Total Variation Flow and Sign Fast Diffusion in one dimension

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Sign-fast diffusion VS total variation flow

Sign-Fast Diffusion Equation (SFDE) as "limit" of Fast Diffusion Eq. (FDE)

$$\partial_t v = \Delta\left(v^m\right) \quad \text{(FDE)} \quad 0 < m < 1 \quad \xrightarrow[m \to 0^+]{} \quad \partial_t v = \Delta\left(\operatorname{sign}(v)\right) \quad \text{(SFDE)}$$

[Note that $v^m = |v|^{m-1} v$.]

Total Variation Flow (TVF) as "limit" of (parabolic) p-Laplacian

$$\partial_t u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) \quad \text{(p-Laplacian)} \quad 1 < p < 2 \quad \xrightarrow[p \to 1^+]{} \quad \partial_t u = \operatorname{div}\left(\frac{Du}{|Du|}\right) \quad \text{(TVF)}$$

Relation between TVF and SFDE in 1 spatial Dimension

If v solves the SFDE, then $u(x) := \int_0^x v(y)\, \mathrm{d}y$ **solves the TVF**

- The above limits and relations are formal and will be justified later.
- We will consider the 1-dimensional case.

Definition of solutions

A function $u \in L^\infty([0,\infty), BV(I)) \cap W^{1,2}_{loc}([0,\infty), L^2(I))$ is a strong solution of the TVF, $\partial_t = \partial_x(Du/|Du|)$, if there exists $z \in L^2_{loc}([0,\infty), W^{1,2}(I))$, with $\|z\|_\infty \le 1$, such that

$$\partial_t u = \partial_x z \qquad \text{on} \qquad (0,\infty) \times I\,,$$

and

$$\int_0^T \int_I z(t,x)\, Du(t,x)\, dt \;\, \mathrm{d}x = \int_0^T \int_I |Du(t,x)|\, \mathrm{d}x\, dt \qquad \forall\, T > 0.$$

- Roughly speaking, the above condition says that $z = Du/|Du|$.
- There is a huge literature on this topic, we refer to the book
  - F. Andreu, V. Caselles, J. M. Mazon, *Parabolic quasilinear equations minimizing linear growth functionals*, Progress in Mathematics, **223**, Birkhäuser Verlag, Basel.

  for a discussion on the different concepts of solution to the TVF.
  (entropy solutions, mild solutions, semigroup solution, ...)
- For the moment, we do not specify any boundary condition. The following discussion could be applied to the Cauchy problem in $\mathbb{R}$, as well as the Dirichlet or the Neumann problem on an interval.
- Works on TVF by: (hopeless to quote everybody, I am really sorry if I forgot someone) L. Ambrosio, F. Andreu, C. Ballester, G. Bellettini, V. Caselles, A. Chambolle, J. I. Diaz, M.-H. Giga, Y. Giga, R. Kobayashi, R. Kohn, S. Masnou, J. M. Mazon, J.-M. Morel, M. Novaga, P. Rybka, ...

Time Discretization

Strong solution u of the TVF is generated via Crandall-Ligget's Theorem, namely the limit of solutions of a time-discretized problem, given by the implicit Euler scheme

$$\frac{u(t_{i+1}) - u(t_i)}{t_{i+1} - t_i} = \partial_x \left(\frac{Du(t_{i+1})}{|Du(t_{i+1})|} \right) . \tag{1}$$

the time-discretized solution u_h with $h = t_{i+1} - t_i > 0$ (fixed) can be characterized by

$$u_h = \operatorname{argmin}\left[\Phi_h(u)\right], \qquad \text{where} \qquad \Phi_h(u) = \int_I |Du| + \frac{1}{2h} \int_I |u - u_0|^2 \, \mathrm{d}x \, .$$

Φ_h is strictly convex (unique minimizer), and (1) is the Euler-Lagrange eq. for Φ_h. u_0 , $u_h \in BV(I)$ implies $\partial_x z_h \in BV(I) \subset L^\infty(I)$, so that z_h is Lipschitz, therefore differentiable outside a countable set of points:

$$N(z_h) := \left\{ x \in \mathbb{R} \;\middle|\; \lim_{\varepsilon \to 0} \frac{z_h(x+\varepsilon) - z_h(x)}{\varepsilon} \text{ does not exists} \right\} . \tag{2}$$

Finally, equation (1) is equivalent to

$$\begin{cases} h\,\partial_x z_h(x) = u_h(x) - u_0(x) & \text{for all } x \in \mathbb{R} \setminus N(z_h) \\ |z_h(x)| \le 1 \, , & \text{for all } x \in \mathbb{R} \\ z_h(x) = \pm 1 \, , & \text{for } |Du_h| - \text{a.e.} \end{cases}$$

Some properties of Solutions

Behaviour near continuity points

If u_h is different from u_0 at some common continuity point x, then it is constant in an open neighborhood of x.

