Hoja de problemas 3: Weak Solutions and Linear Elliptic Equations.

1. Let a, b, c be smooth functions, with a and c strictly positive. Let u be a solution to the boundary value problem

$$-au'' + bu' + cu = f \quad \text{en } I = (0,1), \qquad u(0) = u(1) = 0.$$

Show that u solves an equation of the form $-(\tilde{a}(x)u')' + \tilde{c}(x)u = \tilde{f}$: write the corresponding weak formulation and show that there exists a unique solution.

Solution. Multiply the equation by the factor g, chosen in such a way that $gau'' + bgu' = (\tilde{a}u')'$. Expand this expression to see that we need to impose $\tilde{a} = ga$, $\tilde{a}' = bg$. As a consequence g satisfies $g' = (b - a')g/a$, so that we can take

$$g(x) = \mathrm{e}^{\int_0^x \frac{b-a'}{a}}.$$

Define now $\tilde{a} = ga$, $\tilde{c} = gc$ y $\tilde{f} = gf$, and recall that $\tilde{a}$ and $\tilde{c}$ are positive smooth functions.

The weak formulation is easily obtained, multiplying by a test function v and integrating by parts: $u \in H_0^1(I)$ is a weak solution if

$$\underbrace{\int_0^1 \tilde{a}u'v' + \int_0^1 \tilde{c}uv}_{a(u,v)} = \underbrace{\int_0^1 \tilde{f}v}_{F(v)} \qquad \forall v \in H_0^1(I).$$

It is asy to check that the bilinear form $a(u, v)$ is continuous and coercive on $H_0^1(I)$,

$$|a(u,v)| \le \|\tilde{a}\|_{L^\infty(I)} \int_0^1 |u'|\,|v'| + \|\tilde{c}\|_{L^\infty(I)} \int_0^1 |u|\,|v| \le C\|u\|_{H_0^1(I)}\|v\|_{H_0^1(I)},$$

$$a(u,u) \ge \min_I \tilde{a} \int_0^1 (u')^2 + \min_I \tilde{c} \int_0^1 u^2 \ge \alpha \|u\|^2_{H_0^1(I)}, \qquad \alpha = \min\{\min_I \tilde{a}, \min_I \tilde{c}\} > 0.$$

On the other hand, if $f \in L^2(I)$, the linear functional $F(v) = \int_0^1 \tilde{f}v$ is continuous on $H_0^1(I)$. The existence of a unique weak solution follows by applying Lax-Milgram's Theorem.

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2. Consider the boundary value problem

$$-u'' + ku' + u = f \quad \text{en } I = (0,1), \qquad u'(0) = u'(1) = 0.$$

Write the variational formulation and show that for k sufficiently small there is no unique solution. Find (at least) a value of k and (at least) a function $v \in H^1$, with $v \not\equiv 0$ such that $a(v, v) = 0$.

Solution. The weak formulation is easily obtained, multiplying by a test function $v \in H^1(I)$ and integrating by parts: $u \in H^1(I)$ is a weak solution if

$$\underbrace{\int_0^1 u'v' + k\int_0^1 u'v + \int_0^1 uv}_{a(u,v)} = \underbrace{\int_0^1 fv}_{F(v)} \qquad \forall v \in H^1(I).$$

Continuity of a and F is straightforward. To be able to apply the Lax-Milgram Theorem, the only difficulty is to check coercivity of a. Indeed,

$$a(u,u) = \int_0^1 (u')^2 + \int_0^1 u^2 + k\int_0^1 u'u \ge \left(1 - \frac{|k|}{2}\right) \|u\|^2_{H^1(I)}.$$

As a consequence, a is coercive if $|k| < 2$, so that existence and uniqueness holds for such values of k.

As for the counter-example, take $v(x) = x$: a simple calculation reveals that $a(v, v) = 1 + \frac{1}{3} + \frac{k}{2}$, which is zero if we take $k = -8/3$.

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3. Consider the problem

$$-u''(x) = f(x) \quad \text{en } I = (0,1), \qquad u'(0) - u(0) = 0, \; u'(1) + u(1) = 0.$$

(a) Define a classical solution of the problem, when $f \in C([0,1])$.

(b) Prove that classical solutions are weak i.e. they satisfy

$$u(0)v(0) + u(1)v(1) + \int_0^1 u'v' = \int_0^1 fv, \qquad \forall v \in H^1(I).$$

Define a weak solution to the problem as a function $u \in H^1(I)$ satisfying the above equality.

