# Stability in Gagliardo-Nirenberg-Sobolev inequalities: nonlinear flows, regularity and the entropy method

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## Stability in Gagliardo-Nirenberg-Sobolev inequalities: nonlinear flows, regularity and the entropy method

A joint project with


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Bruno Nazaret
▷ *Université Paris 1 Panthéon-Sorbonne and Mokaplan team*

Nikita Simonov
▷ *LaMMe-Universitè d'Évry Val d'Essone*

# Stability in Gagliardo-Nirenberg-Sobolev inequalities: nonlinear flows, regularity and the entropy method

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   - A special family of Gagliardo-Nirenberg-Sobolev inequalities
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   - The threshold time
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# Gagliardo-Nirenberg-Sobolev inequalities by variational methods

Consider the following family of inequalities

**A special family of Gagliardo-Nirenberg-Sobolev inequalities**

(GNS) $$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p}$$

with

$$\theta = \frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}\,,\qquad p \in \begin{cases} (1,+\infty) & \text{if } d=1,2 \\ (1,p^*] & \text{if } d\ge 3\,, \end{cases}\quad p^* = \frac{d}{d-2} = \frac{2^*}{2}$$

▷ The validity of the inequality (no sharp constant) is due to [Sobolev 1938], [Gagliardo, Nirenberg 1958], but also DeGiorgi, Hardy, Ladyzenskaya, Littlewood, …
▷ The family contains the classical Sobolev Inequality: $p = p^*$

$$\boxed{\mathsf{S}_d\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}$$

## Optimal functions . . .

(GNS) $$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p}$$

$$\theta = \frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}\,, \qquad p \in \begin{cases} (1,+\infty) & \text{if } d=1 \\ (1,p^*] & \text{if } d\ge 3\,, \end{cases} \quad p^* = \frac{d}{d-2} = \frac{2^*}{2}$$

▷ Up to translations, multiplications by a constant and scalings, *there is a unique optimal function* which also provides the value of the optimal constant.

$$\boxed{\mathsf{g}(x) = \left(1+|x|^2\right)^{-\frac{1}{p-1}}}$$

▷ The Sobolev Case $p = p*$ was obtained by [Aubin, Talenti (1976)]. . .

. . . and (before) by [Rodemich (1966)], while the general case was established in 2002

**Theorem (Optimal GNS** [Del Pino - Dolbeault (2002)])

*Equality case in* (GNS) *is achieved if and only if*

$$f \in \mathfrak{M} := \left\{ g_{\lambda,\mu,y} : (\lambda,\mu,y) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d \right\}$$

*Aubin-Talenti functions:* $\boxed{g_{\lambda,\mu,y}(x) := \mu\,\mathsf{g}\left((x-y)/\lambda\right)}$.

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*and optimal constant*

$$\mathscr{C}_{\rm GNS}(p) = \frac{\left(\frac{4\,d}{p-1}\,\pi\right)^{\frac{\theta}{2}} \left(2\,(p+1)\right)^{\frac{1-\theta}{p+1}}}{\left(d+2-p\,(d-2)\right)^{\frac{d-p\,(d-4)}{2\,p\,(d+2-p\,(d-2))}}}\,\Gamma\left(\frac{2\,p}{p-1}\right)^{-\frac{\theta}{d}}\,\Gamma\left(\frac{2\,p}{p-1}-\frac{d}{2}\right)^{\frac{\theta}{d}}\,.$$

## The stability result of G. Bianchi and H. Egnell

In Sobolev's inequality (with optimal contant $\mathsf{S}_d$),

$$\delta[f] := \mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq 0$$

*is there a natural way to bound the l.h.s. from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions* when $d \geq 3$ ?
A question raised in [Brezis-Lieb (1985)]

▷ [Bianchi-Egnell (1991)] There is a positive constant $\alpha$ such that

$$\mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \alpha \inf_{\varphi \in \mathcal{M}} \|\nabla f - \nabla \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants $\alpha$ and $\kappa$ and $f \mapsto \lambda(f)$ such that

$$\mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \left(1 + \kappa \, \lambda(f)^\alpha\right) \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

▷ $L^q$-norm of gradient [Figalli, Maggi, Pratelli (2010,13)], [Figalli, Neumayer (2018)], [Neumayer (2020)], [Figalli, Zhang (2020)]
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## Deficit, scale invariance

**Deficit functional** *(Non-scale invariant Gagliardo-Nirenberg-Sobolev inequalities)*

$$\delta[f] := \underbrace{(p-1)^2}_{a} \|\nabla f\|_2^2 + \underbrace{4\,\frac{d-p\,(d-2)}{p+1}}_{b} \|f\|_{p+1}^{p+1} - \mathcal{K}_{\mathrm{GNS}} \|f\|_{2\,p}^{2\,p\gamma}$$

**Lemma**

(GNS) *is equivalent to* $\delta[f] \ge 0$ *if and only if*

$$\mathcal{K}_{\mathrm{GNS}} = C(p,d)\,\mathscr{C}_{\mathrm{GNS}}^{2\,p\,\gamma}$$

*where* $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$ *and* $C(p,d)$ *is an explicit positive constant*

Take $f_\lambda(x) = \lambda^{\frac{d}{2p}} f(\lambda\,x)$ and optimize on $\lambda > 0$

$$\delta[f] \ge \delta[f] - \inf_{\lambda>0} \delta[f_\lambda] = C(p,d) \left[ \|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \right]^{2p\gamma} - C(p,d)\,\mathscr{C}_{\mathrm{GNS}}^{2p\gamma}(p)\, \|f\|_{2\,p}^{2\,p\gamma} \ge 0\,.$$

▷ A simplification: $\delta[f] = \delta[|f|]$ so we shall assume that $f \ge 0$ a.e.

