1. Study Hölder regularity of the functions for all $\alpha > 0$

$$f_\alpha(x) = \begin{cases} x^\alpha \operatorname{sen}(1/x), & 0 < x \leq 1, \\ 0, & x = 0. \end{cases}$$

Solution. A simple calculation shows that when $\alpha \geq 2$ we have $f_\alpha \in C^1([0,1])$, in particular f_α is Lipschitz. Therefore we consider the case $\alpha \in (0,2)$.
Let $x_n = \frac{1}{(1+\frac{1}{n})\pi}$, $y_n = \frac{1}{n\pi}$, so that

$$\frac{|f(x_n) - f(y_n)|}{|x_n - y_n|^\gamma} = 2^\gamma \pi^{\gamma-\alpha} n^{2\gamma-\alpha} \left(1 + \frac{1}{2n}\right)^{\gamma-\alpha} \to \infty \quad \text{cuando } n \to \infty \quad \text{si } \gamma > \alpha/2.$$

As a consequence, the Hölder exponent has to be at most $\alpha/2$. Let us check that it is exactly $\alpha/2$: consider the function

$$\begin{aligned} \phi(x) &= \left(x^\alpha \sin \tfrac{1}{x} - a^\alpha \sin \tfrac{1}{a}\right)^{2/\alpha} = \phi(x) - \phi(a) \\ &= \tfrac{2}{\alpha} \left(\xi^\alpha \sin \tfrac{1}{\xi} - a^\alpha \sin \tfrac{1}{a}\right)^{\frac{2}{\alpha}-1} \left(\alpha \xi^{\alpha-1} \sin \tfrac{1}{\xi} - \xi^{\alpha-2} \cos \tfrac{1}{\xi}\right)(x-a), \end{aligned}$$

where we have used the Mean Value Theorem with $0 \leq a < \xi < x \leq 1$. As a consequence,

$$\begin{aligned} \frac{\phi(x)}{x-a} &= \tfrac{2}{\alpha} \left(\alpha^{\frac{\alpha}{2-\alpha}} \xi^{\frac{2}{2-\alpha}} \sin^{\frac{2}{2-\alpha}} \tfrac{1}{\xi} - \left(\tfrac{a}{\xi}\right)^\alpha \xi^{\frac{2}{2-\alpha}} \sin \tfrac{1}{a} \sin^{\frac{\alpha}{2-\alpha}} \tfrac{1}{\xi}\right)^{\frac{2-\alpha}{\alpha}} \\ &\quad - \tfrac{2}{\alpha} \left(\xi^{\frac{\alpha}{2-\alpha}} \sin \tfrac{1}{\xi} \cos^{\frac{\alpha}{2-\alpha}} \tfrac{1}{\xi} - \left(\tfrac{a}{\xi}\right)^\alpha \sin \tfrac{1}{a} \cos^{\frac{\alpha}{2-\alpha}} \tfrac{1}{\xi}\right)^{\frac{2-\alpha}{\alpha}} \leq C, \end{aligned}$$

the result follows.

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2. Let $\alpha \in (0,1)$ and consider the function

$$u(x) = (1+x^2)^{-\alpha/2} (\log(2+x^2))^{-1}, \qquad x \in \mathbb{R}.$$

Show that $u \in W^{1,p}(\mathbb{R})$ for any $p \in [1/\alpha, \infty]$, and that $u \notin L^q(\mathbb{R})$ when $q \in [1, 1/\alpha)$.

Solution. We will use the following statements, whose proof is left as an (easy) exercise

$$\int_1^\infty \frac{dx}{x^\alpha} < \infty \Leftrightarrow \alpha > 1, \qquad \int_2^\infty \frac{dx}{x \log^\beta x} < \infty \Leftrightarrow \beta > 1.$$

On one hand, the function u is clearly bounded for any $\alpha \in (0,1)$, $u \in L^\infty(\mathbb{R})$.
On the other hand, if $p \in (1/\alpha, \infty)$,

$$\int_{\mathbb{R}} |u|^p = 2\int_0^1 |u|^p + 2\int_1^\infty |u|^p = C + 2\int_1^\infty \frac{dx}{(1+x^2)^{\alpha p/2} (\log(2+x^2))^p} \leq C + \frac{2}{\log^p 2} \int_1^\infty \frac{dx}{x^{\alpha p}} < \infty.$$

The critical case, $p = 1/\alpha$,

$$\int_{\mathbb{R}} |u|^p = C + 2\int_2^\infty \frac{dx}{(1+x^2)^{1/2} (\log(2+x^2))^{1/\alpha}} \leq C + \frac{2}{2^{1/\alpha}} \int_2^\infty \frac{dx}{x \log^{1/\alpha} x} < \infty.$$

Finally, the derivatives

$$u'(x) = -\alpha(1+x^2)^{-\frac{\alpha}{2}-1} x (\log(2+x^2))^{-1} - (1+x^2)^{-\alpha/2} (\log(2+x^2))^{-2} \frac{2x}{2+x^2},$$

has a decay at infinity which is faster than the function u, therefore it lies (at least) in the same $L^p(\mathbb{R})$ space as u. We can conclude that $u \in W^{1,p}(\mathbb{R})$, $p \in [1/\alpha, \infty]$.

