EDPs en Ciencia e ingenieria. M.M.A. de la UAM *Curso 2016/17*

Hoja de problemas 1: LAPLACE AND POISSON EQUATIONS

1. Prove that Laplace equation $\Delta u = 0$ is invariant under rotations: let O be an orthogonal matrix $n \times n$ and define
$$v(x) := u(Ox), \qquad x \in \mathbb{R}^N.$$
Show that $\Delta v = 0$.

Solution. By the chain rule,
$$\frac{\partial v}{\partial x_i}(x) = \sum_{j=1}^{N} \frac{\partial u}{\partial x_j}(Ox) O_{ji}, \qquad \frac{\partial^2 v}{\partial x_i^2}(x) = \sum_{k,j=1}^{N} \frac{\partial^2 u}{\partial x_k \partial x_j}(Ox) O_{ki} O_{ji}.$$
Then use the fact that since O is a orthogonal matrix we have $\sum_{i=1}^{N} O_{ki} O_{ji} = \delta_{kj}$, so that
$$\begin{aligned} \Delta v(x) &= \sum_{i=1}^{N} \frac{\partial^2 v}{\partial x_i^2}(x) = \sum_{k,j=1}^{N} \frac{\partial^2 u}{\partial x_k \partial x_j}(Ox) \sum_{i=1}^{N} O_{ki} O_{ji} \\ &= \sum_{k,j=1}^{N} \frac{\partial^2 u}{\partial x_k \partial x_j}(Ox) \delta_{kj} = \sum_{k=1}^{N} \frac{\partial^2 u}{\partial x_k^2}(Ox) = \Delta u(Ox) = 0. \end{aligned}$$

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2. Let u be an harmonic function and let $\phi : \mathbb{R} \mapsto \mathbb{R}$ be a smooth convex function. Prove that $v := \phi(u)$ is a subharmonic function.

Solution. Let us calculate
$$\frac{\partial v}{\partial x_i} = \phi'(u) \frac{\partial u}{\partial x_i}, \qquad \frac{\partial^2 v}{\partial x_i^2} = \phi''(u) \left(\frac{\partial u}{\partial x_i} \right)^2 + \phi'(u) \frac{\partial^2 u}{\partial x_i^2}.$$
Hence we have
$$\Delta v = \sum_{i=1}^{N} \frac{\partial^2 v}{\partial x_i^2} = \phi''(u) \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i} \right)^2 + \phi'(u) \Delta u.$$
Recall that u is harmonic, i.e. $\Delta u = 0$, and that ϕ is convex and smooth, so that $\phi'' \geq 0$; therefore we get
$$\Delta v = \phi''(u) \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i} \right)^2 \geq 0.$$

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3. Show that $x \mapsto \log |x|$ is a subharmonic function in the domain $\mathbb{R}^N \setminus \{0\}$ if $N \geq 2$.

Solution. Let us calculate
$$\frac{\partial u}{\partial x_i}(x) = \frac{x_i}{|x|^2}, \qquad \frac{\partial^2 u}{\partial x_i^2}(x) = \frac{|x|^2 - 2x_i^2}{|x|^4}.$$
As a consequence,
$$\Delta u(x) = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}(x) = \frac{n-2}{|x|^2} \geq 0 \quad \text{si } n \geq 2.$$

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4. Show that $v := |Du|^2$ a subharmonic function if u is harmonic.

Solution. Let us calculate directly, starting from the expression $v = \sum_{i=1}^N \left(\frac{\partial u}{\partial x_i}\right)^2$. Indeed,

$$\frac{\partial v}{\partial x_j} = \sum_{i=1}^N 2\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i \partial x_j}, \qquad \frac{\partial^2 v}{\partial x_j^2} = 2\sum_{i=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\partial^2 u}{\partial x_j^2}\right).$$

As a consequence, $-\Delta v \leq 0$:

$$\Delta v = \sum_{j=1}^N \frac{\partial^2 v}{\partial x_j^2} = 2\sum_{i,j=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 + \sum_{i=1}^N \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}(\Delta u) = 2\sum_{i,j=1}^N \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)^2 \geq 0,$$

since u is harmonic.