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## An abstract stability result

Relative entropy

$$\mathscr{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left( f^{p+1} - g^{p+1} - \frac{1+p}{2p} g^{1-p} \left( f^{2p} - g^{2p} \right) \right) dx$$

Deficit functional

$$\delta[f] := a \, \|\nabla f\|_2^2 + b \, \|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}} \, \|f\|_{2p}^{2p\gamma} \geq 0$$

**Theorem** (Abstract Stability for GNS [BDNS (2020)])

*Let* $d \geq 1$ *and* $p \in (1, p^*)$. *There is a* $\mathscr{C} > 0$ *such that*

$$\delta[f] \geq \mathscr{C} \, \mathscr{F}[f]$$

*for any* $f \in \mathscr{W} := \{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2 \, dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx) \}$ *such that*

$$\int_{\mathbb{R}^d} f^{2p}(1,x) \, dx = \int_{\mathbb{R}^d} |g|^{2p}(1,x) \, dx$$

## Relative entropy, relative Fisher information

Idea of the proof of the Abstract Stability result:

▷ *Free energy* or *relative entropy functional*

$$\mathcal{E}[f|g] := \frac{2\,p}{1-p}\int_{\mathbb{R}^d}\Big(f^{p+1} - g^{p+1} - \tfrac{1+p}{2\,p}\,g^{1-p}\,\big(f^{2\,p} - g^{2\,p}\big)\Big)\,\mathrm{d}x$$

▷ *Relative Fisher information*

$$\mathcal{J}[f|g] := \frac{p+1}{p-1}\int_{\mathbb{R}^d}\big|(p-1)\,\nabla f + f^p\,\nabla g^{1-p}\big|^2\,\mathrm{d}x$$

It turns out that the GNS is nothing but a Entropy - Entropy Production inequality:

**Lemma** (Entropy - Entropy Production inequality [Del Pino - Dolbeault (2002)])

$$\frac{p+1}{p-1}\,\delta[f] = \mathcal{J}\big[f|g_f\big] - 4\,\mathcal{E}\big[f|g_f\big] \ge 0$$

## A weak stability result and the entropy controls $L^1$ distance

**Lemma** (A weak stability result [Dolbeault-Toscani (2016)])

$$\delta[f] \ge \delta_\star[f] \gtrsim \mathcal{E}[f|g]^2$$

$$\text{If} \quad \int_{\mathbb{R}^d} f^{2p}\,(1,x,|x|^2)\,\mathrm{d}x = \int_{\mathbb{R}^d} g^{2p}\,(1,x,|x|^2)\,\mathrm{d}x\,, \quad g \in \mathfrak{M}$$

$$\text{then} \quad \mathcal{E}[f|g] = \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1}-g^{p+1}\right)\mathrm{d}x \quad \text{and} \quad \delta_\star[f] \gtrsim \mathcal{E}[f|g]^2$$

**Lemma** (Csiszár-Kullback inequality [BDNS (2020)])

*Let $d \ge 1$ and $p > 1$. There exists a constant $C_p > 0$ such that*

$$\left\|f^{2p}-g^{2p}\right\|^2_{\mathrm{L}^1(\mathbb{R}^d)} \le C_p\,\mathcal{E}[f|g] \quad \textit{if} \quad \|f\|_{2p} = \|g\|_{2p}$$

▷ The proof uses also:

- the Carré du Champ method (nonlinear version of Bakry-Emery)
- Concentration Compactness (that is where “we lose the constant”). □

## A constructive stability result by the "flow method"

The relative entropy

$$\mathscr{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left( f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\, \mathrm{g}^{1-p} \left( f^{2p} - \mathrm{g}^{2p} \right) \right) \mathrm{d}x$$

The deficit functional

$$\delta[f] := a \, \|\nabla f\|_2^2 + b \, \|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}} \, \|f\|_{2p}^{2p\gamma} \geq 0$$

**Theorem** (Constructive Stability for GNS — BDNS (2020))

*Let $d \geq 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is an explicit constant $\mathscr{C} = \mathscr{C}(d, p, A, G) > 0$ such that*

$$\delta[f] \geq \mathscr{C} \, \mathscr{F}[f]$$

*for any $f \in \mathscr{W} := \left\{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2 \, \mathrm{d}x) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}x) \right\}$ such that*

$$\int_{\mathbb{R}^d} f^{2p} \, \mathrm{d}x = \int_{\mathbb{R}^d} |\mathrm{g}|^{2p} \, \mathrm{d}x \,, \quad \int_{\mathbb{R}^d} x \, f^{2p} \, \mathrm{d}x = 0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} f^{2p} \, dx \leq A \quad \text{and} \quad \mathscr{F}[f] \leq G$$