(c) Prove existence and uniqueness of weak solutions to the above problem.
Hint: Prove and use the following Poincaré-type inequality

$$\int_0^1 u^2 \leq C\left((u(0))^2 + (u(1))^2 + \int_0^1 (u')^2\right) \qquad \forall u \in H^1(I).$$

(d) Prove that $f \in C(\bar{I})$ implies $u \in C^2(\bar{I})$.

(e) Show that any weak solution which is $C^2(\bar{I})$ is indeed a classical solution.

Solution. (a) A classical solution is a function $u \in C^2(\overline{I})$ that satisfies the equation at every point of I and the boundary conditions.

(b) Multiply the equation by a smooth function v and integrating by parts, gives

$$-u'(1)v(1) + u'(0)v(0) + \int_0^1 u'v' = \int_0^1 fv.$$

The result follows using the boundary conditions to eliminate the values of the derivative at $x = 0$ y $x = 1$. By approximation, the above formual can be extended to any $v \in H^1(I)$.

(c) Let us first prove the Poincaré-type inequality. We begin with $u(x) - u(0) = \int_0^x u'$. Using Hölder inequality, we get $(u(x) - u(0))^2 \leq \int_0^1 (u')^2$. Expanding the squares and integrating over I we get

$$\int_0^1 u^2 \leq -u^2(0) + 2u(0)\int_0^1 u + \int_0^1 (u')^2.$$

Recall now that $2u(0)u(x) \leq \varepsilon u^2(x) + \frac{u^2(0)}{\varepsilon}$, to get

$$(1-\varepsilon)\int_0^1 u^2 \leq \left(\frac{1}{\varepsilon} - 1\right)u^2(0) + \int_0^1 (u')^2.$$

We can take $\varepsilon < 1$ to get the result (indeed a slightly finer result).

Existence and uniqueness follow by Lax-Milgram Theorem, applied to the bilinear form $a(u,v) = u(0)v(0) + u(1)v(1) + \int_0^1 u'v'$ and to the linear functional $F(v) = \int_0^1 fv$: both are defined on $H^1(I)$. If $f \in L^2(I)$, clearly F is continuous. Continuity of a follows easily by trace inequality. Coercivity follows by the above Poincaré-type inequality.

(d) Take $v \in C_c^\infty(I)$, we can easily see that $-u'' = f$ in I in the distributional sense. Since $f \in C(\overline{I})$, then it follows that $u \in C^2(\overline{I})$.

(e) Let us begin by taking test functions $v \in C_c^\infty(I)$ in the weak formulation. Integration by parts then gives $\int_0^1 (-u'' - f)v = 0$, from which we deduce that $-u'' = f$ in I. Take now a test function v such that $v(0) = 1$, $v(1) = 0$. Integration by parts together with the fact that u satisfies the equation, we see that $u'(0) = u(0)$. Finally, taking a test function v such that $v(0) = 0$, $v(1) = 1$, integration by parts together with the fact that u satisfies the equation allows to conclude that $-u'(1) = u(1)$.

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4. Consider the boundary value problem

$$u''''(x) = f(x) \quad \text{in } I = (0,1), \qquad u(0) = u'(0) = u(1) = u'(1) = 0.$$

Here, u represents, for instance, deflection of a bar fixed at the extremals and under the influence of a transversal force of intensity f. Given $f \in C(\overline{I})$:

(a) Define classical solutions.

(b) Define weak solutions (the correct functional space is $H_0^2(I)$).

(c) Show that every classical solution is a weak solution.

(d) Prove that there exists a unique weak solution.

(e) Prove that if $f \in C(\bar{I})$, then $u \in C^4(\bar{I})$.

(f) Prove that if a weak solution is in $C^4(\bar{I})$, then it is a classical solution.

Solution. (a) A classical solution is a function $u \in C^4(\overline{I})$ that satisfies the equation at every point of I and the boundary conditions.

(b) A weak solution is a function $u \in H_0^2(I)$ such that

$$\underbrace{\int_0^1 u''v''}_{a(u,v)} = \underbrace{\int_0^1 fv}_{F(v)} \qquad \forall v \in H_0^2(I).$$

(c)Multiply the equation by any $v \in C_c^\infty(I)$ and integrate by parts. Boundary terms disappear because of the boundary conditions. By approximation, the result is true for all $v \in H_0^2(I)$.

(d) The only difficulty to apply Lax-Milgram Theorem, is coercivity. We use Poincaré inequality applied to the first derivative u':

$$\int_0^1 ((u')')^2 \geq C \int_0^1 (u')^2, \quad C > 0.$$

Next we apply Poincaré inequality to u, to conclude

$$a(u,u) = \int_0^1 (u'')^2 \geq \alpha \|u\|^2_{H_0^2(I)}, \qquad \alpha > 0.$$

(e) This follows since $u'''' = f$ in the distributional sense (i.e. tested with C_c^∞ functions), since u is a weak solution and f is continuous.

