ON THE COHERENCE OF THE GROUP ALGEBRAS OF ONE-RELATOR GROUPS

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Abstract. Let G be a one-relator group without torsion. We show that G is homologically coherent and if K is a field of characteristic 0, then the group algebra $K[G]$ is coherent.

1. Introduction

A group G is **coherent** if every finitely generated subgroup is finitely presented and a ring R is (left) **coherent** if any finitely generated left ideal is finitely presented as an R-module. There seems to be a relation between G and $\mathbb{Z}[G]$ being coherent. They are equivalent for elementary amenable groups [21] (see also [5]) and they are commensurability invariants. However, in general there is no known implication between these two properties. Both properties imply a weaker property of being homologically coherent. A group G is **homologically coherent** if any finitely generated subgroup is of type FP_2.

We will not describe here the known examples of coherent groups. The reader may consult an excellent recent survey on coherent groups by Wise [37]. The only result that is no described in the survey is a result of Louder and Wilton [31] who proved the coherence of a one-relator group whose two-generated subgroups are free. The boldest conjecture in this area is that posed by Baumslag [3]: all one-relator groups are coherent.

There are no many papers about coherent group rings. For example we could not find a reference for the coherence of $K[G]$, where G is a free-by-cyclic group and K is a field. Although this result can be extracted easily from the main result of [17]. In Theorem 3.3 we prove the coherence of group algebras of an ascending HNN extension of free groups. Other examples of coherent group rings come from general results on coherent rings. If k is a commutative Noetherian ring, by [6], $k[G]$ is coherent if G is the direct product of a free group and an abelian group and, by [1] and [27], $k[G]$ is coherent if G belongs to the smallest family of groups containing all virtually polycyclic groups and closed under amalgamated products and HNN extension with virtually polycyclic edge subgroups.

We would like to mention a conjecture for graded algebras: every graded algebra with a single defining relation is graded coherent (see [33] for partial results). In this paper we study the coherence of group algebras of one-relator groups.

Theorem 1.1. *Let K be a field of characteristic 0 and let G be a one-relator group without torsion. Then the group algebra $K[G]$ is coherent.*

Our proof uses the existence of the division ring $\mathcal{D}_{K[G]}$ (see Subsection 2.3). When K has characteristic 0, its existence follows from the solution of the strong

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Atiyah conjecture for one-relator groups [24]. If we knew that this division ring exists for an arbitrary K, then we would have the same result for this K as well. The conclusion of Theorem 1.1 holds also for one-relator groups with torsion and an arbitrary field K, since by a results of Kielak and Linton [22], these groups are virtually free-by-cyclic.

Our method does not allow to show that $\mathbb{Z}[G]$ is coherent if G is a one-relator group without torsion. However we can obtain a weak version of this.

Theorem 1.2. *Let G be a one-relator group without torsion. Then G is homologically coherent.*

Since by [13, Theorem 7.9], a subgroup of type FP_2 of a hyperbolic group of cohomological dimension 2 is hyperbolic, and, in particular finitely presented, we obtain the following corollary.

Corollary 1.3. *Let G be a hyperbolic one-relator group. Then G is coherent.*

As we have mentioned above, our proof uses properties of the division ring $\mathcal{D}_{K[G]}$. It is the Hughes-free division $K[G]$-ring whose uniqueness in the case of locally indicable groups was proved by Hughes in [20]. The existence of $\mathcal{D}_{K[G]}$ for an arbitrary field K is an open problem. Recall that, if R is a ring, the **weak dimension** of a right R-module M is the largest i for which there exists a left R-module N such that $\mathrm{Tor}_i^R(M,N) \neq 0$. A new property that we have discovered in the case of one-relator groups is the following result.

Theorem 1.4. *Let K be a field and G a one-relator group without torsion. Assume that $\mathcal{D}_{K[G]}$ exists. Then as a right $K[G]$-module, $\mathcal{D}_{K[G]}$ is of weak dimension at most* 1.

This property was known for free, limit and free-by-cyclic groups G. It does not hold for locally indicable groups in general: there are locally indicable groups with non-trivial second L^2-Betti number (for example, if G is the direct product of two non-abelian free groups).

As an immediate consequence of Theorem 1.4 we obtain that the second L^2-Betti number of a subgroup of one-relator group vanishes. The case of one-relator groups with torsion follows from [22].

Corollary 1.5. *Let G be a one-relator group and H is a subgroup of G. Then the right $\mathbb{Q}[H]$-module $\mathcal{D}_{\mathbb{Q}[H]}$ is of weak dimension at most* 1*. In particular, $b_2^{(2)}(H) = 0$.*

The paper is organized as follows. In Section 2 we describe all the preliminary results that we need in the paper. In particular, we define the division ring $\mathcal{D}_{K[G]}$ and describe its main properties. In Section 3 we present a method that allows to prove that under certain conditions a finitely generated module is finitely presented. Our first application is Theorem 3.3 where we prove the coherence of $K[G]$, where K is a field and G is an ascending HNN extension of a free group. We finish Section 3 explaining how Theorems 1.1 and 1.2 follow from Theorem 1.4. Section 4 is the core of the paper. There we show that certain $K[F]$-module, where F is a free group, is flat. This can be used in calculations of different Tor groups. As an example, in Section 5, we give a new proof of the 1-rank Hanna Neumann conjecture of Wise [36]. Finally, in Section 6 we prove Theorem 1.4.

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2. Preliminaries

2.1. **General notations.** All rings in this paper are associative and have the identity element. All ring homomorphisms send the identity to the identity. By an R**-ring** we understand a ring homomorphism $\varphi : R \to S$. We will often refer to S as R-ring and omit the homomorphism φ if φ is clear from the context. Two R-rings $\varphi_1 : R \to S_1$ and $\varphi_2 : R \to S_2$ are said to be **isomorphic** if there exists a ring isomorphism $\alpha : S_1 \to S_2$ such that $\alpha \circ \varphi_1 = \varphi_2$.

For a ring R, a left R-module M and $a \in R$, we put

$$\mathrm{Ann}_M(a) = \{m \in M : a \cdot m = 0\}.$$

Let G be a group and k a commutative ring. We denote by $I_{k[G]}$ the augmentation ideal of $k[G]$. If H is a subgroup of G we denote by $I^G_{k[H]}$ the left ideal of $k[G]$ generated by $I_{k[H]}$. Given two left $k[G]$-modules M and N, $M \otimes_k N$ becomes a left $k[G]$-module if we define $g(m \otimes_K n) = gm \otimes_K gn$.

