)^\circ$*.*

Proof. Clearly $|_{C^*}$ maps G-$\mathbf{Mod}^{\mathrm{gfg}}(E)^\circ$ into G-$\mathbf{Mod}(E^*)^\circ$. Also, $|_{U_p}$ maps objects in G-$\mathbf{Mod}^{\mathrm{gfg}}(E)^\circ$ into G-$\mathbf{Mod}(U_p)^\circ$: if $Li_Y^* \mathcal{A}_{\sigma_1} = \mathrm{Id}$, then we use the following identity, which comes from applying Definition 2.1 and Remark 2.3:

$$Li^*_{Y_g}\mathcal{A}_{g^{-1}\sigma_1 g} = Li^*_{Y_1}(g^{-1})^*\left(g^*\mathcal{A}_{g^{-1}\sigma_1} \circ \mathcal{A}_g\right) =$$

$$= Li^*_{Y_1}\left(\mathcal{A}_{g^{-1}\sigma_1} \circ (g^{-1})^*\mathcal{A}_g\right) = Li^*_{Y_1}\left(\sigma_1^*\mathcal{A}_{g^{-1}} \circ \mathcal{A}_{\sigma_1} \circ (g^{-1})^*\mathcal{A}_g\right).$$

Therefore, if $Li^*_{Y_1}\mathcal{A}_{\sigma_1} = \mathrm{Id}$, we have that

$$Li^*_{Y_g}\mathcal{A}_{g^{-1}\sigma_1 g} = Li^*_{Y_1}\left(\sigma_1^*\mathcal{A}_{g^{-1}} \circ (g^{-1})^*\mathcal{A}_g\right) \overset{\sigma_1\circ i_1 = i_1}{=} Li^*_{Y_1}\left(\mathcal{A}_{g^{-1}} \circ (g^{-1})^*\mathcal{A}_g\right) = Li^*_{Y_1}\mathcal{A}_1 = \mathrm{Id}.$$

Applying Theorem 1.1, G-$\mathbf{Mod}^{\mathrm{gfg}}(E) \cong$ G-$\mathbf{Mod}(U_p) \times_{\text{G-}\mathbf{Mod}(U_p^*)}$ G-$\mathbf{Mod}^{\mathrm{gfg}}(E^*) \supseteq$ G-$\mathbf{Mod}(U_p)^\circ \times_{\text{G-}\mathbf{Mod}(U_p^*)}$ G-$\mathbf{Mod}^{\mathrm{gfg}}(E^*)^\circ$ as a full subcategory, which itself contains G-$\mathbf{Mod}^{\mathrm{gfg}}(E)^\circ$ by the discussion above. It only remains to prove that G-$\mathbf{Mod}^{\mathrm{gfg}}(E)^\circ \supseteq$ G-$\mathbf{Mod}(U_p)^\circ \times_{\text{G-}\mathbf{Mod}(U_p^*)}$ G-$\mathbf{Mod}^{\mathrm{gfg}}(E^*)^\circ$. Since we are dealing with full subcategories, we only need to check the containment of objects: we need to prove that for $M \in$ G-$\mathbf{Mod}^{\mathrm{gfg}}(E)$, if $Li^*_Y\mathcal{A}_{\sigma_1}$ acts as the identity on both $Li^*_Y M|_{E^*}$ and $Li^*_{Y_g} M|_{U_p}$ for every g, then $Li^*_Y\mathcal{A}_{\sigma_1}$ is the identity on $Li^*_Y M$ as well.

Let us show this: Let K be the image of $Li^*_Y\mathcal{A}_{\sigma_1} - \mathrm{Id}$. Since $(Li^*_Y\mathcal{A}_{\sigma_1} - \mathrm{Id})|_{E^*} = 0$ and $|_{E^*}$ is an exact functor, K is supported away from E^* i.e. on Gp. Since $|_{U_p}$ is an exact functor, we also have that the formal fiber K_p vanishes. We are left with the points gp in the orbit of p. If gp is fixed by σ_1, we use the equation above: $\mathrm{Id} = Li^*_{Y_g}\mathcal{A}_{g^{-1}\sigma_1 g} = Li^*_{Y_1}(\mathcal{A}_{g^{-1}} \circ \mathcal{A}_{\sigma_1} \circ (g^{-1})^*\mathcal{A}_g)$, which implies that $Li^*_{Y_1}\mathcal{A}_{\sigma_1} = \mathrm{Id}$ at the stalk around gp as well. Therefore, all the stalks of K vanish, so indeed $M \in$ G-$\mathbf{Mod}^{\mathrm{gfg}}(E)^\circ$. □