(f) Test the equation with $v \in C_c^\infty(I)$. Integrate by parts (in the unusual direction)to get $\int_0^1 (u'''' - f)v = 0$, hence u satisfies the equation. Boundary conditions are contained in the properties of the space H_0^2, where u lies.

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5. Let $I = (0,1)$. Show that the functional $F : H^1(I) \mapsto \mathbb{R}$ defined by $F(u) = u(0)$ is linear and continuous. Show next that there exists a unique $v_0 \in H^1(I)$ such that

$$u(0) = \int_0^1 (u'v_0' + uv_0) \qquad \forall u \in H^1(I).$$

Show that v_0 is solution to a certain differential equation with suitable boundary conditions.
Find an explicit expression for v_0.

Solution. The functional F is clearly linear. Continuity follows by trace inequality. We already know that the bilinear form $a(v_0, u) = \int_0^1 (u'v_0' + uv_0)$ is continuous and coercive on $H_0^1(I)$, hence existence and uniqueness are granted by Lax-Milgram Theorem.

Taking test functions $u \in C_c^\infty(I)$ we can show that $-u'' + u = 0$ in I. Next, taking test functions such that $u(0) = 0$, $u(1) = 1$, we see that $v_0'(1) = 0$. Finally taking test functions so that $u(0) = 1$, $u(1) = 0$, we obtain $v_0'(0) = -1$.

Finding v_0 is now an basic exercise of ordinary differential equations:

$$u(x) = \frac{\mathrm{e}^x + \mathrm{e}^{2-x}}{\mathrm{e}^2 - 1}.$$

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6. Find a function $u \in C^2([0,1/2])$ con $u(0) = u(1/2) = 0$ such that for any $v \in C^2([0,1/2])$ we have

$$\int_0^{1/2} (u'v' + (4u-1)v) = 0.$$

Solution. The function u is a weak solution to

$$-u'' + 4u = 1 \quad \text{en } (0,1/2), \qquad u(0) = 0 = u(1/2).$$

We will find a classical solution with ODEs techniques. Since classical solutions are also weak solutions, and weak solutions are unique (for this problem) such classical solution will be the solution we were looking for.

A general expression for classical solutions to the homogeneous equations is $u_\mathrm{h}(x) = A\mathrm{e}^{2x} + B\mathrm{e}^{-2x}$. A particular solution of the inhomogeneous equation is $u_\mathrm{p}(x) = 1/4$. Hence, the general classical solution will have the form

$$u_\mathrm{g}(x) = A\mathrm{e}^{2x} + B\mathrm{e}^{-2x} + \frac{1}{4}.$$

To obtain the solution, we impose boundary conditions, which gives the linear system of equations

$$A + B + \frac{1}{4} = 0, \qquad A\mathrm{e} + \frac{B}{\mathrm{e}} + \frac{1}{4} = 0,$$

whose solutions are $A = -\frac{1}{4(1+\mathrm{e})}$, $B = -\frac{\mathrm{e}}{4(1+\mathrm{e})}$.

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7. Consider the boundary value problem $u'' = 2$, $u(1) = u(-1) = 0$, whose solution is given by $\bar{u}(x) = x^2 - 1$; write the variational formulation to conclude that for all $u \in C^2$ with $u(1) = u(-1) = 0$ we have

$$\frac{8}{3} + \int_{-1}^{1} \big((u')^2 + 4u\big) \geq 0.$$

Solution. a function $\bar{u} \in H_0^1((-1,1))$ is a weak solution of the problem if $\int_{-1}^1 \bar{u}'v' + 2\int_{-1}^1 v = 0$ for all $v \in H_0^1(I)$. The unique weak solution $\bar{u}$ is also solution to the minimization problem

$$F(\bar{u}) = \min_{u \in H_0^1((-1,1))} F(u), \qquad F(u) = \frac{1}{2}\int_{-1}^1 (u')^2 + 2\int_{-1}^1 u.$$

We already know the weak solution to the problem, which happens to be the classical one in the statement: $\bar{u}(x) = x^2 - 1$. A simple calculation shows that $F(\bar{u}) = -4/3$, whence the result for all $u \in H_0^1((-1,1))$, in particular for all $u \in C^2$ with $u(1) = u(-1) = 0$.

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8. (Hardy Inequality in dimension $N = 1$). Let $I = (0, 1)$.