## The fast diffusion equation and the entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m$$

Letting

$$u = f^{2p} \qquad \text{so that} \qquad u^m = f^{p+1}$$

we have

$$p = \frac{1}{2m-1} \in (1, p^*] \quad \Longleftrightarrow \quad m = \frac{p+1}{2p} \in [m_1, 1)$$

- ▷ The Rényi entropy powers and the Gagliardo-Nirenberg inequalities: Nonlinear Carré du Champ method in original variables.
- ▷ Selfsimilar variables: the Nonlinear Fokker-Plank FDE Self-similar solutions and the entropy-entropy production method
- ▷ Large time asymptotics: spectral analysis (Hardy-Poincaré inequality) and improved rates of convergence to equilibrium.
- ▷ The initial time layer improvement: backward estimate. Bringing the asymptotic improvement as $t \to \infty$ back to $t = 0$.

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Consider the fast diffusion equation in $\mathbb{R}^d$, $d \geq 1$, $m \in (0,1)$

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with initial datum $u(t=0,x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \,\mathrm{d}x = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \,\mathrm{d}x < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$B(t,x) := \frac{1}{\left(\kappa\, t^{1/\mu}\right)^d} \mathscr{B}\left(\frac{x}{\kappa\, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$, $\kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$ and $\mathscr{B}$ is the Barenblatt profile

$$\mathscr{B}(x) := \left(C + |x|^2\right)^{-\frac{1}{1-m}}$$

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## Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \geq m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\,\mathrm{d}x = 0$$

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## From the carré du champ method to stability results

▷ ***Carré du champ method*** (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m\,, \quad \frac{d\mathsf{E}}{dt} = -\mathsf{I}\,, \quad \frac{d\mathsf{I}}{dt} \le -\Lambda\,\mathsf{I}$$

deduce that $\mathsf{I} - \Lambda\,\mathsf{F}$ is monotone non-increasing with limit 0

$$\boxed{\mathsf{I}[u] \ge \Lambda\,\mathsf{F}[u]}$$

**Consequence:** $\boxed{\mathsf{I} - \Lambda\,\mathsf{F} \ge 0}$ is equivalent to sharp GNS $\boxed{\delta[f] \ge 0}$

▷ ***Improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star \ge \Lambda$ such that

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We use linearization and *improved Hardy-Poincaré Inequalities*

▷ ***Improved entropy – entropy production inequality*** (weaker form)

$$\mathsf{I} \ge \Lambda\,\psi(\mathsf{F})$$

for some $\psi$ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

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## Self-similar variables: entropy-entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\Big(\tau,\frac{x}{\kappa\,R}\Big)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu}\,,\quad\tau(t):=\tfrac12\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

(1) $$\frac{\partial v}{\partial \tau}+\nabla\cdot\Big[v\left(\nabla v^{m-1}-2\,x\right)\Big]=0$$

*Generalized entropy (free energy)* and *Fisher information*

$$\mathscr F[v]:=-\frac1m\int_{\mathbb R^d}\left(v^m-\mathscr B^m-m\,\mathscr B^{m-1}\,(v-\mathscr B)\right)\mathrm dx$$

$$\mathscr I[v]:=\int_{\mathbb R^d}v\left|\nabla v^{m-1}+2\,x\right|^2\mathrm dx$$

are such that $\boxed{\mathscr I[v]\ge 4\,\mathscr F[v]}$ by (GNS) [Del Pino-Dolbeault (2002)] so that

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## Spectral gap: sharp asymptotic rates of convergence

[Blanchet, MB, Dolbeault, Grillo, Vázquez, BBDGV (2009) and BDGV (2010)]

(H) $$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,\mathrm{d}x$, for $\alpha<0$

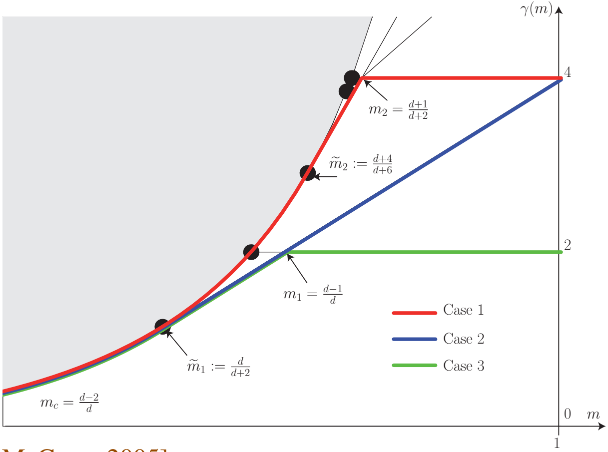
**Lemma ([BBDGV (2009), BDGV (2010)])**

*Under assumption* (H),

$$\mathcal{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\; t\ge 0\,, \quad \gamma(m):=(1-m)\,\Lambda_{1/(m-1),d}$$

