

**Author's note:** This article may use ideas you haven't learned yet, and might seem overly complicated. It is not. Understanding Stirling's formula is not for the faint of heart, and requires concentrating on a sustained mathematical argument over several steps.

Even if you are not interested in all the details, I hope you will still glance through the article and find something to pique your curiosity. If you are interested in the details, but don't understand something, you are urged to pester your mathematics teacher for help.

## Factorials!

Unbelievably large numbers are sometimes the answers to innocent looking questions. For instance, imagine that you are playing with an ordinary deck of 52 cards. As you shuffle and re-shuffle the deck you wonder: How many ways could the deck be shuffled? That is, how many different ways can the deck be put in order? You reason that there are 52 choices for the first card, then 51 choices for the second card, then 50 for the third card, etc. This gives a total of

$$52 \times 51 \times 50 \times \cdots \times 2 \times 1$$

ways to order a deck of cards. We call this number "52 factorial" and write it as the numeral 52 with an exclamation point: 52! This number turns out to be the 68 digit monster

80658175170943878571660636856403766975289505440883277824000000000000

which means that if every one on earth shuffled cards from now until the end of the universe, at a rate of 1000 shuffles per second, we wouldn't even scratch the surface in getting all possible orders. Whew! No wonder we use exclamation marks!

For any positive integer  $n$  we calculate " $n$  factorial" by multiplying together all integers up to and including  $n$ , that is,  $n! = 1 \times 2 \times 3 \times \cdots \times n$ . Here are some more examples of factorial numbers:

$$\begin{array}{lllll} 1! = 1 & 2! = 2 & 3! = 6 & 4! = 24 & 5! = 120 \\ 6! = 720 & 7! = 5040 & 8! = 40320 & 9! = 362880 & 10! = 3628800 \end{array}$$

**Stirling's formula** Factorials start off reasonably small, but by 10! we are already in the millions, and it doesn't take long until factorials are unwieldy behemoths like 52! above. Unfortunately there is no shortcut formula for  $n!$ , you have to do all of the multiplication. On the other hand, there is a famous approximate formula, named after the Scottish mathematician James Stirling (1692-1770), that gives a pretty accurate idea about the size of  $n!$ .

$$\text{Stirling's formula: } n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Before we continue, let's take a moment to contemplate the fact that  $n$  factorial involves nothing more sophisticated than ordinary multiplication of whole numbers, which Stirling's formula relates to an expression involving square roots,  $\pi$  (the area of a unit circle), and  $e$  (the base of the natural logarithm). Such are the surprises in store for students of mathematics.

Here is Stirling's approximation for the first ten factorial numbers:

$$\begin{array}{cccccc} 1! \approx 0.92 & 2! \approx 1.92 & 3! \approx 5.84 & 4! \approx 23.51 & 5! \approx 118.02 & \\ 6! \approx 710.08 & 7! \approx 4980.39 & 8! \approx 39902.39 & 9! \approx 359536.87 & 10! \approx 3598695.62 & \end{array}$$

You can see that the larger  $n$  gets, the better the approximation proportionally. In fact the approximation  $1! \approx 0.92$  is accurate to 0.08, while  $10! \approx 3598695.62$  is only accurate to about 30,000. But the proportional error for  $1!$  is  $(1! - .92)/1! = .0800$  while for  $10!$  it is  $(10! - 3598695.62)/10! = .0083$ , ten times smaller. This is the correct way to understand Stirling's formula, as  $n$  gets large, the proportional error  $(n! - \sqrt{2\pi n}(n/e)^n)/n!$  goes to zero.

Developing approximate formulas is something of an art. You need to know when to be sloppy and when to be precise. We will make two attempts to understand Stirling's formula, the first uses easier ideas but only gives a sloppy version of the formula. We will follow that with a more sophisticated attack that uses knowledge of calculus and the natural log function. This will give us Stirling's formula up to a constant.