Behaviour at discontinuity points (jumps decrease size in time)

Let $u_0 \in BV(I)$. Then, the following inequalities hold for any $x \in I$:

$$\begin{array}{llll} \text{if} & u_h(x^-) \le u_h(x^+) & \text{then} & u_0(x^-) \le u_h(x^-) < u_h(x^+) \le u_0(x^+) \\ \text{if} & u_h(x^+) \le u_h(x^-) & \text{then} & u_0(x^+) \le u_h(x^+) < u_h(x^-) \le u_0(x^-)\,. \end{array}$$

Moreover,

$$\begin{array}{lll} u_h(x^-) < u_h(x^+) & \text{implies} & z_h(x) = 1 \\ u_h(x^-) > u_h(x^+) & \text{implies} & z_h(x) = -1\,. \end{array}$$

Local continuity

Let $x \in I$. If u_0 is continuous at x, then u_h is continuous at x.

The dynamics of local step functions I. The time discretized case. Maximum steps.

- Let us fix an interval $I = I_1 \cup I_2 \cup I_3$
- Assume that $u_0 = \alpha_1\chi_1 + \alpha_2\chi_2 + \alpha_3\chi_3$ on I with $\alpha_2 > \max\{\alpha_1, \alpha_3\}$ and $\chi_k = \chi_{I_k}$ is the char. funct. of $I_k = (x_{k-1}, x_k)$.
- We make no assumptions on u_0 outside I. Fix $h > 0$ small.

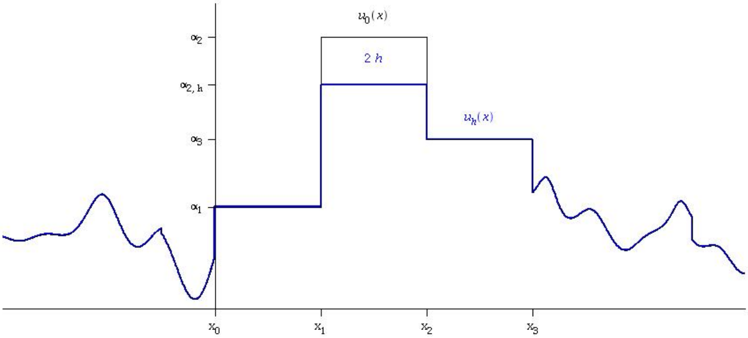


Figure: *Dynamics of a maximum step. This figure shows the dynamic only inside the interval* $[x_0, x_3]$.

The dynamics of local step functions I. The time discretized case

Evolution of a general step function.

- Let $\alpha_k \in \mathbb{R}$ for $k = 0, \ldots, N+1$, and $\chi_k = \chi_{I_k}$ is the char. funct. of $I_k = (x_{k-1}, x_k)$ (also the values $x_0 = -\infty$ and $x_{N+1} = +\infty$ are allowed)
- If $0 < \ell h < \min_{j=0,\ldots,N} \left\{ \left|\alpha_j - \alpha_{j+1}\right| \min\left\{|I_j|, |I_{j+1}|\right\} \right\}$, the discrete solution after ℓ steps is given by

$$u_0 = \sum_{k=0}^{N+1} \alpha_k \chi_k \qquad \text{gives} \qquad u_{\ell h} = \sum_{k=0}^{N+1} \alpha_{k,\ell h} \chi_k \quad \text{on } I,$$

where we are able to explicitly get the values of $\alpha_{k,\ell h}$ for $k = 1, \ldots N$, and some information on $\alpha_{0,\ell k}$ and $\alpha_{N+1,\ell k}$: for $k = 1, \ldots N$

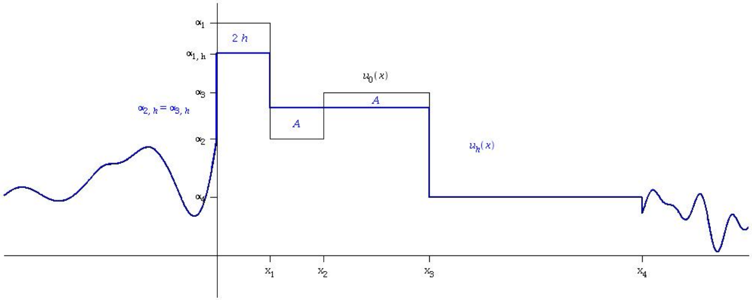
$$\alpha_{k,\ell h} = \begin{cases} \alpha_k\,, & \text{if } \alpha_{k-1} < \alpha_k < \alpha_{k+1} \text{ or if } \alpha_{k+1} < \alpha_k < \alpha_{k-1} \\ \alpha_k - \dfrac{2\ell h}{|I_k|}\,, & \text{if } \alpha_k > \max\left\{\alpha_{k-1}\,,\ \alpha_{k+1}\right\} \\ \alpha_k + \dfrac{2\ell h}{|I_k|}\,, & \text{if } \alpha_k < \min\left\{\alpha_{k-1}\,,\ \alpha_{k+1}\right\} \end{cases}$$

(3)

The dynamics of local step functions I. The time discretized case

A concluding remark on the smallness of the time step h.

