Asymptotics of the Fast Diffusion Equation via Entropy methods

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(Joint work with *A. Blanchet, J. Dolbeault, G. Grillo* and *J. L. Vázquez*)

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The Cauchy Problem for the Fast Diffusion Equation in $\mathbb{R}^d$

$$\begin{cases} \partial_\tau u = \Delta\left(\dfrac{u^m}{m}\right) = \nabla\cdot\left(u^{m-1}\nabla u\right)\,, & (\tau,y)\in(0,T)\times\mathbb{R}^d \\ u(0,\cdot)=u_0\,, & u_0\in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \end{cases}$$

for any $m < 1$ (i.e. *Fast Diffusion*, FDE)

- We consider non-negative initial data and solutions.
- Note that $m \leq 0$ is included and $m = 0$ corresponds to *logarithmic diffusion.*
- Existence and uniqueness of weak solutions by Herrero and Pierre (1985).
- Solutions have different behaviour if $m_c < m < 1$ and if $m < m_c$, where

$$m_c := \frac{d-2}{d}\,, \qquad \text{and} \qquad m_c > 0 \iff d \geq 3$$

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Basic Properties

- For $m > m_c$ *the mass* $\int_{\mathbb{R}^d} u(y,t)\,\mathrm{d}y$ *is preserved in time if* $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$.
  Non-negative solutions are positive and smooth for all $x \in \mathbb{R}^d$ and $t > 0$.

- If $m < m_c$ *mass is* NOT *preserved* and *solutions may extinguish in finite time.*

$$u_0 \in \mathrm{L}^{p_c}(\mathbb{R}^d)\ ,\quad p_c = \frac{d\,(1-m)}{2} \quad \Longrightarrow \quad \exists\ T = T(u_0)\ :\quad u(\tau,\cdot) \equiv 0 \quad \forall\ t \geq T$$

Semigroup Properties

For any two non-negative solutions u_1 and u_2 of the FDE defined on a time interval $[0,T)$, with initial data in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, we have

1. **L^1-Contractivity**

$$\int_{\mathbb{R}^d} |u_1(t_2) - u_2(t_2)|\,\mathrm{d}x \leq \int_{\mathbb{R}^d} |u_1(t_1) - u_2(t_1)|\,\mathrm{d}x\ ,$$

   for any $0 \leq t_1 \leq t_2 \leq T$.

2. **Comparison Principle**

$$u_{01}(x) \leq u_{02}(x)\ \text{ a.e.} \qquad \Rightarrow \qquad u_1(t,x) \leq u_2(t,x)\ \text{ a.e.}$$

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Barenblatt and Pseudo-Barenblatt Solutions

- When $m < m_c$, assume that u extinguish in finite time T.
- When $m_c < m < 1$, T is a free parameter to be suitably chosen later.

Self-similar Structure

$$U_{D,T}(\tau,y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,m}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}$$

Time Scaling

$$\begin{cases} R(\tau) := \big[d\,(m-m_c)\,(T+\tau)\big]^{\frac{1}{d\,(m-m_c)}} & \text{if } m_c < m < 1, \quad \text{Super-Critical Range} \\ R(\tau) := e^{T+\tau} & \text{if } m_c = m\,, \quad \text{First Critical Exp.} \\ R(\tau) := \big[d\,(m_c-m)\,(T-\tau)\big]^{-\frac{1}{d\,(m_c-m)}} & \text{if } m < m_c\,, \quad \text{Sub-Critical Range} \end{cases}$$

Assumption on the Data

(H1) $u_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, non-negative and there exist positive constants T and $D_0 > D_1$

$$U_{D_0,T}(0,y) \le u_0(y) \le U_{D_1,T}(0,y) \quad \forall\, y \in \mathbb{R}^d\,.$$

(H2) There exist $D_* \in [D_1, D_0]$ and $f \in \mathrm{L}^1(\mathbb{R}^d)$ such that

$$u_0(y) = U_{D_*,T}(0,y) + f(y) \quad \forall\, y \in \mathbb{R}^d\,.$$

- When $m_c < m < 1$, $(H1)$ implies $(H2)$. Moreover in this range we have the

Theorem. *Global Harnack principle* (M.B. - J.L. Vazquez, 2006)

Any solution with non-negative initial data $u_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ that decays at infinity like $u_0(y) = O(|y|^{2/(1-m)})$, is trapped for all $t > 0$ between two Barenblatt solutions.

- If $m_c < m < 1$ we can replace $(H1)$ and $(H2)$ by

$$u_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d) \quad \text{and} \quad u_0(y) = O\big(|y|^{2/(1-m)}\big) \quad \text{when} \quad |y| \to \infty$$

Thus if $m_c < m < 1$, assumption (H1) is less restrictive than one could think.

- When $m < m_c$, by Comparison Principle, (H1) implies that the extinction of $u(t,\cdot)$ occurs exactly at time T .

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- When $m < m_c$, by Comparison Principle, (H1) implies that the extinction of $u(t,\cdot)$ occurs exactly at time T .