5.3 Relation to other discrete equations and to D-modules

Elliptic equations generalize discrete equations such as difference equations, i.e. sheaves equivariant under $z \mapsto z + 1$, and q-equations, i.e. sheaves equivariant under $z \mapsto qz$, where $q \in k^\times$ is fixed (note that up to a change of coordinates on $\mathbb{P}^1$ these are all the automorphisms). This happens when the curve E is reducible, in which case its components have degree $(1, 1)$ (since they are not allowed to be fibers), and therefore each component is the graph Γ_τ of an automorphism τ of $\mathbb{P}^1$. Since E is preserved by interchanging the coordinates there are two possiblities: either the components are interchanged, in which case they are the graph of an automorphism τ and its inverse (which must be different from τ, so $\tau^2 \neq 1$); or they are both preserved, in which case we have the graphs of two different involutions, one of which could possibly be the identity.

In the case where $E = \Gamma_\tau \cup \Gamma_{\tau^{-1}}$, elliptic equations are strongly related to τ-equivariant sheaves on $\mathbb{P}^1$, which are difference equations if τ is $z \mapsto z+1$ and q-equations if τ is $z \mapsto qz$ (note that these are the only possibilities up to a change of coordinates). Away from the fixed points of τ, the notions of an E-module and a τ-equivariant sheaf are equivalent, and this equivalence can be extended over the special points for flat sheaves, as Proposition 5.12 shows.

Notice that the fixed **geometric** points of τ are the images of the singular geometric points of $\Gamma_\tau \cup \Gamma_{\tau^{-1}}$. In the situation where $E = \Gamma_{\tau_1} \cup \Gamma_{\tau_2}$, the singular geometric points are the preimages of the points p for which $\tau_1 p = \tau_2 p$, or equivalently fixed geometric points of $\tau_1\tau_2$.

Proposition 5.12. *Let k be perfect and not of characteristic 2. Suppose $\tau \in \mathrm{Aut}(\mathbb{P}^1)$ is such that $\tau^2 \neq 1$. Let $E = \Gamma_\tau \cup \Gamma_{\tau^{-1}}$ and let Z be the fixed scheme of τ. Then the following categories are equivalent:*

1. *τ-equivariant sheaves on $\mathbb{P}^1$ which are flat at Z.*
2. *E-modules on the curve $E = \Gamma_\tau \cup \Gamma_{\tau^{-1}}$ which are flat at Z.*

Suppose we are given $\tau_1 \neq \tau_2 \in \mathrm{Aut}(\mathbb{P}^1)$ such that $\tau_j^2 = \mathrm{Id}$, and $E = \Gamma_{\tau_1} \cup \Gamma_{\tau_2}$. Let $\widetilde{G}$ be the infinite dihedral group generated by τ_1 and τ_2, acting on $\mathbb{P}^1$ (the action is not necessarily faithful, for example if $\tau_1 = \mathrm{Id}$). Let Z be the fixed scheme of $\tau_1\tau_2$. Then the following categories are equivalent:

1. *$\widetilde{G}$-equivariant sheaves on $\mathbb{P}^1$ which are flat at Z.*
2. *E-modules on the curve $E = \Gamma_{\tau_1} \cup \Gamma_{\tau_2}$ which are flat at Z.*

Proof. Applying Proposition 5.5, we have that E-modules which are flat at the singular points are equivalent to modules equivariant for the action of the dihedral group, and which are flat at the preimages of these singular points. The condition that $Li^*_{\widetilde{Y}}\mathcal{A}_{\sigma_1} = \mathrm{Id}$ doesn't come into play, because in this case σ_1 acts freely on $\Gamma_\tau \sqcup \Gamma_{\tau^{-1}}$, since it interchanges the two components.

