Sharp Extinction Rates for Fast Diffusion Equations on Generic Bounded Domains

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The Fields Institute, Toronto, Canada
Online Event, October 19 - 23, 2020
http://www.fields.utoronto.ca/activities/20-21/fieldsmedalsym

2010: BIRS Banff

2011: UIMP Santander

2016: The Application!

2012: The Theory…

... and after-after the Medal!

2018: Rio, after the Medal...

2019: Barcelona DHC

The Dirichlet Problem for Diffusion Equations in $\Omega \subset \mathbb{R}^N$

We consider, in a bounded and smooth domain $\Omega \subset \mathbb{R}^N$ solutions to

$$\begin{cases} \partial_\tau u = \frac{1}{m}\Delta\left(u^m\right) = \nabla \cdot \left(u^{m-1}\nabla u\right) , & \text{in } (0,+\infty)\times\Omega \\ u(0,\cdot) = u_0 \ , & \text{in } \Omega \\ u = 0 \ , & \text{on } (0,+\infty)\times\partial\Omega \end{cases}$$

Existence and uniqueness of weak solutions for this problem is well understood:

- $m = 1$: Heat Equation. (Fourier)
- $m > 1$: Porous Medium regime (slow diffusion, finite speed of propagation)
- $0 < m < 1$: Fast Diffusion regime (extinction in finite time)
  Complete references for PME/FDE: Vázquez books (2006-07).
- $m \le 0$: Ultra Fast Diffusion regime. (possible non-existence)
  - **Dirichlet problem: solutions fail to exists**, Vázquez (1992).
  - Cauchy problem: non-existence for L^1-data, Vázquez (1992).
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About Nonlinear Diffusions

$$u_t = \nabla\cdot\left(a(t,x)\nabla u\right) \qquad \underline{u \geq 0}$$

DIFF. COEFF

- $a = 1$ Heat Eq.
- $a \simeq 1$ Uniform parabolic

Density dep. diff. a may depend on u.

$$a(t,x) = u^{m-1}$$

$u \simeq 0$ $m > 1$ $a \simeq 0$ Diffusion "stop" $\rightsquigarrow$ F.S. Propag.

$u \simeq 0$ $m < 1$ $a \simeq +\infty$ super diffusive. $\rightsquigarrow$ Extinction in finite time

Asymptotic behaviour of the Heat Equation $u_t = \Delta u$. (Fourier ∼1822)

Let $u_0 \geq 0$ and let (λ_k, ϕ_k) be the eigen-elements of the Dirichlet Laplacian:

$$u(\tau,x) = \sum_{k=1}^{\infty} \mathrm{e}^{-\lambda_k \tau} \hat{u}_{0,k}\, \phi_k(x) \qquad \text{where} \qquad \hat{u}_{0,k} = \int_\Omega u_0 \phi_k \,\mathrm{d}x$$

From the above formula it is quite simple to deduce that

$$\mathrm{e}^{\lambda_1 \tau} u(\tau,\cdot) \xrightarrow[\tau\to\infty]{} \hat{u}_{0,1}\phi_1 \qquad \text{in } \mathrm{L}^p(\Omega),\ \forall p \in [1,\infty]$$

Actually, one can do better:

Relative Error Convergence

$$\left\| \frac{u(\tau,\cdot)}{\mathrm{e}^{\lambda_1 \tau}\hat{u}_{0,1}\phi_1} - 1 \right\|_{\mathrm{L}^\infty} \leq \mathrm{e}^{-(\lambda_2-\lambda_1)\tau} \underbrace{\sum_{k=2}^{\infty} \mathrm{e}^{-(\lambda_k-\lambda_2)\tau} \left|\hat{u}_{0,k}\right| \left\| \frac{\phi_k}{\phi_1} \right\|_{\mathrm{L}^\infty(\Omega)}}_{<+\infty}$$

- Essential ingredients: second spectral gap $\lambda_2 - \lambda_1 > 0$, and boundary behaviour of eigenfunctions: $|\phi_k| \lesssim |\phi_1|$
- Sharp result. Relies on representation formula, Fourier and Spectral analysis
- More general linear operators can be treated essentially in the same way.