2.2. **Ordered groups.** A total order $\preceq$ on a group G is **right-invariant** if for any $a, b, g \in G$, if $a \preceq b$ then $ag \preceq bg$. We say that $\preceq$ is **Conradian** if for all elements $f, g \succeq 1$, there exists a natural number n such that $g^n f \succ g$. Recall that a group G is **locally indicable** if every finitely generated non-trivial subgroup of G has an infinite cyclic quotient. A useful characterization of locally indicable groups says that they are the groups admitting a Conradian order ([9]).

2.3. **Hughes-free, Linnell and Dubrovin division rings.** Let G be a group. In this subsection we will assume that K is a field and to simplify the exposition we will only consider the group algebra $K[G]$. However,we want to underline that all the definitions and results can be easily extended to the case of crossed products $E * G$, where E is a division ring.

Let $\phi : K[G] \to \mathcal{D}$ be a division $K[G]$-ring. Let $N \le H \le G$ be subgroups of G. We denote by $\mathcal{D}_H$ the division closure of $\phi(K[H])$ in $\mathcal{D}$. We say that $\mathcal{D}$ is (left) (N, H)**-free** if the map

$$\mathcal{D}_N \otimes_{K[N]} K[H] \to \mathcal{D},\ d \otimes a \mapsto d\phi(a),$$

is injective. An alternative reformulation of (N, H)-freeness is the following: if $q_1, \ldots, q_n \in H$ are in different right N-cosets, then for any non-zero elements $d_1, \ldots, d_n \in \mathcal{D}_N$, the sum $\sum_{i=1}^n d_i\phi(q_i)$ is not equal to zero. It is clear that if $N \le H_1 \le H_2 \le G$ and $\mathcal{D}$ is (N, H_2)-free, then it is also (N, H_1)-free.

This property appeared in the work of Hughes [20] in the context of locally indicable groups. Let G be a locally indicable group and $\phi : K[G] \to \mathcal{D}$ a division $K[G]$-rings. We say that $\phi : K[G] \to \mathcal{D}$ (or simply $\mathcal{D}$, when ϕ is clear from the context) is **Hughes-free** if $\mathcal{D}$ is epic (i.e. $\mathcal{D} = \mathcal{D}_G$) and (N, H)-free for any pair $N \trianglelefteq H$ of subgroups of G with $H/N \cong \mathbb{Z}$. A very important contribution of Hughes is the following result proved in [20] (see also, [10] and [24, Theorem 5.2]).

Theorem 2.1 (Hughes)**.** *Let K be a field and G a locally indicable group. Then up to $K[G]$-isomorphism there exists at most one Hughes-free division $K[G]$-ring.*

In view of this results if G is locally indicable and the Hughes free division $K[G]$-ring exists, we will denote it by $\mathcal{D}_{K[G]}$. It was conjectured that $\mathcal{D}_{K[G}$ always exists and this was proven in [24, Corollary 6.7] in the case where K is of characteristic 0. The reader can consult [25] to see what is known at this moment about the general case.

In order to generalizes the notion of Hughes-free division ring to an arbitrary torsion-free group, Linnell proposed in [28] the notion of strongly Hughes-free division $K[G]$-ring. We say that an epic division $K[G]$-ring $\phi: K[G] \to \mathcal{D}$ is **strongly Hughes-free** if it is (N, H)-free for any pair $N \trianglelefteq H$ of subgroups of G.

There is another instance where (N, H)-freeness appeared. An equivalent formulation of the strong Atiyah conjecture over $\mathbb{Q}$ for torsion-free groups G says that the division closure $\mathcal{D}(G)$ of $\mathbb{Q}[G]$ in the ring of affiliated operators $\mathcal{U}(G)$ is a division ring (see, [29, Proposition 1.2]). The ring $\mathcal{D}(G)$ is a called **Linnell** ring. By the discussion after [28, problem 4.5], the Linnell ring (if it is a division ring) is (N, G)-free for any subgroup N of G. In particular, by Theorem 2.1, if G is locally indicable, $\mathcal{D}(G) \cong \mathcal{D}_{\mathbb{Q}[G]}$ as $\mathbb{Q}[G]$-rings. One consequence of this fact is that for locally indicable groups the L^2-Betti numbers can be computed in a pure algebraic way:

$$b_k^{(2)}(G) = \dim_{\mathcal{D}_{\mathbb{Q}[G]}} \operatorname{Tor}_k^{\mathbb{Q}[G]}(\mathcal{D}_{\mathbb{Q}[G]}, \mathbb{Q}).$$

This also leads us to the following definition. Let G be a torsion-free group. We say that an epic division $K[G]$-ring $\phi: K[G] \to \mathcal{D}$ is **Linnell** if it is (N, G)-free for any subgroup N of G. In view of the previous discussion it is tempting to propose the following variation of the strong Atiyah conjecture for torsion-free groups.

Conjecture 1. *Let K be a field and G a torsion-free group. Then a Linnell division $K[G]$-ring exists and it is unique up to $K[G]$-isomorphism.*

The strong Atiyah conjecture indicates that in the case, where K is a subfield of $\mathbb{C}$, a candidate for a Linnell division $K[G]$-ring is the division closure of $K[G]$ in $\mathcal{U}(G)$. If K is an arbitrary field and G has a right-invariant order, Dubrovin proposed such a candidate [11] (see also [4, 15]).

Let G be a group with a right-invariant order $\preceq$. **The space of Malcev-Neumann series** $\mathcal{MN}_{\preceq}(K[G])$ is the abelian group consisting of formal infinite sums $m = \sum_{g \in G} k_g g$, with $k_g \in K$, such that the support of m,

$$\operatorname{supp}\left(\sum_{g \in G} k_g g\right) = \{g \in G : \ k_g \neq 0\}$$

is a well-ordered subset of G. We denote by $\mathcal{E}_{\preceq}(K[G])$ the ring of endomorphisms of the abelian group $\mathcal{MN}_{\preceq}(K[G])$ and we will use the notation where the elements of $\mathcal{E}_{\preceq}(K[G])$ act on $\mathcal{MN}_{\preceq}(K[G])$ on the right side.

Since the order $\preceq$ is right-invariant, the ring $K[G]$ is embedded into $\mathcal{E}_{\preceq}(K[G])$ and we will identify the elements of $K[G]$ with their images. The **Dubrovin** ring $\mathcal{D}_{\preceq}(K[G])$ is the division closure of $K[G]$ in $\mathcal{E}_{\preceq}(K[G])$. Dubrovin conjectured that this ring is a division ring. The following result has appeared in the literature in slightly different contexts and we include its proof for the readers' convenience.