While analyzing $(H1)$ and $(H2)$, it naturally arises the new exponent

The New Critical Exponent m_*

$$m_* = \frac{d-4}{d-2} < m_c = \frac{d-2}{d}$$

- When $m > m_*$, (H2) follows from (H1) since the difference of two Barenblatt solutions is always integrable. For $m \leq m_*$, (H2) is an additional restriction.
- In the range $m \leq m_c$, the pseudo-Barenblatt solutions are not integrable.
- For $m < m_c$ many solutions vanish in finite time and have various asymptotic behaviors depending on the initial data.
  - Solutions with bounded and integrable initial data are described by self-similar solutions with so-called *anomalous exponents*.
  - Even for solutions with initial data not so far from a pseudo-Barenblatt solution, the asymptotic behavior may significantly differ from the behavior of a pseudo-Barenblatt solution.
  - When $m \leq m_c$, assumption (H1)-(H2) are more restrictive than for $m > m_c$.

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Proposition - Relative Conservation of Mass

Let $m < 1$. Consider a solution u of the FDE with initial data u_0 satisfying (H1)-(H2). If for some $D > 0$, $\int_{\mathbb{R}^d} \left[u_0 - U_{D,T}(0,\cdot)\right]\,\mathrm{d}y$ is finite, then

$$\int_{\mathbb{R}^d} \left[u(\tau,y) - U_{D,T}(\tau,y)\right]\mathrm{d}y = \int_{\mathbb{R}^d} \left[u_0(y) - U_{D,T}(0,y)\right]\mathrm{d}x \quad ,\forall\, \tau \in (0,T)\ .$$

The map $D \mapsto \int_{\mathbb{R}^d}(v_0 - V_D)\,\mathrm{d}x$ is continuous and monotone increasing.

We can define a unique $D_* \in [D_1, D_0]$ such that

- If $m > m_*$, then

$$\int_{\mathbb{R}^d} \left[u(\tau,y) - U_{D_*,T}(\tau,y)\right]\mathrm{d}y = 0 \quad \forall\, t > 0\ .$$

- If $m \in (0, m_*]$, integrals are infinite unless $D = D_*$ and then,

$$\int_{\mathbb{R}^d} \left[u_0 - U_{D_*,T}(0,\cdot)\right]\mathrm{d}y = \int_{\mathbb{R}^d} f\,\mathrm{d}x \quad \forall\, t > 0\ .$$

  The perturbation $f \in \mathrm{L}^1(\mathbb{R}^d)$ of $U_{D_*,T}$, can be with nonzero mass.

- In both cases, we summarize the fact that $\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\left[u_0 - U_{D_*,T}(0,\cdot)\right]\,\mathrm{d}y = 0$ by saying that the *relative mass is conserved.*

Theorem - Intermediate asymptotics

Let $d \geq 3$, $m < 1$, $m \neq m_*$. Consider a solution u of the FDE, with initial data satisfying (H1)-(H2). For τ *large* enough, for any $q \in (q_*, \infty]$, there exists a positive constant C such that

$$\|u(\tau) - U_{D_*}(\tau)\|_q \leq C\, R(\tau)^{-\alpha}$$

where the *optimal rate* is given by

$$\alpha = \Lambda_{m,d} + d\,(q-1)/q$$

and $\Lambda_{m,d}$ is the inverse of the Hardy-Poincaré constant $\mathcal{C}_{m,d} = \Lambda_{m,d}^{-1}$.

Large means $\boxed{\tau \to T}$, if $m < m_c$, and $\boxed{\tau \to \infty}$, if $m \geq m_c$.

The Hardy-Poincaré constant

For any $m < 1$, $m \neq m_*$, we define

$$\Lambda_{m,d} := \frac{1}{\mathcal{C}_{m,d}} := \inf_h \frac{\int_{\mathbb{R}^d} |\nabla h|^2\, V_{D_*}\, dx}{\int_{\mathbb{R}^d} |h - \bar{h}|^2\, V_{D_*}^{2-m}\, dx}\,.$$

We shall prove that $\Lambda_{m,d}$ *is strictly positive and independent of* D_*.

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We shall prove that $\Lambda_{m,d}$ *is strictly positive and independent of* D_*.

Recall that $m_* = (d-4)/(d-2)$ and $V_D = (D + |x|^2)^{-\frac{1}{1-m}}$.