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SEP. VAR. $\quad u(\tau,x) = T(\tau)\,\phi(x)$

$$\partial_\tau u = T'\phi = T\,\Delta\phi = \Delta u$$

$$\begin{cases} T' = -\lambda T \rightsquigarrow T(\tau) = T_0 e^{-\lambda\tau} \\ -\Delta\phi = \lambda\phi \overset{\text{SPEC. AN.}}{\rightsquigarrow} (\lambda_k, \phi_k) \end{cases} \qquad \phi_k \text{ is a basis in } L^2$$

- Projections / Orthog. Cond.

$$\frac{d}{dt}\langle u(\tau), \phi_k\rangle_{L^2} = \int_\Omega \partial_\tau u \cdot \phi_k \underset{(EQ)}{=} \int \Delta u\,\phi_k \underset{(PARTS)}{=} \int u\,\Delta\phi_k$$

$$= -\lambda_k \int u(\tau)\,\phi_k \implies \int u(\tau)\,\phi_k\,dx \overset{(*)}{=} e^{-\lambda_k\tau}\int u_0\,\phi_k.$$

$$u(\tau,x) = \sum_{k\geq 1}\langle u(\tau),\phi_k\rangle_{L^2}\,\phi_k(x) \overset{(*)}{=} \sum_{k\geq 1}\langle u_0,\phi_k\rangle_{L^2}\,e^{-\lambda_k\tau}\,\phi_k(x) \qquad \boxed{*}$$

Asymptotic behaviour of the Porous Medium Equation $u_t = \Delta u^m$, $m > 1$

The asymptotic behaviour is described in terms of the "Friendly Giant":

$$\mathcal{U}(\tau, x) = S(x)\,\tau^{-1/(m-1)}\,, \qquad \text{with} \qquad \mathcal{U}(0,x) = +\infty\,.$$

Here, S is the *unique* nonnegative solution to the stationary problem

$$\text{(EDP)} \qquad -\Delta S^m = \mathsf{c}\,S \quad \text{in } \Omega, \qquad S = 0 \quad \text{on } \partial\Omega,$$

with $\mathsf{c} = 1/(m-1) > 0$, since $m > 1$.

Logarithmic time rescaling:

$$t = \log(\tau + 1) \qquad \text{and} \qquad w(t,x) = \tau^{1/(m-1)} u(\tau, x)\,,$$

we transform the parabolic problem into

$$\begin{cases} \partial_t w(t,x) = \Delta w^m(t,x) + \mathsf{c}\,w(t,x)\,, & (t,x) \in (0,\infty) \times \Omega\,, \\ w(t,x) = 0\,, & (t,x) \in (0,\infty) \times \partial\Omega\,, \\ w(0,x) = u_0(x)\,, & x \in \Omega\,. \end{cases}$$

The separation of variables solution $\mathcal{U}$ becomes stationary.

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Asymptotic behaviour of the Porous Medium Equation (continued)

Relative Error Convergence, rescaled variables

$$\left\| \frac{w(t,\cdot)}{S} - 1 \right\|_{L^\infty(\Omega)} \lesssim e^{-t} \qquad \text{for all } t \gg 1$$

Relative Error Convergence, original variables

$$\left\| \frac{u(\tau,\cdot)}{\mathcal{U}(\tau,)} - 1 \right\|_{L^\infty(\Omega)} \le \frac{2}{m-1}\frac{t_0}{t_0+\tau} \qquad \text{for all } \tau \ge t_0$$

- $t_0 \sim \left(\int_\Omega u_0\phi_1 \, dx\right)^{-(m-1)}$, where ϕ_1 is the first eigenfunction of the Laplacian.
- The decay rate $1/\tau$ is sharp: it is realized by

$$\mathcal{U}(\tau+1,x) = \frac{S(x)}{(1+\tau)^{\frac{1}{m-1}}} \qquad \text{with initial datum} \qquad \mathcal{U}(1,x) = S(x).$$

- Result obtained first by Aronson-Peletier (1981), then generalized by Vázquez (2004). New (quantitative) proof for more general (even nonlocal) diffusion equations of PME-type, by M.B.-Figalli-Sire-Vázquez (2015-18).