If $q \in [1, 1/\alpha)$, taking $\varepsilon > 0$ so that $\alpha q + \varepsilon < 1$ (we can do it since $\alpha q < 1$),

$$\int_{\mathbb{R}} |u|^q = C + 2\int_2^\infty \frac{dx}{(1+x^2)^{\alpha q/2}(\log(2+x^2))^q} \geq C + C\int_2^\infty \frac{dx}{x^{\alpha q+\varepsilon}} = \infty.$$

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3. Let $\Omega = \{x \in \mathbb{R}^2 : |x_1| < 1, |x_2| < 1\}$ and

$$u(x) = \begin{cases} 1 - x_1 & \text{si } x_1 > 0,\ |x_2| < x_1, \\ 1 + x_1 & \text{si } x_1 < 0,\ |x_2| < -x_1, \\ 1 - x_2 & \text{si } x_2 > 0,\ |x_1| < x_2, \\ 1 + x_2 & \text{si } x_2 < 0,\ |x_1| < -x_2. \end{cases}$$

Find the values of p, $1 \leq p \leq \infty$, such that $u \in W^{1,p}(\Omega)$.

Solution #1. It is trivial to check that $u \in L^\infty(\Omega)$, with $\|u\|_{L^\infty(\Omega)} = 1$. Since Ω is a bounded domain, then $u \in L^p(\Omega)$, for all $1 \leq p \leq \infty$.

Given $\phi \in C_c^\infty(\Omega)$ we have that

$$\int_\Omega u\partial_{x_1}\phi = \sum_{j=1}^4 \int_{T_j} u\partial_{x_1}\phi = -\sum_{j=1}^4 \int_{T_j} \phi\partial_{x_1}u + \sum_{j=1}^4 \int_{\partial T_j} u\phi \mathrm{e}_1 \cdot \nu^j,$$

where ν^j is the unit exterior normal to T_j in ∂T_j. On one hand we get

$$-\sum_{j=1}^4 \int_{T_j} \phi\partial_{x_1}u = -\int_\Omega (-\chi_{T_1} + \chi_{T_2})\phi.$$

On the other hand, since ϕ is compactly supported in Ω, and observing that T_i and T_j have a common a side, it follows that $\nu^i = -\nu^j$ on that common side,

$$\sum_{j=1}^4 \int_{\partial T_j} u\phi \mathrm{e}_1 \cdot \nu^j = 0.$$

We conclude

$$\int_\Omega u\partial_{x_1}\phi = -\int_\Omega (-\chi_{T_1} + \chi_{T_2})\phi,$$

that is

$$\partial_{x_1}u = -\chi_{T_1} + \chi_{T_2}$$

in the distributional sense. We notice that $\partial_{x_1}u \in L^\infty(\Omega)$. As a consequence, since the domain is bounded, $\partial_{x_1}u \in L^p(\Omega)$, $1 \leq p \leq \infty$. The same holds for $\partial_{x_2}u$ (simply by switching x_1 and x_2), we conclude that $u \in W^{1,p}(\Omega)$ for all $1 \leq p \leq \infty$.

Solution #2. It is easy to check that $\text{mín}\{f,g\} = -\{f-g\}_+ + f$. We know that $h \in W^{1,p}(\Omega)$, therefore $\{h\}_+ \in W^{1,p}(\Omega)$ (cf. Problem 11), and we can conclude that if $f, g \in W^{1,p}(\Omega)$, then $\text{mín}\{f,g\} \in W^{1,p}(\Omega)$.

The function u satisfies

$$u(x_1,x_2)) = \text{mín}\{1-x_1, 1+x_1, 1-x_2, 1+x_2\}.$$

Being the minimum of $W^{1,\infty}(\Omega)$ functions, it lies in the same space, hence in all $W^{1,p}(\Omega)$, with $1 \leq p \leq \infty$, since Ω is bounded.

Remark. The same holds true also for $\text{máx}\{f,g\} = \{f-g\}_+ + g$: indeed, if $f, g \in W^{1,p}(\Omega)$, then $\text{máx}\{f,g\} \in W^{1,p}(\Omega)$.