Theorem - Spectral Gap: Hardy-Poincaré Inequalities

Let $d \geq 1$ and $D > 0$. There exists $\Lambda_{m,d}$, not depending on D, such that

POINCARÉ CASE. If $m \in (0,1)$ and $1 \leq d \leq 4$, or $m \in (m_*, 1)$ and $d \geq 5$, then

$$\Lambda_{m,d} \int_{\mathbb{R}^d} |g - \overline{g}|^2 \; V_D^{2-m} \, dx \leq \int_{\mathbb{R}^d} |\nabla g|^2 \; V_D \, dx \,, \qquad \overline{g} = \frac{\int_{\mathbb{R}^d} g \; V_D^{2-m} \, dx}{\int_{\mathbb{R}^d} V_D^{2-m} \, dx} \,.$$

HARDY CASE. In case $d \geq 3$ and $m < m_*$, we have

$$\Lambda_{m,d} \int_{\mathbb{R}^d} g^2 \; V_D^{2-m} \, dx \leq \int_{\mathbb{R}^d} |\nabla g|^2 \; V_D \, dx \,,$$

with *optimal constant*

$$\Lambda_{m,d} = \begin{cases} \dfrac{2}{1-m} \,, & \text{if} \quad \dfrac{d-1}{d} < m < 1 \\ 2\dfrac{2-d(1-m)}{1-m} \,, & \text{if} \quad \dfrac{d}{d+2} < m < \dfrac{d-1}{d} \\ \dfrac{[(d-2)\,(m-m_*)]^2}{4\,(1-m)^2} \,, & \text{if} \quad m < \dfrac{d}{d+2}, \quad m \neq m_* \end{cases}$$

Note: $\Lambda_{m,d} = 0$ when $m = m_*$. No Spectral Gap!! Other functional ineq. needed!!

Some Remarks

- We observe that the weight is a power of the Barenblatt and has a “fat tail”
$$V_D^{2-m} \sim V_D/|x|^2 \,, \qquad \text{as } |x| \to \infty$$
- $m < m_*$, *Hardy-type*: the weight V_D^{2-m} is *not integrable*, no average, the infimum of the spectrum is positive, and $\mathcal{C}_{m,d}$ *is the best constant*.
- $m_* < m < 1$, *Poincaré-type:* the weight V_D^{2-m} is integrable, and the spectral gap inequality involves the average as the classical Poincarè inequality, but with *weights.*
- We have calculated the *complete spectrum for any* $m < 1$. Our spectral analysis completes/complements previous works by Denzler-McCann in the range $m > m_c$.
- The optimal rate of convergence has been calculated by DelPino-Dolbeault when $m > (d-1)/d$, by Carrillo-Vázquez when $m > m_c$ while no other results where known for $m < m_c$.
- Our Spectral Gap Theorem is an explicit example for which *weighted Poincaré inequality holds*, while the corresponding *weighted Logarithmic Sobolev inequality does not hold*, even in dimension $d = 1$, c.f. Barthe-Roberto.

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Passing to Self-Similar Variables: Nonlinear Fokker-Plank Equation

- When $m < m_c$, assume that u extinguish in finite time T.
- When $m_c < m < 1$, T is a free parameter to be suitably chosen later.

Let $a = (1-m)/2[d(1-m)-2]$. Define the rescaled function v by

$$v(t,x) := R^d(\tau)\, u(\tau,y)\,, \qquad t := a \log\left(\frac{R(\tau)}{R(0)}\right)\,, \qquad x := \sqrt{a}\,\frac{y}{R(\tau)}\,.$$

Non-linear Fokker-Planck equation (NLFP)

The function v is solution to the *non-linear Fokker-Planck equation*:

$$\begin{cases} \dfrac{\partial v}{\partial t} = \Delta(v^m) + \dfrac{2}{1-m}\nabla(x\,v) = \nabla\cdot\left[v\nabla\left(\dfrac{v^{m-1}-V_D^{m-1}}{m-1}\right)\right] & \text{in } (0,+\infty)\times\mathbb{R}^d, \\ v(0,\cdot) = v_0 = R(0)^d u_0(\cdot\, R(0)) & \text{in } \mathbb{R}^d\,, \end{cases}$$

- T disappeared from the equation but is still in the change of variable.
- The stationary solution is the (pseudo)-Barenblatt solution:

$$V_D(x) := \left(D + |x|^2\right)^{-\frac{1}{1-m}}\,, \qquad \text{we leave } D \text{ as a free “mass” parameter.}$$

Assumption on the Data in Self-similar Variables

(H1) $u_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, non-negative and there exist positive constants $D_0 > D_1$

$$V_{D_0}(x) \le v_0(x) \le V_{D_1}(x) \quad \forall\, x \in \mathbb{R}^d .$$

(H2) There exist $D_* \in [D_1, D_0]$ and $f \in \mathrm{L}^1(\mathbb{R}^d)$ such that

$$v_0(x) = V_{D_*}(x) + f(x) \quad \forall\, x \in \mathbb{R}^d .$$

- The center of mass of the initial datum is not fixed. Fixing the center of mass improves the rate in the range $m_c < m < 1$.
- Remember that the “mass” parameter D_* is fixed by *conservation of relative mass.*

Main theorem if $m \neq m_*$ (Convergence with rate)

