APPROXIMATION BY SUBGROUPS OF FINITE INDEX AND THE HANNA NEUMANN CONJECTURE

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Abstract. Let F be a free group (pro-p group) and U and W two finitely generated subgroups (closed subgroups) of F. The Strengthened Hanna Neumann conjecture says that

$$\sum_{x\in U\backslash F/W} \overline{\mathrm{rk}}(U\cap xWx^{-1}) \le \overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W), \text{ where } \overline{\mathrm{rk}}(U)=\max\{\mathrm{rk}(U)-1,0\}.$$

This conjecture was proved in the case of abstract groups independently by J. Friedman and I. Mineyev in 2011.

In this paper we give the proof of the conjecture in the pro-p context and also present a new proof in the abstract case. We also show that the Lück approximation conjecture holds for free groups.

1. Introduction

Let F be a free group and U and W two finitely generated subgroups of F. In 1954, A. G. Howson [17] showed that

$$\overline{\mathrm{rk}}(U\cap W) \le 2\overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W)+\overline{\mathrm{rk}}(U)+\overline{\mathrm{rk}}(W), \text{ where } \overline{\mathrm{rk}}(U)=\max\{\mathrm{rk}(U)-1,0\}.$$

Three years later H. Neumann [29] improved the Howson bound and proved that

$$\overline{\mathrm{rk}}(U\cap W) \le 2\overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W).$$

She also conjectured that the factor of 2 in the above inequality is not necessary and that one always has

$$\overline{\mathrm{rk}}(U\cap W) \le \overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W).$$

This statement became known as the Hanna Neumann conjecture. It received a lot of attention since then.

In 1990, W. D. Neumann [30] conjectured that, in fact, the following inequality holds

$$\sum_{x\in U\backslash F/W} \overline{\mathrm{rk}}(U\cap xWx^{-1}) \le \overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W).$$

This conjecture received the name of the Strengthened Hanna Neumann conjecture. It was proved independently by J. Friedman [16] and I. Mineyev [28] only in 2011 and these proofs were also the first proofs of the original Hanna Neumann conjecture. Later W. Dicks gave a simplification for both proofs (see [8, 9]).

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It is clear that we can ask a similar question in the context of pro-p groups. Let U and W be a finitely generated closed subgroups of a free pro-p group. Is it true that

$$\sum_{x\in U\backslash F/W} \overline{\mathrm{rk}}(U\cap xWx^{-1}) \le \overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W)? \tag{1}$$

A variation of this question is considered in [32, Open Question 9.1.21]. In [23, Proposition 3.6] A. Lubotzky proved that free pro-p groups have Howson's property, i.e. the intersection of two closed finitely generated subgroups is again finitely generated. However, Lubotzky's proof was not constructive and did not provide any upper bound on the rank of the intersection $U\cap V$ in terms of the ranks of U and V. The main result of this paper is to prove the inequality (1).

Theorem 1.1. *The Strengthened Hanna Neumann conjecture holds in a free pro-p group.*

After some preparations in Sections 2 and 3, we prove Theorem 1.1 in Section 4.

Note that Friedman's and Mineyev's proofs of the Strengthened Hanna Neumann conjecture for abstract free groups use combinatorial methods of actions of groups on graphs. Unfortunately, we do not have analogous techniques in the case of pro-p groups. However, as in many other similar situations (like, for example, in the proof of the pro-p analog of the Kurosh subgroup theorem [27]), we can use (co)homological methods. It turns out that our homological approach works also in the abstract case and this leads us to a new proof of the original Strengthened Hanna Neumann conjecture.

We will provide all the details of this new proof. Thus, the reader interested only in the Strengthened Hanna Neumann conjecture for abstract free groups may skip Sections 2, 3 and 4 and start reading the paper from Section 5.

Our hope is that this new method can be used in the proof of the Hanna Neumann conjecture (and probably of the Strengthened Hanna Neumann conjecture) for non-abelian surface groups. The best result for surface groups is due to T. Soma [37] (see also [36]) and says that

$$\overline{\mathrm{rk}}(U\cap V) \le 1161\overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(V)$$

for two finitely generated subgroups U and V of a surface group.

One of the tools that we use in our proof in the abstract case is the Lück approximation conjecture for free groups. We believe that this result is also of independent interest.

Theorem 1.2. *The Lück approximation conjecture holds for free groups.*

In Section 5 the reader will find all the definitions and a brief historical account of this conjecture and the proof of Theorem 1.2. The Strengthened Hanna Neumann conjecture for abstract free groups is proved in Section 7 after some preparatory work in Section 6.

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2. Approximation by chains of open subgroups in free pro-p groups

Let G be a pro-p group. Denote by $\mathbb{F}_p[[G]]$ the completed group algebra of G over $\mathbb{F}_p$. Right or left pro-p $\mathbb{F}_p[[G]]$-modules together with their (continuous) morphisms form two abelian categories (in [32] they are denoted by $\mathbf{PMod}(\mathbb{F}_p[[G]])$ and $\mathbf{PMod}(\mathbb{F}_p[[G]]^{op})$ respectively). In this paper the modules from $\mathbf{PMod}(\mathbb{F}_p[[G]])$ and $\mathbf{PMod}(\mathbb{F}_p[[G]]^{op})$ will be called simply G**-modules**.

Let $\phi : \mathbb{F}_p[[G]]^n \to \mathbb{F}_p[[G]]^m$ be a homomorphism of left G-modules. There exists a matrix $A_\phi \in \mathrm{Mat}_{n\times m}(\mathbb{F}_p[[G]])$ such that ϕ is realized as the multiplication by A_ϕ:

$$\phi(a_1,\dots,a_n) = (a_1,\dots,a_n)A_\phi \text{ for any } (a_1,\dots,a_n)\in \mathbb{F}_p[[G]]^n.$$

Let H be an open normal subgroup of G. We denote by $\phi_{G/H}$ the map induced by ϕ on $\mathbb{F}_p[[G/H]]^n$:

$$\phi_{G/H}(a_1,\dots,a_n) = (a_1,\dots,a_n)A_\phi \text{ for any } (a_1,\dots,a_n)\in \mathbb{F}_p[G/H]^n.$$

We put

$$k_{G/H}(\phi) = \frac{\dim_{\mathbb{F}_p}\ker\phi_{G/H}}{|G:H|} \text{ and } c_{G/H}(\phi) = \frac{\dim_{\mathbb{F}_p}\mathrm{coker}\phi_{G/H}}{|G:H|}.$$

Note that $k_{G/H}(\phi) - c_{G/H}(\phi) = n-m$.

Let $G = G_1 > G_2 > \dots$ be a chain of normal open subgroups of G with trivial intersection. In this section we will study the behavior of the sequence $\{k_{G/G_i}(\phi)\}$ where G is a free finitely generated pro-p group.

Our first result is well-known and its variation has appeared for example in [1, Lemma 4.1].

Lemma 2.1. *Let G be a pro-p group, $\phi : \mathbb{F}_p[[G]]^n \to \mathbb{F}_p[[G]]^m$ a homomorphism of left G-modules and $H_1 \le H_2$ two open normal subgroups of G. Then*

$$c_{G/H_1}(\phi) \le c_{G/H_2}(\phi) \text{ and } k_{G/H_1}(\phi) \le k_{G/H_2}(\phi).$$

Proof. Note that

$$\mathrm{coker}\phi_{G/H_2} \cong \mathbb{F}_p \otimes_{\mathbb{F}_p[H_2/H_1]} \mathrm{coker}\phi_{G/H_1}.$$

Hence by [1, Lemma 4.1], $c_{G/H_1}(\phi) \le c_{G/H_2}(\phi)$. Thus, we also obtain that

$$k_{G/H_1}(\phi) = n-m+c_{G/H_1}(\phi) \le n-m+c_{G/H_2}(\phi) = k_{G/H_2}(\phi).$$

□

The following consequence is almost immediate.

Corollary 2.2. *Let G be a pro-p group and $G = G_1 > G_2 > \dots$ a chain of normal open subgroups of G with trivial intersection. Then there exist the limits*

$$k_G(\phi) = \lim_{i\to\infty} k_{G/G_i}(\phi) \text{ and } c_G(\phi) = \lim_{i\to\infty} c_{G/G_i}(\phi)$$

and they do not depend on the chain $G = G_1 > G_2 > \dots$.

Proof. By Lemma 2.1 the sequence $\{k_{G/G_i}(\phi)\}$ is decreasing. This implies that there exists the limit $\lim_{i\to\infty} k_{G/G_i}(\phi)$.

Let $G = G_1' > G_2' > \dots$ be another chain of normal open subgroups of G with trivial intersection. Then for any i there exists j such that $G_j \le G_i'$ and $G_j' \le G_i$. Hence by Lemma 2.1,

$$k_{G/G_k'}(\phi) \le k_{G/G_i}(\phi) \text{ and } k_{G/G_k}(\phi) \le k_{G/G_i'}(\phi) \text{ for every } k\ge j.$$

This implies that the limit $\lim_{i\to\infty} k_{G/G_i}(\phi)$ does not depend on the choice of the chain $G = G_1 > G_2 > \ldots$. Clearly the same holds for the limit $\lim_{i\to\infty} c_{G/G_i}(\phi)$. □

Now we assume that G is a finitely generated free pro-p group. We want to show that $k_G(\phi)$ is always an integer in this case. The reader may compare this result with [1, Theorem 4.3]. We will use a similar strategy in our proof.