**Attempt 1.** To warm up, let's look at an approximation for the exponential function  $e^x$ . The functions  $1 + y$  and  $e^y$  have the same value and the same slope when  $y = 0$ . This means that  $1 + y \approx e^y$  when  $y$  is near zero, either positive or negative. Applying this approximation to  $x/n$ , for any  $x$  but large  $n$ , gives  $1 + x/n \approx e^{x/n}$ . Now if we take  $n - 1$ st power on both sides, we get the approximation

$$\left(1 + \frac{x}{n}\right)^{n-1} \approx e^{(n-1)x/n} \approx e^x.$$

Returning to factorials, we begin with an obvious upper bound. The number  $n!$  is the product of  $n$  integers, none bigger than  $n$ , so that  $n! \leq n^n$ . With a bit more care, we can write  $n!$  precisely as a fraction of  $n^n$  as follows:

$$n! = \left(1 - \frac{1}{2}\right)^1 \left(1 - \frac{1}{3}\right)^2 \cdots \left(1 - \frac{1}{n}\right)^{n-1} n^n.$$

I won't deprive you of the pleasure of working out the algebra to confirm that this formula is really correct. Using the approximation for the exponential function  $e^x$  we can replace each of the factors  $(1 - 1/k)^{k-1}$  by  $e^{-1}$  and arrive at  $n! \approx e(n/e)^n$ . Because of cumulative errors, the formula  $e(n/e)^n$  sorely underestimates  $n!$ , but it does have the right order of magnitude and explains where the factor "e" comes from.

**Attempt 2.** Mathematically, addition is easier to handle than multiplication so our next attempt to get Stirling's formula converts it into an addition problem by taking logs. Our

warmup problem this time is an approximate formula for the natural log function. We start with the series expansion

$$\frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$$

Substitute  $x = 1/(2j + 1)$  and rearrange to get

$$\left( j + \frac{1}{2} \right) \ln \left( 1 + \frac{1}{j} \right) - 1 = \frac{1}{3(2j+1)^2} + \frac{1}{5(2j+1)^4} + \frac{1}{7(2j+1)^6} + \dots$$

Now replacing the sequence of odd numbers 3, 5, 7, ... by the value 3 in the denominator makes the result bigger, so we have the inequality

$$\left( j + \frac{1}{2} \right) \ln \left( 1 + \frac{1}{j} \right) - 1 \leq \frac{1}{3} \left( \frac{1}{(2j+1)^2} + \frac{1}{(2j+1)^4} + \frac{1}{(2j+1)^6} + \dots \right)$$

The sum on the right takes the form of the famous “geometric series”

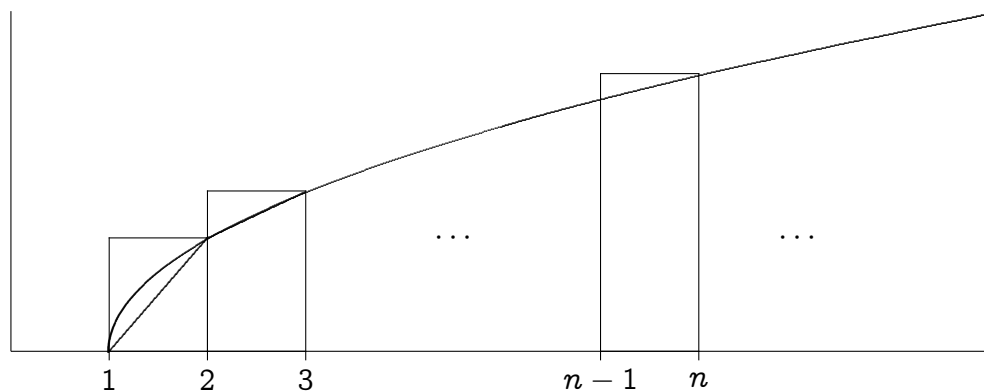
$$\rho + \rho^2 + \rho^3 + \dots = \frac{\rho}{1-\rho}.$$