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5. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $u \in C^2(\Omega) \cap C(\overline{\Omega})$ a solution to

$$\Delta u = -1 \quad \text{en } \Omega, \qquad u|_{\partial\Omega} = 0.$$

Prove that $\forall x_0 \in \Omega$ we have that

$$u(x_0) \geq \frac{1}{2N} \min_{x \in \partial\Omega} |x - x_0|^2.$$

Solution. Fixed $x_0 \in \Omega$, it is easy to check that the function $v(x) = \frac{|x-x_0|^2}{2N}$ satisfies $\Delta v = 1$. As a consequence $\Delta(u+v) = 0$. By the maximum principle $u+v$ has minimum on $\partial\Omega$, namely

$$(u+v)(x) \geq \min_{x\in\partial\Omega}(u+v)(x) = \min_{x\in\partial\Omega} v(x) = \frac{1}{2n}\min_{x\in\partial\Omega}|x-x_0|^2.$$

Since $v(x_0) = 0$, the result follows.

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6. Let u be a classical solution to

$$-\Delta u = f \quad \text{en } B_1(0), \qquad u = g \quad \text{en } \partial B_1(0).$$

Show that there exists a constant $C > 0$, independent of u, such that

$$\max_{B_1(0)} |u| \leq C(\max_{\partial B_1(0)} |g| + \max_{B_1(0)} |f|).$$

Solution. On one hand, we first let $v(x) = \|g\|_{L^\infty(\partial B_1(0))} + \frac{\|f\|_{L^\infty(B_1(0))}}{2n}(1 - |x|^2)$. It is easy to see that $\Delta v = -\|f\|_{L^\infty(B_1(0))}$. Therefore we have

$$\begin{cases} -\Delta(u-v) = f - \|f\|_{L^\infty(B_1(0))} \leq 0 & \text{en } B_1(0), \\ u - v = g - \|g\|_{L^\infty(\partial B_1(0))} \leq 0 & \text{en } \partial B_1(0). \end{cases}$$

By the maximum principle, $u \leq v$ en $B_1(0)$.

On the other hand,

$$\begin{cases} -\Delta(u+v) = f + \|f\|_{L^\infty(B_1(0))} \geq 0 & \text{en } B_1(0), \\ u + v = g + \|g\|_{L^\infty(\partial B_1(0))} \geq 0 & \text{en } \partial B_1(0). \end{cases}$$

Again by the maximum principle, $u \geq -v$ en $B_1(0)$.

Summing up, $|u| \leq v$ in $B_1(0)$, hence the result holds with $C = 1$.

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7. Let u be a positive harmonic function in $B_r(0)$. Use Poisson formula to show that

$$r^{N-2}\frac{r-|x|}{(r+|x|)^{N-1}}u(0) \le u(x) \le r^{N-2}\frac{r+|x|}{(r-|x|)^{N-1}}u(0).$$

This is an explicit form of the Harnack inequality.

Solution. Let us recall Poisson formula for harmonic functions:

$$u(x) = \frac{r^2-|x|^2}{\omega_N r}\int_{\partial B_r(0)} \frac{u(y)}{|x-y|^N}\, d\sigma_y,$$

where ω_N is the area (or (N-1)-dimensional volume) of the unit sphere of $\mathbb{R}^N$. Next we use that $|x-y| \ge ||y|-|x||$, to get

$$u(x) \le \frac{r^2-|x|^2}{(r-|x|)^N}\, r^{N-2} \fint_{\partial B_r(0)} u\, d\sigma.$$

Using the the mean value property for u (which is harmonic) and simplifying the fraction, we obtain

$$u(x) \le r^{N-2}\frac{r+|x|}{(r-|x|)^{N-1}}u(0).$$

A similar argument allows to prove the analogous estimates from below, using $|x-y| \le |y|+|x|$.

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8. Consider the problem

$$\begin{cases} \Delta u(x) + c(x)u(x) = 0, & x \in \Omega, \\ u(x) = g(x), & x \in \partial\Omega, \end{cases}$$

where we assume $c(x) < 0$. Show that this problem has a unique solution. Show by an example that when $c(x) > 0$ uniqueness fails.