- Since we are mainly interested in the limit $h \to 0$, condition on smallness of h is always fulfilled.
- Anyway it is interesting to observe that the dynamic becomes more complicated to understand for general values of h, since the "locality" property is lost.
- Figure below shows a situation when a maximum and a minimum disappear in one step (for this to happen, the area A has to be less than $2h$). Of course one can construct much more complicated examples.
- We can observe that the value of u_h inside $[x_1, x_2]$ depends on the values of u_0 on both $[x_1, x_2]$ and $[x_2, x_3]$.



The dynamics of local step functions II. The continuous time case.

Letting $h \to 0^+$ in the time discretized solution we obtain

$$u_0(x) = \sum_{k=0}^{N+1} \alpha_k \chi_{I_k}(x) \quad \rightsquigarrow \quad u(t,x) = u_0(x) + t \sum_{k=0}^{N+1} \beta_{k,\ell h} \chi_k(x) \quad \text{on } [0,t_1] \times I,$$

with $t_1 < \min_{j=0,\dots,N} \left\{ \left|\alpha_j - \alpha_{j+1}\right| \min\left\{|I_j|, |I_{j+1}|\right\} \right\}$, and

$$\beta_{k,\ell h} := \begin{cases} 0\,, & \text{if } \alpha_{k-1} < \alpha_k < \alpha_{k+1} \text{ or if } \alpha_{k+1} < \alpha_k < \alpha_{k-1} \\ -\dfrac{2}{|I_k|}\,, & \text{if } \alpha_k > \max\left\{\alpha_{k-1}\,,\, \alpha_{k+1}\right\} \\ \dfrac{2}{|I_k|}\,, & \text{if } \alpha_k < \min\left\{\alpha_{k-1}\,,\, \alpha_{k+1}\right\} \end{cases}$$

for $k = 1, \dots, N$, and

$$\beta_{0,\ell h} \begin{cases} \geq 0\,, & \text{if } \alpha_0 < \alpha_1 \\ \leq 0\,, & \text{if } \alpha_0 > \alpha_1 \end{cases} \qquad \beta_{N+1,\ell h} \begin{cases} \geq 0\,, & \text{if } \alpha_N > \alpha_{N+1} \\ \leq 0\,, & \text{if } \alpha_N < \alpha_{N+1}. \end{cases}$$

On I_0 and I_{N+1} it is monotonically increasing/decreasing, depending on the value on I_1 and I_N.

- This formula will then continue to hold until a maximum/minimum disappear.
- After repeating this at most N times, all the maxima and minima inside I disappear, and $u(t)$ is monotonically decreasing/increasing on I.
- For instance, if $I = \mathbb{R}$ and the initial data is a compactly supported step function, then $u \equiv 0$ after some finite time T (which we call *extinction time*).
- If u_0 is an increasing (resp. decreasing) step function, then it will remain constant in time.

The dynamics of local step functions II. The continuous time case.

Letting $h \to 0^+$ in the time discretized solution we obtain

$$u_0(x) = \sum_{k=0}^{N+1} \alpha_k \chi_{I_k}(x) \quad \rightsquigarrow \quad u(t,x) = u_0(x) + t \sum_{k=0}^{N+1} \beta_{k,\ell h} \chi_k(x) \quad \text{on } [0,t_1] \times I,$$

with $t_1 < \min_{j=0,\ldots,N} \left\{ \left|\alpha_j - \alpha_{j+1}\right| \min\left\{ |I_j|, |I_{j+1}| \right\} \right\}$, and

$$\beta_{k,\ell h} := \begin{cases} 0\,, & \text{if } \alpha_{k-1} < \alpha_k < \alpha_{k+1} \text{ or if } \alpha_{k+1} < \alpha_k < \alpha_{k-1} \\ -\dfrac{2}{|I_k|}\,, & \text{if } \alpha_k > \max\left\{\alpha_{k-1}\,,\ \alpha_{k+1}\right\} \\ \dfrac{2}{|I_k|}\,, & \text{if } \alpha_k < \min\left\{\alpha_{k-1}\,,\ \alpha_{k+1}\right\} \end{cases}$$

for $k = 1, \ldots, N$, and

$$\beta_{0,\ell h} \begin{cases} \geq 0\,, & \text{if } \alpha_0 < \alpha_1 \\ \leq 0\,, & \text{if } \alpha_0 > \alpha_1 \end{cases} \qquad \beta_{N+1,\ell h} \begin{cases} \geq 0\,, & \text{if } \alpha_N > \alpha_{N+1} \\ \leq 0\,, & \text{if } \alpha_N < \alpha_{N+1}. \end{cases}$$

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The dynamics of local step functions II. The continuous time case

- This analysis can be extended to the case of suitable initial value problems on intervals with boundary condition:
- The dynamic of the **Dirichlet problem** is analogous to the one described above for the Cauchy problem with compactly supported initial data.
- The **Neumann problem** on some closed interval $[a, b] = I_0 \cup \ldots \cup I_{N+1}$. The dynamics on $I_1 \cup \ldots \cup I_N$ is known by our analysis (which, as we observed before, is "local"). Neumann condition at the level of discretized problem allows to uniquely characterize the value of u in I_0 and I_{N+1}.