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4. Let $N > 1$. Check that the unbounded fucntion $u(x) = \log\log\left(1 + \frac{1}{|x|}\right)$ lies in $W^{1,n}(B_1(0))$.

Solution. Change to polar coordinates:

$$\int_{B_1(0)} |u|^n = C\int_0^1 r^{n-1}|\log\log(1+\frac{1}{r})|^n\,dr < \infty \quad \text{si } n \geq 2,$$

the function under integral has a continuous extension on the whole interval $[0,1]$, since its limit as $r \to 0^+$ is 0).

A simple calculation shows that

$$\partial_{x_i} u(x) = -\frac{x_i}{(|x|^3 + |x|^2)\log(1+\frac{1}{|x|})} \quad \text{si } x \neq 0.$$

Change again to polar coordinates

$$\int_{B_1(0)} \left|\frac{x_i}{(|x|^3 + |x|^2)\log(1+\frac{1}{|x|})}\right|^n dx \leq C\int_0^1 \frac{r^{n-1}}{\left((r^2+r)\log(1+\frac{1}{r})\right)^n}\,dr \leq C - \int_0^{1/2} \frac{dr}{r\log^n r} < \infty,$$

if $N \geq 2$.

Let $T_k u(x) = \min\{u(x), k\}$. for each constant $k \geq 0$ this function is in $W^{1,n}(B_1(0))$, its weak derivatives are

$$\partial_{x_i} T_k u(x) = -\chi_{\{u<k\}}\frac{x_i}{(|x|^3 + |x|^2)\log(1+\frac{1}{|x|})},$$

which are functions belonging to $L^n(B_1(0))$ uniformly in k, and also to $L^1(B_1(0))$. By Dominated convergence, the limit as $k \to \infty$ becomes

$$\int_{B_1(0)} T_k u \partial_{x_i}\phi = -\int_{B_1(0)} \phi\partial_{x_i} T_k u$$

from which we deduce

$$\partial_{x_i} u(x) = -\frac{x_i}{(|x|^3 + |x|^2)\log(1+\frac{1}{|x|})} \quad \text{en } \mathcal{D}'(B_1(0)),$$

which concludes the proof.

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5. Let $\Omega \subseteq \mathbb{R}^N$ be open and connected and $u \in W^{1,p}(\Omega)$. Show that if $Du = 0$ a.e. in Ω, then u is constant a.e. in Ω.

Solution. For any $\varepsilon > 0$ consider the regularization $u^\varepsilon = \eta_\varepsilon \star u$, and we know that $u^\varepsilon : \Omega_\varepsilon \mapsto \mathbb{R} \in C^\infty(\Omega_\varepsilon)$. Its first order derivatives, $\partial^\alpha u^\varepsilon = \eta_\varepsilon \star \partial^\alpha u$, $|\alpha| = 1$, are also zero on Ω_ε. As a consequence u is constant on each connected component of Ω_ε.

Let $x, y \in \Omega$. Since Ω open and connected, there is a continuous path $\Gamma \subset \Omega$ joining x and y. Let $\delta = \min_{z\in\Gamma} \operatorname{dist}(z, \partial\Omega)$. For all $\varepsilon < \delta$ the whole path Γ lies in Ω_ε, hence x and y lie in the same connected component of Ω_ε. Therefore, $u^\varepsilon(x) = u^\varepsilon(y)$.

Let $\tilde{u}(x) = \lim_{\varepsilon\to 0} u^\varepsilon(x)$. As a consequence of the above results, $\tilde{u}$ is constant in Ω. We also know that $\tilde{u}(x) = u(x)$ a.e. in Ω, and the proof is concluded.

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6. (Fundamental Theorem of Calculus) Let $I \subset \mathbb{R}$ an interval (not necessarily bounded). Let $g \in L^1_{\text{loc}}(I)$. For any fixed $y_0 \in I$ we define

$$v(x) = \int_{y_0}^x g(t)\,dt, \quad x \in I.$$

Prove that $v \in C(I)$ and that

$$\int_I v\varphi' = -\int_I g\varphi \quad \text{for any } \varphi \in C^1_c(I).$$

Solution. We have that

$$\begin{aligned} \int_I v\varphi' &= \int_I \left(\int_{y_0}^x g(t)\,dt\right)\varphi'(x) \\ &= -\int_a^{y_0}\left(\int_x^{y_0} g(t)\varphi'(x)\,dt\right)dx + \int_{y_0}^b\left(\int_{y_0}^x g(t)\varphi'(x)\,dt\right)dx. \end{aligned}$$

By Fubini's Theorem,

$$\begin{aligned} \int_I v\varphi' &= -\int_a^{y_0} g(t)\left(\int_a^t \varphi'(t)\,dx\right)dt + \int_{y_0}^b g(t)\left(\int_t^b \varphi'(x)\,dx\right)dt \\ &= -\int_I g(t)\varphi(t)\,dt. \end{aligned}$$