Theorem 2.3. *Let F be a finitely generated free pro-p group and $\phi : \mathbb{F}_p[[F]]^n \to \mathbb{F}_p[[F]]^m$ a homomorphism of left F-modules. Then $k_F(\phi)$ is an integer.*

Proof. Let $\gamma_1(F) = F$ and $\gamma_{n+1}(F) = [\gamma_n(F), F]$. We fix n for a moment and put $G = F/\gamma_n(F)$. Then G is finitely generated torsion free nilpotent pro-p group. Hence $\mathbb{F}_p[[G]]$ is a Noetherian domain without zero divisors. Denote by $Q(\mathbb{F}_p[[G]])$ its skew field of quotients. [1, Theorem 2.1] (see also the proof of [15, Lemma 7.1]) says that for some chain $G = G_1 > G_2 > \ldots$ of normal open subgroups of G with trivial intersection

$$c_G(\phi) = \lim_{k\to\infty} c_{G/G_k}(\phi) = \dim_{Q(\mathbb{F}_p[[G]])} Q(\mathbb{F}_p[[G]]) \otimes_{\mathbb{F}_p[[F]]} \mathrm{coker}\phi.$$

In particular, $c_G(\phi)$ is an integer because $Q(\mathbb{F}_p[[G]])$ is a skew field.

Observe that $F/F^{p^k}\gamma_n(F)$ is a finite group and $\cap_{k=1}^{\infty} F^{p^k}\gamma_n(F) = \gamma_n(F)$. Thus, by Corollary 2.2, we obtain that $\lim_{k\to\infty} c_{F/F^{p^k}\gamma_n(F)}(\phi)$ is also equal to $c_{F/\gamma_n(F)}(\phi)$.

Now, we will construct inductively a chain $F_1 > F_2 > F_3 > \ldots$ of normal open subgroups of F such that

(1) $\gamma_i(F) \le F_i \le \gamma_{i-1}(F)F^{p^i} \cap F_{i-1}$ and
(2) $|c_{F/F_i}(\phi) - c_{F/\gamma_i(F)}(\phi)| \le \frac{1}{i}$.

The group F_1 is F. Assume that we have constructed F_{i-1} satisfying (1) and (2). Since

$$\lim_{k\to\infty} c_{F/F^{p^k}\gamma_i(F)}(\phi) = c_{F/\gamma_i(F)}(\phi),$$

there exists $k \ge i$ such that $|c_{F/\gamma_i(F)}(\phi) - c_{F/F^{p^k}\gamma_i(F)}(\phi)| \le \frac{1}{i}$. Put $F_i = F_{i-1} \cap F^{p^k}\gamma_i(F)$. Clearly (1) holds. Note that by Lemma 2.1 $c_{F/F_i}(\phi) \le c_{F/F^{p^k}\gamma_i(F)}$. Also, by Corollary 2.2, $c_{F/\gamma_i(F)}(\phi) \le c_{F/F_i}(\phi)$. Hence

$$|c_{F/F_i}(\phi) - c_{F/\gamma_i(F)}(\phi)| \le |c_{F/F^{p^k}\gamma_i(F)}(\phi) - c_{F/\gamma_i(F)}(\phi)| \le \frac{1}{i},$$

and so (2) also holds.

From the construction it follows that $\cap_i F_i = 1$ and $F_i < F_{i-1}$. Therefore,

$$k_F(\phi) = n - m + \lim_{i\to\infty} c_{F/F_i}(\phi) = n - m + \lim_{i\to\infty} c_{F/\gamma_i(F)}(\phi)$$

is integer because $c_{F/\gamma_i(F)}(\phi)$ are integers. □

Remark. *The previous theorem remains true if we only assume that F is an inverse limit of torsion free p-adic analytic pro-p groups.*

3. Finitely presented modules of a free pro-p group

Let G be a pro-p group. We denote by $\hat{\otimes}$ the complete tensor product of G-modules (see [32, Section 5.5]). The category of left (right) G-modules has enough projectives (see for example, [32, Proposition 5.4.2]). This allows us to define

$\mathrm{Tor}_n^{\mathbb{F}_p[[G]]}(N,-)$ as the n-th derived functor of $N\hat{\otimes}_{\mathbb{F}_p[[G]]}-$ (see [32, Section 6]). For simplicity we will write Tor_n^G instead of $\mathrm{Tor}_n^{\mathbb{F}_p[[G]]}$.

In this section F will denote a finitely generated free pro-p group. The following result is well-known for specialist, but we was not able to find an exact reference for it. Therefore, we include an indication for its proof.

Lemma 3.1. *Let P be a right F-submodule of $\mathbb{F}_p[[F]]^d$. Then P is free as a right F-module. In particular we obtain the following.*

1. *If P is a finitely generated right submodule of $\mathbb{F}_p[[F]]^d$, then $P \cong \mathbb{F}_p[[F]]^r$ for some non-negative integer r.*
2. *Every finitely presented right F-module is isomorphic to P_1/P_2, where*
$$\mathbb{F}_p[[F]]^r \cong P_2 \le P_1 \cong \mathbb{F}_p[[F]]^d$$
for some non-negative integers r and d.

Proof. Let $N = \mathbb{F}_p[[F]]^d/P$. Since F is a free pro-p group, by [3, Theorem 4.1], $\mathrm{Tor}_2^F(N,\mathbb{F}_p)$ is equal to zero. Hence, by [32, Proposition 6.1.9], the map $\mathrm{Tor}_1^F(P,\mathbb{F}_p) \to \mathrm{Tor}_1^F(\mathbb{F}_p[[F]],\mathbb{F}_p) = 0$ is injective, and so, $\mathrm{Tor}_1^F(P,\mathbb{F}_p) = 0$. From [3, Corollary 3.2], we obtain that P is projective. Since $\mathbb{F}_p[[F]]$ is a local ring, P is free. □

From now on we fix a chain $F = F_1 > F_2 > \dots$ of normal open subgroups of F with trivial intersection. We put

$$\beta_1^F(N,M) = \limsup_{i\to\infty} \frac{\dim_{\mathbb{F}_p} \mathrm{Tor}_1^{F_i}(N,M)}{|F:F_i|}.$$

Remark. *We do not know whether the value of $\beta_1^F(N,M)$ depends or not on the chain $F = F_1 > F_2 > \dots$. In Proposition 3.2 we will show that it does not if M is finitely presented and N is F-trivial. Also we will see later that*

$$\left\{ \frac{\dim_{\mathbb{F}_p} \mathrm{Tor}_1^{F_i}(N,M)}{|F:F_i|} \right\}$$

are uniformly bounded if N and M are finitely presented. Thus, $\beta_1^F(N,M)$ is finite in this case.

For simplicity we put $\beta_1^F(N) = \beta_1^F(N,\mathbb{F}_p)$ and $\beta_1^F(M) = \beta_1^F(\mathbb{F}_p,M)$.

Proposition 3.2. *Let M be a left finitely presented F-module. Then*

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p} \mathrm{Tor}_1^{F_i}(\mathbb{F}_p,M)}{|F:F_i|}$$

exists, it does not depend on the chain $\{F_i\}$ and it is an integer.

Proof. Since M is a finitely presented F-module, by Lemma 3.1, there exists a homomorphism of F-modules $\phi : \mathbb{F}_p[[F]]^n \to \mathbb{F}_p[[F]]^m$ such that the sequence

$$0 \to \mathbb{F}_p[[F]]^n \xrightarrow{\phi} \mathbb{F}_p[[F]]^m \to M \to 0$$

is exact. Then tensoring this sequence with $\mathbb{F}_p$ over $\mathbb{F}_p[[F_i]]$, we obtain the exact sequence

$$0 \to \mathrm{Tor}_1^{F_i}(\mathbb{F}_p,M) \to \mathbb{F}_p[[F/F_i]]^n \xrightarrow{\phi_{F/F_i}} \mathbb{F}_p[[F/F_i]]^m \to \mathrm{Tor}_0^{F_i}(\mathbb{F}_p,M) \to 0.$$

Hence

$$k_{F/F_i}(\phi) = \frac{\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p, M)}{|F : F_i|}.$$

Thus, by Corollary 2.2 there exists

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p, M)}{|F : F_i|} = k_F(\phi)$$

and it does not depend on the sequence $\{F_i\}$. Since F is a free pro-p group, Theorem 2.3 implies that this limit is an integer. □

The functions β_1 inherits many properties of Tor_1. For example, we have the following result.

Lemma 3.3. *Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be an exact sequence of right F-modules and let M be a left F-module. Then*

$$\beta_1^F(N_1, M) \le \beta_1^F(N_2, M) \le \beta_1^F(N_1, M) + \beta_1^F(N_3, M).$$

Remark. *A similar statement holds if we interchange the roles of the left and right modules.*

Proof. Since every closed subgroup U of F is free, $\operatorname{Tor}_2^U(N_3, M) = 0$. Thus, by [32, Proposition 6.1.9], the short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of right H-modules induces the following exact sequence of $\mathbb{F}_p$-spaces:

$$0 \to \operatorname{Tor}_1^{F_i}(N_1, M) \to \operatorname{Tor}_1^{F_i}(N_2, M) \to \operatorname{Tor}_1^{F_i}(N_3, M).$$

This give the desired inequalities. □

In [19, Theorem 2], J. Lewin proved that a finitely presented module over a free algebra over a field contains a free submodule of finite codimension. In the following proposition we prove a similar results for F-modules. In what follows in this and the next sections the dimension means the dimension over $\mathbb{F}_p$.