$\star$ For example, if $u_0 = \sum_{k=1}^{N+1} \alpha_k \chi_{I_k}$ with $\alpha_1 \leq \ldots \leq \alpha_{N+1}$ (i.e. u_0 is monotonically increasing), then

$$u(t) = u_0 + t\left(\frac{1}{|I_0|}\chi_{I_0} - \frac{1}{|I_{N+1}|}\chi_{N+1}\right)$$

(i.e. the value on I_0 increases, while the one on I_{N+1} decreases).
This holds true until a jump disappears, and then one simply repeat the construction.

The dynamics of local step functions II. The continuous time case

Theorem. (Local continuity)

Assume that u_0 is continuous on some open interval I. Then also the corresponding solution $u(t)$ is continuous on the same interval I and the oscillation is contractive, namely

$$\sup_I u(t) - \inf_I u(t) =: \operatorname{osc}_I \left(u(t)\right) \leq \operatorname{osc}_I \left(u_0\right).$$

The above theorem still holds if u_0 is not continuous on I:
if $u^+(t)$ and $u^-(t)$ are the solution starting respectively from

$$u^+(x) := \begin{cases} u_0(x) & \text{if } x \notin I; \\ \operatorname{esssup}_I u_0 & \text{if } x \in I; \end{cases} \qquad u^-(x) := \begin{cases} u_0(x) & \text{if } x \notin I; \\ \operatorname{essinf}_I u_0 & \text{if } x \in I; \end{cases}$$

then $u^-(t) \leq u(t) \leq u^+(t)$, $u^+(t)$ and $u^-(t)$ are both constant on I, and

$$\|u^-(t,x) - u^-(t,x)\|_{L^\infty(I)} \qquad \text{is decreasing in time.}$$

Further Local Properties of solutions of the TVF.

Arguing by approximation, using the stability in L^p, $1 \le p \le \infty$ we deduce
local properties of the TVF, valid on any subinterval I where the solution $u(t)$ is considered.

1. The set of discontinuity points of $u(t)$ is contained in the set of discontinuity points of u_0, i.e. "the TVF does not create new discontinuities".
2. The number of maxima and minima decreases in time.
3. If u_0 is monotone on I, then $u(t)$ has the same monotonicity as u_0 on I.
   If u_0 is monotone on $\mathbb{R}$, then it is a stationary sol. to the Cauchy problem.
4. $C^{0,\alpha}$-regularity is preserved along the flow for any $\alpha \in (0,1]$.
   Similar results for the denoising problem and for the Neumann problem for the TVF in V. Caselles, A. Chambolle, M. Novaga, Rev. Mat. Iberoamericana **27**, (2011).
   Moreover, if $u_0 \in W^{1,1}(\mathbb{R})$, then $u(t) \in W^{1,1}(\mathbb{R})$
   (this is a consequence of the fact that the oscillation does not increase on any subinterval).
5. If $u_0 \in BV_{loc}(\mathbb{R})$, a priori we do not have a well-defined semigroup. However, in this case u_0 is locally bounded and the set of its discontinuity points is countable, and so in particular has Lebesgue measure zero. Then, by approximation we can still define a dynamics, which will still be contractive in any L^p space.

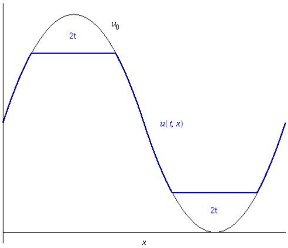
Behaviour near maxima and minima. Assume that u_0 has a local maximum at x_0. Then, at least for short time, the solution is explicitly given near x_0 by

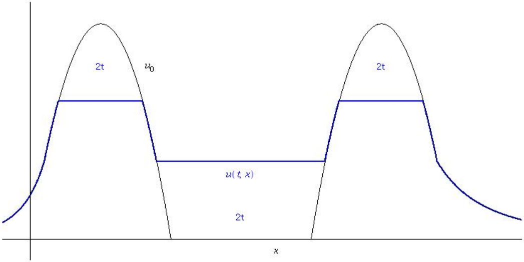
$$u(t,x) = \min\{u_0(x), h(t)\},$$

where the constant value $h(t)$ is implicitly defined by

$$\int_{I_0} \left[u_0(x) - h(t)\right]_+ \, dx = 2t\,,$$

I_0 being the connected component of $\{u_0 > h(t)\}$ containing x_0. The dynamics goes on in this way until a local minimum "merges" with a local maximum, and then one can simply start again the above description starting from the new configuration. For a minimum point the argument is analogous.