*Moreover* $\gamma(m):=2$ *if* $\frac{d-1}{d}=m_1\le m<1$

## Spectral gap




[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [Dolbeault, Toscani, 2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

## The asymptotic time layer improvement

▷ *Linearized free energy* and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,dx$$

[ *Weighted linearization: consider* $v = \mathscr{B} + \mathsf{h}\mathscr{B}^{2-m} g$ *as* $\mathsf{h} \to 0$ ]

▷ *Hardy-Poincaré inequality*. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in L^2(\mathbb{R}^d, \mathscr{B}^{2-m}\,dx)$ such that $\nabla g \in L^2(\mathbb{R}^d, \mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,dx = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,dx = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

**Proposition (Asymptotic time layer improvement** [BDNS (2021)]**)**

*Let* $m \in (m_1, 1)$ *if* $d \ge 2$, $m \in (1/3, 1)$ *if* $d = 1$, $\eta = 2\,(d\,m - d + 1)$ *and* $\chi = m/(266 + 56\,m)$. *If* $\int_{\mathbb{R}^d} v\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ *and*

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

*for some* $\varepsilon \in (0, \chi\,\eta)$, *then*

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# The initial time layer improvement: backward estimate

▷ A hint: for some strictly convex function $\psi$ with $\psi(0)=\psi'(0)=0$, we have

$$\mathscr{I}-4\,\mathscr{F}\ge 4\left(\psi(\mathscr{F})-\mathscr{F}\right)\ge 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement

▷ Rephrasing the *carré du champ* method:

$$\mathscr{Q}[v]:=\frac{\mathscr{I}[v]}{\mathscr{F}[v]}\qquad\text{is such that}\qquad \boxed{\frac{d\mathscr{Q}}{dt}\le\mathscr{Q}\,(\mathscr{Q}-4)}$$

**Lemma** (Initial time layer improvement [BDNS (2021)])

*Assume that $m>m_1$ and $v$ is a solution to (1) with nonnegative initial datum $v_0$. If for some $\eta>0$ and $T>0$, we have $\mathscr{Q}[v(T,\cdot)]\ge 4+\eta$, then*

$$\boxed{\mathscr{Q}[v(t,\cdot)]\ge 4+\frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}}\qquad\forall\,t\in[0,T]}$$

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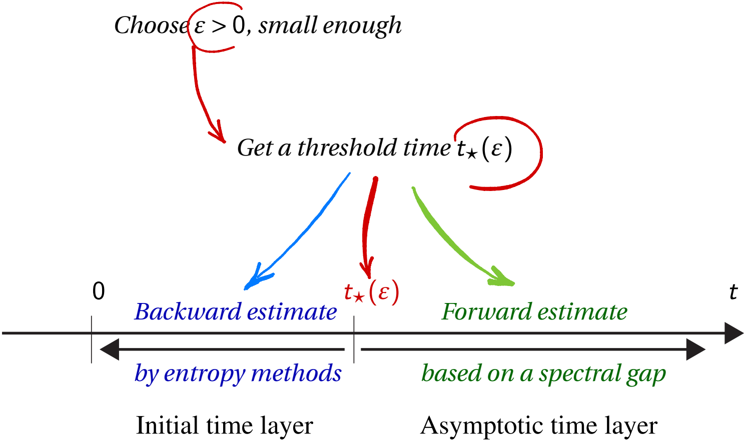
## Stability in (subcritical) Gagliardo-Nirenberg inequalities

*Our strategy*



## The threshold time and the uniform convergence in relative error (UCRE)

**Theorem** (Uniform convergence in relative error [BDNS (2021)])

*Assume that* $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$ *and let* $\varepsilon \in (0, 1/2)$, *small enough,* $A > 0$, *and* $G > 0$ *be given. There exists an explicit threshold time* $t_\star \geq 0$ *such that, if* $u$ *is a solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

*with nonnegative initial datum* $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *satisfying*

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \tag{$\mathrm{H}_A$}$$

$\int_{\mathbb{R}^d} u_0\,\mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{B}\,\mathrm{d}x = \mathcal{M}$ *and* $\mathcal{F}[u_0] \leq G$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{\mathcal{B}(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq t_\star$$

## The threshold time

**Proposition (Explicit threshold time [BDNS (2021)])**

*Let* $m\in(m_1,1)$ *if* $d\ge2$, $m\in(1/3,1)$ *if* $d=1$, $\varepsilon\in(0,\varepsilon_{m,d})$, $A>0$ *and* $G>0$

$$t_\star=\mathsf{c}_\star\,\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

*where* $\mathsf{a}=\frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}$, $\alpha=d\,(m-m_c)$ *and* $\vartheta=\nu/(d+\nu)$

$$\mathsf{c}_\star=\mathsf{c}_\star(m,d)=\sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\,\varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\,\varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m):=\max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1},\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m):=\frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\quad\text{and}\quad\kappa_3(\varepsilon,m):=\frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

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## Global Harnack Principle and Uniform Convergence in Relative Error

The proof of UCRE requires various constructive regularity estimates:

**Theorem (Characterization of GHP and UCRE** [MB-Simonov (2021)]**)**

*Assume that* $m \in (m_c, 1)$ *where* $m_c := \frac{d-2}{d}$, *and if* $u$ *is a solution to the Cauchy problem for* (FDE). *Then the following assertions are equivalent*
*(i) The initial datum satisfies the tail condition* $\mathrm{H}_A$, *namely*

(H$_A$) $$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u_0 \, dx < \infty$$

*(i') The solution satisfies the tail condition* $\mathrm{H}_A$, *at some time* $t_0 \in [0,\infty)$.
*(ii) The Global Harnack Principle holds true:* $\exists \tau_1, \tau_2, M_1, M_2 > 0$ *such that*

(GHP) $$\mathcal{B}_{M_1}(t-\tau_1, x) \le u(t,x) \le \mathcal{B}_{M_2}(t+\tau_2, x),$$

*(iii) The solution "converges" uniformly in relative error to the Barenblatt solution with the same mass:*

(UCRE) $$\lim_{t\to\infty} \left\| \frac{u(t) - \mathcal{B}_M(t)}{\mathcal{B}_M(t)} \right\|_{L^\infty(\mathbb{R}^d)} = 0$$

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## More about Global Harnack Principle

▷ If the tail condition $\mathrm{H}_A$ is not satisfied, GHP and UCRE are not true:

$$\mathcal{B}_M(t,x) \lneqq u_0(x) = \frac{1}{(1+|x|^2)^{\frac{m}{1-m}}},$$

then the solution $u(t,x)$ with initial data $u_0$ satisfies

$$\mathcal{B}_M(t,x) \lneqq \frac{1}{\left[(c\,t+1)^{\frac{1}{1-m}}+|x|^2\right]^{\frac{m}{1-m}}} \le u(t,x) \le \frac{(1+t)^{\frac{m}{1-m}}}{\left(1+t+|x|^2\right)^{\frac{m}{1-m}}},$$

Recall that $\mathcal{B}_M(t,x) \sim |x|^{-\frac{2}{1-m}}$

▷ the GHP was first proven by [Vazquez (2003)] for radial functions, then by [MB-Vazquez (2006)] under non-sharp conditions on the data. [Carrillo-Vazquez (2003)] introduced a condition a posteriori equivalent to $\mathrm{H}_A$ and conjectured that it was sharp.

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# Ideas of the proof of the constructive UCRE and of the threshold time

▷ The GHP implies UCRE on "outer cylinders" of the type $\{|x| \geq C t^{\vartheta}\}$.

▷ To obtain UCRE on "inner cylinders" of the type $\{|x| \leq C t^{\vartheta}\}$, we pass to selfsimilar variables, so that the cylider becomes $\{|x| \leq C\}$ and we have

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$$v_t = \mathrm{div}\,(A(t,x)\,\nabla v) \qquad \text{with} \qquad 0<\lambda_0 \le A(t,x) = u^{m-1}(t,x) \le \lambda_1$$

  Explicit constant in Moser's Harnack ineq. [Moser (1964,71)], explicit Hölder exponents.

## Improved entropy – entropy production inequality: already a stability result

**Theorem** (Improved entropy – entropy production inequality [BDNS (2021)])

*Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number $\zeta$ such that*

$$\mathscr{I}[v] \ge (4+\zeta)\,\mathscr{F}[v]$$

*for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v] = G$, $\int_{\mathbb{R}^d} v\,\mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,\mathrm{d}x = 0$ and $v$ satisfies $(H_A)$*

We have the *asymptotic time layer estimate*

$$\varepsilon \in (0, 2\varepsilon_\star)\,, \quad \varepsilon_\star := \frac{1}{2}\min\{\varepsilon_{m,d}, \chi\,\eta\} \quad \text{with} \quad T = \frac{1}{2}\log R(t_\star)$$

$$(1-\varepsilon)\,\mathscr{B} \le v(t,\cdot) \le (1+\varepsilon)\,\mathscr{B} \quad \forall\, t \ge T$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)] \ge (4+\zeta)\,\mathscr{F}[v(t,.)] \quad \forall\, t \in [0,T]\,, \quad \text{where} \quad \zeta = \frac{4\,\eta\,e^{-4T}}{4+\eta-\eta\,e^{-4T}}$$

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## Two consequences

$$\zeta = \mathrm{Z}\big(A, \mathcal{F}[u_0]\big), \quad \mathrm{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\,\frac{2}{\alpha}} + G}, \quad \zeta_\star := \frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^{\mathrm{a}}}{2\,\alpha\,\mathrm{c}_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

**Corollary** (Improved rates of convergence [BDNS (2021)])

*Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If $v$ is a solution of (1) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,\mathrm{d}x = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,\mathrm{d}x = 0$ and $v_0$ satisfies $(H_A)$, then*

$$\mathcal{F}[v(t,.)] \le \mathcal{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \ge 0$$

▷ The ***stability in the entropy - entropy production estimate:*** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \ge \zeta\,\mathcal{F}[v]$ also holds in a stronger sense (bounded below by Fisher information)

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## Constructive stability results I - Subcritical Case - Entropy Version