   (a) Given $u \in L^p(I)$, show that

$$\left\| \frac{1}{x} \int_0^x u(t)\, dt \right\|_{L^p(I)} \leq \frac{p}{p-1} \|u\|_{L^p(I)}.$$

   Hint. Begin with $u \in C_c(I)$ by defining $\varphi(x) = \int_0^x u(t)\, dt$. Check that $|\varphi|^p \in C^1(\bar{I})$ and calculate the derivative. Finally, use the formula

$$\int_0^1 |\varphi(x)|^p \frac{dx}{x^p} = \frac{1}{p-1} \int_0^1 |\varphi(x)|^p d\left(-\frac{1}{x^{p-1}}\right)$$

   and integrate by parts.

   (b) Let $u \in W^{1,p}(I)$, $1 < p < \infty$. Show that if $u(0) = 0$, then

$$\left\| \frac{u(x)}{x} \right\|_{L^p(I)} \leq \frac{p}{p-1} \|u'\|_{L^p(I)}.$$

Solution. (a) Let $Tu(x) = \varphi(x)/x$. Begin with $u \in C_c(I)$. Recall that in this case $\lim_{x\to 0} \varphi(x) = 0$ and $\lim_{x\to 0} \varphi'(x) = 0$. We have that

$$\int_0^1 |Tu(x)|^p\, dx = -\frac{1}{p-1}|\varphi(1)|^p + \frac{p}{p-1} \int_0^1 |\varphi(x)|^{p-1}(\text{sign } \varphi(x))\varphi'(x) \frac{dx}{x^{p-1}}.$$

By Hölder inequality we get

$$\int_0^1 |Tu(x)|^p\, dx \leq \frac{p}{p-1} \left(\int_0^1 \left| \frac{\varphi(x)}{x} \right|^p dx \right)^{\frac{p-1}{p}} \left(\int_0^1 |\varphi'(x)|^p\, dx \right)^{\frac{1}{p}},$$

that is

$$\|Tu\|^p_{L^p(I)} \leq \frac{p}{p-1} \|Tu\|^{p-1}_{L^p(I)} \|u\|_{L^p(I)}.$$

The general case follows by density, using Fatou's Lemma to take limits on the left-hand side on the inequality.

(b) It is sufficient to recall that $u(x) = \int_0^x u'(t)\, dt$ and use part (a).

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9. (A problem with Hardy-type weights) Let $I = (0, 1)$ and $V = \{v \in H^1(I) : v(0) = 0\}$.

   (a) Given $f \in L^2(I)$ such that $\frac{1}{x} f(x) \in L^2(I)$, show that there exists a unique $u \in V$ satisfying

$$\int_0^1 u'(x)v'(x)\, dx + \int_0^1 \frac{u(x)v(x)}{x^2}\, dx = \int_0^1 \frac{f(x)v(x)}{x^2}\, dx \qquad \forall v \in V. \tag{1}$$

   (b) Write the minimization problem associated to (1)

   (c) Here and in part (d) we will assume that $\frac{1}{x^2} f(x) \in L^2(I)$. Letting $v(x) = \frac{u(x)}{(x+\varepsilon)^2}$, $\varepsilon > 0$, show that

$$\int_0^1 \left| \frac{d}{dx} \left(\frac{u(x)}{x+\varepsilon} \right) \right|^2 dx \leq \int_0^1 \frac{f(x)}{x^2} \frac{u(x)}{(x+\varepsilon)^2}\, dx.$$

   (d) Prove that $\frac{u(x)}{x^2} \in L^2(I)$, $\frac{u(x)}{x} \in H^1(I)$ y $\frac{u'(x)}{x} \in L^2(I)$.

   (e) As a consequence of part (d) show that $u \in H^2(I)$ and that

$$-u''(x) + \frac{u(x)}{x^2} = \frac{f(x)}{x^2} \quad \text{a.e. en } I, \qquad u(0) = u'(0) = 0, \quad u'(1) = 0. \tag{2}$$

   (f) Viceversa, show that if $u \in H^2(I)$ with $\frac{u(x)}{x^2} \in L^2(I)$ satisfies equation (2), hence it satisfies (1).