Under the assumptions of Theorem 1, if $m \neq m_*$, there exists $t_0 \geq 0$, $C > 0$ such that, for all $q > q_*$ with $q_* := \dfrac{2\,d\,(1-m)}{2\,(2-m)+d\,(1-m)}$ one has

$$\|v(t) - V_{D_*}\|_{\mathrm{L}^q(\mathbb{R}^d)} \leq C_q\, \mathrm{e}^{-\Lambda_{m,d}\,t} \quad \forall\, t \geq t_0 \,.$$

where $\Lambda_{m,d}$ is the eigenvalue in the Hardy-Poincaré inequality. Moreover, for all $p \geq d/2$ one has convergence in relative error, namely

$$\left\| \frac{v(t)}{V_{D_*}} - 1 \right\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq C_p\, \mathrm{e}^{-\Lambda_{m,d}\,t} \quad \forall\, t \geq t_0$$

Finally, uniform convergence of all derivatives also hold

$$\|v(t) - V_{D_*}\|_{C^k(\mathbb{R}^d)} \leq C_k\, \mathrm{e}^{-\Lambda_{m,d}\,t} \quad \forall\, t \geq t_0\,, \quad \forall\, k \geq 1.$$

We choose D_* by relative conservation of mass. The function $\boxed{w = v/V_{D_*}}$ satisfies

$$\text{(NLOU)} \qquad w_t = \frac{1}{V_{D_*}} \nabla \cdot \left[w \, V_{D_*} \nabla \left(\frac{w^{m-1} - 1}{m-1} \; V_{D_*}^{m-1} \right) \right] \qquad \text{in } (0,+\infty) \times \mathbb{R}^d \,.$$

i.e. the NonLinear Ornstein-Uhlenbeck equation, whenever v satisfies (NLFP).

Relative entropy/entropy production

Define the nonlinear *relative entropy*

$$\mathcal{F}[w] := \int_{\mathbb{R}^d} \left[\frac{1}{m-1} (w^m - 1) - \frac{m}{m-1} (w-1) \right] V_{D_*}^m \, \mathrm{d}x$$

and the nonlinear *relative entropy production* functional (or *Fisher information*)

$$\mathcal{I}[w] := \int_{\mathbb{R}^d} \left| \nabla \left[\left(\frac{w^{m-1} - 1}{m-1} \right) V_{D_*}^{m-1} \right] \right|^2 w \, V_{D_*} \, \mathrm{d}x \,.$$

If v is solution to (NLFP) or, equivalently, if $w = v/V_{D_*}$ satisfies (NLOU) then

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}[w] = -\mathcal{I}[w] \,.$$

A weighted linearization

Define the function g by

$$w(t,x) = 1 + \varepsilon \, \frac{g(t,x)}{V_{D_*}^{m-1}(x)} \quad \forall \, t > 0 \, , \qquad \forall \, x \in \mathbb{R}^d \, ,$$

Letting $\varepsilon \to 0$ we formally get a linear evolution equation for g, namely

$$g_t = V_{D_*}^{m-2} \, \nabla \cdot [V_{D_*} \, \nabla g] \; .$$

Define the functional

$$\mathsf{F}[g] := \frac{1}{2} \int_{\mathbb{R}^d} |g|^2 \, V_{D_*}^{2-m} \, \mathrm{d}x$$

and notice that its time derivative (along linear flow) is

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{F}[g] = -\mathsf{I}[g] := -\int_{\mathbb{R}^d} |\nabla g|^2 \, V_{D_*} \, \mathrm{d}x \; .$$

We use the spectral gap to obtain the convergence with rate for the linearized flow

$$2 \, \mathsf{F}[g(t)] \leq \frac{1}{\Lambda_{m,d}} \mathsf{I}[g] \quad \Longrightarrow \quad \mathsf{F}[g(t)] \leq \mathsf{F}[g(0)] \, \mathrm{e}^{-\,2\,\Lambda_{m,d}\,t} \quad \forall \, t \geq 0 \; .$$

Comparing Linear and Nonlinear quantities

Define

$$h = h(t) = \max\left\{ \sup_{x\in\mathbb{R}^d} w(t,x)\,,\ \left[\inf_{x\in\mathbb{R}^d} w(t,x)\right]^{-1} \right\}$$

If t is sufficiently large, then

$$h^{m-2}\,\mathsf{F}[g] \le 2\mathcal{F}[w] \le h^{2-m}\,\mathsf{F}[g]$$

and

$$\mathsf{I}[g] \le \left[1 + X(h)\right] \mathcal{I}[w] + Y(h)\,\mathsf{F}[g]$$

with $g := (w-1)\,V_{D_*}^{m-1}$.

Notice that $h(t) \to 1$ as $t \to \infty$, and

$$0 < X(h) + 1 = h^{5-2m} \to 1 \quad \text{as } t \to +\infty$$

$$0 < Y(h) = d(1-m)\left[h^{4(2-m)} - 1\right] \to 0 \quad \text{as } t \to +\infty.$$

This is a consequence of convergence without rate, proved separately.