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$$\left\| \frac{w(t,\cdot)}{S} - 1 \right\|_{L^\infty(\Omega)} \lesssim e^{-t} \qquad \text{for all } t \gg 1$$

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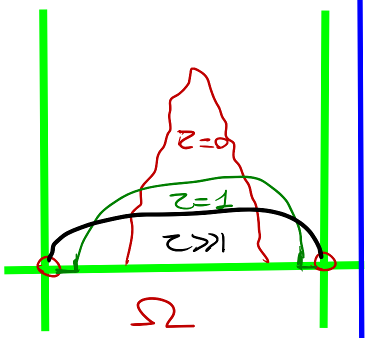
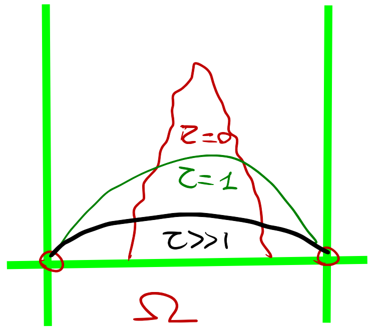
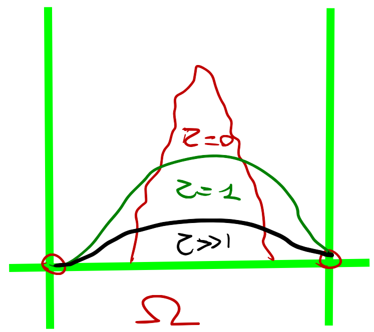
$\Omega \subset \mathbb{R}^N$ — $u_0 = u(\tau=0) \geq 0$ compact support $u_0 \in C_c^\infty(\Omega)$.

$m>1$ Porous Medium	$m=1$ Heat Eq.	$m<1$ Fast Diff.
		
Small times	**Small times**	
• Not positive everywhere in Ω (Finite speed of Propagation) (Free bdry) • Bounded: $\|u(\tau)\|_\infty \leq C \frac{\|u_0\|_1^{2\vartheta_1}}{\tau^{N\vartheta_1}}$, $\vartheta_1 = \frac{1}{2+N(m-1)}$	• Inst. positivity in Ω (No Free bdry) • "Remembers" the Cauchy Pb. (Bounded) $\|u(\tau)\|_\infty \leq C \frac{\|u_0\|_1}{\tau^{N/2}}$	• Inst. Positivity in Ω (?) Boundedness: Problem! $r > \frac{N(1-m)}{2}$ $\|u(\tau)\|_\infty \leq C \frac{\|u_0\|_r^{r\vartheta_r}}{\tau^{N\vartheta_r}}$, $\vartheta_r = \frac{1}{2r+N(m-1)}$
Large times.	**Large times**	
• Absolute Bounds: $\|u(t)\|_\infty \leq \frac{C}{t^{1/(m-1)}}$ • Positive everywhere in Ω • Behaves like Sep-Var: $u(\tau,x) \underset{\tau\to+\infty}{\approx} \frac{S(x)}{{}^{1/(m-1)}}$	• Exp. time decay! $\|u(t)\|_\infty \leq C e^{-\lambda_1 t}$ • Behaves like sep-var. $u(\tau,x) \underset{\tau\to+\infty}{\approx} e^{-\lambda_1 \tau}\, \phi_1(x)$	• EXTINCTION IN FINITE TIME! $T = T(u_0)$ • Does it behaves like sep. var?? $u(\tau,x) \overset{??}{\underset{\tau\to T^-}{\approx}} (T-\tau)^{1/(1-m)}\, S(x)$

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Asymptotic behaviour of the Fast Diffusion Equation, $m < 1$

The rescaled Problem and stationary solutions

$$\begin{cases} u_\tau = \Delta(u^m)\,, \\ u(0,\cdot) = u_0, \\ u_{|\partial\Omega} \equiv 0\,, \end{cases} \xrightarrow[\text{Time-Rescaling}]{} \begin{cases} w_t = \Delta(w^m) + \mathtt{c}w, \\ w(0,\cdot) = u_0, \\ w_{|\partial\Omega} \equiv 0\,, \end{cases}$$

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$$u(\tau,x) = \left(\frac{T-\tau}{T}\right)^{\frac{1}{1-m}} w(t,x) \qquad \text{with} \qquad t = T\log\left(\frac{T}{T-\tau}\right).$$

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We are interested in describing the behaviour near the extinction time:

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Recall that positive bounded stationary solution exist only when $m \in (m_s, 1)$, with $m_s = (N-2)/(N+2)$. Hence the analysis will be resticted to this range.