Loss of mass and extinction time

Let $u(t)$ be the solution to the Cauchy problem in $\mathbb{R}$ for the TVF, starting from a non-negative compactly supported initial datum $u_0 \in L^1(\mathbb{R})$. Then the following estimates hold:

$$\int_{\mathbb{R}} u(t,x)\, \mathrm{d}x = \int_{\mathbb{R}} u_0(x)\, \mathrm{d}x - 2t = 2(T-t) \qquad \text{for all } t \geq 0\,,$$

and the extinction time for u is given by $\qquad T = T(u_0) = \dfrac{1}{2} \int_{\mathbb{R}} u_0(x)\, \mathrm{d}x\,.$

Remark. There is no general explicit formula for the extinction time when u_0 changes sign.

The rescaled flow.

We now are interested in describing the behavior of the solution near the extinction time.
We perform a logarithmic time rescaling, mapping the interval $[0,T)$ into $[0,+\infty)$, where T is the extinction time corresponding to the initial datum u_0. We define

$$w(s,x) = \frac{T}{T-t}\, u\,(t,x)\;, \qquad Z(s,x) = z(t,x)\,, \qquad s = T\log\left(\frac{T}{T-t}\right)$$

where $u(t)$ is a solution to the TVF. Then

$$\partial_s w(s,x) = \partial_x Z + \frac{w}{T}, \qquad Z\cdot D_x w = |D_x w|\,, \qquad w(0,x) = u_0(x).$$

Stationary solutions, Separate Variable Solutions

Stationary solutions $S(x)$ for the rescaled equation for w correspond to separation of variable solutions in the original variable, namely

$$-\partial_x Z = \frac{S}{T} \qquad \text{provides the separate variable solution} \qquad U_T(t,x) := \frac{T-t}{T} S(x) \,.$$

The "extended support"of a function f is the smallest interval that includes the support of f:

$$\mathrm{supp}^*(f) = \inf\left\{[a,b] \mid \mathrm{supp}(f) \subseteq [a,b]\right\} .$$

Theorem. Stationary solutions

All compactly supported solutions of the equation $-\partial_x Z = \frac{S}{T}$, $\quad Z \cdot D_x S = |D_x S|$, are of the form

$$S(x) = \frac{2T}{b-a}\, \chi_{[a,b]}(x) \,, \qquad \text{with } [a,b] \subseteq \mathbb{R}.$$

Proposition. Mass conservation for rescaled solutions

Let $w(s)$ be the rescaled solution, corresponding to $0 \le u_0 \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$. Then

$$\int_{\mathbb{R}} w(s,x)\, \mathrm{d}x = \int_{\mathbb{R}} u_0(x)\, \mathrm{d}x \,.$$

Corollary. Separate variable solutions

All compactly supported solutions of the TVF obtained by separation of variables are of the form

$$U_T(t,x) = 2\frac{T-t}{b-a}\chi_{(a,b)}(x)\,, \qquad \text{where } T>0 \text{ and } [a,b]\subseteq\mathbb{R}.$$

Proposition. Stationary solutions are asymptotic profiles

Let $w(s,x)$ be a solution to the rescaled TVF corresponding to a non-negative initial datum $u_0 \in BV(\mathbb{R})\cap L^1(\mathbb{R})$. Then there exists a subsequence $s_n \to \infty$ such that $w(s_n,\cdot) \to S$ in $L^1(I)$ as $n\to\infty$ where S is a stationary solution as in (1).
Equivalently we have that there exists a sequence of times $t_n \to T$ as $n\to\infty$ such that

$$\left\| \frac{u(t_n,\cdot)}{T-t_n} - \frac{S}{T} \right\|_{L^1} \xrightarrow[n\to\infty]{} 0\,.$$

where S is a stationary solution.

The above result has been proved by F. Andreu, V. Caselles, M. Mazon in a series of paper and in their book, for the Cauchy, Dirichlet or Neumann problem.

Theorem. Extinction profile for solutions to the TVF

Let $u(t,x)$ be a solution to the TVF corresponding to a non negative initial datum $u_0 \in BV(\mathbb{R})$ with $\mathrm{supp}^*(u_0) = [a,b]$, and set

$$T = \frac{1}{2}\int_a^b u_0(x)\,\mathrm{d}x\,.$$

Then $\mathrm{supp}(u(t)) = [a,b]$ for all $t \in (0,T)$ and

$$\left\| \frac{u(t,\cdot)}{T-t} - 2\frac{\chi_{[a,b]}}{b-a} \right\|_{L^1([a,b])} \xrightarrow[t\to T]{} 0\,.$$

Remarks. The above theorem shows to important facts:

(i) The support of the solution becomes instantaneously the “extended support” of the initial datum, which is the support of the extinction profile.