Making the replacement  $\rho = 1/(2j + 1)^2$  and a little algebra yields

$$\left( j + \frac{1}{2} \right) \ln \left( 1 + \frac{1}{j} \right) - 1 \leq \frac{1}{3} \left( \frac{(2j+1)^{-2}}{1-(2j+1)^{-2}} \right) = \frac{1}{12} \left( \frac{1}{j} - \frac{1}{j+1} \right). \quad (1)$$

All that work to show that  $(j + 1/2) \ln(1 + 1/j) - 1$  is pretty close to zero. If you are inclined, you could program your computer to calculate both sides of (1) for various values of  $j$ , just to check that the right hand side really is bigger than the left. Note that we have an upper bound in (1), instead of an approximate formula. This means that the values on the two sides are not necessarily close together, only that the value on the right is bigger.

You will be relieved to hear that we are finally ready to return to Stirling’s approximation for  $n!$ . Taking the natural log on both sides of  $n! = 1 \times 2 \times \dots \times n$ , turns the multiplication into addition:  $\ln(n!) = \ln(1) + \ln(2) + \dots + \ln(n)$ . This sum, in turn, is the area of the first  $n - 1$  rectangles pictured here. The curve in the picture is  $\ln(x)$ , and it reminds us that  $\ln(1) = 0$ .



The area of each rectangle is the area under the curve, plus the area of the triangle at the top, minus the overlap. In other words, using the definitions below we have  $r_j = c_j + t_j - \varepsilon_j$ .

$$\begin{aligned} \text{rectangle} &:= r_j = \ln(j+1) \\ \text{curve} &:= c_j = \int_j^{j+1} \ln(x) dx \\ \text{triangle} &:= t_j = \frac{1}{2}[\ln(j+1) - \ln(j)] \\ \text{overlap} &:= \varepsilon_j = \left(j + \frac{1}{2}\right) \ln\left(1 + \frac{1}{j}\right) - 1 \end{aligned}$$

The overlap  $\varepsilon_j$  is a small sliver shaped region that is barely visible in the picture, except in the first rectangle. Using the inequality (1) we worked so hard to establish, we add up on both sides and see that the infinite series satisfies  $\sum_{j=n}^{\infty} \varepsilon_j < 1/(12n)$ , for any  $n = 1, 2, 3, \dots$

To approximate  $\ln(n!) = \sum_{j=1}^{n-1} r_j$ , we begin by splitting  $r_j$  into parts

$$\ln(n!) = \sum_{j=1}^{n-1} c_j + \sum_{j=1}^{n-1} t_j - \sum_{j=1}^{n-1} \varepsilon_j.$$

Since  $\sum_{j=1}^{n-1} c_j$  is an integral over the range 1 to  $n$ , and  $\sum_{j=1}^{n-1} t_j$  is a telescoping sum, this simplifies to

$$\begin{aligned} \ln(n!) &= \int_1^n \ln(x) dx + \frac{1}{2} \ln(n) - \sum_{j=1}^{n-1} \varepsilon_j \\ &= n \ln(n) - n + 1 + \frac{1}{2} \ln(n) - \left(\sum_{j=1}^{\infty} \varepsilon_j - \sum_{j=n}^{\infty} \varepsilon_j\right). \end{aligned}$$

Taking the exponential gives

$$n! = e^{1 - \sum_{j=1}^{\infty} \varepsilon_j} \sqrt{n} \left(\frac{n}{e}\right)^n e^{\sum_{j=n}^{\infty} \varepsilon_j}$$