**Theorem (Constructive stability I for Gagliardo-Nirenberg [BDNS (2020)])**

*Let* $d \geq 1$, $p \in (1, p^*)$, *where* $p^* = +\infty$ *if* $d = 1$ *or* $2$, $p^* = \frac{d}{d-2}$ *if* $d \geq 3$.

$$\text{If } f \in \mathscr{W}_p(\mathbb{R}^d) := \left\{ f \in \mathrm{L}^{2p}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\, f^p \in \mathrm{L}^2(\mathbb{R}^d) \right\},$$

$$\left( \|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \right)^{2p\gamma} - \left( \mathscr{C}_{\mathrm{GN}} \, \|f\|_{2p} \right)^{2p\gamma} \geq \mathfrak{S}[f] \, \|f\|_{2p}^{2p\gamma} \, \mathsf{E}[f]$$

where

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1} \, \frac{\mathsf{Z}\big(\mathsf{A}[f], \mathsf{E}[f]\big)}{C(p,d)} = \frac{\mathsf{k}_{p,d}\, \zeta_\star}{1 + \mathsf{A}[f]^{(1-m)\frac{2}{\alpha}} + \mathsf{E}[f]}$$

$$\mathsf{E}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left( \frac{\kappa[f]^{p+1}}{\lambda[f]^{d\frac{p-1}{2p}}} f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p} \mathrm{g}^{1-p} \left( \frac{\kappa[f]^{2p}}{\lambda[f]^2} f^{2p} - \mathrm{g}^{2p} \right) \right) dx$$

$$\mathsf{A}[f] := \frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}} \, \|f\|_{2p}^{2p}} \sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x + x_f)|^{2p} \, dx$$

$$\lambda[f] := \left( \frac{2d\kappa[f]^{p-1}}{p^2-1} \, \frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2} \right)^{\frac{2p}{d-p(d-4)}}, \qquad \kappa[f] := \frac{\mathscr{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

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## Constructive stability results II - Subcritical Case - Gradient Version

With $\mathscr{K}_{\mathrm{GNS}} = C(p,d)\,\mathscr{C}_{\mathrm{GNS}}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GNS}}\,\|f\|_{2p}^{2p\gamma}$$

**Theorem** (Constructive stability II for Gagliardo-Nirenberg [BDNS (2020)])

*Let $d \ge 1$ and $p \in (1,p^*)$. There is an explicit $\mathscr{C} = \mathscr{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathrm{A}\big[f^{2p}\big] < \infty$,*

$$\delta[f] \ge \mathscr{C}[f]\,\inf_{\varphi\in\mathfrak{M}}\int_{\mathbb{R}^d}\left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 \mathrm{d}x$$

▷ The dependence of $\mathscr{C}[f]$ on $\mathrm{A}\big[f^{2p}\big]$ and $\mathscr{F}\big[f^{2p}\big]$ is explicit and does not degenerate if $f \in \mathfrak{M}$

▷ Can we remove the condition $\mathrm{A}\big[f^{2p}\big] < \infty$ ? Not with this method :(

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## A constructive stability result by the “flow method” (from the beginning)

The relative entropy

$$\mathscr{F}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\,\mathrm{g}^{1-p}\left(f^{2p} - \mathrm{g}^{2p}\right)\right)dx$$

The deficit functional

$$\delta[f] := a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}}\,\|f\|_{2p}^{2p\gamma} \ge 0$$

**Theorem** (Constructive Stability for GNS — BDNS (2020))

*Let $d \ge 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is an explicit constant $\mathscr{C} > 0$ such that*

$$\boxed{\delta[f] \ge \mathscr{C}\,\mathscr{F}[f]} \qquad \textit{with} \qquad \boxed{\mathscr{C} = \frac{\mathsf{k}_{p,d}}{1 + \mathsf{A}^a + G}}$$

*for any* $f \in \mathscr{W} := \left\{f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2\,dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx)\right\}$ *such that*

$$\int_{\mathbb{R}^d} f^{2p}\,dx = \int_{\mathbb{R}^d} |\mathrm{g}|^{2p}\,dx, \quad \int_{\mathbb{R}^d} x\,f^{2p}\,dx = 0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} f^{2p}\,dx \le A \quad \textit{and} \quad \mathscr{F}[f] \le G$$

## Constructive Stability in Sobolev's inequality (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathscr{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\ |x|\, f^{p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

**Theorem** (Constructive stability for Sobolev [BDNS (2021)])

*Let $d \ge 3$ and $A > 0$. Then for any nonnegative $f \in \mathscr{W}_{p^\star}(\mathbb{R}^d)$ such that*

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, \mathrm{d}x = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, \mathrm{d}x \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, \mathrm{d}x \le A$$

*we have*

$$\delta[f] := \|\nabla f\|_2^2 - \mathsf{S}_d^2\, \|f\|_{2^*}^2 \ge \frac{\mathscr{C}_\star(A)}{4 + \mathscr{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 \mathrm{d}x$$

$\mathscr{C}_\star(A) = \mathfrak{C}_\star \left(1 + A^{1/(2\,d)}\right)^{-1}$ *and $\mathfrak{C}_\star > 0$ depends only on $d$*