Solution. Given two solutions u and v, then $w = u - v$ satisfies

$$\Delta w(x) + c(x)w(x) = 0, \quad x \in \Omega, \qquad w(x) = 0, \quad x \in \partial\Omega.$$

Let us first show an "energy method" to prove uniqueness: multiply the equation by w and integrate by parts to get

$$0 = -\int_\Omega \left(|\nabla w|^2 + c(x)w^2\right) \le 0,$$

this clearly shows that $w = 0$ in Ω.

This proof requires that w is C^1 up to the boundary $\partial\Omega$, for the integration by parts to be true. This requirement seems not natural for the problem at hand, since usually we just have $C^2(\Omega) \cap C^0(\overline{\Omega})$. This can be avoided by using weak solutions and Sobolev functions; on the other hand, there is another proof, that uses the Maximum Principle arguments.

Assume that w is positive at some point of Ω. Therefore w will have a positive maximum at a point $x_0 \in \Omega$. Hence,

$$0 \ge \Delta w(x_0) = -c(x_0)w(x_0) > 0,$$

which is a contradiction, therefore we have $w \le 0$. An analogous argument allow to conclude that $w \ge 0$, hence we conclude that $w = 0$.

The non-uniqueness can be shown by means of an example in dimension $N = 1$, on the interval $(0, \pi)$ and with $c(x) = 1$. All the functions of the form $u_c(x) = c \sin x$ are clearly solutions to the problem.

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9. (Schwartz Reflection Principle) Consider the open semiball $U^+ = \{x \in \mathbb{R}^N : \ |x| < 1, x_N > 0\}$. Let $u \in C^2(U^+)$ be harmonic in U^+ with $u = 0$ on $\partial U^+ \cap \{x_N = 0\}$. Given $x \in U = B_1(0)$ we define

$$v(x) := \begin{cases} u(x) & \text{si } x_N \geq 0, \\ -u(x_1, \ldots, x_{n-1}, -x_N) & \text{si } x_N < 0. \end{cases}$$

Show that v is harmonic in U.

Solution. The function is clearly harmonic both when $x_N > 0$ and when $x_N < 0$. Hence the Mean Value Property is satisfied in balls centered at points of those sets, provided the radii are small enough. Let us show that the same holds for sufficiently small balls centered at points with $x_N = 0$. Indeed, taking $r > 0$ small enough, $B_r(x) \subset B_1(0)$, and

$$⨍_{\partial B_r(x)} v = \frac{1}{|B_r(0)|} \left(\int_{\partial B_r(x) \cap \{y_N > 0\}} v(y)\, d\sigma_y + \int_{\partial B_r(x) \cap \{y_N < 0\}} v(y)\, d\sigma_y \right) = 0 = v(x).$$

Since v is continuous, we can conclude that v is harmonic in $B_1(0)$.

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10. Let $\Omega \subset \mathbb{R}^N$, be a domain, $N \geq 2$, and $x_0 \in \Omega$. Let u be a *bounded* harmonic function in $\Omega_0 := \Omega \backslash \{x_0\}$. Show that we can define a value $u(x_0)$ such that the extended function is harmonic on the whole Ω.

Solution. Let $\rho > 0$ be such that $B_\rho(_0) \subset \Omega$. Consider the unique solution v to the problem

$$\Delta v = 0 \quad \text{in } B_\rho(x_0), \qquad v = u \quad \text{on } \partial B_\rho(x_0).$$

We will see that $v = u$ in $B_\rho(x_0) \backslash \{x_0\}$. Let us prove the result for $N \geq 3$ (the case $N = 2$ is analogous)

$$M_\varepsilon = \|u - v\|_{L^\infty(\partial B_\varepsilon(x_0))} \leq K < \infty.$$

Here, $\varepsilon \in (0, \rho)$ is arbitrary. The two functions

$$w^\pm = M_\varepsilon \left(\frac{\varepsilon}{|x - x_0|} \right)^{N-2} \pm (u - v)$$

are harmonic in the annulus $\{\varepsilon < |x - x_0| < \rho\}$. Moreover, they are both nonnegative for $|x - x_0| = \rho$, since $u = v$ on $\partial B_\rho(x_0)$. On the sphere $|x - x_0| = \varepsilon$ we have

$$w^\pm \Big|_{|x-p|=\varepsilon} = M_\varepsilon \pm (u - v) \Big|_{|x-p|=\varepsilon} \geq 0.$$