It remains to check that G-equivariant sheaves on $\Gamma_\tau \sqcup \Gamma_{\tau^{-1}}$ (which are flat at $\pi^{-1}(Z)$) are equivalent to τ-equivariant sheaves on $\mathbb{P}^1$ (which are flat at Z). Given such an equivariant sheaf M on $\mathbb{P}^1$, we may pull it back by the projection $\pi_1 : \Gamma_\tau \sqcup \Gamma_{\tau^{-1}} \to \mathbb{P}^1$, and it automatically becomes $\frac{\mathbb{Z}}{2\mathbb{Z}}\langle\sigma_1\rangle$-equivariant (Lemma 5.3). The action of σ is given by $\sigma|_{\Gamma_{\tau^{\pm 1}}} = \sigma_1 \circ \tau^{\pm 1} : \Gamma_{\tau^{\pm 1}} \to \Gamma_{\tau^{\mp 1}}$, and therefore we must define

$$\mathcal{A}_\sigma = (\tau^{\pm 1})^* \mathcal{A}_{\sigma_1} \circ \mathcal{A}_{\tau^{\pm 1}} \quad \text{on } \Gamma_{\tau^{\pm 1}}.$$

It is straightforward to check that indeed $\sigma^*\mathcal{A}_\sigma \circ \mathcal{A}_\sigma = \mathrm{Id}$, so $\pi_1^* M$ is G-equivariant. If M is flat at Z, then $\pi_1^* M$ is flat at $\pi^{-1}(Z)$. Going back, if we start with N on $\Gamma_\tau \sqcup \Gamma_{\tau^{-1}}$ which is G-equivariant, we can get a sheaf on $\mathbb{P}^1$ by taking $M = \pi_{1*}(N|_{\Gamma_\tau})$. Then on M we let $\mathcal{A}_\tau = (\sigma^*\mathcal{A}_{\sigma_1} \circ \mathcal{A}_\sigma)|_{\Gamma_\tau}$. If N is flat at $\pi^{-1}(Z)$, then M is flat at Z. It is straightforward to check that these constructions are mutually inverse.

In the second situation, we proceed analogously: we must show that G-equivariant sheaves on $\Gamma_{\tau_1} \sqcup \Gamma_{\tau_2}$ are equivalent to $\widetilde{G}$-equivariant sheaves on $\mathbb{P}^1$, and that the flatness condition is preserved. As above, given a $\widetilde{G}$-equivariant sheaf M on $\mathbb{P}^1$, we consider $\pi_1^* M$ as a $\mathbb{Z}/2\mathbb{Z}\langle\sigma_1\rangle$-equivariant sheaf. This time, the action of σ on Γ_{τ_i} equals τ_i, so we define $\mathcal{A}_\sigma = \mathcal{A}_{\tau_i}$ on Γ_{τ_i}. As before, $\pi_1^* M$ becomes G-equivariant and it is flat at the fixed points of $\tau_2\tau_1$ if $\widetilde{M}$ is as well. The inverse of this functor is given as follows: starting with an equivariant sheaf N on $\Gamma_{\tau_1} \sqcup \Gamma_{\tau_2}$, we let $M = \pi_{1*}(N|_{\tau_1})$. The $\widetilde{G}$-equivariant structure is given by $\mathcal{A}_{\tau_1} = \mathcal{A}_\sigma|_{\Gamma_{\tau_1}}$, and $\overline{\tau_2} = \overline{\sigma_1} \circ \overline{\sigma} \circ \overline{\sigma_1}$, since the analogous relations hold for the action of $\widetilde{G}$ on $\mathbb{P}^1$. Again, flatness at the specified points is preserved and one can check that the constructions are mutual inverses.

□

Recall that the flatness condition cannot be completely removed, as the example in Remark 5.8 shows.

Further, if the components of E coincide so that E becomes the double diagonal, then E-modules become strongly related to D-modules on $\mathbb{P}^1$. This is very similar to Grothendieck's definition of a connection, see [7, I §2].

Proposition 5.13. *Let $\tau \in \mathrm{Aut}(\mathbb{P}^1)$ be such that $\tau^2 = \mathrm{Id}$. Let I be the ideal sheaf of the graph of τ in $\mathbb{P}^1 \times \mathbb{P}^1$, and let E be the subscheme cut out by I^2 and let Δ be the diagonal. Then E-modules are equivalent to the following:*

- *If $\tau = \mathrm{Id}$, E-modules are equivalent to ordered pairs of D-modules on $\mathbb{P}^1$, i.e.* E-**Mod** $\cong \mathcal{D}$-**Mod**$(\mathbb{P}^1) \oplus \mathcal{D}$-**Mod**$(\mathbb{P}^1)$. *The full subcategory of E-modules such that $\mathcal{A}|_\Delta = \mathrm{Id}$ is equivalent to* $\mathcal{D}$-**Mod**$(\mathbb{P}^1)$.