By the Hardy-Poincaré inequality

$$\mathsf{F}[g] \leq \frac{1}{\Lambda_{m,d}}\,\mathsf{I}[g] \leq \frac{1}{\Lambda_{m,d}}\left[(1+X(h))\,\mathcal{I}[w] + Y(h)\,\mathsf{F}[g]\right],$$

we deduce that

$$\mathcal{F}[w] \leq \frac{h^{2-m}}{2}\,\mathsf{F}[g] \leq \frac{h^{2-m}\left[1+X(h)\right]}{2\left[\Lambda_{m,d}-Y(h)\right]}\mathcal{I}[w]$$

as soon as $0 < h < h_* := \min\left\{h > 0 : \Lambda_{m,d} - Y(h) \geq 0\right\}$. Moreover

$$0 \leq h - 1 \leq C\mathcal{F}[w]^{\frac{1-m}{d+2-(d+1)m}}$$

for a suitable constant $C > 0$. Recall that $h \to 1$, $X(h), Y(h) \to 0$ as $t \to \infty$. When t is large, there exists a suitable $\gamma > 0$:

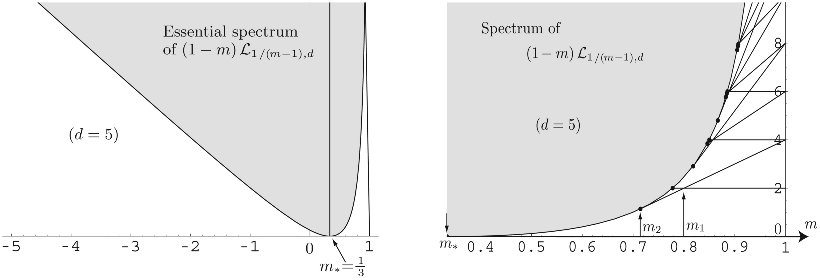
$$\gamma\,\mathcal{F}[w] \leq \mathcal{I}[w] = -\frac{\mathrm{d}\mathcal{F}[w]}{\mathrm{d}t} \quad\Longrightarrow\quad \mathcal{F}[w(t)] \leq \mathcal{F}[w_0]\,\mathrm{e}^{-\gamma\,t}.$$

That is, for the L^2-norm:

$$\|v - V_{D_*}\|_{\mathrm{L}^2}^2 \leq \left\|V_{D_*}^{2-m}\right\|_{\mathrm{L}^\infty}\int |v - V_{D_*}|^2\,V_{D_*}^{m-2}\,\mathrm{d}x = C\,\mathsf{F}[g] \leq C\frac{1}{C_0}\mathcal{F}[w] \leq \tilde{C}\,\mathrm{e}^{-\gamma\,t}.$$

Improvement of convergence: First we prove uniform convergence of w to 1 by an interpolation lemma. Letting then $h(t) = 1 + C\,\mathrm{e}^{-\gamma t}$ in the above estimates, we conclude that γ can be improved up to $\Lambda_{m,d}$. □

Improved Rates in the range $m_1 < m < 1$.



Figures: Spectrum of $\mathcal{L}_{1/(1-m),d}\, g = (1-m) V_D^{m-2} \nabla \cdot \left[V_D \nabla g \right]$ as a function of m, for $d = 5$.

- Under the extra assumption $\int_{\mathbb{R}^d} x g \, dx = 0$, we have an improved optimal constant in the Hardy-Poincaré inequality: $\widetilde{\Lambda}_{m,d} = 2\frac{2-d(1-m)}{1-m} \geq \Lambda_{m,d} = \frac{2}{1-m}$ in

$$\widetilde{\Lambda}_{m,d} \int_{\mathbb{R}^d} |g - \overline{g}|^2 \, V_D^{2-m} \, dx \leq \int_{\mathbb{R}^d} |\nabla g|^2 \, V_D \, dx$$

- As a consequence, under the extra assumption $\int_{\mathbb{R}^d} x v_0 \, dx = 0$, keeping $\int_{\mathbb{R}^d} v_0 - V_{D_*} \, dx = 0$, we have an improvement of the decay rate of the entropy:

$$\mathcal{F}[w(t)] \leq K \, e^{-\widetilde{\Lambda}_{m,d}\, t}$$

The critical case $m = m_*$

If $m = m_*$ there is no spectral gap. One may expect, and gets indeed, a polynomial rate of convergence. In fact we have the following results.

Main theorem in the critical case

Under the running assumption we have:

$$\mathcal{F}[w(t)] \le K\, t^{-1/2}\,, \quad \forall\, t \ge t_0\,. \tag{1}$$

Moreover for any $q \in (1, \infty]$, $j \in \mathbb{N}$:

$$\|v(t) - V_{D_*}\|_q \le K(q)\, t^{-1/4}\,, \quad \forall\, t \ge t_0\,;$$
$$\|v(t) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \le H_j\, t^{-1/4}\,, \quad \forall\, t \ge t_0\,.$$

Rescaling back to the original space–time variables we obtain

Corollary (Intermediate Asymptotics)

For any $q \in (1, \infty]$, there exists a positive constant C such that:

$$\|u(t) - U_{D_*}(t)\|_q \le C\,(T-t)^{\sigma(q)}\, \log\left(T/(T-t)\right)^{-1/4}.$$

with $\sigma(\infty) = d(d-2)/4$ and $\sigma(q) = (q-1)\sigma(\infty)/q$ for $q < \infty$.