Proposition 3.4. *Let M be a finitely presented left F-module. Then M contains a free left F-submodule of finite codimension.*

Remark. *A similar statement holds if we substitute left by right.*

Proof. By Lemma 3.1, M is isomorphic to P_1/P_2, where $P_2 \le P_1$ are finitely generated free left F-modules. Since P_2 is a finitely generated F-module, the submodule $(F-1)P_2$ is open in P_2. Thus, there exists an open submodule P_1' of P_1 such that $P_1' \cap P_2 \le (F-1)P_2$. Let $P_1'' = P_2 + P_1'$. Observe that

$$P_2 \cap (F-1)P_1'' = (F-1)P_2. \tag{2}$$

By Lemma 3.1, P_1'' is a free F-module. Since $\mathbb{F}_p[[F]]$ is a local ring, any lifting of a $\mathbb{F}_p$-basis of $P_1''/(F-1)P_1''$ is a free generating set of P_1''. Hence, from (2) it follows that we can complete a free generating set of P_2 to a free generating set of P_1''. Therefore P_2 is a free summand of P_1'', i.e. there exists a free submodule P_2' such that $P_1'' = P_2' \oplus P_2$. Thus, $P_1''/P_2 \cong P_2'$ is free and of finite codimension in $P_1/P_2 \cong M$. □

Corollary 3.5. *Let N and M be right and left F-modules respectively. Assume that*

(1) *N is finitely presented and*

(2) *M is finitely presented or of finite dimension*

Then $\mathrm{Tor}_1^F(N,M)$ *is finite.*

Proof. First consider the case where M has finite dimension. Since N is finitely presented, there exist an exact sequence of right F-modules

$$0 \to \mathbb{F}_p[[F]]^r \to \mathbb{F}_p[[F]]^d \to N \to 0.$$

Tensoring this sequence with M we obtain the following exact sequence.

$$0 \to \mathrm{Tor}_1^F(N,M) \to M^r \to M^d \to \mathrm{Tor}_0^F(N,M) \to 0.$$

Hence, $\mathrm{Tor}_1^F(N,M)$ is finite.

Now, assume M is finitely presented. By Proposition 3.4, M contains a free submodule M_0 of finite index. Since M_0 is free, $\mathrm{Tor}_1^F(-,M_0)=0$. Hence, applying [32, Proposition 6.1.9], we conclude that

$$\dim_{\mathbb{F}_p} \mathrm{Tor}_1^F(N,M) \le \dim_{\mathbb{F}_p} \mathrm{Tor}_1^F(N,M/M_0)$$

is also finite. □

Let now H be a (closed) finitely generated subgroup of F and $H_i = F_i \cap H$. Let N and M be right and left H-modules respectively. We put

$$\beta_1^H(N,M) = \limsup_{i\to\infty} \frac{\dim_{\mathbb{F}_p} \mathrm{Tor}_1^{H_i}(N,M)}{|H:H_i|}.$$

Since H is also a free pro-p group, all the properties, established for $\beta_1^F(N,M)$, hold for $\beta_1^H(N,M)$ as well.

Note that if H is an open subgroup of F, then starting from some k, $F_i \le H$ if $i \ge k$. Thus, if N and M are also F-modules and H is open in F, we obtain that

$$\beta_1^H(N,M) = \limsup_{i\to\infty} \frac{\dim_{\mathbb{F}_p} \mathrm{Tor}_1^{H_i}(N,M)}{|H:H_i|} = \limsup_{i\to\infty} \frac{|F:H| \dim_{\mathbb{F}_p} \mathrm{Tor}_1^{F_i}(N,M)}{|F:F_i|} = |F:H|\beta_1^F(N,M). \quad (3)$$

Let M be a left (or right) F-module and H a normal open subgroup of F. We say that M is **H-admissible** if there exists a free F-submodule P of M of finite codimension such that M/P is H-trivial.

Observe, that by Proposition 3.4, every finitely presented F-module M contains a free submodule P of finite codimension. If H is equal to the kernel of the action of F on M/P, then we obtain that M is H-admissible. Now we are ready to prove the main result of this section.

Theorem 3.6. *Let F be a free finitely generated pro-p group, H a normal open subgroup of F and M a left H-admissible F-module. Then there is a left H-submodule M' of M such that $\beta_1^H(M')=0$ and $\dim_{\mathbb{F}_p}(M/M') \le \beta_1^F(M)$.*

Remark. *Note that when F is 2-generated, then, in fact, $\dim_{\mathbb{F}_p}(M/M')$ should be equal to $\beta_1^F(M)$ and if F is cyclic then $\beta_1^F(M)=0$ and so $M'=M$.*

Proof. Since M is H-admissible, M contains a free F-submodule M_0 of finite index and H acts trivially on M/M_0. We will prove by induction on the index of M_0 in M that there is a left H-submodule M' of M, containing M_0, such that $\beta_1^H(M')=0$ and $\dim_{\mathbb{F}_p}(M/M')\le \beta_1^F(M)$.

The base of induction when $M=M_0$ is clear, because M_0 is H-free and so $\beta_1^H(M_0)=0$. Assume now that the proposition holds if $\dim_{\mathbb{F}_p} M/M_0<n$ and let us prove it in the case $\dim_{\mathbb{F}_p} M/M_0=n$.

Let M_1 be a F-submodule of M of codimension 1 that contains M_0. Then, since $\dim_{\mathbb{F}_p} M_1/M_0<n$, there exists a H-submodule M_1' of M_1, containing M_0, such that $\beta_1^H(M_1')=0$ and $\dim_{\mathbb{F}_p}(M_1/M_1')\le\beta_1^F(M_1)$.

By Lemma 3.3, $\beta_1^F(M_1)\le\beta_1^F(M)$. By Proposition 3.2, $\beta_1^F(M)$ and $\beta_1^F(M_1)$ are integers. Hence $\beta_1^F(M)\ge\beta_1^F(M_1)+1$ or $\beta_1^F(M)=\beta_1^F(M_1)$. In the first case we simply take $M'=M_1'$ and we have done. Thus, let us assume that $\beta_1^F(M)=\beta_1^F(M_1)$.

Recall that H acts trivially on M/M_1'. Let $a\in M\backslash M_1$ and put $M'=\langle a,M_1'\rangle$. The following exact diagram of H-modules

$$\begin{array}{ccccc} & & & 0 & \\ & & & \downarrow & \\ 0\to M_1'\to & M' & \to & \mathbb{F}_p & \to 0\\ & \downarrow & & \downarrow & \\ 0\to M_1\to & M & \to & \mathbb{F}_p & \to 0 \end{array}$$

induces the following exact diagram of $\mathbb{F}_p$-vector spaces (here we assume that $F_i\le H$):

$$\begin{array}{cccccc} & & & & & 0\\ & & & & & \downarrow\\ 0\to & \mathrm{Tor}_1^{F_i}(\mathbb{F}_p,M_1') & \to & \mathrm{Tor}_1^{F_i}(\mathbb{F}_p,M') & \xrightarrow{\phi_i} & \mathrm{Tor}_1^{F_i}(\mathbb{F}_p,\mathbb{F}_p)\\ & \downarrow & & \downarrow^{\alpha_i} & & \downarrow^{\beta_i}\\ 0\to & \mathrm{Tor}_1^{F_i}(\mathbb{F}_p,M_1) & \to & \mathrm{Tor}_1^{F_i}(\mathbb{F}_p,M) & \xrightarrow{\psi_i} & \mathrm{Tor}_1^{F_i}(\mathbb{F}_p,\mathbb{F}_p) \end{array}$$

Assume that $\beta_1^H(M')\ne 0$. Hence

$$\lim_{i\to\infty}\frac{1}{|F:F_i|}\dim_{\mathbb{F}_p}\mathrm{Tor}_1^{F_i}(\mathbb{F}_p,M')\ne 0.$$

Since

$$\beta_1^H(M_1')=\lim_{i\to\infty}\frac{1}{|F:F_i|}\dim_{\mathbb{F}_p}\mathrm{Tor}_1^{F_i}(\mathbb{F}_p,M_1')=0,$$

we obtain that

$$\lim_{i\to\infty}\frac{1}{|F:F_i|}\dim_{\mathbb{F}_p}\mathrm{Im}\,\phi_i\ne 0.$$

Taking into account that $\beta_i\circ\phi_i=\psi_i\circ\alpha_i$, we obtain that $\beta_i(\mathrm{Im}\,\phi_i)\le\mathrm{Im}\,\psi_i$ and since $\ker\beta_i=0$, we conclude that

$$\lim_{i\to\infty}\frac{1}{|F:F_i|}\dim_{\mathbb{F}_p}\mathrm{Im}\,\psi_i\ne 0.$$

Hence

$$\beta_1^F(M) - \beta_1^F(M_1) = \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p, M) - \dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p, M_1)}{|F : F_i|} = \lim_{i\to\infty} \frac{\dim_{\mathbb{F}_p} \operatorname{Im} \psi_i}{|F : F_i|} \neq 0,$$

a contradiction. Thus, $\beta_1^H(M') = 0$ and we are done. □

4. The Strengthened Hanna Neumann conjecture for pro-p groups

The main result of this section is the following theorem.

Theorem 4.1. *Let F be a finitely generated free pro-p group and let N and M be right and left finitely presented F-modules respectively. Then*

$$\beta_1^F(N, M) \leq \beta_1^F(N)\beta_1^F(M).$$

Proof. By Proposition 3.4, N contains a free submodule N_0 of finite index. Since M is H-admissible for some open normal subgroup H of F, by Theorem 3.6, there exists a H-submodule M_0 of M such that $\beta_1^H(M_0) = 0$ and $\dim_{\mathbb{F}_p}(M/M_0) \leq \beta_1^F(M)$. Without loss of generality we may also assume that H acts trivially on N/N_0 and M/M_0.