(First Pioneering Result) ***Berryman-Holland*** **(ARMA 1980)**

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Then for any sequence of times $t_n \to \infty$ as $n \to \infty$, there exist a stationary solution S such that

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DIFFICULTY: **Stationary solutions** need **not** to be **unique!**
Different time sequences can a priori converge to different stationary solutions.
However, the asymptotic profile may be unique also when the set of stationary solutions contains more than one element.

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(Convergence in Relative Error) *M.B.-Grillo-Vázquez* **(JMPA 2012)**

Let $m \in (m_s, 1)$ and let u be the solution to the Dirichlet problem and $T = T(m, d, u_0)$ be its extinction time. Let S be the positive classical solution to the elliptic problem (EDP), such that $\|w(t) - S\|_{\mathrm{L}^\infty(\Omega)} \to 0$ as $t \to \infty$. Then we have

$$\lim_{\tau \to T^-} \left\| \frac{u(\tau, \cdot)}{\mathcal{U}(\tau, \cdot)} - 1 \right\|_{\mathrm{L}^\infty(\Omega)} = \lim_{t \to \infty} \left\| \frac{w(t, \cdot)}{S} - 1 \right\|_{\mathrm{L}^\infty(\Omega)} = 0$$

where $\mathcal{U}(\tau, x) = S(x) \left[(T - \tau)/T \right]^{1/(1-m)}$.

Equivalently, the following improved Global Harnack Principle (GHP) holds

$$\boxed{c(\tau)^{-1}\, S(x)\, (T-\tau)^{1/(1-m)} \le u(\tau, x) \le c(\tau)\, S(x)\, (T-\tau)^{1/(1-m)}.}$$

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$$0 < c(\tau) \xrightarrow[\tau \to T^-]{} 1\,, \qquad \text{and} \qquad S(x) \asymp \mathrm{dist}\big(x, \partial\Omega\big)^{\frac{1}{m}}$$

The GHP (with constants $c(\tau) \not\to 1$) was firstly proven by DiBenedetto-Kwong-Vespri (Indiana 1991) and used as a key-tool for higher regularity estimates. It has been used more recently by Jin-Xiong to prove sharp boundary regularity (Preprint 2020).

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Rates of convergence

Let us define the set

$$\mathcal{O} := \left\{ \Omega \subset \mathbb{R}^N\,,\ \Omega \text{ open } \ : \ \overline{\Omega} \text{ is compact and } \partial\Omega \in C^{2,\alpha} \right\}$$

The topology on $\mathcal{O}$ can be defined through a family of neighborhoods as follows:

$$\mathcal{N}_\varepsilon(\Omega) := \{\Omega' \in \mathcal{O} \ : \ \exists\, \Phi \in C^{2,\alpha}(\mathbb{R}^N;\mathbb{R}^N) \text{ with } \|\Phi - \mathrm{Id}\|_{C^{2,\alpha}} < \varepsilon \text{ s.t. } \Omega' = \Phi(\Omega)\ \}.$$

(Sharp Rates of Convergence) ***M.B.-Figalli*** **(CPAM 2020, to appear)**

There exists a open and dense set in $\mathcal{G} \subset \mathcal{O}$ such that for any domain $\Omega \in \mathcal{G}$ the following holds. Let w be a bounded solution to the rescaled problem, and $m \in (m_s, 1)$. Let S be the stationary solution s. t. $\|w(t) - S\|_{\mathrm{L}^\infty(\Omega)} \to 0$ as $t \to \infty$. Then, there exist $\lambda_m, \kappa > 0$ such that for all $t_0 > 0$ and all $t \ge t_0$