- *If $\tau \neq \mathrm{Id}$, E-modules are equivalent to quasicoherent sheaves M on $\mathbb{P}^1$ with two structures:*

  - *A $\mathbb{Z}/2\mathbb{Z}$-equivariant structure $\mathcal{A}_\tau : M \to \tau^* M$.*
  - *A connection $\nabla : M \to \Omega \otimes M$.*

  These two structures are compatible in the sense that $\tau^*\nabla \circ \mathcal{A}_\tau = (\mathrm{Id}_\Omega \otimes \mathcal{A}_\tau) \circ \nabla$. In other words, given $m \in M$, if we let $\nabla m = \sum \alpha_i \otimes m \in \Omega \otimes M$, then we have that
$$\nabla(\overline{\tau} m) = \sum \tau^* \alpha_i \otimes \overline{\tau} m_i$$

Proof. Let $\tau = 1$ and consider an E-module M. Consider $\mathcal{A}|_\Delta$, where Δ is the diagonal: since $\sigma|_\Delta = \mathrm{Id}$, $\mathcal{A}|_\Delta$ is an endomorphism of M whose square is the identity. M then decomposes as the direct sum of its eigenspaces $M_1 \oplus M_{-1}$. First of all, we claim that $M_{\pm 1}$ are E-submodules.

Consider the restriction $\mathcal{A} : \pi_1^* M_1 \to \pi_2^* M_1 \oplus \pi_2^* M_{-1}$. This map becomes the identity when restricted to Δ, so its image is contained in $\pi_2^* M_1 \oplus I_\Delta \pi_2^* M_{-1}$, where I_Δ is the ideal sheaf cutting out the diagonal. Consider a local section $m \in M_1$, and let $\mathcal{A}(\pi_1^* m) = \pi_2^* m + m_1 + m_{-1}$, where $m_1 \in I_\Delta \pi_2^* M_1$ and $m_{-1} \in I_\Delta \pi_2^* M_{-1}$. Notice that $\overline{\sigma} = \sigma^* \circ \mathcal{A}$

acts as ∓ 1 on $I_\Delta \pi_1^* M_{\pm 1}$: this sheaf is generated by elements of the form $(\pi_1^* f - \pi_2^* f)\pi_1^* n$, for $f \in \mathcal{O}_{\mathbb{P}^1}$ and $n \in M_{\pm 1}$. We have that

$$\begin{aligned} \sigma^* \left(\mathcal{A}((\pi_1^* f - \pi_2^* f)\pi_1^* n)\right) = \sigma^* \left((\pi_1^* f - \pi_2^* f)\mathcal{A}(\pi_1^* n)\right) \in \\ \in \sigma^* \left((\pi_1^* f - \pi_2^* f)(\pm \pi_2^* n + I_\Delta \pi_2^* M)\right) \overset{\text{mod } I_\Delta^2}{\equiv} \\ \equiv \sigma^* \left((\pi_1^* f - \pi_2^* f)(\pm \pi_2^* n)\right) = \\ = (\pi_2^* f - \pi_1^* f)(\pm \pi_1^* n) = \mp (\pi_1^* f - \pi_2^* f)\pi_1^* n \end{aligned} \tag{5.2}$$

Now, given $\mathcal{A}(\pi_1^* m) = \pi_2^* m + m_1 + m_{-1}$, let us write the equation $\sigma^* \mathcal{A}(\mathcal{A}(\pi_1^* m)) = \pi_1^* m$:

$$\pi_1^* m = (\sigma^* \mathcal{A})(\mathcal{A}(\pi_1^* m)) = (\sigma^* \mathcal{A})(\pi_2^* m + m_1 + m_{-1}) = (\sigma^* \circ \mathcal{A} \circ \sigma^*)(\pi_2^* m + m_1 + m_{-1})$$

$$= \sigma^*(\mathcal{A}(\pi_1^* m)) + \sigma^*(\mathcal{A}(\sigma^* m_1 + \sigma^* m_{-1})) =$$

$$= \sigma^*(\pi_2^* m + m_1 + m_{-1}) - \sigma^* m_1 + \sigma^* m_{-1} = \pi_1^* m - 2m_{-1}.$$

This implies that $m_{-1} = 0$, so M_1 is an E-submodule. The same computation shows that M_{-1} is also an E-submodule.