By Lemma 3.3,

$$\beta_1^H(N, M) \leq \beta_1^H(N, M_0) + \dim_{\mathbb{F}_p}(M/M_0)\beta_1^H(N, \mathbb{F}_p) \leq \beta_1^H(N, M_0) + \beta_1^F(M)\beta_1^H(N).$$

If $F_i \leq H$, then $\operatorname{Tor}_1^{F_i}(N_0, M_0) = \{0\}$, because N_0 is a free $\mathbb{F}_p[[F_i]]$-module. Hence, applying again Lemma 3.3, we obtain that

$$\beta_1^H(N, M_0) \leq \beta_1^H(N_0, M_0) + \dim_{\mathbb{F}_p}(N/N_0)\beta_1^H(\mathbb{F}_p, M_0) = 0.$$

Thus, we conclude that

$$\beta_1^H(N, M) \leq \beta_1^F(M)\beta_1^H(N).$$

Therefore, using (3), we obtain that

$$\beta_1^F(N, M) = \frac{1}{|F : H|}\beta_1^H(N, M) \leq \frac{1}{|F : H|}\beta_1^H(N)\beta_1^F(M) = \beta_1^F(N)\beta_1^F(M).$$

□

Now let us see how Theorem 4.1 implies the Strengthened Hanna Neumann conjecture for pro-p groups. In the following lemma $\oplus$ will denote the profinite direct sum (as it is defined in [27, 1.12]).

Lemma 4.2. *Let $U, W \leq F$ be two finitely generated closed subgroups of a finitely generated free pro-p group F. Then*

$$\beta_1^F(\mathbb{F}_p[[U\backslash F]], \mathbb{F}_p[[F/W]]) = \sum_{x\in U\backslash F/W} \overline{\operatorname{rk}}(U \cap xWx^{-1}). \tag{4}$$

In particular, $\beta_1^F(\mathbb{F}_p[[U\backslash F]]) = \overline{\operatorname{rk}}(U)$ and $\beta_1^F(\mathbb{F}_p[[F/W]]) = \overline{\operatorname{rk}}(W)$.

Proof. Let M be a left F-module. Put $U_i = U \cap F_i$.

Consider $\mathbb{F}_p[[U\backslash F]]$ as a right F_i-module. Since F_i is normal in F,

$$\mathbb{F}_p[[U\backslash F]] \cong \oplus_{x\in UF_i\backslash F}\mathbb{F}_p[[x^{-1}U_ix\backslash F_i]]$$

as right F_i-modules. By [32, Lemma 6.10.10],

$$\begin{aligned}\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p[[x^{-1}U_ix\backslash F_i]], M) &= \dim_{\mathbb{F}_p} \operatorname{Tor}_1^{x^{-1}U_ix}(\mathbb{F}_p, M)\\ &= \dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F}(\mathbb{F}_p[[x^{-1}U_ix\backslash F]], M).\end{aligned}$$

Since $\mathbb{F}_p[[x^{-1}U_ix\backslash F]]$ and $\mathbb{F}_p[[U_i\backslash F]]$ are isomorphic as right F-modules, we conclude that

$$\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p[[U\backslash F]], M) = |F : F_iU| \dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p[[U_i\backslash F_i]], M).$$

Thus

$$\frac{\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p[[U\backslash F]], M)}{|F : F_i|} = \frac{\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{U_i}(\mathbb{F}_p, M)}{|U : U_i|}. \tag{5}$$

In particular, if $M = \mathbb{F}_p$, we obtain that $\beta_1^F(\mathbb{F}_p[[U\backslash F]]) = \overline{\operatorname{rk}}(U)$. A symmetric argument also implies that

$$\beta_1^F(\mathbb{F}_p[[F/W]]) = \overline{\operatorname{rk}}(W). \tag{6}$$

Now let $M = \mathbb{F}_p[[F/W]]$. Using that

$$\mathbb{F}_p[[F/W]] \cong \oplus_{x\in U\backslash F/W}\mathbb{F}_p[[U/(U\cap xWx^{-1})]]$$

as left U-modules and that the fuctor $\operatorname{Tor}_1^{U_i}(\mathbb{F}_p, -)$ commutes with the profinite direct sum [27, Lemma 3.3], we conclude that

$$\operatorname{Tor}_1^{U_i}(\mathbb{F}_p, \mathbb{F}_p[[F/W]]) \cong \oplus_{x\in U\backslash F/W} \operatorname{Tor}_1^{U_i}(\mathbb{F}_p, \mathbb{F}_p[[U/(U\cap xWx^{-1})]]). \tag{7}$$

Fix for a moment i (for, example let $i = 1$). Let

$$S = \{x \in U\backslash F/W : \ U \cap xWx^{-1} \neq \{1\}\}.$$

Note that $x \in S$ if and only if $\operatorname{Tor}_1^{U_i}(\mathbb{F}_p, \mathbb{F}_p[[U/(U\cap xWx^{-1})]]) \neq \{0\}$. Thus, since, by Corollary 3.5, the left part of the expression (7) is of finite dimension, we obtain that S is finite. Hence the profinite direct sum in the right part of the expression (7) is finite, and so,

$$\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{U_i}(\mathbb{F}_p, \mathbb{F}_p[[F/W]]) = \sum_{x\in S} \dim_{\mathbb{F}_p} \operatorname{Tor}_1^{U_i}(\mathbb{F}_p, \mathbb{F}_p[[U/(U\cap xWx^{-1})]]).$$

Combining this with the equality (5) we obtain that

$$\begin{aligned}\frac{\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{F_i}(\mathbb{F}_p[[U\backslash F]], \mathbb{F}_p[[F/W]])}{|F : F_i|} &=\\ &\sum_{x\in S} \frac{\dim_{\mathbb{F}_p} \operatorname{Tor}_1^{U_i}(\mathbb{F}_p, \mathbb{F}_p[[U/(U\cap xWx^{-1})]])}{|U : U_i|}.\end{aligned} \tag{8}$$

Now consider the equality (8) when i tends to infinity. Then we obtain that

$$\beta_1^H(\mathbb{F}_p[[U\backslash H]], \mathbb{F}_p[[H/W]]) = \sum_{x\in S} \beta_1^U(\mathbb{F}_p[[U/(U\cap xWx^{-1})]]) \overset{\text{by (6)}}{=} \sum_{x\in S} \overline{\mathrm{rk}}(U\cap xWx^{-1}) = \sum_{x\in U\backslash F/W} \overline{\mathrm{rk}}(U\cap xWx^{-1})$$

and we are done.

□

Corollary 4.3. *Let F be a free pro-p group and U and W two closed finitely generated subgroups of F. Then*

$$\sum_{x\in U\backslash F/W} \overline{\mathrm{rk}}(U\cap xWx^{-1}) \le \overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W).$$

Proof. If the inequality does not hold, then there exists a finite subset S of $U\backslash F/W$ such that $\sum_{x\in S} \overline{\mathrm{rk}}(U\cap xWx^{-1}) > \overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W)$. Thus, substituting, if necessary, F by the subgroup generated by U, W and S, we may assume without loss of generality that F is finitely generated.

Then by Lemma 4.2,

$$\beta_1^F(\mathbb{F}_p[[U\backslash F]]) = \overline{\mathrm{rk}}(U),\ \beta_1^F(\mathbb{F}_p[[F/W]]) = \overline{\mathrm{rk}}(W).$$

and

$$\beta_1^F(\mathbb{F}_p[[U\backslash F]], \mathbb{F}_p[[F/W]]) = \sum_{x\in U\backslash F/W} \overline{\mathrm{rk}}(U\cap xWx^{-1}).$$

Hence Theorem 4.1 implies

$$\sum_{x\in U\backslash F/W} \overline{\mathrm{rk}}(U\cap xWx^{-1}) \le \overline{\mathrm{rk}}(U)\overline{\mathrm{rk}}(W)$$

□

5. The Lück approximation conjecture for free groups

Let G be a discrete group and let $l^2(G)$ denote the Hilbert space with Hilbert basis the elements of G; thus $l^2(G)$ consists of all square sumable formal sums $\sum_{g\in G} a_g g$ with $a_g \in \mathbb{C}$ and inner product

$$\langle \sum_{g\in G} a_g g, \sum_{h\in G} b_h h\rangle = \sum_{g\in G} a_g \overline{b_h}.$$

The group G acts on $l^2(G)$ by the multiplications on the left and right sides. The left action of G on $l^2(G)$ extends to an action of $\mathbb{C}[G]$ on $l^2(G)$ and so we obtain that the group algebra $\mathbb{C}[G]$ acts faithfully as bounded linear operators on $l^2(G)$. In what follows we will simply consider $\mathbb{C}[G]$ as a subspace of $\mathcal{B}(l^2(G))$, the algebra of bounded linear operators on $l^2(G)$. The weak closure of $\mathbb{C}[G]$ in $\mathcal{B}(l^2(G))$ is the group von Neumann algebra $\mathcal{N}(G)$ of G. It is equal to the second centralizer of $\mathbb{C}[G]$ in $\mathcal{B}(l^2(G))$. The ring $\mathcal{N}(G)$ satisfies the Ore conditions and its classical ring of quotients is denoted by $\mathcal{U}(G)$. The ring $\mathcal{U}(G)$ can be described as the ring of densely defined (unbounded) operators on $l^2(G)$ which commute with the right action of G. A more detailed account of the subject is given in [2].