Proposition 2.2. *Let K be a field and G a group with a right-invariant order $\preceq$. If the Dubroving ring $\mathcal{D}_{\preceq}(K[G])$ is a division ring, then it is also a Linnell division $K[G]$-ring.*

Proof. Let us denote $\mathcal{D}_{\preceq}(K[G])$ by $\mathcal{D}$. Let N be a subgroup of G. We want to show that if $q_1, \dots, q_n \in G$ are in different right N-cosets, then for any non-zero elements $d_1, \dots, d_n \in \mathcal{D}_N$, the sum $\sum_{i=1}^n d_i q_i \in \mathcal{D}$ is not equal to zero. Assume that $\sum_{i=1}^n d_i q_i \in \mathcal{D} = 0$. Without loss of generality we can also assume that $q_1 = 1$.

Given a subset T of G, we denote by π_T the element of $\mathcal{E}_{\preceq}(K[G])$ defined by means of

$$(\sum_{g\in G} k_g g) \cdot \pi_T = \sum_{g\in T} k_g g \ (k_g \in K).$$

Observe that elements of $K[N]$ commutes with π_N. Hence, the elements of $\mathcal{D}_N$ commutes with π_N too. Thus, for each $i = 1, \dots, n$,

$$\operatorname{supp}(1 \cdot d_i q_i) = \operatorname{supp}(1 \cdot \pi_N d_i q) = \operatorname{supp}(1 \cdot d_i \pi_N q_i) \leq N q_i.$$

Therefore,

$$1 \cdot d_1 = (1 \cdot \sum_{i=1}^n d_i q_i)\pi_N = 0.$$

Since d_1 is invertible in $\mathcal{E}_{\preceq}(K[G])$, we obtain a contradiction. □

As we have mentioned above if G is locally indicable and K is of characteristic zero, then the Hughes-free division ring $\mathcal{D}_{K[G]}$ is Linnell. Gräter proved in [14] the same result for arbitrary field K if $\mathcal{D}_{K[G]}$ exists.

Theorem 2.3 (Gräter)**.** *Let K be a field and G a locally indicable group. Let $\preceq$ be a Conradian order on G. If $\mathcal{D}_{K[G]}$ exists, then the Dubrovin ring $\mathcal{D}_{\preceq}(K[G])$ is a division ring. In particular, $\mathcal{D}_{K[G]}$ is the unique Linnell division $K[G]$-ring.*

Proof. By [14, Corollary 8.3, Theorem 8.1], $\mathcal{D}_{\preceq}(K[G])$ is a division ring and it is isomorphic to $\mathcal{D}_{K[G]}$. By Proposition 2.2, $\mathcal{D}_{\preceq}(K[G])$ is Linnell. Theorem 2.1 implies that $\mathcal{D}_{K[G]}$ is the unique Linnell division $K[G]$-ring. □

3. A criterion for the coherence of a ring and the proof of Theorems 1.1 and 1.2

In this section we obtain Theorems 1.1 and 1.2 as consequences of Theorem 1.4.

Let R be a ring and M a left R-module. We say that M is of **projective dimension** at most k if M has a projective resolution of length k. The following result is a variation of the argument of the proof of [26, Lemma 4].

Proposition 3.1. *Let R be a ring and assume that R can be embedded in a division ring $\mathcal{D}$. Let M be a finitely generated left R-module of projective dimension at most 1. If $\dim_{\mathcal{D}} \operatorname{Tor}_1(\mathcal{D}, M)$ is finite, then M is finitely presented.*

Proof. Since M is a finitely generated left R-module of projective dimension at most 1 there exists an exact sequence of left R-modules

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_0 and P_1 projective and P_0 finitely generated. We want to show that P_1 is also finitely generated. The calculation of $\operatorname{Tor}_1^R(\mathcal{D}, M)$ gives us the exact sequence of left $\mathcal{D}$-modules

$$\operatorname{Tor}_1^R(\mathcal{D}, M) \to \mathcal{D} \otimes_R P_1 \to \mathcal{D} \otimes_R P_0.$$

Since $\dim_{\mathcal{D}} \operatorname{Tor}_1(\mathcal{D}, M)$ and $\dim_{\mathcal{D}} \mathcal{D} \otimes_R P_0$ are finite, $n = \dim_{\mathcal{D}} \mathcal{D} \otimes_R P_1$ is finite as well. Since $\mathcal{D}$ is division ring, there are $m_1, \dots, m_n \in P_1$ such that

$$\mathcal{D} \otimes_R P_1 = \sum_{i=1}^{n} \mathcal{D}(1 \otimes m_i).$$

Since P_1 is projective, there exists a free left R-module L such that $L = P_1 \oplus P_2$. Thus, any element $l \in L$ can be uniquely written as $l = p_1 + p_2$, where $p_1 \in P_1$ and $p_2 \in P_2$. We denore $\pi_{P_1} : L \to P_1$ by $\pi_{P_1}(l) = p_1$.

Put $N = \sum Rm_i$. Since N is finitely generated, there exists a finitely generated free summand L_1 of L that contains N. Consider the natural map $\tau : P_1/N \to L/L_1$. Since $\mathcal{D} \otimes_R (P_1/N) = 0$, $\mathcal{D} \otimes_R \operatorname{Im} \tau = 0$. Thus, since $\operatorname{Im} \tau \le L/L_1$ is a submodule of a free R-module, $\operatorname{Im} \tau = \{0\}$. This implies that $P_1 \le L_1$, and so

$$P_1 = \pi_{P_1}(P_1) = \pi_{P_1}(L_1)$$

is finitely generated. □

A ring R is of (left) **global dimension** k, if every left R-module has a projective resolution of length k. If G is a torsion-free one-relator group, then G is of cohomological dimension 2, and so for any field K, $K[G]$ is of global dimension 2.

Corollary 3.2. *Let R be a ring and assume that R can be embedded in a division ring $\mathcal{D}$. Assume that R is of global dimension at most 2 and the right R-module $\mathcal{D}$ is of weak dimension at most 1. Then R is coherent.*

Proof. Let I be a finitely generated left ideal of R. Let $0 \to P \to R^k \to I \to 0$ be an exact sequence of left R-modules. Hence we obtain the exact sequence

$$0 \to P \to R^k \to R \to R/I \to 0.$$

Since R is of global dimension at most 2, the projective dimension of R/I is at most 2. Hence, by [34, Proposition 8.6(2)], P is projective, and so, I is of projective dimension at most 1. Since $\operatorname{Tor}_1(\mathcal{D}, I) = \operatorname{Tor}_2(\mathcal{D}, R/I)$ and the right R-module $\mathcal{D}$ is of weak dimension at most 1, $\operatorname{Tor}_1(\mathcal{D}, I) = 0$. Thus, by Proposition 3.1, I is finitely presented, and so, R is coherent. □

As a first application we show that the group algebras of an ascending HNN extension of free groups are coherent.

Theorem 3.3. *Let K be a field and G an ascending HNN extension of free groups. Then the group algebra $K[G]$ is coherent.*

Proof. Let us briefly explain the construction of $\mathcal{D}_{K[G]}$. For details see, for example, [25].