(ii) On $[a,b] = \mathrm{supp}^*(u_0)$ we consider the quotient $u(t,x)/U_T(t,x)$, where U_T is the separate variable solution. Then the above convergence result can be rewritten as

$$\left\| \frac{u(t,\cdot)}{U_T(t,\cdot)} - 1 \right\|_{L^1([a,b])} \xrightarrow[t\to T]{} 0\,. \qquad \textit{convergence in relative error}$$

Equivalently, L^1-norm of the difference decays at least with the rate

$$\left\| u(t,\cdot) - U_T(t,\cdot) \right\|_{\mathrm{L}^1(\mathbb{R})} \le o(T-t)\,.$$

We will show that the $o(1)$ appearing in the above rate cannot be quantified/improved, so that the above convergence result is sharp, as we will see in the next slide.

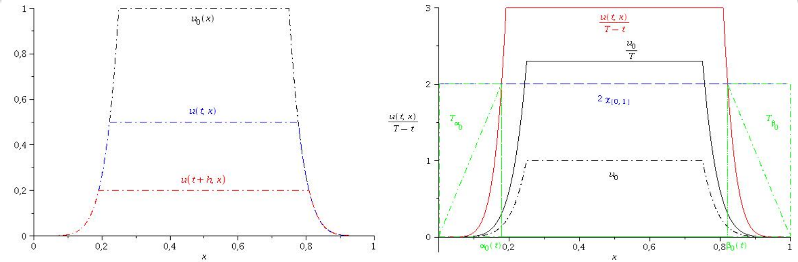
Definition. Let $\xi : [0,\infty) \to [0,\infty)$ be a continuous increasing function, with $\xi(0) = 0$. We say that ξ is a *rate function* if, for any solution $u(t)$ of the TVF,

$$\left\| \frac{u(t)}{T-t} - \frac{S}{T} \right\|_{L^1(I)} \leq \xi(T-t) \qquad \text{for any } t \text{ close to the extinction time } T\,.$$

Theorem. Absence of universal convergence rates

For any rate function $\xi : [0,\infty) \to [0,\infty)$, there exists an initial datum $u_0 \in BV(\mathbb{R})$, with $\mathrm{supp}^*(u_0) = [0,1]$, such that

$$2\,\xi(T-t) \leq \left\| \frac{u(t)}{T-t} - 2\chi_{[0,1]} \right\|_{L^1(I)}, \qquad \text{for any } 0 \leq T-t \leq 1.$$



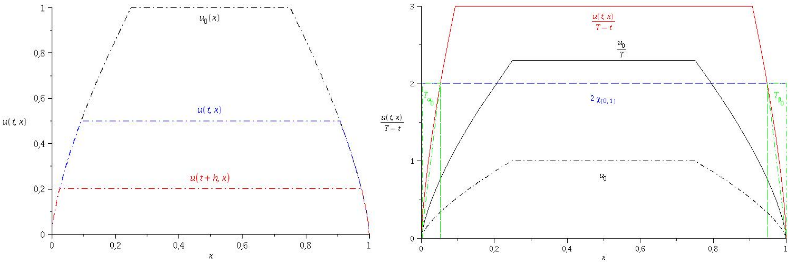
Left: *Dynamic of* $u(t)$*:* black: $u_0(x)$, blue: $u(t,x)$, red: $u(t+h)$. **Right:** *Rescaled dynamic:* black: $u_0(x)$ (dashdot) and u_0/T (cont.), blue: $S(x) = 2\chi_{[0,1]}$, red: $u(t,x)/(T-t)$,

Remark. The above Theorem shows that there cannot be universal rates of convergence. A similar construction will provide (nontrivial) initial data for which the convergence is as fast as desired.

Fast decaying initial data

For any rate function $\xi : [0,\infty) \to [0,\infty)$, there exists an initial datum $u_0 \in L^1(I)$ such that the corresponding solution $u(t)$ satisfies

$$\left\| \frac{u(t)}{T-t} - 2\chi_{[0,1]} \right\|_{L^1(I)} \leq \xi\big(8(T-t)\big), \qquad \text{for any } 0 \leq T-t \leq 1. \tag{4}$$



Left: *Dynamic of* $u(t)$*:* black: $u_0(x)$, blue: $u(t,x)$, red: $u(t+h,x)$. **Right:** *Rescaled dynamic:* black: $u_0(x)$, blue: $S(x) = 2\chi_{[0,1]}$, red: $u(t,x)/(T-t)$,

