EDPs en ciencia e ingeniería. M.M.A. de la UAM

Hoja de problemas 3: Weak Solutions and Linear Elliptic Equations.

1. Let a, b, c be smooth functions, with a and c strictly positive. Let u be a solution to the boundary value problem

-au'' + bu' + cu = f en I = (0, 1), u(0) = u(1) = 0.

Show that u solves an equation of the form $-(\tilde{a}(x)u')' + \tilde{c}(x)u = \tilde{f}$: write the corresponding weak formulation and show that there exists a unique solution.

2. Consider the boundary value problem

$$-u'' + ku' + u = f$$
 en $I = (0, 1),$ $u'(0) = u'(1) = 0$

Write the variational formulation and show that for k sufficiently small there is no unique solution. Find (at least) a value of k and (at least) a function $v \in H^1$, with $v \neq 0$ such that a(v, v) = 0.

3. Consider the problem

$$-u''(x) = f(x)$$
 en $I = (0,1),$ $u'(0) - u(0) = 0, u'(1) + u(1) = 0.$

- (a) Define a classical solution of the problem, when $f \in C([0, 1])$.
- (b) Prove that classical solutions are weak i.e. they satisfy

$$u(0)v(0) + u(1)v(1) + \int_0^1 u'v' = \int_0^1 fv, \quad \forall v \in H^1(I).$$

Define a weak solution to the problem as a function $u \in H^1(I)$ satisfying the above equality.

(c) Prove existence and uniqueness of weak solutions to the above problem. *Hint:* Prove and use the following Poincaré-type inequality

$$\int_0^1 u^2 \le C\left((u(0))^2 + (u(1))^2 + \int_0^1 (u')^2\right) \qquad \forall u \in H^1(I).$$

- (d) Prove that $f \in C(\overline{I})$ implies $u \in C^2(\overline{I})$.
- (e) Show that any weak solution which is $C^2(\bar{I})$ is indeed a classical solution.
- 4. Consider the boundary value problem

$$u'''(x) = f(x)$$
 in $I = (0, 1),$ $u(0) = u'(0) = u(1) = u'(1) = 0.$

Here, u represents, for instance, deflection of a bar fixed at the extremals and under the influence of a transversal force of intensity f. Given $f \in C(\overline{I})$:

- (a) Define classical solutions.
- (b) Define weak solutions (the correct functional space is $H_0^2(I)$).
- (c) Show that every classical solution is a weak solution.
- (d) Prove that there exists a unique weak solution.
- (e) Prove that if $f \in C(\overline{I})$, then $u \in C^4(\overline{I})$.
- (f) Prove that if a weak solution is in $C^4(\overline{I})$, then it is a classical solution.

5. Let I = (0,1). Show that the functional $F : H^1(I) \to \mathbb{R}$ defined by F(u) = u(0) is linear and continuous. Show next that there exists a unique $v_0 \in H^1(I)$ such that

$$u(0) = \int_0^1 (u'v'_0 + uv_0) \qquad \forall u \in H^1(I).$$

Show that v_0 is solution to a certain differential equation with suitable boundary conditions. Find an explicit expression for v_0 .

6. Find a function $u \in C^2([0, 1/2])$ con u(0) = u(1/2) = 0 such that for any $v \in C^2([0, 1/2])$ we have

$$\int_0^{1/2} (u'v' + (4u - 1)v) = 0.$$

7. Consider the boundary value problem u'' = 2, u(1) = u(-1) = 0, whose solution is given by $\bar{u}(x) = x^2 - 1$; write the variational formulation to conclude that for all $u \in C^2$ with u(1) = u(-1) = 0 we have

$$\frac{8}{3} + \int_{-1}^{1} \left((u')^2 + 4u \right) \ge 0.$$

- 8. (Hardy Inequality in dimension N = 1). Let I = (0, 1).
 - (a) Given $u \in L^p(I)$, show that

$$\left\|\frac{1}{x}\int_0^x u(t)\,dt\right\|_{L^p(I)} \le \frac{p}{p-1}\|u\|_{L^p(I)}.$$

Hint. Begin with $u \in C_c(I)$ by defining $\varphi(x) = \int_0^x u(t) dt$. Check that $|\varphi|^p \in C^1(\overline{I})$ and calculate the derivative. Finally, use the formula

$$\int_0^1 |\varphi(x)|^p \frac{dx}{x^p} = \frac{1}{p-1} \int_0^1 |\varphi(x)|^p d\left(-\frac{1}{x^{p-1}}\right)$$

and integrate by parts.