Solution. (a) Existence and uniqueness of a weak solution follow by Lax-Milgram Theorem. The bilinear form $a(u,v) = \int_0^1 u'(x)v'(x)\,dx + \int_0^1 \frac{u(x)v(x)}{x^2}\,dx$ is continuous as a consequence of part (b) of the previous problem:

$$|a(u,v)| \le \|u'\|_{L^2(I)}\|v'\|_{L^2(I)} + \left\|\frac{u(x)}{x}\right\|_{L^2(I)} \left\|\frac{v(x)}{x}\right\|_{L^2(I)} \le C\|u'\|_{L^2(I)}\|v'\|_{L^2(I)} \le \|u\|_{H^1(I)}\|v\|_{H^1(I)}.$$

Coercivity easily follows:

$$a(u,u) = \int_0^1 (u')^2 + \int_0^1 \frac{u(x)}{x^2}\,dx \ge \int_0^1 (u')^2 + \int_0^1 u^2 = \|u\|^2_{H^1(I)}.$$

Again using part (b) of the previous exercise, we see that $F(v) = \int_0^1 \frac{f(x)v(x)}{x^2}\,dx$ is continuous,

$$|F(v)| \le \left\|\frac{f(x)}{x}\right\|_{L^2(I)} \left\|\frac{v(x)}{x}\right\|_{L^2(I)}.$$

(b) The minimization problem associated consists in finding $u \in V$ such that $\varphi(u) = \min_{v\in V} \varphi(v)$, where

$$\varphi(v) = \frac{1}{2}\int_0^1 \left((v'(x))^2 + \frac{v^2(x)}{x^2}\right)dx - \int_0^1 \frac{f(x)v(x)}{x^2}\,dx.$$

The proof of the above statement follows by studying the limit (the first variation or Gateaux derivative is zero at critical points.)

$$\lim_{\varepsilon\to 0} \frac{\varphi(u+\varepsilon v) - \varphi(u)}{\varepsilon}.$$

(c) A simple calculation shows that

$$\begin{aligned} u'(x)\left(\frac{u(x)}{(x+\varepsilon)^2}\right)' + \frac{u^2(x)}{(x+\varepsilon)^2x^2} &= \left(\frac{d}{dx}\left(\frac{u(x)}{x+\varepsilon}\right)\right)^2 + \frac{u^2(x)}{(x+\varepsilon)^2}\left(\frac{1}{x^2} - \frac{1}{(x+\varepsilon)^2}\right) \\ &\ge \left(\frac{d}{dx}\left(\frac{u(x)}{x+\varepsilon}\right)\right)^2, \end{aligned}$$

and the result follows.

(d) Let $g(x) = \frac{u(x)}{x+\varepsilon}$. It is easy to check that $g \in V$. Applying part (b) of the previous exercise, we get

$$\int_0^1 \left(\frac{u(x)}{(x+\varepsilon)x}\right)^2 dx = \left\|\frac{g(x)}{x}\right\|^2_{L^2(I)} \le 4\|g'\|^2_{L^2(I)} = 4\left\|\left(\frac{u(x)}{x+\varepsilon}\right)'\right\|^2_{L^2(I)}.$$

Combining the above result with part (c) of this problem, we obtain by Hölder inequality:

$$\begin{aligned} \left\|\frac{u(x)}{(x+\varepsilon)^2}\right\|^2_{L^2(I)} &\le \int_0^1 \left(\frac{u(x)}{(x+\varepsilon)x}\right)^2 dx \le 4\left\|\left(\frac{u(x)}{x+\varepsilon}\right)'\right\|^2_{L^2(I)} \\ &\le 4\int_0^1 \frac{f(x)}{x^2}\frac{u(x)}{(x+\varepsilon)^2}\,dx \le \left\|\frac{f(x)}{x^2}\right\|_{L^2(I)} \left\|\frac{u(x)}{(x+\varepsilon)^2}\right\|_{L^2(I)}, \end{aligned}$$

so that

$$\left\|\frac{u(x)}{(x+\varepsilon)^2}\right\|_{L^2(I)} \le 4\left\|\frac{f(x)}{x^2}\right\|_{L^2(I)}.$$

Letting $\varepsilon \to 0$ we obtain that $u(x)/x^2$ belongs to $L^2(I)$.

The uniform estimate for $\left\|\frac{u(x)}{(x+\varepsilon)^2}\right\|_{L^2(I)}$ that we have obtained above, show that $g_\varepsilon(x) = u(x)/(x+\varepsilon)$ satisfy $\|g'_\varepsilon\|_{L^2(I)} \le C$. Therefore there exists a subsequence $\{g'_{\varepsilon_k}\}$ weakly convergent in $L^2(I)$ to a function in $L^2(I)$. Since $g_\varepsilon \to u(x)/x$ strongly in $L^2(I)$, we deduce that the limit of g'_{ε_k}, which belongs to $L^2(I)$, is precisely $(u(x)/x)'$.

We finally show that $u'(x)/x$ is also in $L^2(I)$, since

$$\frac{d}{dx}\left(\frac{u(x)}{x}\right) = \frac{u'(x)}{x} - \frac{u(x)}{x^2}.$$

(e) Weak formulation tells us that $u'' = \frac{u}{x^2} - \frac{f}{x^2}$ in the distributional sense. On the other hand, the right-hand side of the equality is in $L^2(I)$, therefore $u \in H^2(I)$.

