1. Let $\mathcal{M} = \left\{ u \in C([0,1]) : \int_0^{1/2} u - \int_{1/2}^1 u = 1 \right\}$ and $I : \mathcal{M} \mapsto \mathbb{R}$ given by $I(u) = \|u\|_\infty$.

   (i) Show that $\inf_{u\in\mathcal{M}} I(u) = 1$.

   (ii) Show that there is no function $u \in \mathcal{M}$ such that $I(u) = 1$.

   The problem is that this space is not reflexive.

Solution. For any $h \in (0, 1/2)$ consider the function

$$u_h(x) = \begin{cases} 1, & x \in \left[0, \frac{1}{2} - h\right), \\ \dfrac{\frac{1}{2} - x}{h} & x \in \left[\frac{1}{2} - h, \frac{1}{2} + h\right), \\ -1, & x \in \left[\frac{1}{2} + h, 1\right]. \end{cases}$$

Hence $u_h \in C([0,1])$ and

$$\int_0^{\frac{1}{2}} u_h - \int_{\frac{1}{2}}^1 u_h = 2\left(\int_0^{\frac{1}{2}-h} dx + \int_0^h \frac{x}{h}\,dx \right) = 2\left(\frac{1}{2} - h + \frac{h}{2} \right) = 1 - h.$$

As a consequence, $v_h = u_h/(1-h) \in \mathcal{M}$. Letting $h \to 0^+$ we obtain $\inf_{u\in\mathcal{M}} I(u) \le 1$, since $I(v_h) = 1/(1-h)$.

Next, if $u \in \mathcal{M}$ and $I(u) \le 1$, since

$$\int_0^{\frac{1}{2}} u \le \frac{1}{2}, \qquad -\int_{\frac{1}{2}}^1 u \le \frac{1}{2},$$

we will obtain equality for $u \in \mathcal{M}$, and this implies $u = 1$ a.e. in $(0, 1/2)$, $u = -1$ a.e. in $(1/2, 1)$, and this is in contradiction $u \in C([0,1])$. This shows that $\inf_{u\in\mathcal{M}} I(u) \ge 1$, so that the minimum is not attained.

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2. Let $\mathcal{M}$ the convex closed subset of $H^1([0,1])$ given by $\mathcal{M} = \{u \in H^1([0,1]) : u(0) = 1, u(1) = 0\}$. Consider the functional $I : \mathcal{M} \mapsto \mathbb{R}$ defined by $I(u) = \int_0^1 x|u'(x)|^2\,dx$.

   (i) Show that $\inf_{u\in\mathcal{M}} I(u) = 0$.

   (ii) Show that there does not exist any $u \in \mathcal{M}$ such that $I(u) = 0$.

   In this case the problem is that the functional is not coercive.

Solution. (i) Let

$$u_h(x) = \frac{\log((1-h)x + h)}{\log h}.$$

So that $u_h \in \mathcal{M}$ and

$$u_h'(x) = \frac{1-h}{\log h}\,\frac{1}{(1-h)x + h}.$$

As a consequence, changing variables $y = (1-h)x + h$ (or equivalently, $x = (y-h)/(1-h)$),

$$\begin{aligned} I(u_h) &= \frac{(1-h)^2}{\log^2 h} \int_0^1 \frac{x\,dx}{((1-h)x+h)^2} = \frac{1}{\log^2 h} \int_h^1 \frac{(y-h)\,dy}{y^2} = \frac{1}{\log^2 h}\left(-\log h + h\left(1 - \frac{1}{h}\right)\right) \\ &= -\frac{1}{\log h} + \frac{h-1}{\log^2 h} \to 0 \quad \text{when } h \to 0^+, \end{aligned}$$

so that $\text{ínf}_{u\in\mathcal{M}} I(u) = 0$.

(ii) If $I(u) = 0$, therefore we have $u' = 0$ a.e., that implies u =constant. This is not possible for $u \in \mathcal{M}$.

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3. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $\beta : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function such that there exist a and b such that

$$0 < a \le \beta'(z) \le b \quad \text{for all } z \in \mathbb{R},$$

and $f \in L^2(\Omega)$.