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The rate are exactly the one which can be forecasted for the linearized equation.
How to prove this? We introduce a geometrization of the linearized problem.
Consider the operator given on $C_c^\infty(\mathbb{R}^d)$ $(d \geq 3)$by

$$L_m v = (D + |x|^2)^{(2-m)/(1-m)} \nabla \cdot \left(\frac{\nabla v}{(D + |x|^2)^{1/(1-m)}} \right) = V_D^{m-2} \nabla \cdot (V_D \nabla v)\,.$$

We shall think of this operator as acting on the Hilbert space $H_m = \mathrm{L}^2(\mathbb{R}^d, V_D^{2-m}\,\mathrm{d}x)$.
Its associated quadratic form, or Dirichlet form, is

$$\mathsf{I}[v] = \int_{\mathbb{R}^d} \frac{|\nabla v(x)|^2}{(D + |x|^2)^{1/(1-m)}} \mathrm{d}x.$$

Consider the manifold $M = \mathbb{R}^d$ endowed with the Riemannian, conformally flat metric defined, in Euclidean (global) coordinates, by

$$\mathbf{g}(x) = (D + |x|^2)^{-1} \mathbf{I}\,,$$

where $\mathbf{I}$ is the Euclidean metric and $|\cdot|$ is the Euclidean norm. We denote by $\mu_{\mathbf{g}}$ the Riemannian measure and by $\Delta_{\mathbf{g}}$ the Laplace-Beltrami operator, defined on $\mathrm{L}^2(\mu_{\mathbf{g}})$.

Lemma

The Laplace-Beltrami operator $\Delta_{\mathbf{g}}$ coincides with L_m precisely when $m = m_* := (d-4)/(d-2)$, both as concerns its explicit expression (in Euclidean coordinates) and as concerns the Hilbert space it acts on.

The manifold has a cigar–like structure. Its Ricci curvature of $(M, \mathbf{g})$ is computable explicitly. In fact:

Lemma. Ricci Curvature

One has the explicit expression

$$R_{ij} = \frac{(2-d)x_i x_j + \delta_{ij}[(d-2)|x|^2 + 2(d-1)]}{(1+|x|^2)^2}$$

In particular the Ricci tensor is positive definite and bounded .

As a consequence, by the celebrated Li–Yau theory we have

Proposition. Li-Yau estimates

For all small positive ε there exists positive constants c_1, c_2 such that

$$\frac{c_1(\varepsilon)}{\mathrm{Vol}[B(x,\sqrt{t})]} e^{-\frac{d^2(x,y)}{(4-\varepsilon)t}} \le K(t,x,y) \le \frac{c_2(\varepsilon)}{\mathrm{Vol}[B(x,\sqrt{t})]} e^{-\frac{d^2(x,y)}{(4+\varepsilon)t}}$$

for all $x, y \in M$, $t > 0$. Here Vol is the Riemannian volume and d is the Riemannian distance.

As a first corollary we get a crucial on–diagonal heat kernel bound, proved by estimating explicitly distances and volumes.

Lemma

The bound

$$K(t,x,x) \approx \frac{1}{t^{\frac{1}{2}} + \log(1+|x|)}, \qquad \forall t \geq 1, \forall x \in \mathbb{R}^d,$$

holds true.

Here $f_1(t) \approx f_2(t)$ means: there exists two constants $c_1, c_2 > 0$ so that $c_1 f_1 \leq f \leq c_2 f_2$ near t.

• In particular this implies that the semigroup is recurrent, so it follows that no Hardy–type inequality can hold and hence the approach used in the noncritical case is not applicable.

Gagliardo–Nirenberg inequalities

Let $v \in L^2(\mathbb{R}^d, d\mu_*) \cap \mathrm{Dom}(\mathsf{I}_{m_*})$ be such that $0 < \mathsf{I}_{m_*}[v]/\|v\|_1^2 < \infty$. Recall that

$$\|v\|_p^p = \int_{\mathbb{R}^d} \frac{|v(x)|^p}{(D+|x|^2)^{d/2}}\,dx \qquad \text{and} \qquad \mathsf{I}_{m_*}[v] = \int_{\mathbb{R}^d} \frac{|\nabla v(x)|^2}{(D+|x|^2)^{(d-2)/2}}\,dx$$

(i) If $\mathsf{I}_{m_*}[v]/\|v\|_1^2 > 1$ we have the following inequality ($\mathbb{R}^d$-like)

$$\|v\|_2^{2(d+2)} \le c_1\ \mathsf{I}_{m_*}[v]^d\, \|v\|_1^{4d},$$

(ii) If $\mathsf{I}_{m_*}[v]/\|v\|_1^2 \le 1$ we have the following inequality (1-dimensional-like)

$$\|v\|_2^6 \le c_2\ \mathsf{I}_{m_*}[v]\ \|v\|_1^4,$$

Moreover the constants c_i depends on d, but not on v.