Using compactly supported smooth test functions and integrating by parts we easily deduce that u satisfies the equation a.e. Next we consider test functions v such that $v(1) = 0$ and $v(0) = 0$ to conclude that $u'(1) = 0$. The condition $u(0) = 0$ is included in the definition of the Hilber space V.

(f) This is standard. Multiply the equation by a smooth test function and integrate by parts. The result follows by a density argument.

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10. Let $I = (0, 1)$ and let us fix a constant $k > 0$.

(a) Given $f \in L^1(I)$, show that there is a unique $u \in H_0^1(I)$ such that

$$\int_I u'v' + k \int_I uv = \int_I fv \qquad \forall v \in H_0^1(I). \tag{3}$$

(b) Prove that $u \in W^{2,1}(I)$.

(c) Prove that

$$\|u\|_{L^1(I)} \leq \frac{1}{k}\|f\|_{L^1(I)}.$$

Hint. Fix a function $\gamma \in C^1(\mathbb{R}, \mathbb{R})$ so that $\gamma'(t) \geq 0$ for all $t \in \mathbb{R}$, $\gamma(0) = 0$, $\gamma(t) = 1$ and all $t \geq 1$ and such that $\gamma(t) = -1$ for all $t \leq -1$. Take $v = \gamma(n\,u)$ in (3) and let $n \to \infty$.

(d) Assume now $f \in L^p(I)$, $p \in (1, \infty)$. Show that there exists $\delta > 0$ independent of k and p such that

$$\|u\|_{L^p(I)} \leq \frac{1}{k + \delta/pp'}\|f\|_{L^p(I)}.$$

Hint. If $p \in [2, \infty)$, take $v = \gamma(u)$ in (3), with $\gamma(t) = |t|^{p-1}\operatorname{sign} t$. If $p \in (1, 2)$, use duality.

(e) if $f \in L^\infty(I)$, show that

$$\|u\|_{L^\infty(I)} \leq C_k\|f\|_{L^\infty(I)},$$

and find the best constant C_k. *Hint.* Find the explicit solution to (3) corresponding to $f \equiv 1$.

Solution. (a) Continuity and coercivity of $a(u, v) = \int_I u'v' + k\int_I uv$ are immediate. Continuity of $F(v) = \int_0^1 fv$, follows by

$$|F(v)| \leq \|v\|_{L^\infty(I)}\|f\|_{L^1(I)} \leq C\|f\|_{L^1(I)}\|v\|_{H_0^1(I)}.$$

(b) Use $u'' = ku - f$ in the distributional sense, and use $L^2(I) \subset L^1(I)$.

(c) Following the hint:

$$k\int_0^1 u\gamma(nu) \leq \int_0^1 f\gamma(nu) \leq \int_0^1 |f|.$$

We obtain the result by taking the limit as $n \to \infty$.

(d) We take the test function $v = \gamma(u) = |u|^{p-1}\operatorname{sign} u$, to obtain

$$\int_0^1 \gamma'(u)(u')^2 + k\int_0^1 |u|^p = \int_0^1 f\gamma(u) \leq \|f\|_{L^p(I)}\|\gamma(u)\|_{L^{p'}(I)} = \|f\|_{L^p(I)}\|u\|^p_{L^p(I)}.$$

Using Poincaré inequality in H_0^1,

$$\int_0^1 \gamma'(u)(u')^2 = \frac{4(p-1)}{p^2}\int_0^1 \left(\left(u^{\frac{p}{2}}\right)'\right)^2 \geq \frac{4C_I}{pp'}\int_0^1 |u|^p,$$

where C_I is the best constant in Poincaré inequality. Combining the above two results allows to conclude.

(e) Letting $p \to \infty$ in the estimate of part (d) we obtain the result with $C_k = 1/k$.

To get the best constant we use the hint: the explicit solution corresponding to $f = 1$ is given by

$$\bar{u}_k(x) = -\frac{1}{k}\frac{\mathrm{e}^{\sqrt{k}/2}}{1+\mathrm{e}^{\sqrt{k}}}\left(\mathrm{e}^{\sqrt{k}(x-\frac{1}{2})} + \mathrm{e}^{-\sqrt{k}(x-\frac{1}{2})}\right) + \frac{1}{k},$$

and has a maximum at $x = 1/2$ that is

$$\|\bar{u}_k\|_{L^\infty(I)} = \frac{1}{k}\frac{\left(1-\mathrm{e}^{\sqrt{k}/2}\right)^2}{1+\mathrm{e}^{\sqrt{k}}}.$$

Given a solutio to the problem with a fixed k and a given f, by the maximum principle and the linearity of the problem, we have that

$$\|u\|_{L^\infty(I)} \le \|\bar{u}_k\|_{L^\infty(I)}\|f\|_{L^\infty(I)}.$$

The best constant is therefore $C_k = \|\bar{u}_k\|_{L^\infty(I)}$.