(b) Let $u \in W^{1,p}(I)$, 1 . Show that if <math>u(0) = 0, then

$$\left\|\frac{u(x)}{x}\right\|_{L^{p}(I)} \leq \frac{p}{p-1} \|u'\|_{L^{p}(I)}.$$

- 9. (A problem with Hardy-type weights) Let I = (0, 1) and $V = \{v \in H^1(I) : v(0) = 0\}$.
 - (a) Given $f \in L^2(I)$ such that $\frac{1}{x}f(x) \in L^2(I)$, show that there exists a unique $u \in V$ satisfying

$$\int_0^1 u'(x)v'(x)\,dx + \int_0^1 \frac{u(x)v(x)}{x^2}\,dx = \int_0^1 \frac{f(x)v(x)}{x^2}\,dx \qquad \forall v \in V.$$
(1)

- (b) Write the minimization problem associated to (1)
- (c) Here and in part (d) we will assume that $\frac{1}{x^2}f(x) \in L^2(I)$. Letting $v(x) = \frac{u(x)}{(x+\varepsilon)^2}$, $\varepsilon > 0$, show that

$$\int_0^1 \left| \frac{d}{dx} \left(\frac{u(x)}{x+\varepsilon} \right) \right|^2 \, dx \le \int_0^1 \frac{f(x)}{x^2} \frac{u(x)}{(x+\varepsilon)^2} \, dx.$$

- (d) Prove that $\frac{u(x)}{x^2} \in L^2(I), \frac{u(x)}{x} \in H^1(I)$ y $\frac{u'(x)}{x} \in L^2(I).$
- (e) As a consequence of part (d) show that $u \in H^2(I)$ and that

$$-u''(x) + \frac{u(x)}{x^2} = \frac{f(x)}{x^2} \quad \text{a.e. en } I, \qquad u(0) = u'(0) = 0, \quad u'(1) = 0.$$
(2)

(f) Viceversa, show that if $u \in H^2(I)$ with $\frac{u(x)}{x^2} \in L^2(I)$ satisfies equation (2), hence it satisfies (1).

- 10. Let I = (0, 1) and let us fix a constant k > 0.
 - (a) Given $f \in L^1(I)$, show that there is a unique $u \in H^1_0(I)$ such that

$$\int_{I} u'v' + k \int_{I} uv = \int_{I} fv \qquad \forall v \in H_0^1(I).$$
(3)

- (b) Prove that $u \in W^{2,1}(I)$.
- (c) Prove that

$$||u||_{L^1(I)} \le \frac{1}{k} ||f||_{L^1(I)}.$$

Hint. Fix a function $\gamma \in C^1(\mathbb{R}, \mathbb{R})$ so that $\gamma'(t) \ge 0$ for all $t \in \mathbb{R}$, $\gamma(0) = 0$, $\gamma(t) = 1$ and all $t \ge 1$ and such that $\gamma(t) = -1$ for all $t \le -1$. Take $v = \gamma(n u)$ in (3) and let $n \to \infty$.

(d) Assume now $f \in L^p(I)$, $p \in (1, \infty)$. Show that there exists $\delta > 0$ independent of k and p such that

$$||u||_{L^p(I)} \le \frac{1}{k + \delta/pp'} ||f||_{L^p(I)}.$$

Hint. If $p \in [2, \infty)$, take $v = \gamma(u)$ in (3), with $\gamma(t) = |t|^{p-1} \operatorname{sign} t$. If $p \in (1, 2)$, use duality.

(e) if $f \in L^{\infty}(I)$, show that

$$||u||_{L^{\infty}(I)} \leq C_k ||f||_{L^{\infty}(I)}$$

and find the best constant C_k . *Hint*. Find the explicit solution to (3) corresponding to $f \equiv 1$.

11. Let I = (0, 1).

(a) Prove that for any $\varepsilon > 0$ there exists a constant C_{ε} such that

$$|u(1)|^{2} \leq \varepsilon ||u'||^{2}_{L^{2}(I)} + C_{\varepsilon} ||u||^{2}_{L^{2}(I)} \qquad \forall u \in H^{1}(I).$$

(b) Show that if the constant k > 0 is big enough, then for all $f \in L^2(I)$ there exists a unique $u \in H^2(I)$ satisfying

-u'' + ku = f en I, u'(0) = 0, u'(1) = u(1).

Write both the associated weak formulation and the minimization problem.

- 12. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $h \in C^{\infty}(\partial \Omega)$ be such that $\int_{\partial \Omega} h = 0$.
 - (a) Define a reasonable concept of weak solution to the problem

$$\Delta u = 0$$
 en Ω , $\partial u / \partial n = h$ en $\partial \Omega$.

- (b) Prove that there exists a unique weak solution such that $\int_{\Omega} u = 0$ and check that the difference between two arbitrary weak solutions has to be constant in Ω .
- 13. Let $\Omega \subset \mathbb{R}^N$ be a bounded connected domain with smooth boundary.
 - (a) Define weak solutions for the Poisson equation with Robin boundary conditions:

$$-\Delta u = f$$
 en Ω , $u + \frac{\partial u}{\partial n} = 0$ sobre $\partial \Omega$,

Check that any classical solution to the problem is a weak solution, and that every weak solution which is also smooth enough, is a classical solution.

(b) Show existence and uniqueness of weak solutions to the problem, for any $f \in L^2(\Omega)$. *Hint.* Use Friedrichs' Inequality.