Let now $\phi : \mathbb{C}[G]^n \to \mathbb{C}[G]^m$ be a homomorphism of left $\mathbb{C}[G]$-modules. There exists a matrix $A_\phi \in \mathrm{Mat}_{n\times m}(\mathbb{C}[G])$ such that ϕ is realized as the multiplication on the right side by A_ϕ:

$$\phi(a_1,\dots,a_n) = (a_1,\dots,a_n)A_\phi \text{ for any } \ (a_1,\dots,a_n) \in \mathbb{C}[G]^n.$$

Let H be a normal subgroup of G. The multiplication on the right side by A_ϕ induces also a bounded linear operator $\phi_{G/H} : (l^2(G/H))^n \to (l^2(G/H))^m$. Let

$$\mathrm{proj}_{\ker \phi_{G/H}} : (l^2(G/H))^n \to (l^2(G/H))^n$$

be the orthogonal projection onto $\ker \phi_{G/H}$. We put

$$\dim_{G/H} \ker \phi_{G/H} := \mathrm{Tr}_{G/H}(\mathrm{proj}_{\ker \phi_{G/H}}) := \sum_{i=1}^n \langle \mathrm{proj}_{\ker \phi_{G/H}} \mathbf{1}_i, \mathbf{1}_i\rangle_{(l^2(G/H))^n},$$

where $\mathbf{1}_i$ is the element of $(l^2(G/H))^n$ having 1 in the ith entry and 0 in the rest of the entries. The number $\dim_{G/H}(\ker \phi_{G/H})$ is the von Neumann dimension of $\ker \phi_{G/H}$.

Now, let $G = G_1 > G_2 > \dots$ be a chain of normal subgroups of G with trivial intersection. The Lück approximation conjecture [24] says the following:

Conjecture. *For every group G and every chain $G = G_1 > G_2 > \dots$ of normal subgroups of G with trivial intersection*

$$\lim_{k\to\infty} \dim_{G/G_k} \ker \phi_{G/G_k} = \dim_G \ker \phi_G. \tag{9}$$

W. Lück gave a very elegant proof of the conjecture in the case where G is an arbitrary group, G_k are of finite index and A_ϕ has rational coefficients ([25]). In the case where G/G_k are sofic and the coefficients of A_ϕ are rational the conjecture is proved by G. Elek and E. Szabó in [14]. Combining the ideas of [14] with the methods of [12] one can show that the conjecture holds when G/G_i are sofic and the coefficients of A_ϕ are algebraic over $\mathbb{Q}$. By a result of G. Elek, the conjecture is also known in full generality when G is amenable ([13], see also [31]). We note that one of the two inequalities of the conjecture is well-known to specialists and attributed to D. Kazhdan. Its proof is essentialy contained in [23, Theorem 2.3(1)], see also [35, Proposition 6.7].

Proposition 5.1. *(Kazhdan's inequality) For every group G and every chain $G = G_1 > G_2 > \dots$ of normal subgroups of G with trivial intersection*

$$\limsup_{k\to\infty} \dim_{G/G_k} \ker \phi_{G/G_k} \le \dim_G \ker \phi_G.$$

The main result of this section is the proof of the Lück approximation conjecture for free groups.

Theorem 5.2. *The Lück approximation conjecture holds for free groups.*

The theorem is a consequence of Proposition 5.1 and the following result.

Proposition 5.3. *Assume the previous notation and suppose that G is a free group. Then for every normal subgroup H of G,*

$$\dim_G \ker \phi_G \le \dim_{G/H} \ker \phi_{G/H}.$$

Proof. Let T be a subring of a ring S. We say that T is **division closed** in S if each element of T which is invertible in S is also invertible in T. The **division closure** $\mathcal{D}_S(T)$ of T in S is the smallest subring of S which contains T and it is division closed in S. Linnell's solution of the strong Atyah Conjecture for free groups [20] implies that the division closure $\mathcal{D}(G) = \mathcal{D}_{\mathcal{U}(G)}(\mathbb{C}[G])$ of $\mathbb{C}[G]$ in $\mathcal{U}(G)$ is a division ring (see, for example, [34, Lemma 3]). Also from the proof of [34, Lemma 3], it follows that

$$\dim_G \ker \phi_G = \dim_{\mathcal{D}(G)} \ker \tilde{\phi}$$

where $\tilde{\phi}$ is the map from $(\mathcal{D}(G))^n$ to $(\mathcal{D}(G))^m$ defined as

$$\tilde{\phi}(a_1, \dots, a_n) = (a_1, \dots, a_n)A_\phi \text{ for any } \ (a_1, \dots, a_n) \in (\mathcal{D}(G))^n.$$

Recall that the inner rank $r_R(A)$ of a n by m matrix A over a ring R is defined as follows.

$$r_R(A) = \min\{s : \ A = A_1A_2, \text{ for some } A_1 \in \mathrm{Mat}_{n\times s}(R), \ A_2 \in \mathrm{Mat}_{s\times m}(R)\}.$$

A square matrix is called full if its inner rank coincides with its size. In [10] W. Dicks and E. D. Sontag introduced the notion of **Sylvester domain** that can be characterized as a ring R that can be embedded in a skew field $\mathcal{D}$ in such way that every full matrix over R becomes invertible over $\mathcal{D}$. If, moreover $\mathcal{D} = \mathcal{D}_{\mathcal{D}}(R)$, then the Cohn theory of skew R-fields implies that $\mathcal{D}$ is unique (as a R-ring) and it is called the universal skew R-field (see [7, 4.5]).

It is a classical result due to G. M. Bergman and P. M. Cohn (see, for example, the discussion in [6]) that $\mathbb{C}[G]$ is a Sylvester domain. Let us denote its universal field of fractions by $\mathcal{D}_G$. From a result of I. Hughes [18] (see also [11]) it follows that $\mathcal{D}_G$ and $\mathcal{D}(G)$ are isomorphic as $\mathbb{C}[G]$-rings.

By [10, Proposition 2] (see also [7, Lemma 4.5.7]), every fully-inverting homomorphism to a skew field preserves the inner rank. Thus,

$$r_{\mathbb{C}[G]}(A_\phi) = r_{\mathcal{D}_G}(A_\phi) = r_{\mathcal{D}(G)}(A_\phi).$$

Hence we obtain that

$$\dim_{\mathcal{D}(G)} \ker \tilde{\phi} = n - r_{\mathcal{D}(G)}(A_\phi) = n - r_{\mathbb{C}[G]}(A_\phi).$$

On the other hand, from standard properties of the von Neumann dimension (see, for exampe, [24, Theorem 1.12 (2)]), we obtain that

$$\dim_{G/H} \ker \phi_{G/H} \geq n - r_{\mathbb{C}[G]}(A_\phi).$$

Hence, we conclude that

$$\dim_G \ker \phi_G = \dim_{\mathcal{D}(G)} \ker \tilde{\phi} = n - r_{\mathbb{C}[G]}(A_\phi) \leq \dim_{G/H} \phi_{G/H}.$$

□

Now we give several corollaries of Theorem 5.2. The first two corollaries are about the approximation of a free group by finite quotients. By [5, Corollary 3], we know that if G is free, then $K[G]$ is a fir (i.e. every one-sided ideal is free). Thus every $K[G]$-module M is isomorphic to P_1/P_2 where $P_2 \leq P_1$ are free modules. Moreover, if M is finitely presented, then P_1 and P_2 can be chosen to be finitely generated.

Corollary 5.4. *Let G be a free group and M a finitely presented left $\mathbb{C}[G]$-module. Then*

$$\lim_{i\to\infty} \frac{\dim_{\mathbb{C}} \operatorname{Tor}_1^{\mathbb{C}[G_i]}(\mathbb{C}, M)}{|G : G_i|}$$

exists for every chain $G = G_1 > G_2 > \dots$ of normal subgroups of finite index with trivial intersection. Moreover it does not depend on the chain $\{G_i\}$.

Proof. Let

$$0 \to \mathbb{C}[G]^n \xrightarrow{\phi} \mathbb{C}[G]^m \to M \to 0$$

be the exact sequence of $\mathbb{C}[G]$ modules. The corollary is the consequence of Theorem 5.2, because

$$\dim_{G/G_i} \ker \phi_{G/G_i} = \frac{\dim_{\mathbb{C}} \operatorname{Tor}_1^{\mathbb{C}[G_i]}(\mathbb{C}, M)}{|G : G_i|}.$$

□

In fact, the previous corollary also holds with coefficients in any field of characteristic 0.

Corollary 5.5. *Let K be a field of characteristic 0, G a free group and M a finitely presented left $K[G]$-module. Then*

$$\lim_{i\to\infty} \frac{\dim_K \operatorname{Tor}_1^{K[G_i]}(K, M)}{|G : G_i|}$$

exists for every chain $G = G_1 > G_2 > \dots$ of normal subgroups of finite index with trivial intersection. Moreover it does not depend on the chain $\{G_i\}$.

Remark. *The limit that appears in the corollary will be denoted by $\beta_1^{K[G]}(M)$.*

In order to prove the corollary we need the following lemma.

Lemma 5.6. *Let L/K be a field extension and R a K-algebra. Let N and M be right and left R-modules respectively. Put $S = R \otimes_K L$. Then the following equality holds.*

$$\dim_K \operatorname{Tor}_1^R(N, M) = \dim_L \operatorname{Tor}_1^S(N \otimes_K L, L \otimes_K M).$$

Proof. From the definitions of Tor functors we obtain that

$$\operatorname{Tor}_i^S(N \otimes_K L, L \otimes_K M) \cong L \otimes_K \operatorname{Tor}_i^R(N, M)$$

as L-spaces. This gives the lemma.