We can remove the normalization of $f$, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \left(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\right)$$

the *Bianchi-Egnell type result*

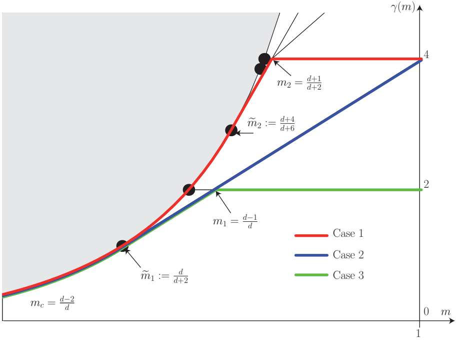
$$\delta[f] \geq \frac{\mathfrak{C}_\star\, Z[f]}{4+Z[f]} \inf_{g\in\mathfrak{M}} \mathscr{J}[f|g]$$

with $x_f$, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

## Idea of the proof: Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathscr{B}_\lambda(x) = \lambda^{-d/2}\,\mathscr{B}\big(x/\sqrt{\lambda}\big)$ in order to match $\int_{\mathbb{R}^d} |x|^2\, v\, dx$ where the function $v$ solves (1) or to further rescale $v$ according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\, w\left(t+\tau(t), \frac{x}{\mathfrak{R}(t)}\right),$$




$$\frac{d\tau}{dt} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d} |x|^2\, v\, dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,, \quad \tau(0) = 0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\,\tau(t)}$$

**Lemma (Delay estimates** [BDNS (2021)]**)**

*$t \mapsto \tau(t)$ is bounded on $\mathbb{R}^+$ (explicit estimates)*

# The End

## Thank You!!!

## Grazie Mille!!!

## Dankeschön!!!

# References

Download slides and papers at: http://verso.mat.uam.es/~matteo.bonforte

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## Entropy growth rate VS Gagliardo-Nirenberg-Sobolev inequalities

*Gagliardo-Nirenberg-Sobolev inequalities*

(GNS) $$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathscr{C}_{\rm GNS}(p) \, \|f\|_{2p}$$

$$p = \frac{1}{2m-1} \in (1, p^*] \quad \Longleftrightarrow \quad m = \frac{p+1}{2p} \in [m_1, 1)$$

$u = f^{2p}$ so that $u^m = f^{p+1}$ and $u\,|\nabla u^{m-1}|^2 = (p-1)^2\,|\nabla f|^2$

$$M = \|f\|_{2p}^{2p}, \quad \mathsf{E}[u] = \|f\|_{p+1}^{p+1}, \quad \mathsf{I}[u] = (p+1)^2 \, \|\nabla f\|_2^2$$

If $u$ solves (FDE) $\frac{\partial u}{\partial t} = \Delta u^m$

$$\mathsf{E}' \geq \frac{p-1}{2p} (p+1)^2 \left(\mathscr{C}_{{\rm GNS}(p)}\right)^{\frac{2}{\theta}} \, \|f\|_{2p}^{\frac{2}{\theta}} \, \|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}} = C_0 \, \mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d} u^m(t,x)\,dx \geq \left(\int_{\mathbb{R}^d} u_0^m\,dx + \frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}} \quad \forall\, t \geq 0$$

Equality case: $u(t,x) = \frac{c_1}{R(t)^d}\,\mathscr{B}\left(\frac{c_2\,x}{R(t)}\right)$, $\mathscr{B}(x) := \left(1+|x|^2\right)^{\frac{1}{m-1}}$

## Nonlinear carré du champ method. Decay of the Fisher information

The $t$-derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\frac{1}{1-m}-1}$ is

$$\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

- *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

- *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$$

If $u$ solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla\mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta\mathsf{P} + |\nabla\mathsf{P}|^2\Big)\,dx$$

$$= -2\int_{\mathbb{R}^d} u^m\left(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta\mathsf{P})^2\right)dx$$

## Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares

$$-\frac{\mathsf{I}}{2\theta}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$
$$=\int_{\mathbb{R}^d}u^m\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2\mathrm{d}x+(m-m_1)\int_{\mathbb{R}^d}u^m\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2\mathrm{d}x$$

▷ Analysis of the asymptotic regime

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^{\theta}\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{M^{\frac{2\theta}{p}}}=\frac{\mathsf{I}[\mathscr{B}]^{\theta}\,\mathsf{E}[\mathscr{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathscr{B}\|_1^{\frac{2\theta}{p}}}=(p+1)^{2\theta}\left(\mathscr{C}_{\mathrm{GNS}}(p)\right)^{2\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^{\theta}\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\geq(p+1)^{2\theta}\left(\mathscr{C}_{\mathrm{GNS}}(p)\right)^{2\theta}M^{\frac{2\theta}{p}}$$

# The constant in Moser's Harnack inequality 1/3

Let $\Omega$ be an open domain and let us consider a nonnegative *weak solution* to

$$\frac{\partial v}{\partial t} = \nabla \cdot \big(A(t,x)\,\nabla v\big) \tag{2}$$

on $\Omega_T := (0,T)\times\Omega$, where $A(t,x)$ is a real symmetric matrix with bounded measurable coefficients satisfying the *uniform ellipticity condition*

$$0 < \lambda_0\,|\xi|^2 \le \xi\cdot(A\,\xi) \le \lambda_1\,|\xi|^2 \quad \forall\,(t,x,\xi)\in\mathbb{R}^+\times\Omega_T\times\mathbb{R}^d\,, \tag{3}$$

where $\xi\cdot(A\xi) = \sum_{i,j=1}^d A_{i,j}\xi_i\xi_j$ and $\lambda_0, \lambda_1$ are positive constants.