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7. Let $I \subset \mathbb{R}$ an interval (not necessarily bounded). Let $u \in W^{1,p}(I)$, $1 \le p \le \infty$. Prove that there exists a function $\tilde{u} \in C(\overline{I})$ such that $u = \tilde{u}$ a.e. in I, and that moreover we have

$$\tilde{u}(x) - \tilde{u}(y) = \int_x^y u'(t)\,dt \quad \text{para todo } x, y \in \overline{I}.$$

Hint. Use the two previous exercises

Solution. Fix $y_0 \in I$ and let $\bar{u}(x) = \int_{y_0}^x u'(t)\,dt$. Thanks to the previous exercise, we have

$$\int_I \bar{u}\varphi' = -\int_I u'\varphi \qquad \forall \varphi \in C^1_c(I).$$

As a consequence, $\int_I (u - \bar{u})\varphi' = 0$ for all $\varphi \in C^1_c(I)$. Thanks to Problem 5, $u - \bar{u} = C$ a.e. in I. The function $\tilde{u} = \bar{u} + C$ has the required properties.

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8. Let $u, v \in H^1(\mathbb{R})$. Show that

$$\int_{\mathbb{R}} uv' = -\int_{\mathbb{R}} u'v.$$

Solution. If $u \in H^1(\mathbb{R})$ and $v \in C^\infty_c(\mathbb{R})$, the identity is nothing but the definition of distributional derivative of u. For the general case, $v \in H^1(\mathbb{R})$, let us take a sequence $\{v_n\} \subset C^\infty_c(\mathbb{R})$ so that $v_n \to v$ en $H^1(\mathbb{R})$. We obtain the result just by taking the limits in

$$\int_{\mathbb{R}} uv_n' = -\int_{\mathbb{R}} u'v_n.$$

Remark. The very same proof works in ANY dimension $N \ge 1$.

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9. (Leibnitz rule in Sobolev Spaces) Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Show that $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and that
$$\partial_{x_i}(uv) = v\partial_{x_i}u + u\partial_{x_i}v, \quad i = 1, \dots, n.$$

Solution. Let $\{u_n\}, \{v_k\} \subset C_c^\infty(\Omega)$ such that $u_n \to u$, $v_k \to v$ en $W^{1,p}_{\text{loc}}(\Omega)$, $\|u_n\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)}$, $\|v_k\|_{L^\infty(\Omega)} \leq \|v\|_{L^\infty(\Omega)}$. We immediately get
$$-\int_\Omega u_n v_k \partial_{x_i}\phi = \int_\Omega \partial_{x_i}(u_n v_k)\phi = \int_\Omega (v_k \partial_{x_i} u_n + u_n \partial_{x_i} v_k)\phi.$$
Taking the limits, first in n then in k at the first and last terms of the above inequality, we obtain
$$-\int_\Omega uv\partial_{x_i}\phi = \int_\Omega (v\partial_{x_i}u + u\partial_{x_i}v)\phi,$$
This means that we satisfy Leibnitz rule for the derivative of a product, in the distributional sense. We then take the limit, recalling that the product of a bounded function with a function of $C_c^\infty(\Omega)$ lies $L^{p'}$.

Once we have checked the identity in the distributional sense, we conclude by recalling that the product of a bounded function (in L^∞) with a function of L^p is still in L^p.

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10. (Chain Rule) Let $F : \mathbb{R} \to \mathbb{R}$ a C^1 function with bounded F' and $F(0) = 0$. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set. Let $u \in W^{1,p}(\Omega)$ for some p, $1 \leq p \leq \infty$. Show that $v = F(u)$ lies in $W^{1,p}(\Omega)$ and that $v_{x_i} = F'(u)u_{x_i}$, $i = 1, \dots, n$.