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11. Let $I = (0,1)$.

   (a) Prove that for any $\varepsilon > 0$ there exists a constant C_ε such that

   $$|u(1)|^2 \le \varepsilon\|u'\|^2_{L^2(I)} + C_\varepsilon\|u\|^2_{L^2(I)} \qquad \forall u \in H^1(I).$$

   (b) Show that if the constant $k > 0$ is big enough, then for all $f \in L^2(I)$ there exists a unique $u \in H^2(I)$ satisfying

   $$-u'' + ku = f \quad \text{en } I, \qquad u'(0) = 0, \quad u'(1) = u(1).$$

   Write both the associated weak formulation and the minimization problem.

Solution. (a) Recall that $u^2(1) - u^2(x) = \int_x^1 (u^2)'$. As a consequence,

$$u^2(1) \le u^2(x) + 2\int_0^1 |u|\,|u'|.$$

Integrating over $(0,1)$, using $2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$ for all $\varepsilon > 0$, we find

$$u^2(1) \le \underbrace{\left(1+\frac{1}{\varepsilon}\right)}_{C_\varepsilon}\|u\|^2_{L^2(I)} + \varepsilon\|u'\|_{L^2(I)}.$$

(b) The weak formulation can be obtained by multiplying by a smooth function, then integrating by parts: the boundary terms disappear because of the boundary conditions. We get that $u \in H^1(I)$ is a weak solution if

$$\underbrace{\int_0^1 (u'v' + kuv) - u(1)v(1)}_{a(u,v)} = \underbrace{\int_0^1 fv}_{F(v)} \qquad \forall v \in H^1(I).$$

To show existence of a unique solution we use the Lax-Milgram Theorem. Continuity of the bilinear form a follows by the trace inequality. Coercivity follows by part (a): we have that

$$a(u,u) \ge (1-\varepsilon)\int_0^1 (u')^2 + (k - C_\varepsilon)\int_0^1 u^2.$$

Just take $\varepsilon < 1$ so that $C_\varepsilon = 1 + \frac{1}{\varepsilon} < k$. This is possible if $k > 2$, by taking $\varepsilon \in (1/(k-1), 1)$.

The associated minimization problem consists in finding $u \in H^1(I)$ so that

$$\varphi(u) \leq \min_{v \in H^1(I)} \varphi(v), \qquad \varphi(v) = \frac{1}{2} \int_0^1 ((v')^2 + kv^2) - \frac{1}{2} v^2(1) - \int_0^1 fv.$$

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12. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary, let $h \in C^\infty(\partial\Omega)$ be such that $\int_{\partial\Omega} h = 0$.

(a) Define a reasonable concept of weak solution to the problem

$$\Delta u = 0 \quad \text{en } \Omega, \qquad \partial u / \partial n = h \quad \text{en } \partial\Omega.$$

(b) Prove that there exists a unique weak solution such that $\int_\Omega u = 0$ and check that the difference between two arbitrary weak solutions has to be constant in Ω.

Solution. (a) Multiply by $v \in C^\infty(\mathbb{R}^N)$ and integrate by parts to get

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} v \frac{\partial u}{\partial n} = 0.$$

Boundary conditions and a density argument allow to conclude that

$$\int_\Omega \nabla u \cdot \nabla v = \int_{\partial\Omega} vh \qquad \forall v \in H^1(\Omega). \tag{4}$$

A weak solution is a function $u \in H^1(\Omega)$ that satisfies (4).

(b) Let $H = \{u \in H^1(\Omega) : \int_\Omega u = 0\}$. H is a closed subspace of $H^1(\Omega)$, therefore an Hilbert space with the $H^1(\Omega)$-norm and scalar product. It is easy to see that the bilinear form $a(u, v) = \int_\Omega \nabla u \cdot \nabla v$ is continuous in H. Coercivity follows by the Poincaré-Wirtinger inequality, since functions of this subspace have zero mean value. On the other hand, if $h \in L^2(\partial\Omega)$, we easily obtain using the trace inequality

$$|F(v)| \leq \|h\|_{L^2(\partial\Omega)} \|v\|_{L^2(\partial\Omega)} \leq C\|v\|_H.$$

The existence of a unique weak solution in H now follows by Lax-Milgram Theorem.