□

Proof of Corollary 5.5. Since M is a finitely presented $K[G]$-module, there are a finitely generated subfield K_0 of K and a homomorphism of $K_0[G]$-modules $\phi : K_0[G]^n \to K_0[G]^m$ such that the sequence

$$0 \to K_0[G]^n \xrightarrow{\phi} K_0[G]^m \to M_0 \to 0$$

is exact and $M \cong K[G] \otimes_{K_0[G]} M_0$. Now embed K_0 in $\mathbb{C}$. Hence by Lemma 5.6,

$$\dim_K \operatorname{Tor}_1^{K[G_i]}(K, M) = \dim_{K_0} \operatorname{Tor}_1^{K_0[G_i]}(K_0, M_0) = \dim_{\mathbb{C}} \operatorname{Tor}_1^{\mathbb{C}[G_i]}(\mathbb{C}, \mathbb{C} \otimes_{K_0} M_0).$$

Thus, Corollary 5.4 implies the desired result. □

The third consequence of Theorem 5.2 is about the approximation of a free group by torsion free amenable quotients. Let K be a field, Z an amenable group and assume that $K[Z]$ has no non-trivial zero-divisors. Then it is well-known that $K[Z]$ satisfies the left and right Ore condition and so it has the classical field of fractions that we denote by $\mathcal{D}(K[Z])$.

Corollary 5.7. *Let K be a field of characteristic 0, G a free group and M a finitely presented left $K[G]$-module. Let $G = G_1 > G_2 > \dots$ be a chain of normal subgroup with trivial intersections such that G/G_i are amenable and the rings $K[G/G_i]$ have no non-trivial zero-divisors. Then there exists i such that*

$$\beta_1^{K[G]}(M) = \dim_{\mathcal{D}(K[G/G_i])} \operatorname{Tor}_1^{K[G]}(\mathcal{D}(K[G/G_i]), M).$$

Proof. As in the proof of Corollary 5.5 without loss of generality we can assume that $K = \mathbb{C}$. Let

$$0 \to (\mathbb{C}[G])^n \xrightarrow{\phi} (\mathbb{C}[G])^m \to M \to 0$$

be an exact sequence of left $\mathbb{C}[G]$-modules. From the remark at the end of the page 422 of [22] it follows that

$$\dim_{G/G_i} \ker \phi_{G/G_i} = n - m + \dim_{G/G_i} l^2(G/G_i) \otimes_{\mathbb{C}[G]} M =$$
$$n - m + \dim_{\mathcal{D}(\mathbb{C}[G/G_i])} \mathcal{D}(\mathbb{C}[G/G_i]) \otimes_{\mathbb{C}[G]} M =$$
$$\dim_{\mathcal{D}(\mathbb{C}[G/G_i])} \operatorname{Tor}_1^{\mathbb{C}[G]}(\mathcal{D}(\mathbb{C}[G/G_i]), M).$$

In particular, $\dim_{G/G_i} \ker \phi_{G/G_i}$ are integers. Hence Theorem 5.2 implies that there exists i such that

$$\beta_1^{\mathbb{C}[G]}(M) = \dim_G \ker \phi_G = \dim_{G/G_i} \ker \phi_{G/G_i} =$$
$$\dim_{\mathcal{D}(\mathbb{C}[G/G_i])} \operatorname{Tor}_1^{\mathbb{C}[G]}(\mathcal{D}(\mathbb{C}[G/G_i]), M).$$

□

6. Finitely presented modules over a free group

In this section we assume that F is a finitely generated free group and K is a field of characteristic 0. Let H be a subgroup of F . Then by the Nielsen-Schreier theorem H is also free. Fix a chain $F = F_1 > F_2 > \dots$ of normal subgroups of F of finite index and assume that

(10) every subgroup of finite index of F contains F_k for some k.

In particular, $\{F_i\}$ intersect trivially. Let N and M be right and left $K[H]$-modules respectively. We put

$$\beta_1^{K[H]}(N, M) = \limsup_{i \to \infty} \frac{\dim_K \operatorname{Tor}_1^{K[F_i \cap H]}(N, M)}{|H : F_i \cap H|}.$$

Observe that $\beta_1^{K[H]}(N) = \beta_1^{K[H]}(N, K)$ and $\beta_1^{K[H]}(M) = \beta_1^H(K, M)$.

From the property (10) of the chain $\{F_i\}$ it follows that if M and N are also F-modules and H is a subgroup of F of finite index, then

$$\beta_1^{K[H]}(N, M) = |F : H| \beta_1^{K[F]}(N, M). \tag{11}$$

This equality will be used only in the proof of Theorem 7.2; in the results of Section 6, it will be enough to assume a weak version of (10): the intersection $\cap_k F_k$ is trivial.

Remark. *As in the pro-p case we know neither*

$$\lim_{i\to\infty} \frac{\dim_K \operatorname{Tor}_1^{K[F_i\cap H]}(N,M)}{|H : F_i \cap H|}$$

exists in general nor whether $\beta^{K[H]}(N,M)$ *depends on the chain* $\{F_i\}$.

Lemma 6.1. *Let* H *be a subgroup of* F,

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

be an exact sequence of right $K[H]$*-modules and* M *a left* $K[H]$*-module. Then*

$$\beta_1^{K[H]}(N_1,M) \le \beta_1^{K[H]}(N_2,M) \le \beta_1^{K[H]}(N_1,M) + \beta_1^{K[H]}(N_3,M).$$

Remark. *A similar statement holds if we interchange the roles of the left and right modules.*

Proof. Since every subgroup U of F is free, $\operatorname{Tor}_2^{K[U]}(N_3,M) = 0$. Thus, by [33, Theorem 8.3], the short exact sequence $0 \to N_1 \to N_2 \to N_3 \to 0$ of right $K[H]$-modules induces the following exact sequence of K-spaces:

$$0 \to \operatorname{Tor}_1^{K[F_i\cap H]}(N_1,M) \to \operatorname{Tor}_1^{K[F_i\cap H]}(N_2,M) \to \operatorname{Tor}_1^{K[F_i\cap H]}(N_3,M).$$

This give the desired inequalities. □

The aim of this section is to prove an analog of Theorem 3.6 for $K[F]$-modules. However, it seems that the straightforward analog does not hold. The main obstacle, for the argument of the proof of Theorem 3.6 to work in the case of $K[F]$-modules, is that $K[F]$ has irreducible modules of dimension greater than 1, and so, we can not use the inductive argument as we have done in the proof of Theorem 3.6. This difficulty has led us to consider modules over a modification of the group ring $K[F]$. The same ring plays an essential role in Dicks' simplification of Friedman's proof [8].

Let L be a field and let $\tau : F \to \operatorname{Aut}(L)$ be a homomorphism. We denote by $L_\tau[F]$ the twisted group ring: its underlying additive group coincides with the ordinary group ring $L[F]$ but the multiplication is defined as follows

$$(\sum_{i=1}^{n} k_i f_i)(\sum_{j=1}^{m} l_j g_j) = \sum_{i=1}^{n}\sum_{j=1}^{m} k_i \tau(f_i)(l_j) f_i g_j,\ k_i, l_j \in L,\ f_i, g_j \in F.$$

Now we are ready to present an analog of Theorem 3.6.

Theorem 6.2. *Let* L *be a field of characteristic 0 and* $\tau : F \to \operatorname{Aut}(L)$ *a homomorphism with finite image. Put* $H = \ker\tau$. *Let* M *be a left* $L_\tau[F]$*-module and assume that there exists a free* $L_\tau[F]$*-submodule* M_0 *such that* $\dim_L M/M_0$ *is finite and* H *acts trivially on* M/M_0. *Then there exists a* $L[H]$*-submodule* M' *of* M, *containing* M_0, *such that*

$$\beta_1^{L[H]}(M') = 0 \text{ and } \dim_L M/M' \le \frac{\beta_1^{L[H]}(M)}{|F : H|}.$$

Remark. *Note that when* F *is 2-generated, then, in fact,* $\dim_L M/M' = \frac{\beta_1^{L[H]}(M)}{|F:H|}$ *and if* F *is cyclic then* $\beta_1^{L[H]}(M) = 0$ *and so* $M' = M$.

Proof. We divide the proof in several claims.

Note that if we define the action of $L_\tau[F]$ on L by means of

$$(\sum_{i=1}^{k} l_i f_i)\cdot l = \sum_{i=1}^{k} l_i\tau(f_i)(l)\ (l, l_i \in L,\ f_i \in F),$$

then L becomes an irreducible left $L_\tau[F]$-module.

Claim 6.3. *L is the unique irreducible left $L_\tau[F]$-module on which H acts trivially.*

Proof. Let J be the ideal of $L_\tau[F]$ generated by $\{h-1|\ h\in H\}$. Put $R = L_\tau[F]/J$. Then any left $L_\tau[F]$-module on which H acts trivially is also an R-module. Let K be the subfield fixed by the automorphims in the image of τ. Note that since the automorphisms from $\mathrm{Gal}(L/K)$ are linearly independent over L, L is a faithful R-module. On the other hand,

$$\dim_K R = |F : H| \dim_K L = (\dim_K L)^2.$$

Thus $R \cong \mathrm{End}_K(L)$ and so L is the unique irreducible R-module up to isomorphism. □

Claim 6.4. *Let P be a finitely presented left $L_\tau[F]$-module. Then $|F : H|$ divides $\beta_1^{L[H]}(P)$.*

Proof. Put $H_i = H\cap F_i$ and let $H_i' = [H_i, H_i]$ be the derived subgroup of H_i. Then H_i' is a normal subgroup of F. It is well-known that F/H_i' is torsion-free. Since τ sends the elements of H_i' to the trivial automorphism of L, abusing slightly the notation we write $R_i = L_\tau[F/H_i']$. By [21, Corollary 4.5], R_i has no non-trivial zero-divisors. Since F/H_i' is virtually abelian, R_i is Noetherian. Thus, R_i has the classical skew field of fractions that we denote by $\mathcal{D}(R_i)$. Also note that

$$\mathcal{D}(R_i) \cong \mathcal{D}(L[H/H_i']) \otimes_{L[H]} L_\tau[F] \tag{12}$$

as $(D(L[H/H_i']), L_\tau[F])$-bimodules, where $D(L[H/H_i'])$ denotes the classical skew field of fractions of $L[H/H_i']$). In particular, if V is a finitely generated $\mathcal{D}(R_i)$-module then

$$\dim_{\mathcal{D}(L[H/H_i'])} V = |F : H| \dim_{\mathcal{D}(R_i)} V$$

is divisible by $|F : H|$.