(i) Define a concept of weak solution for the nonlinear problem

$$-\Delta u = f \quad \text{en } \Omega, \qquad \partial u/\partial n + \beta(u) = 0 \quad \text{in } \partial\Omega.$$

(ii) Prove that there exists a weak solution (and is unique).

Solution. Define a weak solution as a function $u \in H^1(\Omega)$ such that

$$\int_\Omega (\nabla u \cdot \nabla v - fv) + \int_{\partial\Omega} \beta(u)v = 0 \qquad \forall v \in H^1(\Omega). \tag{S1}$$

The boundary term is well defined. Indeed, since $\partial\Omega$ is bounded, trace inequality gives

$$\int_{\partial\Omega} |\beta(u)v| \le \int_{\partial\Omega} (|\beta(0)| + b|u|)|v| \le C\|v\|_{H^1(\Omega)} + \|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

If u is a classical solution, it is sufficient multiply by a test function $v \in C^\infty(\overline{\Omega})$ and intergate by parts to get (S1) for all $v \in C^\infty(\overline{\Omega})$. By density we can then extend the result to $v \in H^1(\Omega)$, just by noticing that for a fixed $u \in H^1(\Omega)$ the linear functional $\int_{\partial\Omega} \beta(u)v$ is continuous in v. This is given by the above estimate.

If $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a weak solution, we can see that is a classical solution as it has been done in the previous exercises (fix the details as an exercise): first test with compactly suppeorted fucntions, so that the equality holds in the sense od distributions, hence a.e., then at every points by continuity. Once we did thay, take general test functions to check the boundary conditions.

(ii) Let $B(z) = \int_0^z \beta(t)\,dt$, $z \in \mathbb{R}$. Define, for any $u \in H^1(\Omega)$,

$$F(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - fu\right) + \int_{\partial\Omega} B(u).$$

The second term is well-defined for any $u \in H^1(\Omega)$. Indeed, using that

$$0 \le \frac{a}{2}u^2 \le B(u) - \beta(0)u \le \frac{b}{2}u^2$$

together with trace inequality, we get

$$\int_{\partial\Omega} |B(u)| \le \int_{\partial\Omega} |\beta(0)u| + \frac{b}{2}\int_{\partial\Omega} u^2 \le C\|u\|_{H^1(\Omega)} + C\|u\|^2_{H^1(\Omega)}.$$

Once the derivation under the integral sign is justified (do it as an exercise), it is easy to see that

$$\lim_{t\to 0} \frac{1}{t}\left(\int_{\partial\Omega} B(u+tv) - \int_{\partial\Omega} B(u)\right) = \int_{\partial\Omega} \beta(u)v.$$

As a consequence, $F(u)$ is differentiable (easy to check) and $F'(u)v = 0$ if and only if it satisfies (S1).

To see that $F(u)$ is coercive, we can use Friedrichs' inequality with $\Gamma = \partial\Omega$,

$$\|u\|^2_{H^1(\Omega)} \le C\left(\|u\|^2_{L^2(\partial\Omega)} + \|Du\|^2_{L^2(\Omega)}\right) \qquad \forall\, u \in H^1(\Omega),$$

together with $B(u) \geq \beta(0)u + \frac{a}{2}u^2$, to get that

$$\begin{aligned} F(u) &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)} - \varepsilon\int_\Omega u^2 + \int_{\partial\Omega} \beta(0)u + \frac{a}{2}\int_{\partial\Omega} u^2 \\ &\geq \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)} - \varepsilon\int_\Omega u^2 - |\beta(0)|\frac{1}{4\varepsilon'} - \varepsilon'\int_{\partial\Omega} u^2 + \frac{a}{2}\int_{\partial\Omega} u^2 \\ &\geq C_1\|u\|^2_{H^1(\Omega)} - \frac{1}{4\varepsilon}\|f\|^2_{L^2(\Omega)} - \varepsilon\int_\Omega u^2 - C \\ &\geq \frac{C_1}{2}\|u\|^2_{H^1(\Omega)} - C, \end{aligned}$$

choosing $\varepsilon' = a/4$ and $\varepsilon = C_1/2$.

