Hoja de problemas 2: SOBOLEV SPACES

1. Study Hölder regularity of the functions for all $\alpha > 0$

$$f_\alpha(x) = \begin{cases} x^\alpha \operatorname{sen}(1/x), & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$$

2. Let $\alpha \in (0,1)$ and consider the function

$$u(x) = (1+x^2)^{-\alpha/2}(\log(2+x^2))^{-1}, \qquad x \in \mathbb{R}.$$

Show that $u \in W^{1,p}(\mathbb{R})$ for any $p \in [1/\alpha, \infty]$, and that $u \notin L^q(\mathbb{R})$ when $q \in [1, 1/\alpha)$.

3. Let $\Omega = \{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$ and

$$u(x) = \begin{cases} 1 - x_1 & \text{si } x_1 > 0,\ |x_2| < x_1, \\ 1 + x_1 & \text{si } x_1 < 0,\ |x_2| < -x_1, \\ 1 - x_2 & \text{si } x_2 > 0,\ |x_1| < x_2, \\ 1 + x_2 & \text{si } x_2 < 0,\ |x_1| < -x_2. \end{cases}$$

Find the values of p, $1 \leq p \leq \infty$, such that $u \in W^{1,p}(\Omega)$.

4. Let $N > 1$. Check that the unbounded fucntion $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$ lies in $W^{1,n}(B_1(0))$.

5. Let $\Omega \subseteq \mathbb{R}^N$ be open and connected and $u \in W^{1,p}(\Omega)$. Show that if $Du = 0$ a.e. in Ω, then u is constant a.e. in Ω.

6. (Fundamental Theorem of Calculus) Let $I \subset \mathbb{R}$ an interval (not necessarily bounded). Let $g \in L^1_{\text{loc}}(I)$. For any fixed $y_0 \in I$ we define

$$v(x) = \int_{y_0}^{x} g(t)\,dt, \quad x \in I.$$

Prove that $v \in C(I)$ and that

$$\int_I v\varphi' = -\int_I g\varphi \quad \text{for any } \varphi \in C^1_{\text{c}}(I).$$

7. Let $I \subset \mathbb{R}$ an interval (not necessarily bounded). Let $u \in W^{1,p}(I)$, $1 \leq p \leq \infty$. Prove that there exists a function $\tilde{u} \in C(\overline{I})$ such that $u = \tilde{u}$ a.e. in I, and that moreover we have

$$\tilde{u}(x) - \tilde{u}(y) = \int_x^y u'(t)\,dt \quad \text{para todo } x, y \in \overline{I}.$$

Hint. Use the two previous exercises

8. Let $u, v \in H^1(\mathbb{R})$. Show that
$$\int_{\mathbb{R}} uv' = -\int_{\mathbb{R}} u'v.$$

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9. (Leibnitz rule in Sobolev Spaces) Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Show that $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and that
$$\partial_{x_i}(uv) = v\partial_{x_i}u + u\partial_{x_i}v, \quad i = 1, \dots, n.$$

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10. (Chain Rule) Let $F : \mathbb{R} \to \mathbb{R}$ a C^1 function with bounded F' and $F(0) = 0$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Let $u \in W^{1,p}(\Omega)$ for some p, $1 \leq p \leq \infty$. Show that $v = F(u)$ lies in $W^{1,p}(\Omega)$ and that $v_{x_i} = F'(u)u_{x_i}$, $i = 1, \dots, n$.

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11. Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, and let $1 \leq p \leq \infty$.
    (a) Prove that $u \in W^{1,p}(\Omega)$, implies $|u| \in W^{1,p}(\Omega)$.
    (b) Prove that $u \in W^{1,p}(\Omega)$ implies $u^+, u^- \in W^{1,p}(\Omega)$, with
$$Du^+ = \begin{cases} Du & \text{a.e. in } \{u > 0\}, \\ 0 & \text{a.e. in } \{u \leq 0\}, \end{cases}$$
$$Du^- = \begin{cases} 0 & \text{a.e. in } \{u \geq 0\}, \\ -Du & \text{a.e. in } \{u < 0\}. \end{cases}$$

    Hint. $u^+ = \lim_{\varepsilon \to 0} F_\varepsilon(u)$, where
$$F_\varepsilon(z) = \begin{cases} (z^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if } z \geq 0, \\ 0 & \text{if } z < 0. \end{cases}$$

    (c) Prove that if $u \in W^{1,p}(\Omega)$, then $Du = 0$ a.e. on the set $\{u = 0\}$.