By Corollary 5.7, there exists i such that

$$\beta_1^{L[H]}(P) = \dim_{\mathcal{D}(L[H/H_i'])} \mathrm{Tor}_1^{L[H]}(D(L[H/H_i']), P).$$

Note that $L_\tau[F]$ is a left free $L[H]$-module. Thus every free resolution of $L_\tau[F]$-module is also a free resolution of $L[H]$-modules. Thus, taking into account (12), we can calculate $\mathrm{Tor}_1^{L[H]}(\mathcal{D}(L[H/H_i']), P)$ in the following way

$$\mathrm{Tor}_1^{L[H]}(\mathcal{D}(L[H/H_i']), P) = \mathrm{Tor}_1^{L_\tau[F]}(\mathcal{D}(L[H/H_i']) \otimes_{L[H]} L_\tau[F], P) = \mathrm{Tor}_1^{L_\tau[F]}(\mathcal{D}(R_i), P).$$

Thus

$$\beta_1^{L[H]}(P) = \dim_{\mathcal{D}(L[H/H_i'])} \mathrm{Tor}_1^{L_\tau[F]}(\mathcal{D}(R_i), P) = |F : H| \dim_{\mathcal{D}(R_i)} \mathrm{Tor}_1^{L_\tau[F]}(\mathcal{D}(R_i), P).$$

This finishes the proof of the claim. □

Now we are ready to finish the proof of the theorem. We will prove the theorem by induction on $\dim_L M/M_0$. The base of induction, when $M = M_0$, is clear, because M_0 is a free $L[H]$-module, and so, $\beta_1^{L[H]}(M_0) = 0$. Assume now that the proposition holds if $\dim_L M/M_0 < n$ and let us prove it in the case where $\dim_L M/M_0 = n$.

Let M_1 be a maximal $L_\tau[F]$-submodule of M that contains M_0. By Claim 6.3, $M/M_1 \cong L$. Then, since $\dim_L M_1/M_0 < n$, there exists a $L[H]$-submodule M_1' of M_1, containing M_0, such that $\beta_1^{L[H]}(M_1') = 0$ and

$$\dim_L(M_1/M_1') \le \frac{\beta_1^{L[H]}(M_1)}{|F:H|}.$$

By Lemma 6.1,

$$\beta_1^{L[H]}(M_1) \le \beta_1^{L[H]}(M).$$

By Claim 6.4, $\beta_1^{L[H]}(M)$ and $\beta_1^{L[H]}(M_1)$ are divisible by $|F:H|$. Hence $\beta_1^{L[H]}(M) \ge \beta_1^{L[H]}(M_1) + |F:H|$ or $\beta_1^{L[H]}(M) = \beta_1^{L[H]}(M_1)$. In the first case we simply take $M' = M_1'$ and we have done. Thus, let us assume that $\beta_1^{L[H]}(M) = \beta_1^{L[H]}(M_1)$.

Take $a \in M\backslash M_1$ and let M' be $L[H]$ submodule generated by a and M_1'. Since H acts trivially on M/M_0, $\dim_L M'/M_1' = 1$. The following exact diagram of $L[H]$-modules

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 \to M_1' \to & & M' & \to & L & \to 0 \\ & & \downarrow & & \downarrow & & \\ 0 \to M_1 \to & & M & \to & L & \to 0 \end{array}$$

induces the following exact diagram of L-vector spaces:

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \downarrow \\ 0 \to & \mathrm{Tor}_1^{L[H_i]}(L, M_1') & \to & \mathrm{Tor}_1^{L[H_i]}(L, M') & \xrightarrow{\phi_i} & \mathrm{Tor}_1^{L[H_i]}(L, L) \\ & \downarrow & & \downarrow^{\alpha_i} & & \downarrow^{\beta_i} \\ 0 \to & \mathrm{Tor}_1^{L[H_i]}(L, M_1) & \to & \mathrm{Tor}_1^{L[H_i]}(L, M) & \xrightarrow{\psi_i} & \mathrm{Tor}_1^{L[H_i]}(L, L) \end{array}$$

Assume that $\beta_1^{L[H]}(M') \ne 0$. Hence

$$\lim_{i\to\infty} \frac{1}{|H:H_i|} \dim_L \mathrm{Tor}_1^{L[H_i]}(L, M') \ne 0.$$

Since

$$\beta_1^{L[H]}(M_1') = \lim_{i\to\infty} \frac{1}{|H:H_i|} \dim_L \mathrm{Tor}_1^{L[H_i]}(L, M_1') = 0,$$

we obtain that

$$\lim_{i\to\infty} \frac{1}{|H:H_i|} \dim_L \mathrm{Im}\,\phi_i \ne 0.$$

Since $\beta_i \circ \phi_i = \psi_i \circ \alpha_i$, we obtain that $\beta_i(\mathrm{Im}\,\phi_i) \le \mathrm{Im}\,\psi_i$ and since $\ker \beta_i = 0$ we conclude that

$$\lim_{i\to\infty} \frac{1}{|H:H_i|} \dim_L \mathrm{Im}\,\psi_i \ne 0.$$

Hence

$$\beta_1^{L[H]}(M) - \beta_1^{L[H]}(M_1) = \lim_{i\to\infty} \frac{\dim_L \operatorname{Tor}_1^{L[H_i]}(L, M) - \dim_L \operatorname{Tor}_1^{L[H_i]}(L, M_1)}{|H : H_i|} =$$

$$\lim_{i\to\infty} \frac{\dim_L \operatorname{Im} \psi_i}{|H : H_i|} \neq 0,$$

a contradiction. Thus, $\beta_1^{L[H]}(M') = 0$ and we are done. □

7. The Strengthened Hanna Neumann conjecture for free groups

Let F be a finitely generated free group and K a field of characteristic 0. The main result of this section is an analog of Theorem 4.1.

Let M be a left (or right) $K[F]$-module and H a normal subgroup of F of finite index. We say that M is **H-admissible** if there exists a free submodule P of M of finite K-codimension such that M/P is H-invariant. We say that M is **admissible** if it is H-admissible for some normal subgroup H of F of finite index. The modules that will interest us are admissible.

Proposition 7.1. *Let U be a finitely generated subgroup of F. Then the left $K[F]$-module $K[F/U]$ is admissible.*

Proof. Let $M = K[F/U]$. Since U is finitely generated, there exists a subgroup H of F of finite index such that H is equal to the free product of U and some subgroup W of F (see [26, Proposition 3.10]). Let $u_1, \ldots, u_l$ and $w_1, \ldots, w_k$ be sets of free generators of U and W respectively. Then the left ideal P of $K[F]$ generated by $u_1 - 1, \ldots, u_l - 1, w_1 - 1, \ldots, w_k - 1$ is free on these generators. Hence $\bar{P} = P/K[F](U-1)$ is free. Note that $\bar{P}$ has finite codimension in M and $M/\bar{P} \cong K[F/H]$ is $\cap_{f\in F} H^f$-invariant. □

The following result is the main result of this section.

Theorem 7.2. *Let F be a finitely generated free group, K a field of characteristic 0 and let N and M be right and left finitely presented admissible $K[F]$-modules respectively. Then*

$$\beta_1^{K[F]}(N, M) \leq \beta_1^{K[F]}(N)\beta_1^{K[F]}(M).$$

Proof. Let H be a normal subgroup of finite index of F such that M and N are H-admissible. We put $G = F/H = \{g_1, \ldots, g_t\}$. Let $L = K(x_g | g \in G)$ be the field of rational functions on t variables over K. Then we can define $\tau : F \to \operatorname{Aut}(L)$ as follows

$$\tau(f)(p(x_{g_1}, \ldots, g_t)) = p(x_{fg_1}, \ldots, x_{fg_t}),\ p(x_{g_1}, \ldots, g_t) \in L.$$

Put $L_0 = L^{\tau(F)}$. Then L/L_0 is a Galois extension with Galois group isomorphic to G. We put $\tilde{N} = N \otimes_{K[F]} L_\tau[F]$ and $\tilde{M} = L_\tau[F] \otimes_{K[F]} M$. Since τ sends H to the identity, the bimodules ${}_{K[F]}L_\tau[F]_{L[H]}$ and ${}_{K[F]}L[F]_{L[H]}$ are isomorphic. Hence we have the following isomorphisms of right $L[H]$-modules:

$$\tilde{N} = N \otimes_{K[F]} L_\tau[F] \cong N \otimes_{K[F]} L[F] \cong N \otimes_K L.$$

Analogously, we obtain that $\tilde{M} \cong L \otimes_K M$ as left $L[H]$-modules. Thus, by Lemma 5.6,

$$\beta_1^{K[H]}(N,M) = \beta_1^{L[H]}(\tilde{N},\tilde{M}),\ \beta_1^{K[H]}(M) = \beta_1^{L[H]}(\tilde{M}) \text{ and } \beta_1^{K[H]}(N) = \beta_1^{L[H]}(\tilde{N}). \tag{13}$$

Since M is H-admissible, M contains a free $K[F]$-submodule M_0 of finite codimension such that H acts trivially on M/M_0. Since $L_\tau[F]$ is a free (and so a flat) $K[F]$-module, $\tilde{M}_0 = L_\tau[F] \otimes_{K[F]} M_0$ is a free $L_\tau[F]$-submodule of $\tilde{M}$. Moreover H acts trivially on $\tilde{M}/\tilde{M}_0$. A similar argument implies also that the right $L_\tau[F]$-module $\tilde{N}$ contains a free submodule $\tilde{N}_0$ of finite codimension.

