Hoja de problemas 1: LAPLACE AND POISSON EQUATIONS

1. Prove that Laplace equation $\Delta u = 0$ is invariant under rotations: let O be an orthogonal matrix $n \times n$ and define
$$v(x) := u(Ox), \qquad x \in \mathbb{R}^N.$$
Show that $\Delta v = 0$.

2. Let u be an harmonic function and let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be a smooth convex function. Prove that $v := \phi(u)$ is a subharmonic function.

3. Show that $x \mapsto \log |x|$ is a subharmonic function in the domain $\mathbb{R}^N \setminus \{0\}$ if $N \geq 2$.

4. Show that $v := |Du|^2$ a subharmonic function if u is harmonic.

5. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution to
$$\Delta u = -1 \quad \text{en } \Omega, \qquad u|_{\partial\Omega} = 0.$$
Prove that $\forall x_0 \in \Omega$ we have that
$$u(x_0) \geq \frac{1}{2N} \min_{x \in \partial\Omega} |x - x_0|^2.$$

6. Let u be a classical solution to
$$-\Delta u = f \quad \text{en } B_1(0), \qquad u = g \quad \text{en } \partial B_1(0).$$
Show that there exists a constant $C > 0$, independent of u, such that
$$\max_{B_1(0)} |u| \leq C(\max_{\partial B_1(0)} |g| + \max_{B_1(0)} |f|).$$

7. Let u be a positive harmonic function in $B_r(0)$. Use Poisson formula to show that
$$r^{N-2} \frac{r - |x|}{(r + |x|)^{N-1}} u(0) \leq u(x) \leq r^{N-2} \frac{r + |x|}{(r - |x|)^{N-1}} u(0).$$
This is an explicit form of the Harnack inequality.

8. Consider the problem
$$\begin{cases} \Delta u(x) + c(x)u(x) = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases}$$
where we assume $c(x) < 0$. Show that this problem has a unique solution. Show by an example that when $c(x) > 0$ uniqueness fails.

9. (Schwartz Reflection Principle) Consider the open semiball $U^+ = \{x \in \mathbb{R}^N : \ |x| < 1, x_N > 0\}$. Let $u \in C^2(U^+)$ be harmonic in U^+ with $u = 0$ on $\partial U^+ \cap \{x_N = 0\}$. Given $x \in U = B_1(0)$ we define
$$v(x) := \begin{cases} u(x) & \text{si } x_N \geq 0, \\ -u(x_1, \dots, x_{n-1}, -x_N) & \text{si } x_N < 0. \end{cases}$$
Show that v is harmonic in U.

10. Let $\Omega \subset \mathbb{R}^N$, be a domain, $N \geq 2$, and $x_0 \in \Omega$. Let u be a *bounded* harmonic function in $\Omega_0 := \Omega \setminus \{x_0\}$. Show that we can define a value $u(x_0)$ such that the extended function is harmonic on the whole Ω.

11. Let $\Omega \subset \mathbb{R}^N$, be a bounded domain, $N \geq 2$, and let $x_0 \in \Omega$. Define $\Omega_0 := \Omega \setminus \{x_0\}$ and let u and v be two harmonic functions in Ω_0, continuous in $\Omega_1 = \Omega_0 \cup \partial\Omega$ and such that: (i) $u(x) \leq v(x)$ for all $x \in \partial\Omega$; (ii) $|u(x)| \leq M$, $|v(x)| \leq M$ for all $x \in \Omega_1$. Use the Maximum Principle to show that $u(x) \leq v(x)$ for all $x \in \Omega_1$.

12. Find an expression for the Green function of the Dirichlet problem for the Laplace equations in an annular region $B_R(x_0) \setminus B_r(x_0)$, with $0 < r < R$.

13. Show that a solution to $\Delta u - u^2 = 0$ in a domain Ω cannot attain its maximum in Ω, except if $u \equiv 0$.

14. Let $u \in C^2(B_1(0)) \cap C(\overline{B_1(0)})$ be a solution to the Dirichlet problem

$$\begin{cases} \Delta u = u^2 + f(|x|), & x \in B_1(0), \\ u(x) = 1, & x \in \partial B_1(0), \end{cases}$$

where $f(|x|) \geq 0$ is of class $C^1(\Omega)$. Calculate the maximum of u in $\overline{B_1(0)}$ and show that it does not depend on f.