The action of $\mathcal{A}$ on M_1 is the same as a connection by the definition of Grothendieck, see for example [7]. Similarly, the action of $-\mathcal{A}$ on M_{-1} is a connection. Therefore, E-modules consist of the direct sum of two D-modules. If we impose the condition that $\mathcal{A}|_\Delta = \mathrm{Id}$, then $M_{-1} = 0$, so we just obtain one D-module.

Now suppose that $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ is an involution, and letting E be the doubled graph of τ, consider an E-module M. Let us write the second projection as $\pi_2 = \tau \circ \pi_3$, so we have that $\pi_1 \circ \sigma = \tau \circ \pi_3$. The E-module structure is an isomorphism $\mathcal{A} : \pi_1^* M \to \pi_3^* \tau^* M$, such that $\sigma^* \mathcal{A} = \mathcal{A}^{-1}$. If we embed E in $\mathbb{P}^1 \times \mathbb{P}^1$ by (π_1, π_3), it becomes the double diagonal. Let Δ be the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$, embedded by (π_1, π_3). Consider $\mathcal{A}|_\Delta : M = \pi_1^* M|_\Delta \to \pi_3^* \tau^* M|_\Delta = \tau^* M$, which gives M a τ-equivariant structure. Since $\sigma^* \mathcal{A} \circ \mathcal{A} = 1$, M is $\mathbb{Z}/2\mathbb{Z}$-equivariant, and not just $\mathbb{Z}$-equivariant. Let $\mathcal{A}_\tau = \mathcal{A}|_\Delta : M \to \tau^* M$.

Consider the adjunction map $J_{\tau^* M} : \tau^* M \to \pi_{3*} \pi_3^* \tau^* M$. Since $\pi_{3*} \pi_3^* \tau^* M = \pi_{1*} \pi_3^* \tau^* M$ as sheaves of groups, $J_{\tau^* M}$ can be seen as a k-linear map to $\pi_{1*} \pi_3^* \tau^* M$. This is the map that to a section assigns its first order jet, and analogously we have $J = J_M : M \to \pi_{1*} \pi_3^* M$. We define the following composition ∇:

$$-\nabla : M \xrightarrow{\mathcal{A} - J_{\tau^* M} \circ \mathcal{A}_\tau} \pi_{1*} \pi_3^* \tau^* M \xrightarrow{\pi_{1*} \pi_3^* \tau^* \mathcal{A}_\tau} \pi_{1*} \pi_3^* M.$$

So we have that $\mathcal{A} = J_{\tau^* M} \circ \mathcal{A}_\tau - (\pi_{1*} \pi_3^* \tau^* \mathcal{A}_\tau)^{-1} \circ \nabla = J_{\tau^* M} \circ \mathcal{A}_\tau - \pi_{1*} \pi_3^* \mathcal{A}_\tau \circ \nabla$ (we implicitly identify $\mathcal{A}$ with $\pi_{1*} \mathcal{A}|_M$). Notice now that $J_{\tau^* M} \circ \mathcal{A}_\tau = \pi_{1*} \pi_3^* \mathcal{A}_\tau \circ J$: both are equal as maps $M \to \pi_{3*} \pi_3^* \tau^* M$ due to the adjunction relation, and $\pi_{1*} \pi_3^* \tau^* M = \pi_{3*} \pi_3^* \tau^* M$ as sheaves of groups. Therefore,

$$\pi_{1*} \mathcal{A}|_M = \pi_{1*} \pi_3^* \mathcal{A}_\tau \circ (J - \nabla).$$

Let us call $D : \pi_1^* M \to \pi_3^* M$ the map obtained from $J - \nabla$ from the adjunction $\pi_1^* \vdash \pi_{1*}$. We obtain the relation $\mathcal{A} = \pi_3^* \mathcal{A}_\tau \circ D$. Now, $D = \pi_3^* \tau^* \mathcal{A}_\tau \circ \mathcal{A}$ is $\mathcal{O}$-linear, and further $D|_\Delta = \tau^* \mathcal{A}_\tau \circ \mathcal{A}_\tau = \mathrm{Id}$. Therefore, ∇ is a covariant derivative, i.e. a linear connection on M, again by the reasoning in [7]: any such $\mathcal{O}$-linear map D which restricts to the identity on Δ gives a linear connection $\nabla = J - D$.