The group G has a locally free normal subgroup N such that $G/N \cong \mathbb{Z}$. Let $t \in G$ be such that G/N is generated by tN and let $\tau : K[N] \to K[N]$ be the automorphism induced by the conjugation by t: $\tau(a) = tat^{-1}$. Then $K[G]$ is naturally isomorphic to the ring of twisted Laurent polynomials $K[N][t^{\pm 1}, \tau]$. Since N is locally free, there exists $\mathcal{D}_{K[N]}$. Moreover, $\operatorname{Tor}_2^{K[N]}(\mathcal{D}_{K[N]}, M) = 0$ for any left $K[N]$-module M. Since $\mathcal{D}_{K[N]}$ is unique, we can extend τ to an automorphism $\mathcal{D}_{K[N]} \to \mathcal{D}_{K[N]}$, which we will also call τ. Then $\mathcal{D}_{K[G]}$ is isomorphic to the Ore classical ring of fractions of $\mathcal{D}_{K[N]}[t^{\pm 1}, \tau]$.

Let M be a left $K[G]$-module. Then by Shapiro's lemma we obtain

$$\operatorname{Tor}_2^{K[G]}(\mathcal{D}_{K[N]}[t^{\pm 1}, \tau], M) = \operatorname{Tor}_2^{K[N]}(\mathcal{D}_{K[N]}, M) = 0.$$

Thus, $\mathrm{Tor}_2^{K[G]}(\mathcal{D}_{K[G]}, M) = 0$ as well. Hence the right $K[G]$-module $\mathcal{D}_{K[G]}$ is of weak dimension at most 1.

Since G is a HNN extension of free groups, it is of cohomological dimension at most 2. Hence $K[G]$ is of global dimension at most 2. Therefore, $K[G]$ is coherent by Corollary 3.2. □

Proof of Theorem 1.1. By [7] (see also, [18, 19]), G is locally indicable. By [24, Corollary 6.7], there exist $\mathcal{D}_{K[G]}$. By Theorem 1.4 the right $K[G]$-module $\mathcal{D}_{K[G]}$ is of weak dimension at most 1. Since G is of cohomological dimension at most 2, $K[G]$ is of global dimension at most 2. Thus, we can apply Corollary 3.2 and obtain that $K[G]$ is coherent. □

Proof of Theorem 1.2. Let H be a finitely generated subgroup of G. Since G is of cohomological dimension 2, H is also of cohomological dimension 2 and so the left $\mathbb{Z}[H]$-module $I_{\mathbb{Z}[H]}$ is of projective dimension 1. Observe that

$$\mathrm{Tor}_1^{\mathbb{Z}[H]}(\mathcal{D}_{\mathbb{Q}[H]}, I_{\mathbb{Z}[H]}) \cong \mathrm{Tor}_2^{\mathbb{Z}[H]}(\mathcal{D}_{\mathbb{Q}[H]}, \mathbb{Z}) \cong \\ \mathrm{Tor}_2^{\mathbb{Z}[G]}(\mathcal{D}_{\mathbb{Q}[H]} \otimes_{\mathbb{Z}[H]} \mathbb{Z}[G], \mathbb{Z}) \cong \mathrm{Tor}_2^{\mathbb{Z}[G]}(\mathcal{D}_{\mathbb{Q}[H]} \otimes_{\mathbb{Q}[H]} \mathbb{Q}[G], \mathbb{Z}).$$

By Proposition 2.3, $\mathcal{D}_{\mathbb{Q}[H]} \otimes_{\mathbb{Q}[H]} \mathbb{Q}[G]$ can be seen as a submodule of $\mathcal{D}_{\mathbb{Q}[G]}$. Thus, since $\mathbb{Q}[G]$ is of global dimension at most 2 and, by Theorem 1.4, $\mathcal{D}_{\mathbb{Q}[G]}$ is of weak dimension at most 1, the right $\mathbb{Q}[G]$-module $\mathcal{D}_{\mathbb{Q}[H]} \otimes_{\mathbb{Q}[H]} \mathbb{Q}[G]$ is of weak dimension at most 1. Hence, $\mathrm{Tor}_1^{\mathbb{Z}[H]}(\mathcal{D}_{\mathbb{Q}[H]}, I_{\mathbb{Z}[H]}) = 0$. Thus, Proposition 3.1 implies that $I_{\mathbb{Z}[H]}$ is finitely presented, and so H is of type FP_2. Therefore, G is homologically coherent. □

4. A key result

In this section we will prove the key result of this paper.

Theorem 4.1. *Let K be a field, F a free group and $w \in F$ a word which is not a proper power. Let $G = F/\langle w^F \rangle$. Assume that $\mathcal{D}_{K[G]}$ exists. Then the left $K[F]$-module*

$$\mathcal{D}_{K[G]} \otimes_K (I_{K[F]}/K[F](w-1))$$

is flat.

Before proving the theorem we need some auxiliary results.

Lemma 4.2. *Let K be a field and F a free group. Let V be a left $K[F]$-module and $G = F/N$ be a locally indicable group. Let*

$$0 \neq \alpha = \sum_{i=1}^{n} c_i \cdot f_i \in K[F] \ (0 \neq c_i \in K,\ f_i \in F).$$

Assume that all $g_i = f_i N \in G$ are different. Then

$$\mathrm{Ann}_{V \otimes_K \mathcal{D}_{K[G]}}(\alpha) = \{m \in V \otimes_K \mathcal{D}_{K[G]} \colon \alpha \cdot m = 0\}$$

is trivial.

Proof. We prove the lemma by induction on n. If $n = 1$, the statement is clear.

Consider now the case where $n > 1$ and assume that $\mathrm{Ann}_{V \otimes_K \mathcal{D}_{K[G]}}(\alpha) \neq 0$. Without loss of generality we can assume that $f_1 = 1$. Let $H = \langle g_2, \ldots, g_n \rangle$. Since $n > 1$ and all g_i are different, H is not trivial. Let $\widetilde{H} = \langle f_2, \ldots, f_n \rangle$. Since G

is locally indicable, there exists an epimorphism $\phi : H \to \mathbb{Z}$, which induces an epimorphism $\widetilde{\phi} : \widetilde{H} \to \mathbb{Z}$ satisfying $\widetilde{\phi}(x) = \phi(xN)$.

Let $s \in \widetilde{H}$ be such that $\widetilde{\phi}(s) = 1$. Then we can write

$$\alpha = \sum_{j=a}^{b} \alpha_j \cdot s^j, \text{ with } \alpha_j \in K[\ker \widetilde{\phi}] \text{ and } \alpha_b \neq 0.$$

Observe that if we write

$$\alpha_b = \sum_{k=1}^{l} d_k \cdot h_k \ (0 \neq d_k \in K,\ h_k \in F),$$

then all $h_k N$ are different and $l < n$.