Let us check that $B(u)$ is convex:

$$B(u+v) - B(u) = \int_u^{u+v} \beta(t)\,dt \geq \int_u^{u+v} (\beta(u) + a(t-u))\,dt = B'(u)v + \frac{a}{2}v^2.$$

As a consequence, B is more than convex, it is strictly convex.

Now we check that $\int_{\partial\Omega} B(u)$ is continuous in the strong topology of $H^1(\Omega)$,

$$\begin{aligned} \left|\int_{\partial\Omega}(B(u) - B(v))\right| &\leq \int_{\partial\Omega}\left|\int_u^v \beta(t)\,dt\right| \leq \int_{\partial\Omega}\left(|\beta(0)|\,|u-v| + \frac{b}{2}\left|u^2 - v^2\right|\right) \\ &\leq C\|u-v\|_{H^1(\Omega)} + C\|u+v\|_{H^1(\Omega)}\|u-v\|_{H^1(\Omega)}. \end{aligned}$$

This immediately imply that $F(u)$ is weakly lower semi-continuous.

We can therefore use the direct method of calculus of variations, to show existence of a weak solution. Moreover, it is easy to check that also $F(u)$ is strictly convex (using again Friedrichs' inequality) so that solution is also unique.

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4. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Given $u \in H^1(\Omega)$, we define the *surface of the graphic* of u by

$$F(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\,dx.$$

(i) Prove that the functional F is C^1 in $H^1(\Omega)$.

(ii) Let $g \in H^1(\Omega)$, and $\mathcal{A} = \{u = g + v : v \in H^1_0(\Omega)\}$. Prove that a critical point of F in $\mathcal{A}$ is a weak solution to the *equation of minimal surfaces*:

$$\nabla \cdot \left(\frac{\nabla u}{(1 + |\nabla u|^2)^{1/2}}\right) = 0 \quad \text{en } \Omega, \qquad u = g \quad \text{en } \partial\Omega.$$

The expression on the left of this equality is N-times the *mean curvature of the graphic* of u. Hence a minimal surface has zero mean curvature.

(iii) Check wether or not the direct method of calculus of variations can be used to deduce existence of a minimizer of F in $\mathcal{A}$.

(iv) Let $J(w) = \int_\Omega w\,dx$. assume that u is a smooth minimizer of F in $\mathcal{A} \cap \{w : J(w) = 1\}$. Show that the graphic of u is a minimal surface with constant mean curvature.

Solution. (i) By Taylor expansion,

$$\frac{1}{t}(F(u+tv) - F(u)) = \int_\Omega \left(\frac{\nabla u \cdot \nabla v}{\sqrt{1 + |\nabla(u + \xi v)|^2}} + \frac{t|\nabla v|^2}{\sqrt{1 + |\nabla(u + \xi v)|^2}}\right),$$

with $\xi = \xi(x)$, $\xi \in (0,1)$. When $t \to 0$, using Dominated Convergence, we get that

$$F'(u)v = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1 + |\nabla u|^2}}.$$

Let us prove that F' is continuous. We have that

$$|F'(u)w - F'(v)w| \le \|w\|_{H^1(\Omega)} \left(\int_\Omega \left| \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right|^2 \right)^{1/2}.$$

Next we observe that the function $x/\sqrt{1+|x|^2}$, $x \in \mathbb{R}^N$, is Lipschitz. Indeed,

$$\left| \partial_{x_i} \left(\frac{x_j}{\sqrt{1+|x|^2}} \right) \right| = \left| \frac{\delta_{ij}(1+|x|^2) - x_i x_j}{(1+|x|)^{3/2}} \right| \le \left| \frac{2(1+|x|^2)}{1+|x|^2} \right| \le 2.$$

As a consequence,

$$\int_\Omega \left| \frac{\nabla u}{\sqrt{1+|\nabla u|^2}} - \frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right|^2 \le C \int_\Omega |\nabla u - \nabla v|^2 \le C\|u-v\|^2_{H^1(\Omega)}.$$

This proves continuity of F'.

(ii) If u is a critical point for F in $\mathcal{A}$ with $u \in C^2(\Omega)$, for all $v \in C^\infty_c(\Omega)$ we get

$$0 = F'(u)v = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} = -\int_\Omega v \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right),$$

and we also get

$$\nabla \cdot \left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}} \right) = 0.$$

Viceversa, if u satisfies the equation, we just have to integrate by parts again. As usual, the boundary condition $u = g$ on $\partial\Omega$ is included in the definition of $\mathcal{A}$.