Pause to note that this is an exact equation, not approximate. It gives  $n!$  as the product of an unknown constant, the term  $\sqrt{n} (n/e)^n$ , and a term  $e^{\sum_{j=n}^{\infty} \varepsilon_j}$  that converges to 1 as  $n \rightarrow \infty$ . With the formula  $\sum_{j=n}^{\infty} \varepsilon_j < 1/(12n)$  we now have the bounds

$$C \sqrt{n} \left(\frac{n}{e}\right)^n \leq n! \leq C \sqrt{n} \left(\frac{n}{e}\right)^n e^{\frac{1}{12n}},$$

where  $e^{\frac{11}{12}} \leq C \leq e$ . Once we've identified  $C = \sqrt{2\pi}$ , we get

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n e^{1/12n}$$

If you've made it this far, congratulations! Now you see why Stirling's formula works. The part we skipped, to show that the unknown constant  $C$  is actually equal to  $\sqrt{2\pi}$  is not an easy step. But we leave this aside, and look at some other properties of the number  $n!$ .

**Number of digits** For any  $x > 0$  the formula  $d(x) = \lfloor \log_{10}(x) \rfloor + 1$  gives the number of digits of  $x$  to the left of the decimal point. For moderate sized factorials we can simply plug this formula into a computer to see how many digits  $n!$  has. For example,  $d(52!) = 68$  and  $d(1000000!) = 5565709$ . But suppose we wanted to find the number of digits in a really large factorial, say googol factorial? (Googol means ten raised to the power 100 or  $10^{100}$ ). Even a computer can't calculate googol factorial, so we must use Stirling's formula. Let  $g = 10^{100}$ , substitute into Stirling's formula, and take log (base 10) on both sides to obtain

$$\log_{10}(\sqrt{2\pi g} \left(\frac{g}{e}\right)^g) \leq \log_{10}(g!) \leq \log_{10}(\sqrt{2\pi g} \left(\frac{g}{e}\right)^g e^{1/12g}).$$

Let's concentrate on the left side  $\log_{10}(\sqrt{2\pi g}(g/e)^g)$ . Using the logarithm property and the fact that  $\log_{10}(g) = 100$ , we simplify this to  $\log_{10}(\sqrt{2\pi}) + 50 + g(100 - \log_{10}(e))$ . The hard part of this calculation is to find  $\log_{10}(e)$  to over 100 decimal places, but the computer is happy to do it for us. Once this is accomplished we find that

$$\log_{10}(\sqrt{2\pi g}(g/e)^g) = 995657055180967481723488710810833949177056029941963334338855462168341353507911292252707750506615682567.21202883\dots$$

When we knock off the decimal part and add 1, we get  $d(\sqrt{2\pi g}(g/e)^g)$ . We can be sure that the number of digits in googol factorial is the same by comparing with the upper bound. The right hand side  $\log_{10}(\sqrt{2\pi g}(g/e)^g e^{1/12g})$  exceeds the left hand side only by the minuscule amount  $\log_{10}(e^{1/12g}) = \log_{10}(e)/12g$ . When this is added to the fractional part .21202883..., the first hundred or so digits after the decimal point are not changed. Therefore  $d(\sqrt{2\pi g}(g/e)^g e^{1/12g})$  is equal to  $d(\sqrt{2\pi g}(g/e)^g)$ , and since  $d(g!)$  is in between, it also must be the same.

Raising 10 to the power of the fractional part .21202883... gives us the first few digits of  $g!$ , so we conclude that googol factorial is  $g! = 16294 \dots 00000$ , where the dots stand in for the rest of the exactly

$$d(g!) = 995657055180967481723488710810833949177056029941963334338855462168341353507911292252707750506615682568$$

digits. This explains why no one can or ever will calculate all the digits of googol factorial. Where would you put it? A library filled with books containing nothing but digits? A trillion trillion computer hard drives? None of these puny containers could hold it. This super-monster has more digits than the number of atoms in the universe.

**Trailing zeros** Looking back, you may notice that  $52!$  ends with a stream of zeros. For that matter, all the factorials starting with 5!, have zeros at the end. Let's try to