For simplicity we will write $\mathcal{D}$ instead of $\mathcal{D}_{K[G]}$. We can see $V \otimes_K \mathcal{D}$ as a $(K[F], \mathcal{D})$-bimodule. Therefore, given a basis B of $\mathcal{D}$ as a left $\mathcal{D}_H$-module we obtain that

$$V \otimes_K \mathcal{D} = \oplus_{q \in B} (V \otimes_K \mathcal{D}_H\, q) = \oplus_{q \in B} (V \otimes_K \mathcal{D}_H) q.$$

Thus, since $\mathrm{Ann}_{V \otimes_K \mathcal{D}}(\alpha) \neq 0$, $\mathrm{Ann}_{V \otimes_K \mathcal{D}_H}(\alpha) \neq 0$ as well.

Let $t = sN \in H$. Since $\mathcal{D}$ is Hughes-free, the ring R generated by $\mathcal{D}_{\ker \phi}$ and t is isomorphic to the ring of twisted polynomials $\mathcal{D}_{\ker \phi}[t, \tau]$, where τ is the authomorphism of $\mathcal{D}_{\ker \phi}$ induced by conjugation by t. Observe that $\mathcal{D}_H$ is the Ore ring of fractions of R. Thus, since $\mathrm{Ann}_{V \otimes_K \mathcal{D}_H}(\alpha) \neq 0$, we also obtain that $\mathrm{Ann}_{V \otimes_K R}(\alpha) \neq 0$.

Let $0 \neq m \in \mathrm{Ann}_{V \otimes_K R}(\alpha)$. We can write

$$m = \sum_{j=c}^{d} m_j \cdot t^j, \text{ with } m_j \in V \otimes_K \mathcal{D}_{\ker \phi} \text{ and } m_d \neq 0.$$

Observe that for every j,

$$t^j (V \otimes_K \mathcal{D}_{\ker \phi}) = V \otimes_K t^j\, \mathcal{D}_{\ker \phi} = (V \otimes_K \mathcal{D}_{\ker \phi}) t^j.$$

Thus, since, $\alpha \cdot m = 0$, $\alpha_b t^b \cdot m_d t^d = 0$. Since $m_d \neq 0$, $t^b \cdot m_d t^d \neq 0$. Hence $\mathrm{Ann}_{V \otimes_K \mathcal{D}}(\alpha_d) \neq 0$. But this contradicts the inductive hypothesis. □

Given a left $K[G]$-module M, let M^* be the right $K[G]$-module that coincides with M as a K-vector space and the action of G is given by $m \cdot g = g^{-1} m$. In the same way given a right $K[G]$ module M we can define the left $K[G]$-module M^*. For a K-space V we denote by $V^{n \times 1}$ the space V^n of columns and by $V^{1 \times n}$ the space V^n of rows.

Lemma 4.3. *Let G be a group. The following properties holds.*

(1) *Let $A \in \mathrm{Mat}_{m \times n}(K[G])$ and $M = K[G]^{m \times 1} / A \cdot K[G]^{n \times 1}$. Then*
$$M^* \cong K[G]^{1 \times m} / K[G]^{1 \times n} \cdot A^*,$$
where A^* is obtained from A by changing all group elements $f \in G$ by f^{-1} in the entries of A and then applying the transposition.

(2) *Let $w \in G$. Then we have that $(I_{K[G]} / K[G](w-1))^* \cong I_{K[G]} / (w-1) K[G]$.*

(3) *Given left $K[G]$-modules M and N and a right $K[G]$-module L, we have that for every $k \in \mathbb{N}$.*
$$\mathrm{Tor}_k^{K[G]}(L, M \otimes_K N) \cong \mathrm{Tor}_k^{K[G]}(N^*, M \otimes_K L^*).$$

Proof. (1) is clear, (2) is an application of (1) and the proof of (3) is the same as of [8, Proposition III.2.2]

□

Lemma 4.4. *Let F be a group, $A \in \mathrm{Mat}_{m\times n}(K[F])$ such that $\mathrm{Ann}_{K[F]^{n\times 1}}(A) = 0$ and $M = K[F]^{m\times 1}/A \cdot K[F]^{n\times 1}$. Let L be a left $K[F]$-module. Then*

$$\mathrm{Tor}_1^{K[F]}(M, L) \cong \mathrm{Ann}_{L^{n\times 1}}(A).$$

Proof. This follows directly from the definition of $\mathrm{Tor}_1^{K[F]}$. □

Proof of Theorem 4.1. We will write $\mathcal{D}$ instead of $\mathcal{D}_{K[G]}$. Observe that for any $f \in F$,

$$K[F]/K[F](w^f - 1) \cong K[F]/K[F](w - 1),$$

and so, since $K[F]/K[F](w-1)$ has a unique quotient isomorphic to the trivial module K, $I_{K[F]}/K[F](w^f - 1) \cong I_{K[F]}/K[F](w-1)$ as left $K[F]$-modules. Thus, without loss of generality we can assume that w is cyclically reduced. It is clear that we can also assume that F is finitely generated.

Let $\{x_1, \dots, x_d\}$ be a set of free generators of F. Let $w = x_{i_1}^{\epsilon_1} \dots x_{i_k}^{\epsilon_k}$ be the reduced form of w (here $\epsilon_i = \pm 1$). Without loss of generality we assume that x_1 appears in the reduced form of w. Write

$$w - 1 = (x_1 - 1)\alpha_1 + \dots + (x_d - 1)\alpha_d.$$

The support of α_1 consists of words $x_{i_m}^{\epsilon_m} \dots x_{i_k}^{\epsilon_k}$, where

$$1 \le m \le k+1 \text{ and } x_{i_m}^{\epsilon_m} = x_1^{-1} \text{ or } x_{i_{m-1}}^{\epsilon_{m-1}} = x_1.$$

Observe that, since w is reduced, both cases cannot occur and since w is cyclically reduced, both 1 and w cannot be in the support of α_1. By [35] the image in G of every proper non-trivial subword of w is non-trivial. Thus, the support of α_1 satisfies the hypothesis of Lemma 4.2. Hence for every $\mathbb{C}[F]$-module V

$$\mathrm{Ann}_{\mathcal{D}\otimes_K V}(\alpha_1) = 0. \tag{1}$$

Let L be a right $K[F]$-module. Then by Lemma 4.3,

$$\mathrm{Tor}_1^{K[G]}(L, \mathcal{D} \otimes_K I_F/K[F](w-1)) \cong \mathrm{Tor}_1^{K[G]}(I_F/(w-1)K[F], \mathcal{D} \otimes_K L^*).$$