Solution. Given $\phi \in C_c^\infty(\Omega)$, there is a sequence $\{u_n\} \subset C^\infty(\Omega)$ so that $u_n \to u$ in $W^{1,p}(\text{sop}\,\phi)$ and $u_n \to u$ a.e. in Ω. We then have
$$-\int_\Omega F(u_n)\partial_{x_i}\phi = \int_\Omega \phi F'(u_n)\partial_{x_i}u_n. \tag{1}$$
Moreover,
$$\left|\int_\Omega (F(u_n) - F(u))\partial_{x_i}\phi\, dx\right| \leq \|\partial_{x_i}\phi\|_\infty \sup|F'| \int_{\text{sop}\,\phi} |u_n - u| dx \to 0 \quad \text{cuando } n \to \infty.$$
We also have
$$\begin{aligned}&\left|\int_\Omega (F'(u_n)\partial_{x_i}u_n - F'(u)\partial_{x_i}u)\phi\, dx\right| \\ &\quad \leq \|\phi\|_\infty \sup|F'| \int_{\text{sop}\,\phi} |\partial_{x_i}u_n - \partial_{x_i}u|\, dx + \int_{\text{sop}\,\phi} |F'(u_n) - F'(u)||Du|\, dx \to 0 \quad \text{cuando } n \to \infty.\end{aligned}$$
We have used Dominated Convergence together with the pointwise convergence of $|F'(u_n) - F'(u)|$ to 0, in order to prove the convergence of the second term in the right-hand side. Take the limit in (1), to get
$$-\int_\Omega F(u)\partial_{x_i}\phi = \int_\Omega \phi F'(u)\partial_{x_i}u,$$
which is equivalent to $v_{x_i} = F'(u)\partial_{x_i}u$. Under our assumptions on F and u, we know that the right-hand side is in $L^p(\Omega)$, therefore also $v_{x_i} \in L^p(\Omega)$.

Finally,
$$\int_\Omega |v|^p = \int_\Omega |F(u) - F(0)|^p \leq (\sup|F'|)^p \int_\Omega |u|^p < \infty,$$
which gives the result.

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11. Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain, and let $1 \le p \le \infty$.

(a) Prove that $u \in W^{1,p}(\Omega)$, implies $|u| \in W^{1,p}(\Omega)$.

(b) Prove that $u \in W^{1,p}(\Omega)$ implies $u^+, u^- \in W^{1,p}(\Omega)$, with

$$Du^+ = \begin{cases} Du & \text{a.e. in } \{u > 0\}, \\ 0 & \text{a.e. in } \{u \le 0\}, \end{cases}$$
$$Du^- = \begin{cases} 0 & \text{a.e. in } \{u \ge 0\}, \\ -Du & \text{a.e. in } \{u < 0\}. \end{cases}$$

Hint. $u^+ = \text{lím}_{\varepsilon \to 0} F_\varepsilon(u)$, where

$$F_\varepsilon(z) = \begin{cases} (z^2 + \varepsilon^2)^{1/2} - \varepsilon & \text{if } z \ge 0, \\ 0 & \text{if } z < 0. \end{cases}$$

(c) Prove that if $u \in W^{1,p}(\Omega)$, then $Du = 0$ a.e. on the set $\{u = 0\}$.

Solution. It is sufficient to prove part (b). Parts (a) and (c) follow immediately, since $|u| = u^+ + u^-$ and $u = u^+ - u^-$.

Let us show part (b). It is sufficient to prove it for u^+, since $u^- = (-u)^+$. Following the hint, we use the Chain Rule of Exercise 10, with $\phi \in C_c^\infty(\Omega)$

$$\int_\Omega F_\varepsilon(u)\partial_{x_i}\phi \, dx = -\int_{\{u>0\}} \phi \frac{u\partial_{x_i} u}{(u^2+\varepsilon^2)^{\frac{1}{2}}} \, dx.$$

Letting $\varepsilon \to 0$ and using Dominated Convergence, we get

$$\int_\Omega u^+ \partial_{x_i}\phi \, dx = -\int_{\{u>0\}} \phi \partial_{x_i} u \, dx.$$

This concludes the proof.

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12. Let $\Omega \subset \mathbb{R}^N$ an open set with C^1 boundary. Show by means of an example that $L^p(\Omega)$ functions, with $p \in [1, \infty)$, do not necessarily have a trace on $\partial\Omega$. More precisely, show that there can not exist a linear bounded operator $T : L^p(\Omega) \to L^p(\partial\Omega)$ such that $Tu = u_{|\partial\Omega}$ for all $u \in C(\overline{\Omega}) \cap L^p(\Omega)$.

Solution. Let us show a counterexample in dimension $N = 1$. We want to show that there there does not exists a constant $C > 0$ such that $\|Tu\|_{L^p(\partial\Omega)} \le C\|u\|_{L^p(\Omega)}$ for all $u \in C(\overline{\Omega}) \cap L^p(\Omega)$. Assume by contradiction that this holds true. Choose a family of continuous functions on $\Omega = (0, 1)$ given by

$$f_n(x) = n^{\alpha+1} \left\{ \frac{1}{n^\alpha} - x \right\}_+ .$$

We have that

$$\int_0^1 |f_n|^p \le n^{p-\alpha} = 1 \quad \text{si } \alpha = p.$$

But we also have

$$\|Tf_n\|^p_{L^p(\partial\Omega)} = |f_n(0)|^p + |f_n(1)|^p = n^p,$$

which clearly contradicts the hypothesis.