Take any two weak solutions to the problem u and v. We have that $u - \frac{1}{|\Omega|}\int_\Omega u$ and $v - \frac{1}{|\Omega|}\int_\Omega v$ are two weak solutions which belong to H. By uniqueness of weak solutions in H,

$$u - \frac{1}{|\Omega|} \int_\Omega u = v - \frac{1}{|\Omega|} \int_\Omega v,$$

which means that u and v differ by a constant factor.

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13. Let $\Omega \subset \mathbb{R}^N$ be a bounded connected domain with smooth boundary.

(a) Define weak solutions for the Poisson equation with Robin boundary conditions:

$$-\Delta u = f \quad \text{en } \Omega, \qquad u + \frac{\partial u}{\partial n} = 0 \quad \text{sobre } \partial\Omega,$$

Check that any classical solution to the problem is a weak solution, and that every weak solution which is also smooth enough, is a classical solution.

(b) Show existence and uniqueness of weak solutions to the problem, for any $f \in L^2(\Omega)$.
Hint. Use Friedrichs' Inequality.

Solution. (a) Assume that we have a classical solution to the problem, i.e. $u \in C^2(U) \cap C^1(\overline{U})$ that satisfies the equation for all $x \in U$ and the boundary condition for all $x \in \partial U$. Multiply the equation by $v \in H^1(U)$ and integrate by parts to get

$$\int_U fv = -\int_U v\Delta u = \int_U \nabla u \cdot \nabla v - \int_{\partial U} v\frac{\partial u}{\partial \nu}.$$

Boundary conditions then imply

$$\int_U \nabla u \cdot \nabla v + \int_{\partial U} uv = \int_U fv. \tag{5}$$

This expression makes sense if $u, v \in H^1(U)$. We therefore define a weak solution of the problem to be a function $u \in H^1(U)$ that satisfies (5) for all $v \in H^1(U)$. Recall that the Trace Theorem tells us that the boundary conditions can be taken in the sense of traces.

Viceversa, consider a weak solution u which is smooth enough, namely $u \in C^2(U) \cap C^1(\overline{U})$. We need to prove that it is a classical solution. Take a test $v \in C_c^\infty(U)$. We have that $\int_U \nabla u \cdot \nabla v = \int_U fv$. An integration by parts allow to conclude that $\int_U (-\Delta u - f)v = 0$ for all $v \in C_c^\infty(U)$, i.e. $-\Delta u = f$ in U.

Ahora tomamos un test general $v \in H^1(U)$. Integrando por partes,

$$\int_U (-\Delta u - f)v + \int_{\partial U} \left(u + \frac{\partial u}{\partial \nu}\right) v = 0.$$

Como ya sabemos que $-\Delta u = f$, tenemos por tanto que $\int_{\partial U} \left(u + \frac{\partial u}{\partial \nu}\right) v = 0$ para toda $v \in H^1(U)$, y en particular para toda $v \in C^\infty(\overline{U})$. De ahí es fácil deducir que $u + \frac{\partial u}{\partial \nu} = 0$ en ∂U.

(b) Existence and uniqueness follow by Lax-Milgram Theorem.

The linear functional defined in $H^1(U)$ given by $v \to \int_U fv$ ir clearly linear. It is also continuous, by Hölder inequality:

$$\left|\int_U fv\right| \leq \int_U |fv| \leq \|f\|_{L^2(U)}\|v\|_{L^2(U)} \leq \|f\|_{L^2(U)}\|v\|_{H^1(U)}.$$

Define the bilinear form on $H^1(U) \times H^1(U)$

$$B[u, v] = \int_U Du \cdot Dv + \int_{\partial U} uv.$$

Let us prove that it is continuous and coercive. On one hand, continuity follows by Hölder inequality and trace inequality:

$$\begin{aligned} |B[u,v]| &\leq \int_U |Du \cdot Dv| + \int_{\partial U} |uv| \leq \|Du\|_{L^2(U)}\|Dv\|_{L^2(U)} + \|u\|_{L^2(\partial U)}\|v\|_{L^2(\partial U)} \\ &\leq \|Du\|_{L^2(U)}\|Dv\|_{L^2(U)} + \|u\|_{H^1(U)}\|v\|_{H^1(U)} \leq \widehat{C}\|u\|_{H^1(U)}\|v\|_{H^1(U)}. \end{aligned}$$

Coercivity follows by Friedrics inequality (with $\Gamma = \partial U$):

$$B[u, u] = \int_U |Du|^2 + \int_{\partial U} u^2 \geq C\|u\|^2_{H^1(U)}.$$

We can finally apply Lax-Milgram Theorem to get existence and uniqueness of a weak solution.