Since $I_{K[F]} = (x_1 - 1)K[F] \oplus \dots \oplus (x_d - 1)K[F]$,

$$I_{\mathbb{C}[F]}/(w-1)K[F] \cong K[F]^{1\times d}/(\alpha_1, \dots, \alpha_d)K[F].$$

Therefore, by Lemma 4.4 and taking into account (1), we obtain that

$$\mathrm{Tor}_1^{K[G]}(L, \mathcal{D} \otimes_K I_F/K[F](w-1)) \cong \mathrm{Tor}_1^{K[G]}(I_F/(w-1)K[F], \mathcal{D} \otimes_K L^*) = 0.$$

for every right $K[F]$-module L. Hence $\mathcal{D} \otimes_K (I_{K[F]}/K[F](w-1))$ is flat. □

5. An application of the key result: 1-rank Hanna Neumann conjecture

Let $d(G)$ denote the minimal number of generators of a group G and

$$\overline{d}(G) = \max\{0, d(G) - 1\}.$$

Given two finitely generated subgroups U and W of a free group F the Friedman-Mineev theorem [12, 32] (previously known as the Strengthened Hanna Neumann conjecture) states that

$$\sum_{x\in W\backslash F/U} \overline{d}(xUx^{-1}\cap W)\le \overline{d}(U)\overline{d}(W).$$

This result says nothing about the cyclic intersections $xUx^{-1}\cap W$. Wise proposed a 1-rank version of this conjecture, which was proved independently by Helfer and Wise [16] and Louder and Wilton [30].

Theorem 5.1 (Helfer-Wise, Louder-Wilton)**.** *Let U be a subgroup of a free group F, $w\in F$ a non-proper power element and $W=\langle w\rangle$. Then*

$$\sum_{x\in W\backslash F/U} d(xUx^{-1}\cap W)\le \left\{\begin{array}{ll} d(U) & \textit{if } U\le \langle w^F\rangle \\ \overline{d}(U) & \textit{if } U\not\le \langle w^F\rangle \end{array}\right. .$$

In this section we will prove a generalization of this result. In order to formulate it, we need other interpretation for the sum $\sum_{x\in W\backslash F/U} d(xUx^{-1}\cap W)$ (we follow the approach developed in [23, 2]). Consider $\mathbb{Q}[F/U]$ as a left $\mathbb{Q}[W]$-module. Then

$$\mathbb{Q}[F/U]\cong \bigoplus_{x\in W\backslash F/U} \mathbb{Q}[W/(xUx^{-1}\cap W)].$$

Therefore, the sum that appears in the theorem has the following interpretaion

$$\sum_{x\in W\backslash F/U} d(xUx^{-1}\cap W)=\dim_{\mathbb{Q}} \operatorname{Ann}_{\mathbb{Q}[F/U]}(w-1).$$

If F is a free group, all left ideals of $\mathbb{Q}[F]$ are left free $\mathbb{Q}[F]$-modules of a unique rank. The rank of a free $\mathbb{Q}[F]$-module L we denote by $\operatorname{rk}(L)$ and we also put $\overline{\operatorname{rk}}(L)=\max\{\operatorname{rk}(L)-1,0\}$. Notice that $\mathbb{Q}[F/U]\cong \mathbb{Q}[F]/I^F_{\mathbb{Q}[U]}$ and $\operatorname{rk}(I^F_{\mathbb{Q}[U]})=d(U)$. Thus, the following result is a generalization of Theorem 5.1.

Theorem 5.2. *Let F be a free group, $w\in F$ a non-proper power element, I the ideal of $\mathbb{Q}[F]$ generated by $w-1$ and L a left ideal of $\mathbb{Q}[F]$, then*

$$\dim_{\mathbb{Q}} \operatorname{Ann}_{\mathbb{Q}[F]/L}(w-1)\le \left\{\begin{array}{ll} \operatorname{rk}(L) & \textit{if } L\le I \\ \overline{\operatorname{rk}}(L) & \textit{if } L\not\le I \end{array}\right. .$$

Proof. Let $G=F/\langle w^F\rangle$, $\mathcal{D}=\mathcal{D}_{\mathbb{Q}[G]}$ and $M=\mathbb{Q}[F]/L$. Note that

$$\dim_{\mathbb{Q}} \operatorname{Ann}_M(w-1)=\dim_{\mathbb{Q}} \operatorname{Tor}_1^{\mathbb{Q}[W]}(\mathbb{Q},M)= \\ \dim_{\mathcal{D}} \operatorname{Tor}_1^{\mathbb{Q}[W]}(\mathcal{D},M)=\dim_{\mathcal{D}} \operatorname{Tor}_1^{\mathbb{Q}[F]}(\mathcal{D},\mathbb{Q}[F/W]\otimes_{\mathbb{Q}} M).$$

Taking into account the exact sequence of left $\mathbb{Q}[F]$-modules

$$0\to (I_{\mathbb{Q}[F]}/\mathbb{Q}[F](w-1))\otimes_{\mathbb{Q}} M\to \mathbb{Q}[F/W]\otimes_{\mathbb{Q}} M\to M\to 0,$$

we obtain that

$$\text{(2)}\quad \dim_{\mathcal{D}} \operatorname{Tor}_1^{\mathbb{Q}[F]}(\mathcal{D},\mathbb{Q}[F/W]\otimes_{\mathbb{Q}} M)\le \\ \dim_{\mathcal{D}} \operatorname{Tor}_1^{\mathbb{Q}[F]}(\mathcal{D},(I_{\mathbb{Q}[F]}/\mathbb{Q}[F](w-1))\otimes_{\mathbb{Q}} M)+\dim_{\mathcal{D}} \operatorname{Tor}_1^{\mathbb{Q}[F]}(\mathcal{D},M).$$

Observe that if we see $\mathcal{D}$ as a right $\mathbb{Q}[F]$-module, $\mathcal{D}^* \cong \mathcal{D}$. Thus, from Lemma 4.3 and Theorem 4.1, we obtain that

$$(3) \quad \operatorname{Tor}_1^{\mathbb{Q}[F]}(\mathcal{D}, I_{\mathbb{Q}[F]}/\mathbb{Q}[F](w-1)) \otimes_{\mathbb{Q}} M) = \operatorname{Tor}_1^{\mathbb{Q}[F]}(M^*, \mathcal{D} \otimes_{\mathbb{Q}} (I_{\mathbb{Q}[F]}/\mathbb{Q}[F](w-1))) = 0.$$