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12. Let $\Omega \subset \mathbb{R}^N$ an open set with C^1 boundary. Show by means of an example that $L^p(\Omega)$ functions, with $p \in [1, \infty)$, do not necessarily have a trace on $\partial\Omega$. More precisely, show that there can not exist a linear bounded operator $T : L^p(\Omega) \to L^p(\partial\Omega)$ such that $Tu = u_{|\partial\Omega}$ for all $u \in C(\overline{\Omega}) \cap L^p(\Omega)$.

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13. (a) Show that there does not exists any constant $C > 0$ such that
$$\int_{\mathbb{R}^N} u^2 \leq C \int_{\mathbb{R}^N} |\nabla u|^2 \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

    (b) (Hardy Inequality) For all $N \geq 3$ there exists $C > 0$ such that
$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\, dx \leq C \int_{\mathbb{R}^N} |\nabla u|^2\, dx \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

    Hint. $|\nabla u + \lambda \frac{x}{|x|^2} u|^2 \geq 0$ for all $\lambda \in \mathbb{R}$.

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14. Let $\alpha > 0$. Show that there exists $C = C(N, \alpha) > 0$ so that

$$\int_{B_1(0)} u^2 \le C \int_{B_1(0)} |\nabla u|^2$$

for all $u \in H^1(B_1(0))$ such that $|\{x \in B_1(0) : u(x) = 0\}| \ge \alpha$.

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15. (Friedrichs' Inequality) Let $\Omega \subset \mathbb{R}^N$ be an open connected domain, with smooth boundary and let $\Gamma \subset \partial\Omega$ a set with positive $(N-1)$-dimensional measure. Show that there exists a constant $C > 0$ so that

$$\|u\|_{H^1(\Omega)}^2 \le C\left(\|u\|_{L^2(\Gamma)}^2 + \|\nabla u\|_{L^2(\Omega)}^2\right) \qquad \forall\ u \in H^1(\Omega).$$

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16. Integrate by parts to prove the following inequality

$$\|Du\|_{L^2} \le C\|u\|_{L^2}^{1/2}\|D^2u\|_{L^2}^{1/2} \qquad \text{for all } u \in C_c^\infty(\Omega).$$

Prove also that the inequality holds for $u \in H^2(\Omega) \cap H_0^1(\Omega)$ if Ω is a bounded domain with smooth boundary.

Hint. Take two sequences $\{v_k\}_{k=1}^\infty \subset C_c^\infty(\Omega)$ converging to u in $H_0^1(\Omega)$ and $\{w_k\}_{k=1}^\infty$ converging to u in $H^2(\Omega)$.

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17. (Gagliardo-Nirenberg Inequality – First form, dimension $N = 1$) Let $\Omega = (0, 1)$.

(a) Let $1 \le q < \infty$ and $1 < r \le \infty$. Show that

$$\|u\|_{L^\infty(\Omega)} \le C\|u\|_{W^{1,r}(\Omega)}^a\|u\|_{L^q(\Omega)}^{1-a} \quad \text{para toda } u \in W^{1,r}(\Omega)$$

for some constant $C = C(q, r) > 0$, where $a \in (0, 1)$ is given by

$$a\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q}.$$

Hint. Begin with the case $u(0) = 0$ write $G(u(x)) = \int_0^x G'(u(t))u'(t)\,dt$, where $G(t) = |t|^{\alpha-1}t$ and $\alpha = 1/a$. When $u(0) \ne 0$, use the above inequality with ζu, where $\zeta \in C^1([0, 1])$, $\zeta(0) = 0$, $\zeta(t) = 1$ for all $t \in [1/2, 1]$.

(b) Let $1 \le q < p < \infty$ y $1 \le r \le \infty$. Show that

$$\|u\|_{L^p(\Omega)} \le C\|u\|_{W^{1,r}(\Omega)}^b\|u\|_{L^q(\Omega)}^{1-b} \quad \text{para toda } u \in W^{1,r}(\Omega)$$

for some constant $C = C(p, q, r) > 0$, where $b \in (0, 1)$ is given by

$$b\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q} - \frac{1}{p}.$$

Hint. Write $\|u\|_{L^p(\Omega)}^p = \int_\Omega |u|^q|u|^{p-q} \le \|u\|_{L^q(\Omega)}^q\|u\|_{L^\infty(\Omega)}^{p-q}$ and use part (a) when $r > 1$.

(c) Under the same assumptions as in part (b), show that

$$\|u\|_{L^p(\Omega)} \le C\|u'\|_{L^r(\Omega)}^b\|u\|_{L^q(\Omega)}^{1-b} \quad \text{for all } u \in W^{1,r}(\Omega) \text{ tal que } \int_\Omega u = 0.$$