It remains to check that $\tau^* \nabla \circ \mathcal{A}_\tau = \pi_{1*} \pi_3^* \mathcal{A}_\tau \circ \nabla$. We can repeat the same reasoning from Equation (5.2) to conclude that $(\sigma^*(\tau, \tau)^*) \circ D$ acts as -1 on $I_\Delta \pi_1^* M$. Note that $\sigma \circ (\tau, \tau)$ is the map that interchanges π_1 with π_3, so in this case $\sigma \circ (\tau, \tau)$ plays the role of σ above and π_3 plays the role of π_2. Let us abbreviate $(\tau, \tau) \circ \sigma = \widetilde{\sigma}$. Taking this into account, let us show that $\widetilde{\sigma}^* D = D^{-1}$: $\pi_1^* M$ is generated by elements of the form $\pi_1^* m \in \pi_1^* M$. Then $J(m) = \pi_3^* m$, by definition, and $\nabla m \in I_\Delta \pi_3^* M$, since $J \equiv \mathrm{Id} \mod I_\Delta$. Therefore,

$$(\widetilde{\sigma}^* D)\left(D(\pi_1^* m)\right) = \left(\widetilde{\sigma}^* \circ D \circ \widetilde{\sigma}^*\right)(Jm - \nabla m) = \left(\widetilde{\sigma}^* \circ D \circ \widetilde{\sigma}^*\right)(\pi_3^*(m) - \nabla m) =$$

$$= \widetilde{\sigma}^*(D(\pi_1^* m)) - \widetilde{\sigma}^*(D \widetilde{\sigma}^* \nabla m) = \widetilde{\sigma}^*(\pi_3^* m - \nabla m) + \widetilde{\sigma}^* \nabla m = \pi_1^* m.$$

Then the relation $\sigma^* \mathcal{A} \circ \mathcal{A} = \mathrm{Id}$ implies the following:

$$\begin{aligned} \sigma^* \mathcal{A} = \mathcal{A}^{-1} &\Rightarrow \sigma^* \mathcal{A} = \sigma^*(\pi_3^* \mathcal{A}_\tau \circ D) = \pi_1^* \tau^* \mathcal{A}_\tau \circ \sigma^* D \overset{(\tau,\tau)^* \sigma^* D = D^{-1}}{=} \\ &= \pi_1^* \mathcal{A}_\tau^{-1} \circ (\tau, \tau)^* D^{-1} = \mathcal{A}^{-1} \\ &\Rightarrow \pi_3^* \mathcal{A}_\tau \circ D = (\tau, \tau)^* D \circ \pi_1^* \mathcal{A}_\tau. \end{aligned}$$

The last equality, after applying π_{1*} and restricting to $M \subset \pi_{1*} \pi_1^* M$, reads

$$\pi_{1*} \pi_3^* \mathcal{A}_\tau \circ (J - \nabla) = \tau^*(J - \nabla) \circ \mathcal{A}_\tau.$$

Now we note that $\pi_{1*} \pi_3^* \mathcal{A}_\tau \circ J = \tau^* J \circ \mathcal{A}_\tau$: earlier we showed. Together with the fact that $\tau^* J = J_{\tau^* M}$, the above equality follows. Therefore, $\pi_{1*} \pi_3^* \mathcal{A}_\tau \circ \nabla = \tau^* \nabla \circ \mathcal{A}_\tau$.

The identification between $\Omega \otimes M$ and $\pi_{1*} I_\Delta \pi_3^* M$ takes a generator of the form $df \otimes m$ and maps it to $\pi_{1*}((\pi_3^* f - \pi_1^* f) \pi_3^* m)$. Therefore, for any morphism $\phi : M \to N$

$$(\pi_{1*} \pi_3^* \phi)(df \otimes m) = \pi_{1*}((\pi_3^* f - \pi_1^* f)(\pi_3^* \phi)(\pi_3^* m)) = \pi_{1*}((\pi_3^* f - \pi_1^* f) \pi_3^*(\phi m)) = df \otimes \phi m. \tag{5.3}$$

So applying this to $\phi = \mathcal{A}_\tau$ we see that for a general element $\sum \alpha_i \otimes m$ with $\alpha_i \in \Omega$, the action of $\pi_{1*} \pi_3^* \mathcal{A}_\tau$ by linearity is $\pi_{1*} \pi_3^* \mathcal{A}_\tau \left(\sum \alpha_i \otimes m\right) = \sum \alpha_i \otimes \mathcal{A}_\tau m$.