For the second summand in (2) we have that

$$\dim_{\mathcal{D}} \operatorname{Tor}_1^{\mathbb{Q}[F]}(\mathcal{D}, M) = \operatorname{rk}(L) - 1 + \dim_{\mathcal{D}} \operatorname{Tor}_0^{\mathbb{Q}[F]}(\mathcal{D}, M) = \begin{cases} \operatorname{rk}(L) & \text{if } L \leq I \\ \overline{\operatorname{rk}}(L) & \text{if } L \not\leq I \end{cases}.$$

This finishes the proof of the theorem. □

6. The weak dimension of $\mathcal{D}_{K[G]}$ as a $K[G]$-module

In this section we present a proof Theorem 1.4, which relies on Theorem 4.1. Theorem 4.1 implies the equality (3), and as a consequence the injectivity of the natural homomorphism of $\mathcal{D}_{\mathbb{Q}[G]}$-modules

$$\operatorname{Tor}_1^{\mathbb{Q}[F]}(\mathcal{D}_{\mathbb{Q}[G]}, \mathbb{Q}[F/W] \otimes_{\mathbb{Q}} M) \to \operatorname{Tor}_1^{\mathbb{Q}[F]}(\mathcal{D}_{\mathbb{Q}[G]}, M).$$

The main ingredient in our proof of Theorem 1.4 is an interpretation (see Claims 6.6 and 6.7) of the injectivity of this map.

Proof of Theorem 1.4. For simplicity we will write $\mathcal{D}$ instead of $\mathcal{D}_{K[G]}$. In order to prove that the right $K[G]$-module $\mathcal{D}$ is of weak dimension 1, it is enough to show that $\operatorname{Tor}_2^{K[G]}(\mathcal{D}, M) = 0$, where $M = R/I$ and I is a finitely generated left ideal of $K[G]$. Since

$$\operatorname{Tor}_2^{K[G]}(\mathcal{D}, M) = \operatorname{Tor}_1(\mathcal{D}, I),$$

we want to show that $\operatorname{Tor}_1^{K[G]}(\mathcal{D}, I) = 0$.

Represent G as $G = F/\langle w^F \rangle$, where $w \in F$. Then this induces a homomorphism

$$\phi : K[F] \to K[G],$$

where $\ker \phi$ is the left ideal of $K[F]$ generated by $\{(w-1)t \colon t \in F\}$. Enumerate $F = \{t_i \colon i \in \mathbb{N}\}$. Let L_0 be a finitely generated left ideal of $K[F]$ such that $\phi(L_0) = I$ and inductively

$$L_{k+1} = L_k + K[F](w-1)t_{k+1} \text{ and } M_k = K[F]/L_k.$$

Let $B_k \subset K[F]$ be such that $\{b + L_k \colon b \in B_k\}$ be a K-basis of $\operatorname{Ann}_{M_k}(w-1)$. We put

$$I_k = K[G] \otimes_{K[F]} \left(L_k / (\sum_{b \in B_k} K[F](w-1)b + K[F](w-1)L_k) \right).$$

Claim 6.1. *We have that*

$$I_k \cong K[G] \otimes_{K[F]} L_k / \sum_{b \in B_k} K[G](1 \otimes (w-1)b).$$

Proof. This is clear because the image of $w-1$ in $K[G]$ is zero. □

The modules I_k are related to I in the following way.

Claim 6.2. *We have that* $I \cong \varinjlim_k I_k$.

Proof. let $\theta_k : L_k \to L_{k+1}$ be the embedding map. Since

$$(w-1)t_{k+1} \in \sum_{b\in B_{k+1}} K[F](w-1)b + K[F](w-1)L_{k+1},$$

θ_k induces a surjective map $\overline{\theta_k} : I_k \to I_{k+1}$. Consider the direct limit $\varinjlim_k I_k$ with respect of $\overline{\theta_k}$ and let $\epsilon_k : I_k \to \varinjlim_i I_i$ be the induced map.

The restriction of ϕ on L_k factors through a homomorphism $\overline{\phi_k} : I_k \to I$ of left $K[G]$-modules. Moreover, $\overline{\phi_k} = \overline{\phi_{k+1}} \circ \overline{\theta_k}$. Thus we can define

$$\overline{\phi} = \varinjlim_k \overline{\phi_k} : \varinjlim_k I_k \to I.$$

This is a surjective homomorphism of left $K[G]$-modules. Let $\epsilon_k(x) \in \ker \overline{\phi}$ for some $x \in I_k$. Then $\phi_k(x) = 0$. There exists $y \in L_k$ such that

$$x = 1 \otimes y + \sum_{b\in B_k} K[G](1 \otimes (w-1)b).$$

Thus, since $\phi_k(x) = 0$, $y \in \ker \phi$. Hence there exists $n \geq k$ such that $y \in \sum_{i=1}^n \mathbb{Q}[F](w-1)t_i$. Thus $\overline{\theta_{n-1}} \circ \ldots \circ \overline{\theta_k}(x) = 0$. Therefore, $\epsilon_k(x) = 0$ and ϕ is an isomorphism. □

Observe that, since $\mathrm{Tor}_1^{K[G]}(\mathcal{D}, \cdot)$ commutes with direct limits, in order to show that $\mathrm{Tor}_1^{K[G]}(\mathcal{D}, I) = 0$, it is enough to prove that $\mathrm{Tor}_1^{K[G]}(\mathcal{D}, I_k) = 0$ for every k.

Claim 6.3. *We have that* $\dim_K \mathrm{Tor}_1^{K[\langle w\rangle]}(K, M_k) = |B_k|$.

Proof. Observe that $\mathrm{Tor}_1^{K[\langle w\rangle]}(K, M_k) \cong \mathrm{Ann}_{M_k}(w-1)$. □

Consider the exact sequence

$$0 \to L_k \to K[F] \to M_k \to 0$$

as a sequence of $K[\langle w\rangle]$-modules. It induces an exact sequence of left $\mathcal{D}$-modules

$$0 \to \mathrm{Tor}_1^{K[\langle w\rangle]}(\mathcal{D}, M_k) \xrightarrow{\nu_k} \mathcal{D} \otimes_{K[\langle w\rangle]} L_k \xrightarrow{\mu_k} \mathcal{D} \otimes_{K[\langle w\rangle]} K[F]$$

Claim 6.4. *We have that* $\ker \mu_k = \bigoplus_{b\in B_k} \mathcal{D}(1 \otimes (w-1)b)$.

Proof. First observe that we have an exact sequence

$$0 \to \mathrm{Tor}_1^{K[\langle w\rangle]}(K, M_k) \xrightarrow{\overline{\nu}_k} K \otimes_{K[\langle w\rangle]} L_k \xrightarrow{\overline{\mu}_k} K \otimes_{K[\langle w\rangle]} K[F],$$

and $a = 1 \otimes x \in \ker \overline{\mu}_k$ if and only if $x \in L_k \cap (w-1)K[F]$. Thus

$$x \in \sum_{b\in B_k} K(w-1)b + (w-1)L_k \text{ and } a \in \sum_{b\in B_k} K(1 \otimes (w-1)b).$$

On the other hand, Claim 6.3 implies that the sum $\sum_{b\in B_k} K(1 \otimes (w-1)b)$ is direct.