Solutions to the SFDE VS solutions to the TVF

- Formally: The TVF and SFDE are formally related by the fact that "u solves the TVF if and only if $D_x u$ solves the SFDE".
- In order to make this rigorous, we need first to explain what do we mean by a solution of the SFDE, and then we will prove the above relation by approximating the TVF with the p-Laplacian and the SFDE by the porous medium equation.
- The notion of solution we consider for the SFDE is the one of mild solution. We use: P. Benilan, M. G. Crandall, Indiana Univ. Math. J. **30** (1981), no. 2, 161–177.
- The multivalued graph of the function $r \mapsto \operatorname{sign}(r)$ is maximal monotone (MMG).
- There exists a unique solution $u \in C([0,\infty); L^1(\mathbb{R})) \cap L^\infty([0,\infty) \times \mathbb{R})$ corresponding to the initial datum $u_0 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ that solves the problem

$$\begin{cases} u_t = \Delta\varphi(u)\,, & \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}\big) \\ u(0,x) = u_0(x)\,, & x \in \mathbb{R} \end{cases}$$

  where the first equation is meant in the sense that

$$u_t = \Delta w \qquad \text{in } \mathcal{D}'\big((0,\infty)\times\mathbb{R}\big)\,, \qquad \text{with } w(t,x) \in \varphi(u(t,x)) \qquad \text{a.e. } t, x \in \mathbb{R}\,.$$

- Given an approximating sequence of MMG $\varphi_n \to \varphi$ then one can prove that $u_n \to u$ in $C\big([0,\infty); L^1(\mathbb{R})\big)$.

TVF vs SFDE

Assume u_0 is a smooth compactly supported function. Let $1 < p \leq 2$, and $m = p - 1$.
Then the following diagram is commutative:

$$
\begin{array}{ccc}
T_t^p u_0 \in W^{1,p}(\mathbb{R}) & \xrightarrow{\;p \to 1^+\;} & T_t^1 u_0 \in W^{1,1}(\mathbb{R}) \\
\downarrow \partial_x & & \downarrow \partial_x \\
S_t^m\left(\partial_x u_0\right) \in L^{1+m}(\mathbb{R}) & \xrightarrow[\;m \to 0^+\;]{} & S_t^0\left(\partial_x u_0\right) \in L^1(\mathbb{R}).
\end{array}
$$

Note that the convergence in meant in the sense of distributions

- S_t^m is the semigroup associated to the FDE equation $\partial_t v = \Delta(v^m)$.
  We have that $S_t^m v_0 \to S_t^0 v_0$ as $m \to 0^+$, in $C([0,\infty); L^1(\mathbb{R}))$, for any initial datum $v_0 \in L^1$.
- We can consider the p-Laplacian semigroup T_t^p for $p = 1 + m$.
  If $u_0 \in W^{1,p}(\mathbb{R})$, then $T_t^p u_0 \in W^{1,p}(\mathbb{R})$, so that as $p \to 1^+$, strong solutions to the p-Laplacian converge to strong solutions to the TVF. So that $T_t^p u_0 \to T_t^1 u_0$ in $C\left([0,\infty); L^1(\mathbb{R})\right)$ as $p \to 1^+$, where T_t^1 denotes the TVF-semigroup.
- If $p = 1 + m$, we have that $\partial_x\left(T_t^p u_0\right)$ solves (in the distributional and semigroup sense) the FDE with initial datum $\partial_x u_0$, i.e. $\partial_x\left(T_t^p u_0\right) = S_t^m\left(\partial_x u_0\right)$.
  Hence, by letting $m \to 0^+$, we recover such a relation in the limit $p = 1$ and $m = 0$.

Measures as initial data. Once the correspondence between TVF and SFDE is established for smooth initial data, by stability in L^1 of both semigroups it immediately extends to $u_0 \in W^{1,1}(\mathbb{R})$, and then by approximation to $BV(\mathbb{R}) \cap L^1(\mathbb{R})$ initial data. However, at the level of the SFDE this would correspond to finite measures v_0 such that $\int_{-\infty}^{x} v_0(dy) \in L^1(\mathbb{R})$, which is possible if and only if $\int_{-\infty}^{+\infty} v_0(dy) = 0$. Actually, this class of data correspond exactly to the one for which there is extinction in finite time (as this is the case for L^1 initial data to the TVF). We can remove this unnatural constraint on v_0. Summing up, we have shown that:

- If $v_0 \in L^1(\mathbb{R})$, the unique mild solution of the SFDE is given by

$$S_t^0 v_0 = \partial_x \left(T_t^1 \left(\int_{-\infty}^{x} v_0(dy) \right) \right). \tag{5}$$

- Using (5) we can uniquely extend the generator S_t^0 to measure initial data (actually, since the semigroup T_t^1 is well-defined on $L^2(\mathbb{R})$, one could even extend the SFDE to distributional initial data in $W^{-1,2}(\mathbb{R})$).