Analogous counterexamples can be constructed in any dimension $N > 1$.

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13. (a) Show that there does not exists any constant $C > 0$ such that

$$\int_{\mathbb{R}^N} u^2 \leq C \int_{\mathbb{R}^N} |\nabla u|^2 \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

(b) (Hardy Inequality) For all $N \geq 3$ there exists $C > 0$ such that

$$\int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx \leq C \int_{\mathbb{R}^N} |\nabla u|^2\,dx \quad \text{for all } u \in H^1(\mathbb{R}^N).$$

Hint. $|\nabla u + \lambda \frac{x}{|x|^2} u|^2 \geq 0$ for all $\lambda \in \mathbb{R}$.

Solution. (a) Let $\zeta \in C^\infty(\mathbb{R}^N)$, be so that $\zeta \geq 0$, $\zeta(x) = 1$ if $|x| \leq 1$, $\zeta(x) = 0$ if $|x| \geq 2$. Define $\zeta_k(x) = \zeta(x/k)$. If there would exist $C > 0$ for all functions of $H^1(\mathbb{R}^N)$, we shall have

$$\int_{\mathbb{R}^N} \zeta^2(x/k)\,dx \leq \frac{C}{k^2} \int_{\mathbb{R}^N} |\nabla \zeta|^2(x/k)\,dx \quad \text{for all } k.$$

Changing variables $x = ky$,

$$\int_{\mathbb{R}^N} \zeta^2 \leq \frac{C}{k^2} \int_{\mathbb{R}^N} |\nabla \zeta|^2 \quad \text{for all } k.$$

We can let $k \to \infty$ to get a contradiction.

(b) Follow the hint and expand the square:

$$0 \leq \int_{\mathbb{R}^N} \left| \nabla u + \lambda \frac{x}{|x|^2} u \right|^2 dx = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + \frac{\lambda x \cdot \nabla(u^2)}{|x|^2} + \lambda^2 \frac{u^2}{|x|^2} \right) dx.$$

Recalling that $\nabla \cdot \left(\frac{x}{|x|^2}\right) = \frac{N-2}{|x|^2}$, we obtain

$$0 \leq \int_{\mathbb{R}^N} |\nabla u|^2 - \lambda(N-2) \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx + \lambda^2 \int_{\mathbb{R}^N} \frac{u^2}{|x|^2}\,dx.$$

The (positive) minimum of the quadratic polynomial in λ is attained at $\lambda = (N-2)/2$. Substitute this value in the above inequality, gives the result with $C = 4/(N-2)^2$.

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14. Let $\alpha > 0$. Show that there exists $C = C(N, \alpha) > 0$ so that

$$\int_{B_1(0)} u^2 \leq C \int_{B_1(0)} |\nabla u|^2$$

for all $u \in H^1(B_1(0))$ such that $|\{x \in B_1(0) : u(x) = 0\}| \geq \alpha$.

Solution. Let $B = B_1(0)$ and $A = \{x \in B : u(x) = 0\}$. Using Poincaré inequality, we know that there exists $C > 0$ so that

$$C\|\nabla u\|_{L^2(B)} \geq \left\| u - \frac{1}{|B|}\int_B u \right\|_{L^2(B)} \geq \left| \|u\|_{L^2(B)} - \left\| \frac{1}{|B|}\int_B u \right\|_{L^2(B)} \right|.$$

By Hölder inequality,

$$\left\| \frac{1}{|B|}\int_B u \right\|_{L^2(B)}^2 = \frac{1}{|B|}\left(\int_{B\setminus A} u \right)^2 \leq \frac{|B \setminus A|}{|B|} \|u\|_{L^2(B)}^2.$$

As a consequence,

$$C\|\nabla u\|_{L^2(B)} \geq \|u\|_{L^2(B)} \left(1 - \left(\frac{|B| - \alpha}{|B|} \right)^{1/2} \right).$$

The result follows, since $1 - \left(\frac{|B|-\alpha}{|B|}\right)^{1/2} > 0$.