## The constant in Moser's Harnack inequality 2/3

Let us consider the neighborhoods

$$D_R^+(t_0,x_0) := (t_0 + \tfrac{3}{4}R^2, t_0 + R^2) \times B_{R/2}(x_0)\,,$$
$$D_R^-(t_0,x_0) := \left(t_0 - \tfrac{3}{4}R^2, t_0 - \tfrac{1}{4}R^2\right) \times B_{R/2}(x_0)\,, \tag{4}$$

We claim that the following *Harnack inequality* holds [Moser (1964,71)]:

**Theorem (Parabolic Harnack inequality [BDNS (2020,21)])**

*Let $T>0$, $R\in(0,\sqrt{T})$, and take $(t_0,x_0)\in(0,T)\times\Omega$ such that $\left(t_0-R^2,t_0+R^2\right)\times B_{2R}(x_0)\subset\Omega_T$. Under Assumption (3), if $v$ satisfies*

$$\iint_{(0,T)\times\Omega}\left(-\varphi_t\,v+\nabla\varphi\cdot(A\nabla v)\right)\mathrm{d}x\,\mathrm{d}t=0 \tag{5}$$

*for any $\varphi\in C_c^\infty((0,T)\times\Omega)$, then*

$$\sup_{D_R^-(t_0,x_0)} v \le \overline{\mathsf{h}} \inf_{D_R^+(t_0,x_0)} v\,. \tag{6}$$

▷ This result is known from [Moser (1964,71)]. However, to the best of our knowledge, a complete constructive proof and an expression of $\overline{\mathsf{h}}$ was still missing.

## The constant in Moser's Harnack inequality 3/3

The constant in Moser's Harnack inequality has the expression

$$\overline{\mathsf{h}} := \mathsf{h}^{\lambda_1 + \lambda_0^{-1}}. \tag{7}$$

where

$$\mathsf{h} := \exp\left[2^{d+4}\, 3^d\, d + c_0^3\, 2^{2\,(d+2)+3}\left(1 + \frac{2^{d+2}}{(\sqrt{2}-1)^{2\,(d+2)}}\right)\sigma\right] \tag{8}$$

where

$$c_0 = 3^{\frac{2}{d}}\, 2^{\frac{(d+2)\,(3\,d^2+18\,d+24)+13}{2\,d}} \left(\frac{(2+d)^{1+\frac{4}{d^2}}}{d^{1+\frac{2}{d^2}}}\right)^{(d+1)(d+2)} \mathcal{K}^{\frac{2\,d+4}{d}}, \tag{9}$$

$$\sigma = \sum_{j=0}^{\infty} \left(\tfrac{3}{4}\right)^j \left((2+j)\,(1+j)\right)^{2\,d+4}. \tag{10}$$

The constant $\mathcal{K}$ is the constant in Sobolev embedding (explicit).

## Explicit Hölder continuity exponent

▷ It is well known that Harnack inequalities imply Hölder continuity of solutions.

▷ We obtain a quantitative expression of the Hölder continuity exponent, which only depends on the Harnack constant, *i.e.* on $d$, $\lambda_0$ and $\lambda_1$.

▷ Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$ be bounded domains and let $Q_1 := (T_2, T_3) \times \Omega_1 \subset (T_1, T_4) \times \Omega_2 =: Q_2$, where $0 \le T_1 < T_2 < T_3 < T < 4$. Define the *parabolic distance:*

(11) $$\mathrm{dist}(Q_1, Q_2) := \inf_{\substack{(t,x)\in Q_1 \\ (s,y)\in[T_1,T_4]\times\partial\Omega_2\cup\{T_1,T_4\}\times\Omega_2}} |x-y| + |t-s|^{\frac{1}{2}} .$$

**Theorem (Hölder Continuity with explicit exponents [BDNS (2020,21)])**

*Let $v$ be a nonnegative solution of (2) on $Q_2$ which satisfies (5) and assume that $A(t,x)$ satisfies (3). Then we have*

(12) $$\sup_{(t,x),(s,y)\in Q_1} \frac{|v(t,x)-v(s,y)|}{\left(|x-y|+|t-s|^{1/2}\right)^{\nu}} \le 2\left(\frac{128}{\mathrm{dist}(Q_1,Q_2)}\right)^{\nu} \|v\|_{\mathrm{L}^\infty(Q_2)} .$$

*where*

(13) $$\nu := \log_4\left(\frac{\overline{\mathrm{h}}}{\overline{\mathrm{h}}-1}\right),$$

*and $\overline{\mathrm{h}}$ is as in (7).*