Since $\mathcal{D}$ is trivial $K[\langle w\rangle]$-module,

$$\ker \mu_k \cong \mathcal{D} \otimes_K \ker \overline{\mu}_k \cong \bigoplus_{b\in B_k} \mathcal{D}(1 \otimes (w-1)b).$$

□

Claim 6.5. *Let T be the left transversal of $\langle w\rangle$ in F. Let M be a $K[F]$-module. Then the K-linear map*

$$\tau_M : K[F]\otimes_{K[\langle w\rangle]} M \to K[F/\langle w\rangle]\otimes_K M,\ t\otimes m\mapsto t\langle w\rangle\otimes tm\ (t\in T, m\in M)$$

is an isomorphism of left $K[F]$-modules.

Proof. One can check this by directly applying the definition of the map τ_M. □

Consider an exact sequence of left $K[F]$-modules

$$0\to K[F/\langle w\rangle]\otimes_K L_k\to K[F/\langle w\rangle]\otimes_K K[F]\to K[F/\langle w\rangle]\otimes_K M_k\to 0.$$

It induces an exact sequence of left $\mathcal{D}$-modules

$$0\to \operatorname{Tor}_1^{K[F]}(\mathcal{D}, K[F/\langle w\rangle]\otimes_K M_k)\xrightarrow{\gamma_k}$$
$$\mathcal{D}\otimes_{K[F]}(K[F/\langle w\rangle]\otimes_K L_k)\xrightarrow{\delta_k}\mathcal{D}\otimes_{K[F]}(K[F/\langle w\rangle]\otimes_K K[F]).$$

Claim 6.6. *We have that* $\operatorname{Im}\gamma_k=\ker\delta_k=\bigoplus_{b\in B_k}\mathcal{D}(1\otimes(1\otimes(w-1)b))$.

Proof. Let T be the left transversal of $\langle w\rangle$ in F and assume that $1\in T$. We can use the maps constructed in Claim 6.5. Consider the following commutative diagram of left $\mathcal{D}$-modules.

$$\begin{array}{ccc}\mathcal{D}\otimes_{K[F]}K[F]\otimes_{K[\langle w\rangle]} L_k & \xrightarrow{\mu_k} & \mathcal{D}\otimes_{K[F]}K[F]\otimes_{K[\langle w\rangle]} K[F]\\ \downarrow\operatorname{Id}_{\mathcal{D}}\otimes\tau_{L_k} & & \downarrow\operatorname{Id}_{\mathcal{D}}\otimes\tau_{K[F]}\\ \mathcal{D}\otimes_{K[F]}(K[F/\langle w\rangle]\otimes_K L_k) & \xrightarrow{\delta_k} & \mathcal{D}\otimes_{K[F]}(K[F/\langle w\rangle]\otimes_K K[F])\end{array}.$$

By Claim 6.5, the vertical arrows are isomorphisms. Thus,

$$\ker\delta_k=\operatorname{Id}_{\mathcal{D}}\otimes\tau_{L_k}(\ker\mu_k)\overset{\text{Claim 6.4}}{=}\bigoplus_{b\in B_k}\mathcal{D}(1\otimes(1\otimes(w-1)b)).$$

□

Consider the following commutative diagram of left $K[F]$-modules with exact horizontal sequence.

$$\begin{array}{ccccccc}0\to & K[F/\langle w\rangle]\otimes_K L_k & \to & K[F/\langle w\rangle]\otimes_K K[F] & \to & K[F/\langle w\rangle]\otimes_K M_k & \to 0\\ & \downarrow & & \downarrow & & \downarrow & \\ 0\to & L_k & \to & K[F] & \to & M_k & \to 0\end{array}$$

It induces a commutative diagram of left $\mathcal{D}$-modules with horizontal exact sequences.

$$\begin{array}{cccc}0\to & \operatorname{Tor}_1^{K[F]}(\mathcal{D}, K[F/\langle w\rangle]\otimes_K M_k) & \xrightarrow{\gamma_k} & \mathcal{D}\otimes_{K[F]}(K[F/\langle w\rangle]\otimes_K L_k)\\ & \downarrow\phi_k & & \downarrow\psi_k\\ 0\to & \operatorname{Tor}_1^{K[F]}(\mathcal{D}, M_k) & \xrightarrow{\alpha_k} & \mathcal{D}\otimes_{K[F]}L_k\end{array}. \tag{4}$$

Claim 6.7. *The map ϕ_k is injective, and so the restriction of ψ_k on $\operatorname{Im}\gamma_k$ is injective.*

Proof. As we have explained in the paragraph before the proof of Theorem 1.4, the map ϕ_k is injective by Theorem 4.1. Since γ_k and α_k are injective, the restriction of ψ_k on $\operatorname{Im}\gamma_k$ is injective as well. □

Claim 6.8. *We have that for every $k\in\mathbb{N}$,* $\operatorname{Tor}_1^{K[G]}(\mathcal{D}, I_k)=0$.

Proof. By Claim 6.1, we have the following exact sequence

$$K[G]^{|B_k|} \xrightarrow{\beta_k} K[G] \otimes_{K[F]} L_k \to I_k \to 0,$$

where $\beta_k((x_b : b \in B_k)) = \sum_{b \in B_k} x_b \otimes (w-1)b$.

By Claim 6.6, $\operatorname{Im} \gamma_k = \bigoplus_{b \in B_k} \mathcal{D}(1 \otimes (1 \otimes (w-1)b))$ and by Claim 6.7, the map

$$\psi_k : \bigoplus_{b \in B_k} \mathcal{D}(1 \otimes (1 \otimes (w-1)b)) \to \mathcal{D} \otimes_{K[F]} L_k$$

is injective. Hence the map

$$\mathrm{Id}_{\mathcal{D}} \otimes \beta_k : \mathcal{D} \otimes_{K[G]} K[G]^{|B_k|} \to \mathcal{D} \otimes_{K[G]} (K[G] \otimes_{K[F]} L_k)$$

is also injective. Therefore, $\ker \beta_k = 0$ and $\mathrm{Tor}_1^{K[G]}(\mathcal{D}, I_k) = 0$. □

As we have noticed above $\mathrm{Tor}_1^{K[G]}(\mathcal{D}, \cdot)$ commutes with direct limits, and so by Claims 6.2 and 6.8, $\mathrm{Tor}_1^{K[G]}(\mathcal{D}, I) = 0$. □

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