General properties of the SFDE flow. Arguing by approximation (or again using the direct relation with the TVF), as a consequence we have the following properties of the SFDE flow:

- *Nonnegative initial data.*
  Let $v_0 \geq 0$ be a locally finite measure, and define $u_0(x) := \int_{-\infty}^{x} v_0(dy) \geq 0$.
  Since u_0 is monotone non-decreasing, it does not evolve under the TVF.
  Hence v_0 is a stationary solution to the SFDE.
  (Actually, since monotone profiles are the only stationary state for the TVF, the only stationary solutions for the SFDE are nonnegative/nonpositive initial data.)
- *Only zero mean valued initial data extinguish in finite time.*
  If v_0 is a finite measure, $v(t)$ converges in finite time to a stationary solution $\bar{v}$ such that $\int_{\mathbb{R}} \bar{v}(dy) = \int_{\mathbb{R}} v_0(dy)$.
  Moreover, $\bar{v} \equiv 0$ (i.e. v_0 extinguish in finite time) if and only if $\int_{\mathbb{R}} v_0(dy) = 0$.

Example 1. Delta masses as initial data.

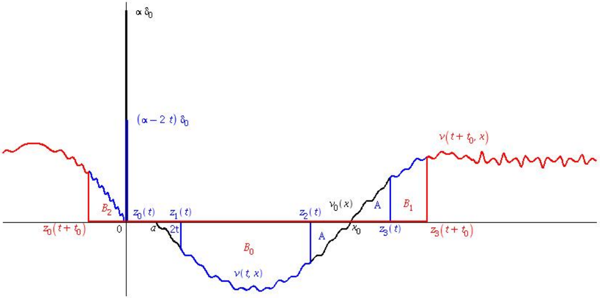
Let us assume that $v_0 = \sum_{i=1}^{N} a_i \delta_{x_i}$, with $x_1 < \cdots < x_N$. Then, for $t > 0$ small (dep. on $|a_i|$)

$$v(t) = \sum_{i=1}^{N} a_i(t) \delta_{x_i}, \qquad \text{with } a_i(0) = a_i \text{ and}$$

$$a_i(t) = \begin{cases} a_i\,, & \text{if } \operatorname{sign}(a_{i-1}) = \operatorname{sign}(a_{i+1}) = \operatorname{sign}(a_i) \\ \operatorname{sign}(a_i)\big(|a_i| - 4t\big)_+\,, & \text{if } \operatorname{sign}(a_{i-1}) = \operatorname{sign}(a_{i+1}) = -\operatorname{sign}(a_i) \\ \operatorname{sign}(a_i)\big(|a_i| - 2t\big)_+\,, & \text{if } \operatorname{sign}(a_{i-1})\ \operatorname{sign}(a_{i+1}) = -1, \end{cases} \tag{6}$$

where we use the convention $\operatorname{sign}(a_0) := \operatorname{sign}(a_1)$ and $\operatorname{sign}(a_{N+1}) := \operatorname{sign}(a_N)$. This formula holds true until one mass disappear at some time $t_1' > 0$, and then it suffices to $v(t_1')$ as initial data and repeat the construction

Example 2. Interaction between a delta and a continuous part.



Sign VS Logarithmic Fast Diffusion.

- Consider the fast diffusion equation $u_t = \Delta u^m$, with $0 < m < 1$, and assume that $v_0 \geq 0$ (so $v(t) \geq 0$ for all $t \geq 0$).
- Changing the time scale $t \mapsto mt$, the above equation can be written in two different ways which lead to two different limiting equations, indeed setting $\rho(t,x) = v(t/m, x)$,

$$\partial_t v = \Delta\big(v^m\big) \xrightarrow[m\to 0^+]{} \partial_t v = \Delta\big(\,\mathrm{sign}(v)\big)$$

$$\partial_t \rho = \mathrm{div}\big(\rho^{m-1}\nabla\rho\big) \xrightarrow[m\to 0^+]{} \partial_t \rho = \mathrm{div}\,\big(\rho^{-1}\nabla\rho\big) = \Delta\big(\log(\rho)\big)\,.$$

- The evolution of $\rho(t,x)$ on $[0,T]$ corresponds to the evolution of $v(t,x)$ on $[0, T/m]$. The diffusion of v is slower than the diffusion of ρ by a factor $1/m$.
- So, when analyzing the limit as $m \to 0^+$, one gets two different limits: the evolution of $\rho(t,x)$ on the time interval $0 \leq t \leq T$ corresponds to the evolution of $v(t,x)$ on the time interval $0 \leq t < \infty$ for every $T > 0$.
- The solution to the SFDE corresponds to an evolution "infinitely slower" than the LFDE.

	Cauchy Problem	Dirichlet Problem
$u_0 \geq 0$	SFDE: trivial$^{(*)}$	SFDE: NON-trivial
$u_0 \geq 0$	LFDE: NON-Trivial	LFDE: trivial$^{(**)}$
$\mathrm{sign}(u_0) = \pm 1$	SFDE: NON-trivial	SFDE: NON-trivial
$\mathrm{sign}(u_0) = \pm 1$	LFDE: NOT-possbile	LFDE: NOT-possbile

$(*)$ $u(t,\cdot) = u_0$, $(**)$ Immediate extinction.

Logarithmic Diffusion has been studied by A. Rodriguez, J.L. Vázquez,... and many other authors (hopeless to quote everybody).

The End

Thank you!!!