Consider now a local section $m \in M$, and let $\nabla m = \sum \alpha_i \otimes m_i \in \Omega \otimes M$. We have that

$$\nabla(\tau^*(\mathcal{A}_\tau m)) = \tau^*(\tau^* \nabla(\mathcal{A}_\tau m)) = \tau^* \left(\pi_{1*} \pi_3^* \mathcal{A}_\tau(\nabla m)\right) =$$

$$= \tau^* \left(\sum \alpha_i \otimes \mathcal{A}_\tau m_i\right) = \sum \tau^* \alpha_i \otimes \tau^*(\mathcal{A}_\tau m_i).$$

So $\nabla \circ \tau^* \circ \mathcal{A}_\tau = \tau^* \circ (\mathrm{Id}_\Omega \otimes \mathcal{A}_\tau) \circ \nabla$, or in other words, $\tau^* \nabla \circ \mathcal{A}_\tau = (\mathrm{Id}_\Omega \otimes \mathcal{A}_\tau) \circ \nabla$. This identity is our claim.

Let us go backwards: to construct an E-connection starting from ∇ and $\mathcal{A}_\tau$, one takes $D = J - \nabla$ as above, and $\mathcal{A} = \pi_3^*\mathcal{A}_\tau \circ D$, and the previous reasoning shows that it is indeed an E-connection if ∇ and $\mathcal{A}_\tau$ commute in the appropriate sense.

It remains to check that these two constructions are functors. In other words, given two sheaves M, N each with $\mathcal{A}, \mathcal{A}_\tau, \nabla$ as above, we would like to show that a morphism $\phi : M \to N$ commutes with $\mathcal{A}$ if and only if it commutes with $\mathcal{A}_\tau$ and ∇.

Suppose ϕ commutes with $\mathcal{A}_\tau$ and ∇, i.e. $\mathcal{A}_\tau \circ \phi = \tau^*\phi \circ \mathcal{A}_\tau$ and $\nabla \circ \phi = (\mathrm{Id}_\Omega \otimes \phi) \circ \nabla$. The latter equation amounts to saying that $D \circ \pi_1^*\phi = \pi_3^*\phi \circ D$: indeed, $J : \pi_1^* m \mapsto \pi_3^* m$ commutes in this way with ϕ, and further if $\nabla m = \sum \alpha_i \otimes m_i$, we can apply (5.3) again to conclude that

$$\pi_3^*\phi(\nabla m) = \pi_3^*\phi\left(\sum \alpha_i \otimes m_i\right) = \sum \alpha_i \otimes \phi m_i = (\mathrm{Id} \otimes \phi)\nabla m.$$

We have that $\pi_{1*}(D \circ \pi_1^*\phi)|_M = (J - \nabla) \circ \phi = \pi_{1*}\pi_3^*\phi \circ (J - \nabla) = \pi_{1*}(\pi_3^*\phi \circ D)|_M$, so by the adjunction we have that $D \circ \pi_1^*\phi = \pi_3^*\phi \circ D$. Finally, we have the desired relation:

$$\mathcal{A} \circ \pi_1^*\phi = \pi_3^*\mathcal{A}_\tau \circ D \circ \pi_1^*\phi = \pi_3^*\mathcal{A}_\tau \circ \pi_3^*\phi \circ D = \pi_3^*\tau^*\phi \circ (\pi_3^*\mathcal{A}_\tau \circ D) = \pi_2^*\phi \circ \mathcal{A}.$$

Conversely, suppose ϕ is such that $\mathcal{A} \circ \pi_1^*\phi = \pi_2^*\phi \circ \mathcal{A}$. Taking this relation restricted to Δ we obtain the equation $\mathcal{A}_\tau \circ \phi = \tau^*\phi = \mathcal{A}_\tau$. Now we can proceed as above:

$$\pi_3^*\mathcal{A}_\tau \circ D \circ \pi_1^*\phi = \mathcal{A} \circ \pi_1^*\phi = \pi_2^*\phi \circ \mathcal{A} = \pi_3^*\tau^*\phi \circ (\pi_3^*\mathcal{A}_\tau \circ D) = \pi_3^*\mathcal{A}_\tau \circ \pi_3^*\phi \circ D.$$

We conclude that $D \circ \pi_1^*\phi = \pi_3^*\phi \circ D$, from which it follows that $\nabla \circ \phi = \pi_3^*\phi \circ \nabla$, by following the reasoning above.

□

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