(iii) Summing up we have proved that there exists a function $\mathcal{N} : (0,\infty) \to (0,\infty)$ such that

$$\frac{\|v\|_2^2}{\|v\|_1^2} \le \mathcal{N}\left(\frac{\mathsf{I}_{m_*}[v]}{\|v\|_1^2}\right),$$

and

$$\lim_{\xi\to\infty} \frac{\mathcal{N}(\xi)}{\xi^{d/(d+2)}} > 0, \qquad \text{and} \qquad \lim_{\xi\to 0+} \frac{\mathcal{N}(\xi)}{\xi^{1/3}} > 0.$$

Remark. Inequality (i) is "stronger" than the Hardy-Poincaré, indeed plugging $\|v_0\|_1^2 \le \mathsf{I}_{m_*}[v_0]$ (strong assumption) gives

$$\|v\|_2^2 \le c_1\, \mathsf{I}_{m_*}[v]\,.$$

To get a hint of how this inequality helps, consider the linear situation. We have:

$$\frac{d\mathsf{F}[v(t)]}{dt} = -\mathsf{I}[v(t)] \leq -c\frac{\mathsf{F}[v(t)]^3}{\|v(t)\|_1^4} = -c\frac{\mathsf{F}[v(t)]^3}{\|v_0\|_1^4}$$

Integrating the above differential inequality, we obtain:

$$\mathsf{F}[v(t)] \leq \tilde{c}\,\frac{\|v_0\|_1^2}{t^{1/2}}.$$

Is the above use of Gagliardo–Nirenberg inequalities allowed?
Yes, provided

$$\mathsf{I}[v(t)]/\|v(t)\|_1^2 \leq c_0,$$

but this is true e.g. for positive data since the L^1 norm is conserved (to be proved) and the energy decreases.

How can we get the same claim in the nonlinear case? Hard to tell briefly, I sketch the main points without any proof.

(1) One has, setting $g = (w-1)V_*^{m-1}$ the following bound:

$$\mathsf{I}[w] \leq k_1 \mathcal{I}[w] + k_2 \underbrace{\int_{\mathbb{R}^d} g^4 V_*^{4-3m}\, \mathrm{d}x}_{R[g]}$$

This does not depend on the evolution but only on the assumption on the data (preserved along it). The remainder term $R[g]$ is hard to estimate when $m = m_*$.

(2) $\mathcal{I}(t) \to 0$ as $t \to +\infty$. This is usually proved via "Bakry–Emery–like" methods, but we have no spectral gap here! We use Benilan–Crandall estimate.

(3) The inequality

$$\|w-1\|^{2+\frac{m}{1-m}}_{\mathrm{L}^{2+\frac{m}{1-m}}(\mathbb{R}^d)} \leq \overline{D}_m\, \mathsf{F}[w]$$

holds true. Hence, the inequality

$$\mathsf{I}[w] \leq k_1 \mathcal{I}[w] + k_3 \mathcal{F}^{1+\sigma}[w] \qquad \left[\ \Rightarrow \qquad \mathsf{I}[w] \to 0\ \right]$$

holds along the evolution. Again this depends only on the assumptions (preserved along the evolution). This inequality is sufficient to conclude when $m \neq m_*$, while it is not sufficient when $m = m_*$: we need a bit more!!

(4) The inequality

$$\mathsf{I}[w(t)] \leq k_4 \mathcal{I}[w(t)]$$

holds true along the nonlinear evolution, when $m = m_*$.
Very Hard to prove:
we have to estimate the reminder term $R[g]$ in terms of $\mathsf{I}[w(t)]^{1+\varepsilon}$ or $\mathcal{F}^{3+\varepsilon}$, but our weighted Gagliardo–Nirenberg inequality is not sufficient, and the other useful functional inequalities do not hold for this special class of weights.

(5) Using again the GN inequality and the previous steps then gives

$$\mathcal{F}^3[w(t)] \leq -\mathcal{K} \frac{\mathrm{d}\mathcal{F}[w(t)]}{\mathrm{d}t}$$

Integrating it gives the claim as concerns the decay of $\mathcal{F}$.

$(4+5)^*$ To be precise, the above steps 5 and 6 can only be done on a family of intervals $[t_k, t_{k+1}]$ of length at least $1/2$, with $t_k \to \infty$ as $k \to \infty$, but this is sufficient to conclude.

(6) Regularity theory for the solution of the equation holds, so that one can prove a priori that $\sup_{t \geq 1} \|w(t)\|_{C^k} \leq A_k < +\infty$. This and some interpolation arguments involving the C^k norms yield the other claims.

The End

Thank you!!!