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15. (Friedrichs' Inequality) Let $\Omega \subset \mathbb{R}^N$ be an open connected domain, with smooth boundary and let $\Gamma \subset \partial\Omega$ a set with positive $(N-1)$-dimensional measure. Show that there exists a constant $C > 0$ so that

$$\|u\|^2_{H^1(\Omega)} \leq C\left(\|u\|^2_{L^2(\Gamma)} + \|\nabla u\|^2_{L^2(\Omega)}\right) \qquad \forall\ u \in H^1(\Omega).$$

Solution. We proceed by contradiction. Suppose that the inequality is false. Therefore, for all $k \in \mathbb{N}$ there exists a a function $u_k \in H^1(U)$ such that

$$\|u_k\|^2_{H^1(U)} \geq k\left(\|u_k\|^2_{L^2(\Gamma)} + \|\nabla u_k\|^2_{L^2(U)}\right).$$

Let $v_k = u_k/\|u_k\|_{H^1(U)}$. As a consequence, $\|v_k\|_{H^1(U)} = 1$ and $\|v_k\|^2_{L^2(\Gamma)} + \|\nabla v_k\|^2_{L^2(U)} < 1/k$. We deduce that $v_k \to 0$ in $L^2(\Gamma)$ and that $\partial_{x_i} v_k \to 0$ in $L^2(U)$, $i = 1, \ldots, N$. Next, since the sequence $\{v_k\}_{k=1}^{\infty}$ is bounded in $H^1(U)$, using Rellich-Kondrachov Theorem, we can extract a subsequence, that we call $\{v_k\}$ for simplicity, which is convergent in $L^2(U)$ to a limit function v. Let us show that v_k also converges to v in $H^1(U)$. Indeed,

$$\|v_m - v_l\|_{H^1(U)} \leq C\left(\|v_m - v_l\|_{L^2(U)} + \|\nabla v_m\|_{L^2(U)} + \|\nabla v_l\|_{L^2(U)}\right).$$

Since $\{v_k\}$ converges in $L^2(U)$, it is a Cauchy sequence in that space, and since its gradient converges to 0, taking sufficiently big m and l we have that $\|v_m - v_l\|_{H^1(U)}$ can be as small as we want. As a consequence, $v_k \to v$ in $H^1(U)$. This implies that $\nabla v_k \to \nabla v$ in $L^2(U)$. But we already know that $\nabla v_k \to (0, \ldots, 0)$ in $L^2(U)$. Since U is connected, hence v is constant in U.

On the other hand, recall that Γ has positive $(N-1)$-dimensional measure, hence, by trace inequality we get $\|v_k - v\|_{L^2(\Gamma)} \leq C\|v_k - v\|_{H^1(U)}$. As a consequence, $v_k \to v$ in $L^2(\Gamma)$. But we have shown that $v_k \to 0$ in $L^2(\Gamma)$, which implies $v = 0$ in Γ in the trace sense. We deduce that $v = 0$ a.e. in U, and that $v_k \to 0$ in $H^1(U)$. This gives a contradiction, since $\|v_k\|_{H^1(U)} = 1$.

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16. Integrate by parts to prove the following inequality

$$\|Du\|_{L^2} \leq C\|u\|^{1/2}_{L^2}\|D^2u\|^{1/2}_{L^2} \qquad \text{for all } u \in C^\infty_{\mathrm{c}}(\Omega).$$

Prove also that the inequality holds for $u \in H^2(\Omega) \cap H^1_0(\Omega)$ if Ω is a bounded domain with smooth boundary.

Hint. Take two sequences $\{v_k\}_{k=1}^{\infty} \subset C^\infty_{\mathrm{c}}(\Omega)$ converging to u in $H^1_0(\Omega)$ and $\{w_k\}_{k=1}^{\infty}$ converging to u in $H^2(\Omega)$.

Solution. We follow the hint, and we integrate by parts and using Hölder inequality,

$$\sum_{i=1}^{n}\int_\Omega \partial_{x_i}v_k\partial_{x_i}w_k = -\sum_{i=1}^{n}\int_\Omega v_k\partial^2_{x_i}w_k \leq \sum_{i=1}^{n}\|v_k\|_{L^2(\Omega)}\|\partial^2_{x_i}w_k\|_{L^2(\Omega)} \leq C\|v_k\|_{L^2(\Omega)}\|D^2w_k\|_{L^2(\Omega)}.$$

Taking the limit as $k \to \infty$ gives the result.

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17. (Gagliardo-Nirenberg Inequality – First form, dimension $N = 1$) Let $\Omega = (0, 1)$.

(a) Let $1 \leq q < \infty$ and $1 < r \leq \infty$. Show that

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|^a_{W^{1,r}(\Omega)}\|u\|^{1-a}_{L^q(\Omega)} \quad \text{para toda } u \in W^{1,r}(\Omega)$$

for some constant $C = C(q, r) > 0$, where $a \in (0, 1)$ is given by

$$a\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q}.$$

Hint. Begin with the case $u(0) = 0$ write $G(u(x)) = \int_0^x G'(u(t))u'(t)\,dt$, where $G(t) = |t|^{\alpha-1}t$ and $\alpha = 1/a$. When $u(0) \neq 0$, use the above inequality with ζu, where $\zeta \in C^1([0,1])$, $\zeta(0) = 0$, $\zeta(t) = 1$ for all $t \in [1/2, 1]$.