By Theorem 6.2, there exists a $L[H]$ -submodule $\tilde{M}'$ of $\tilde{M}$ such that

$$\beta_1^{L[H]}(\tilde{M}') = 0 \text{ and } \dim_L(\tilde{M}/\tilde{M}') \le \frac{\beta_1^{L[H]}(\tilde{M})}{|F:H|}.$$

First observe that $\beta_1^{L[H]}(\tilde{N},\tilde{M}') = 0$. Indeed, since $\mathrm{Tor}_1^{L[H\cap F_i]}(\tilde{N}_0,\tilde{M}')$ vanish for all i, Lemma 6.1 implies that

$$\beta_1^{L[H]}(\tilde{N},\tilde{M}') \le \dim_L(\tilde{N}/\tilde{N}_0)\beta_1^{L[H]}(\tilde{M}') + \beta_1^{L[H]}(\tilde{N}_0,\tilde{M}') = 0.$$

Applying again Lemma 6.1, we obtain that

$$\beta_1^{L[H]}(\tilde{N},\tilde{M}) \le \beta_1^{L[H]}(\tilde{N},\tilde{M}') + \dim_L(\tilde{M}/\tilde{M}')\beta_1^{L[H]}(\tilde{N},L) \le \frac{\beta_1^{L[H]}(\tilde{M})\beta_1^{L[H]}(\tilde{N})}{|F:H|}.$$

Therefore,

$$\beta_1^{K[F]}(N,M) \overset{\text{by (11)}}{=} \frac{\beta_1^{K[H]}(N,M)}{|F:H|} \overset{\text{by (13)}}{=} \frac{\beta_1^{L[H]}(\tilde{N},\tilde{M})}{|F:H|} \le \frac{\beta_1^{L[H]}(\tilde{M})\beta_1^{L[H]}(\tilde{N})}{|F:H|^2} \overset{\text{by (13)}}{=} \frac{\beta_1^{K[H]}(M)\beta_1^{K[H]}(N)}{|F:H|^2} \overset{\text{by (11)}}{=} \beta_1^{K[F]}(M)\beta_1^{K[F]}(N).$$

□

In order to show how the Strengthened Hanna Neumann conjecture follows from Theorem 7.2 we need to prove the following two lemmas.

Lemma 7.3. *Let F be a finitely generated free group and let N and M be right and left finitely presented admissible $K[F]$-modules respectively. Then $\mathrm{Tor}_1^{K[F]}(N,M)$ is finite.*

Proof. Since N and M are admissible, N contains a free submodule N_0 of finite codimension and M contains a free submodule M_0 of finite codimension. Since N_0 and M_0 are free, $\mathrm{Tor}_1^{K[F]}(N_0,-) = \mathrm{Tor}_1^{K[F]}(-,M_0) = 0$. Hence the long exact sequence for Tor-functors ([33, Theorem 8.3]) implies that

$$\dim_K \mathrm{Tor}_1^{K[F]}(N,M) \le \dim_K \mathrm{Tor}_1^{K[F]}(N/N_0, M/M_0)$$

is finite. □

Lemma 7.4. *Let F be a finitely generated free group, U, W finitely generated subgroups of F and K a field. Then*

$$\beta_1^{K[F]}(K[U\backslash F], K[F/W]) = \sum_{x\in U\backslash F/W} \overline{\mathrm{rk}}(U\cap xWx^{-1}). \tag{14}$$

In particular, $\beta_1^{K[F]}(K[U\backslash F]) = \overline{\mathrm{rk}}(U)$ *and* $\beta_1^{K[F]}(K[F/W]) = \overline{\mathrm{rk}}(W)$.

Proof. Let M be a left $K[F]$-module. Put $U_i = U\cap F_i$.

Consider $K[U\backslash F]$ as a right $K[F_i]$-module. Since F_i is normal in F,

$$K[U\backslash F] \cong \oplus_{x\in UF_i\backslash F} K[x^{-1}U_ix\backslash F_i]$$

as right $K[F_i]$-modules. By Shapiro's Lemma (see, for example, [3, Proposition 6.2]),

$$\begin{aligned}\dim_K \mathrm{Tor}_1^{K[F_i]}(K[x^{-1}U_ix\backslash F_i], M) &= \dim_K \mathrm{Tor}_1^{K[x^{-1}U_ix]}(K, M)\\ &= \dim_K \mathrm{Tor}_1^{K[F]}(K[x^{-1}U_ix\backslash F], M).\end{aligned}$$

Since $K[xU_ix^{-1}\backslash F]$ and $K[U_i\backslash F]$ are isomorphic as right $K[F]$-modules, we conclude that

$$\dim_K \mathrm{Tor}_1^{K[F_i]}(K[x^{-1}U_ix\backslash F_i], M) = \dim_K \mathrm{Tor}_1^{K[F_i]}(K[U_i\backslash F_i], M)$$

and so

$$\begin{aligned}\dim_K \mathrm{Tor}_1^{K[F_i]}(K[U\backslash F], M) &= \sum_{x\in UF_i\backslash F} \dim_K \mathrm{Tor}_1^{K[F_i]}(K[x^{-1}U_ix\backslash F_i], M) =\\ |F:F_iU| \dim_K \mathrm{Tor}_1^{K[F_i]}(K[U_i\backslash F_i], M) &= |F:F_iU| \dim_K \mathrm{Tor}_1^{K[U_i]}(K, M).\end{aligned}$$

Thus

$$\frac{\dim_K \mathrm{Tor}_1^{K[F_i]}(K[U\backslash F], M)}{|F:F_i|} = \frac{\dim_K \mathrm{Tor}_1^{K[U_i]}(K, M)}{|U:U_i|}.$$

In particular, if $M = K$, we obtain that $\beta_1^{K[F]}(K[U\backslash F]) = \overline{\mathrm{rk}}(U)$. A symmetric argument also implies that

$$\beta_1^{K[F]}(K[F/W]) = \overline{\mathrm{rk}}(W). \tag{15}$$

Now let $M = K[F/W]$. Using that

$$K[F/W] \cong \oplus_{x\in U\backslash F/W} K[U/(U\cap xWx^{-1})]$$

as left $K[U]$-modules we conclude that

$$\dim_K \mathrm{Tor}_1^{K[U_i]}(K, K[F/W]) = \sum_{x\in U\backslash F/W} \dim_K \mathrm{Tor}_1^{K[U_i]}(K, K[U/(U\cap xWx^{-1})]).$$

Puting all these equalities together we obtain that

$$\frac{\dim_K \mathrm{Tor}_1^{K[F_i]}(K[U\backslash F], K[F/W])}{|F:F_i|} = \sum_{x\in U\backslash F/W} \frac{\dim_K \mathrm{Tor}_1^{K[U_i]}(K, K[U/(U\cap xWx^{-1})]])}{|U:U_i|}. \tag{16}$$

Fix for a moment i (for example, put $i = 1$). Let

$$S = \{x \in U\backslash F/W : \ U \cap xWx^{-1} \neq \{1\}\}.$$

Note that $x \in S$ if and only if $\operatorname{Tor}_1^{K[U_i]}(K, K[U/(U \cap xWx^{-1})]) \neq \{0\}$. Thus, since, by Lemma 7.3, the left part of the equality (16) is finite, we obtain that S is finite.

Now consider the equality (16) when i tends to infinity. Then we conclude from (15) that

$$\beta_1^{K[F]}(K[U\backslash F], K[F/W]) = \sum_{x\in S} \beta_1^{K[U]}(K[U/(U\cap xWx^{-1})]) = \sum_{x\in S} \overline{\operatorname{rk}}(U \cap xWx^{-1}) = \sum_{x\in U\backslash F/W} \overline{\operatorname{rk}}(U\cap xWx^{-1}).$$

□

Corollary 7.5. *Let F be a free group and U and W two finitely generated subgroups of F. Then*

$$\sum_{x\in U\backslash F/W} \overline{\operatorname{rk}}(U\cap xWx^{-1}) \leq \overline{\operatorname{rk}}(U)\overline{\operatorname{rk}}(W).$$

Proof. If the inequality does not hold, then there exists a finite subset S of $U\backslash F/W$ such that $\sum_{x\in S} \overline{\operatorname{rk}}(U\cap xWx^{-1}) > \overline{\operatorname{rk}}(U)\overline{\operatorname{rk}}(W)$. Thus, substituting, if necessary, F by the subgroup generated by U, W and S, we may assume without loss of generality that F is finitely generated.

Let K be a field of characteristic 0. Then by Lemma 7.4,

$$\beta_1^{K[F]}(K[U\backslash F]) = \overline{\operatorname{rk}}(U), \ \beta_1^{K[F]}(K[F/W]) = \overline{\operatorname{rk}}(W).$$

and

$$\beta_1^{K[F]}(K[U\backslash F], K[F/W]) = \sum_{x\in U\backslash F/W} \overline{\operatorname{rk}}(U\cap xWx^{-1}).$$

Hence Proposition 7.1 and Theorem 7.2 imply that

$$\sum_{x\in U\backslash F/W} \overline{\operatorname{rk}}(U\cap xWx^{-1}) \leq \overline{\operatorname{rk}}(U)\overline{\operatorname{rk}}(W)$$

□

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