(iii) The direct method in calculus of variations, cannot be used since F is not weakly lower semi-continuous.

Let us show the latter in the case in which, for instance $\Omega = (-1,1)$, $g = 0$ and $f_k \in \mathcal{A}$ obtained by splitting Ω in 2^{k+1} subintervals of the same length, in which f_k is a line with slope 0 and 2 alternately, if we are in $(-1,0)$, or with slope 0 y -2, if we are in (0,1). Banach-Alaouglu's Theorem, gives that there is a subsequence $\{f_{k_j}\}$ converging to $f(x) = 1-|x|$ in the weak topology of $H^1_0(\Omega)$. Next we observe that $F(f) = 2\sqrt{2}$, while $F(f_k) = \sqrt{5}$. Hence, $F(f_k) < F(f)$, and of course F is not weakly semi-continuous.

Summing up, the direct methods of Calculus of Variations do not apply in a straightforward way.

(iv) The set $\mathcal{A} \cap \{w : J(w) = 1\}$ is an affine submanifold whose tangent space is given by $X = H^1_0(\Omega) \cap \{w : J(w) = 0\}$. As a consequence, the minimizer satisfies

$$0 = F'(u)v = \int_\Omega \frac{\nabla u \cdot \nabla v}{\sqrt{1+|\nabla u|^2}} \qquad \forall v \in X.$$

Letting

$$K = \frac{1}{|\Omega|} \int_\Omega \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right).$$

Hence, if $v \in C^\infty_c(\Omega)$ and $v_\Omega = \frac{1}{|\Omega|}\int_\Omega v$, we obtain that $v - v_\Omega \in X$ and

$$\begin{aligned} 0 &= F'(u)(v - v_\Omega) = \int_\Omega \nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right)(v - v_\Omega) \\ &= \int_\Omega \left[\nabla \cdot \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \right) - K \right] v\,, \end{aligned}$$

which implies that

$$\nabla \cdot \left(\frac{\nabla u}{(1+|\nabla u|^2)^{1/2}} \right) = K,$$

since it holds for any $v \in C^\infty_c(\Omega)$.

5. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Consider the eigenvalue problem for the bi-harmonic opeartor with Dirichlet boundary conditions,

$$\Delta^2 u = \lambda u \quad \text{en } \Omega, \qquad u = \partial u/\partial n = 0 \quad \text{in } \partial\Omega.$$

Show that there exists a non-trivial weak solution (λ, u) to the problem when $\lambda > 0$.

Solution. Letting

$$F(u) = \int_\Omega |\Delta u|^2, \qquad u \in H_0^2(\Omega).$$

Let us minimize $F(u)$ on $H_0^2(\Omega)$ under the condition $\|u\|_{L^2} = 1$.
We are indeed minimizing the Rayleigh quotient $I(u) = F(u)/\|u\|_{L^2}^2$.
(We follow the same ideas of the minimization problem for the equation $-\Delta u = \lambda_1 u$)
We recall the following inequality:

$$\|u\|_{H^2(\Omega)} \le C\|\Delta u\|_{L^2(\Omega)} \qquad \text{for all } u \in H_0^2(\Omega) \tag{1}$$

(the proof is an exercise, integrate by parts, $C > 0$ depends only on N.) As a consequence

$$cF(u) \le \|u\|_{H^2(\Omega)}^2 \le CF(u), \qquad u \in H_0^2(\Omega),$$

(the first inequality is easy). As a consequence, $F(u)$ is coercive and $\sqrt{F(u)}$ is an equivalent norm in $H_0^2(\Omega)$. Hence, $F(u)$ is weakly lower semi-continuous in $H_0^2(\Omega)$. Rellich-Kondrachov's Theorem implies that there is a subsequence converging to a minimizer u.