By the Maximum Principle we have that for all $\varepsilon < |x - x_0| < \rho$,

$$|u - v|(x) \leq \frac{M_\varepsilon \varepsilon^{N-2}}{|x - x_0|^{N-2}} \leq \frac{K \varepsilon^{N-2}}{|x - x_0|^{N-2}}.$$

The result follows by letting $\varepsilon \to 0$.

Remarks. (i) We can substitute the condition of boundedness of u with $\lim_{x \to x_0} |x - x_0|^{N-2} u(x) = 0$ if $N \geq 3$ or $\lim_{x \to x_0} u(x)/\log|x - x_0| = 0$ if $N = 2$. (ii) When $N \geq 3$ and $\lim_{x \to x_0} |x - x_0|^{N-2} u(x) = C$ we have that $u(x) = v + C|x - x_0|^{2-N}$, where v is an harmonic function in Ω. The result for $N = 2$ is also true.

The proof of the remarks is left as an exercise.

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11. Let $\Omega \subset \mathbb{R}^N$, be a bounded domain, $N \geq 2$, and let $x_0 \in \Omega$. Define $\Omega_0 := \Omega \setminus \{x_0\}$ and let u and v be two harmonic functions in Ω_0, continuous in $\Omega_1 = \Omega_0 \cup \partial\Omega$ and such that: (i) $u(x) \leq v(x)$ for all $x \in \partial\Omega$; (ii) $|u(x)| \leq M$, $|v(x)| \leq M$ for all $x \in \Omega_1$. Use the Maximum Principle to show that $u(x) \leq v(x)$ for all $x \in \Omega_1$.

Solution. Using Problem 10, we can extend $u - v$ to an harmonic funtion in Ω. Aa a consequence, since $u - v \leq 0$ in $\partial\Omega$, by the Maximum Principle we have that $u \leq v$ in Ω.

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12. Find an expression for the Green function of the Dirichlet problem for the Laplace equations in an annular region $B_R(x_0) \setminus B_r(x_0)$, with $0 < r < R$.

Solution. Assume that the region is $\Omega = B_R(0) \setminus B_r(0)$. Let

$$\lambda = \frac{r}{R} < 1, \qquad \bar{x} = \frac{R^2 x}{|x|^2}, \qquad \tilde{x} = \frac{r^2 x}{|x|^2}.$$

Consider first

$$h_1 = \Gamma(|x - y|) - \Gamma(\frac{|x|}{R}|\bar{x} - y|), \quad x, y \in \Omega,\ x \neq y.$$

Then you shall look at the web page:

`http://sunlimingbit.wordpress.com/2013/05/24/green-function-for-annular-region/`

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13. Show that a solution to $\Delta u - u^2 = 0$ in a domain Ω cannot attain its maximum in Ω, except if $u \equiv 0$.

Solution. First notice that u is subharmonic, since $-\Delta u = -u^2 \leq 0$. As a consequence, if u attains the maximum in Ω, then necessarily u is a constant function, hence $-\Delta u = 0$. Finally, since u solves $\Delta u = u^2$, it follows that $u \equiv 0$.

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14. Let $u \in C^2(B_1(0)) \cap C(\overline{B_1(0)})$ be a solution to the Dirichlet problem

$$\begin{cases} \Delta u = u^2 + f(|x|), & x \in B_1(0), \\ u(x) = 1, & x \in \partial B_1(0), \end{cases}$$

where $f(|x|) \geq 0$ is of class $C^1(\Omega)$. Calculate the maximum of u in $\overline{B_1(0)}$ and show that it does not depend on f.

Solution. We have that $-\Delta u = -u^2 - f(|x|) \leq 0$, hence u is subharmonic; as a consequence u attains the maximum on the boundary, $\max_\Omega u = \max\limits_{\partial\Omega} u = 1$.