Weighted Hardy Inequalities

We now consider the limit $D \to 0^+$, in the Spectral Gap Theorem . Letting $\alpha := 1/(m-1)$, we obtain the *Weighted Hardy inequality,*

$$\int_{\mathbb{R}^d} \frac{|g|^2}{|x|^2}\,|x|^\alpha\,\mathrm{d}x \le \mathcal{H}_\alpha \int_{\mathbb{R}^d} |\nabla g|^2\,|x|^\alpha\,\mathrm{d}x\,, \quad \forall\, g \in \mathcal{D}(\mathbb{R}^d)\,.$$

with the optimal constant

$$\mathcal{H}_\alpha := \frac{4}{[2\,\alpha + d - 2]^2} = \frac{8\,m\,(1-m)}{[(d-2)\,(m-m_*)]^2} \cdot \frac{1-m}{2\,m}\,.$$

Sketch of Proof.

Such an inequality is easy to establish by the “completing the square method” as follows.

$$\begin{aligned} 0 \quad &\le \quad \int_{\mathbb{R}^d} \left| \nabla g + \lambda\,\frac{x}{|x|^2}\,g \right|^2 |x|^{2\alpha}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} |\nabla g|^2\,|x|^{2\alpha}\,\mathrm{d}x + \left[\lambda^2 - \lambda\,(2\,\alpha + d - 2)\right] \int_{\mathbb{R}^d} \frac{|g|^2}{|x|^2}\,|x|^{2\alpha}\,\mathrm{d}x\,. \end{aligned}$$

An optimization of the right hand side with respect to λ gives the desired inequality. $\square$

The Limit as $m \to 1$

In the limit $m \to 1$, we observe that

$$\lim_{m\to 1^-} D_*^{1/(1-m)}\, V_{D_*} = (2\,\pi\, D_*)^{d/2}\,\mu \quad \text{with} \quad \mu(x) = \frac{e^{-\frac{|x|^2}{2\,D_*}}}{(2\,\pi\, D_*)^{d/2}}\;.$$

so that the equation *formally converges to the Ornstein-Uhlenbeck equation*,

$$g_t \;=\; V_{D_*}^{m-2}(x)\,\nabla\cdot\left[V_{D_*}\,\nabla\, g\right] \quad\longrightarrow\quad g_t = \mu^{-1}\,\nabla\cdot\left(\,\mu\,\nabla g\,\right)\;.$$

Also the spectral gap inequality

$$\int_{\mathbb{R}^d} |g|^2\; V_D^{2-m}\;\mathrm{d}x \le \mathcal{C}_{m,d}\int_{\mathbb{R}^d} |\nabla g|^2\; V_D\;\mathrm{d}x \quad \forall\; g \in C^\infty(\mathbb{R}^d)\text{ such that }\int_{\mathbb{R}^d} g\; V_D^{2-m}\;\mathrm{d}x = 0$$

formally converges to the Gaussian-Poincaré inequality

$$\int_{\mathbb{R}^d} |\phi|^2\,\mathrm{d}\mu \le \int_{\mathbb{R}^d} |\nabla\phi|^2\;\mathrm{d}\mu \quad \forall\;\phi\in C^\infty(\mathbb{R}^d)\;\text{ such that }\int_{\mathbb{R}^d}\phi\;\mathrm{d}\mu = 0\;,$$

where $\mathrm{d}\mu := \mu\;\mathrm{d}x$. In the Gaussian case, a logarithmic Sobolev inequality holds, [Gross]

$$\int_{\mathbb{R}^d} |\phi|^2\;\log\left(\frac{|\phi|^2}{\int_{\mathbb{R}^d}|\phi|^2\,\mathrm{d}\mu}\right)\;\mathrm{d}\mu \;\le\; 2\int_{\mathbb{R}^d}|\nabla\phi|^2\;\mathrm{d}\mu\;,$$

which is stronger than the Gaussian Poincaré inequality. With the measure $V_{D_*}\,\mathrm{d}x$. Although the spectral gap inequality holds true, there is no corresponding logarithmic Sobolev inequality.

Application to the Heat Equation

$$u_\tau = \Delta u$$

Logarithmic time-space rescaling gives the

[Fokker-Plank] $\qquad v_t = \Delta v + \nabla \cdot (x\, v)$

Pass to the quotient $w = \dfrac{v}{\mu}$, where μ is the gaussian, to get

[Ornstein-Uhlenbeck] $\qquad w_t = \mu^{-1}\, \nabla \cdot \left(\mu\, \nabla w \right)$

the Gaussian Poincaré inequality gives then, as before,

$$\int_{\mathbb{R}^d} |w-1|^2 \, d\mu \le e^{-t} \int_{\mathbb{R}^d} |w_0 - 1|^2 \, d\mu \quad \forall\, t \ge 0$$

Then by interpolation, we get

$$\|w-1\|_\infty \le \mathcal{K} e^{-t}$$

that means, once we go back to the original variables,

$$\mu(\tau, x)\, \Lambda_0 \;\le\; u(\tau, x) \;\le\; \Lambda_1\, \mu(\tau, x)$$

which are the well known Heat Kernel Estimates of solution to the Heat Equation.