It only remains to check that the functional is bounded from below: we will use Poincaré inequality together with (1) as follows:

$$F(u) \ge C^{-1}\|u\|_{H^2(\Omega)}^2 \ge C^{-1}S_2^{-2}\|\nabla u\|_{L^{2^*}(\Omega)}^2 \ge C^{-1}S_2^{-2}\,|\Omega|^a\,\|\nabla u\|_{L^2(\Omega)}^2 \ge C^{-1}S_2^{-2}\,|\Omega|^a\,\lambda_1 > 0$$

where we have used the Sobolev inequality applied to $|\nabla u|$, namely

$$\|\nabla u\|_{L^{2^*}(\Omega)} \le S_2\|u\|_{H^2(\Omega)}^2$$

and Hölder inequality

$$\|\nabla u\|_{L^2(\Omega)} \le |\Omega|^{-a}\|\nabla u\|_{L^{2^*}(\Omega)} \qquad \text{with } a = \frac{1}{2^*} - \frac{1}{2}$$

and finally Poincaré inequality for $\|u\|_{L^2(\Omega)} = 1$

$$0 < \lambda_1 = \lambda_1\|u\|_{L^2(\Omega)}^2 \le \|\nabla u\|_{L^2(\Omega)}^2$$

Therefore we have a solution Φ_1 that minimizes $F(u)$ in $H_0^2(\Omega)$ under the condition $\|\Phi_1\|_{L^2(\Omega)} = 1$, hence there exists a minimum Λ_1 of $F(u)$ attained at $u = \Phi_1$, so that

$$0 < C^{-1}S_2^{-2}\,|\Omega|^a\,\lambda_1 < \Lambda_1 = \min_{0\not\equiv u\in H_0^2(\Omega)} \frac{F(u)}{\|u\|_{L^2(\Omega)}^2} = \frac{F(\Phi_1)}{\|\Phi_1\|_{L^2(\Omega)}^2}$$

and of course $\Delta^2\Phi_1 = \Lambda_1\Phi_1$ in a weak sense (minima are critical point), hence Φ_1 is a weak solution.
We conclude that (Λ_1, Φ_1) is a weak solution to the problem.
As a byproduct of the proof, we also get a (non-sharp) lower bound for the first eigenvalue of Δ^2: $\Lambda_1 > C^{-1}S_2^{-2}\,|\Omega|^a\,\lambda_1 > 0$, where S_2 is the Sobolev constant, λ_1 the Poincaré constant and C the constant in inequality (1) which depends only on N.

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6. Let $f \in L^2(\Omega)$. Show that there exists a unique minimizer u of

$$J(w) = \int_\Omega \left(\frac{1}{2} |\nabla w|^2 - f w \right)$$

in $\mathcal{A} = \{w \in H_0^1(\Omega) : |\nabla w| \le 1 \text{ a.e. } \}$. Show that

$$\int_\Omega \nabla u \cdot \nabla (w-u) \ge \int_\Omega f(w-u) \qquad \text{for all } w \in \mathcal{A}.$$

Solution. We already know that $J(w)$ is weakly lower semi-continuous and coercive. Moreover, $\mathcal{A}$ is convex and closed. As usual, this is enough to conclude that there exists a minimizer $u \in \mathcal{A}$ (use Rellich-Kondrachov's Theorem).

Let us calculate,

$$J(w+v) - J(w) = J'(w)v + \frac{1}{2} \int_\Omega |\nabla v|^2.$$

If $v \in H_0^1(\Omega)$ is such that $u + v \in \mathcal{A}$, hence $u + tv \in \mathcal{A}$, $0 \le t \le 1$, because $\mathcal{A}$ is convex. Since u is a minimizer,

$$0 \le \frac{1}{t} \left(J(u+tv) - J(u) \right) = J'(u)v + \frac{t}{2} \int_\Omega |\nabla v|^2.$$

Letting $t \to 0$ we get that $J'(u)v \ge 0$. Hence,

$$J(u+v) - J(u) \ge \frac{1}{2} \int_\Omega |\nabla v|^2.$$

This implies uniqueness of u, the minimizer in $\mathcal{A}$. The inequality

$$\int_\Omega \nabla u \cdot \nabla (w-u) \ge \int_\Omega f(w-u), \qquad w \in \mathcal{A},$$

is the "Euler-Lagrange inequality" associated to J, obtained by $J'(u)v \ge 0$, with $v = w - u$.