(b) Let $1 \leq q < p < \infty$ y $1 \leq r \leq \infty$. Show that

$$\|u\|_{L^p(\Omega)} \leq C\|u\|^b_{W^{1,r}(\Omega)}\|u\|^{1-b}_{L^q(\Omega)} \quad \text{para toda } u \in W^{1,r}(\Omega)$$

for some constant $C = C(p, q, r) > 0$, where $b \in (0, 1)$ is given by

$$b\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q} - \frac{1}{p}.$$

Hint. Write $\|u\|^p_{L^p(\Omega)} = \int_\Omega |u|^q|u|^{p-q} \leq \|u\|^q_{L^q(\Omega)}\|u\|^{p-q}_{L^\infty(\Omega)}$ and use part (a) when $r > 1$.

(c) Under the same assumptions as in part (b), show that

$$\|u\|_{L^p(\Omega)} \leq C\|u'\|^b_{L^r(\Omega)}\|u\|^{1-b}_{L^q(\Omega)} \quad \text{for all } u \in W^{1,r}(\Omega) \text{ tal que } \int_\Omega u = 0.$$

Solution. (a) Following the hint, using that $G'(t) = \alpha|t|^{\alpha-1}$, and Hölder inequality with conjugate exponents r and r', we get

$$|u(x)|^\alpha = |G(u(x))| \leq \int_0^1 |G'(u(t))|\,|u'(t)|\,dt \leq \alpha\|u'\|_{L^r(\Omega)}\|u\|^{\alpha-1}_{L^{(\alpha-1)r'}(\Omega)}.$$

The result for functions such that $u(0) = 0$ follows immediately, taking $q = (\alpha - 1)r'$, and recalling that $\alpha = 1/a$. The definition of q is equivalent to $a\left(\frac{1}{q} + 1 - \frac{1}{r}\right) = \frac{1}{q}$. Let us notice that we actually get something better: instead of the norm $W^{1,r}$ we get L^r norm of the derivative.

The general case follows again by the hint. Apply the previous case to

$$|(\zeta u)(x)| \leq C\|(\zeta u)'\|^a_{L^r(\Omega)}\|\zeta u\|^{1-a}_{L^q(\Omega)}.$$

Recall that $(\zeta u)' = \zeta' u + \zeta u'$, so that

$$\|(\zeta u)'\|_{L^r(\Omega)} \leq C\left(\|\zeta u'\|_{L^r(\Omega)} + \|\zeta' u\|_{L^r(\Omega)}\right) \leq C\left(\|u'\|_{L^r(\Omega)} + \|u\|_{L^r(\Omega)}\right) \leq C\|u\|_{W^{1,r}(\Omega)}.$$

We also have $\|\zeta u\|_{L^q(\Omega)} \leq C\|u\|_{L^q(\Omega)}$, which leads to

$$|u(x)| = |(\zeta u)(x)| \leq C\|u\|^a_{W^{1,r}(\Omega)}\|u\|^{1-a}_{L^q(\Omega)} \quad \text{si } x \in [1/2, 1].$$

To analyze the other half of the interval, let us consider the function $\tilde{u}(x) = u(1 - x)$ and let us apply the result on $[1/2, 1]$ to $\tilde{u}$. We then get

$$|u(x)| = |\tilde{u}(1 - x)| \leq C\|\tilde{u}\|^a_{W^{1,r}(\Omega)}\|\tilde{u}\|^{1-a}_{L^q(\Omega)} \quad \text{si } x \in [0, 1/2].$$

The result follows once we notice that

$$\|\tilde{u}\|_{W^{1,r}(\Omega)} = \|u\|_{W^{1,r}(\Omega)}, \qquad \|\tilde{u}\|_{L^q(\Omega)} = \|u\|_{L^q(\Omega)}.$$

(b) If $r > 1$, let us just follow the hint, taking $b = a\left(1 - \frac{q}{p}\right)$. If $r = 1$, we use again the hint, but instead of part (a) we now use the Sobolev inequality $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W^{1,1}(\Omega)}$ (NOTICE that we are in DIMENSION $N = 1$), and recall that in this case we have $b = 1 - \frac{p}{q}$.

(c) Combine Poincaré-Wirtinger inequality with the result of part (b) as follows:

$$\left\|u - \frac{1}{|\Omega|}\int_\Omega u\right\|_{L^r(\Omega)} \leq C\|u'\|_{L^r(\Omega)},$$

which implies $\|u\|_{L^r(\Omega)} \leq C\|u'\|_{L^r(\Omega)}$; hence we have $\|u\|_{W^{1,r}(\Omega)} \leq C\|u'\|_{L^r(\Omega)}$, which combined with the result of part (b) proves the claim.