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and the decay rate $\lambda_m > 0$ is sharp. Also, for all $t_0 > 0$ and all $t \ge t_0$

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Remark. When m is close to 1, more precisely for $m_\sharp < m < 1$, some first (non sharp) rates of convergence were obtained by M.B.- Grillo-Vázquez (JMPA 2012), by entropy methods based on quantitative continuity with respect to m. The expression of $m_\sharp \gneqq m_s = \frac{N-2}{N+2}$ is explicit, but complicated (it depends on the constant in the elliptic Harnack inequality).

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In what follows we change functions and parameters as follows: let

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so that

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Properties of Stationary Solutions

Consider positive classical solutions V to the homogeneous Dirichlet problem for

$$-\Delta V = \mathsf{c} V^p \qquad \text{with} \qquad 1 < p < \frac{N+2}{N-2}$$

recall that Ω is a regular domain of class $C^{2,\alpha}$.

- Existence is guaranteed in all the exponent range
- Boundedness is guaranteed via DeGiorgi-Nash-Moser techniques. Absolute bounds: there exists $C = C(\Omega) > 0$ such that $\|V\|_{L^\infty} \le C$. (Gidas-Spruck, DeFiguereido-Lions-Nussbaum, '80s)
- Nonnegative solutions are indeed positive in Ω by Harnack inequalities, and have the precise boundary behaviour $V \asymp \mathrm{dist}(\cdot, \partial\Omega)$.
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- Regularity: solutions are classical in the interior, even $C^\infty(\Omega)$, and regular up to the boundary, $C^{2,\alpha}(\overline{\Omega})$.
- Uniqueness depends on the domain.
  - Holds on balls
  - Does not hold on annuli
  - Several conditions are present in the literature.

Properties of Stationary Solutions

Consider positive classical solutions V to the homogeneous Dirichlet problem for

$$-\Delta V = \mathsf{c} V^p \qquad \text{with} \qquad 1 < p < \frac{N+2}{N-2}$$

recall that Ω is a regular domain of class $C^{2,\alpha}$.

- Existence is guaranteed in all the exponent range
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The linearized Problem

$w_0, v_0 \rightsquigarrow S(w_0) = V(v_0)$

Consider the linear parabolic (weighted) equation

$$pV^{p-1}\partial_t f = \Delta f + \mathsf{c}pV^{p-1}f \qquad \left(\xleftarrow[v=V+\varepsilon f]{\varepsilon\to 0} \qquad \partial_t v^p = \Delta v + \mathsf{c}v^p \right)$$

i.e. the linearization around the stationary state of the rescaled nonlinear FDE.

- Notice that *V is not a stationary solution* to the linearized equation: indeed $-\Delta V = \mathsf{c}V^p \neq \mathsf{c}pV^p$, since $p > 1$.
- Natural question: which are the stationary states of such equation?
- Stationary solutions φ must satisfy the homogeneous Dirichlet problem associated to the linear elliptic equation

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The Spectrum of $-\Delta$ on $\mathrm{L}^2_V := \mathrm{L}^2_{V^{p-1}}(\Omega)$

$(-\Delta)$ is a linear unbounded selfadjoint operator on L^2_V, associated to the Dirichlet form $Q(f) = \int_\Omega |\nabla f|^2 \,\mathrm{d}x$, and has a discrete spectrum:

$$0 < \lambda_{V,1} < \lambda_{V,2} < \cdots < \lambda_{V,k} < \lambda_{V,k+1} \to \infty$$

Denote by $\pi_{\mathbb{V}_k} : \mathrm{L}^2_V \to \mathbb{V}_k$ the projections on the eigenspaces $\mathbb{V}_k$, $N_k = \dim(\mathbb{V}_k)$, and by $\{\phi_{k,j}\}_{j=1,\ldots,N_k}$ the basis of $\mathbb{V}_k$ of normalized eigenfunctions.

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Finally, $\lambda_{V,1} = \mathsf{c} > 0$ and $\phi_{1,1} = V/\|V\|_{\mathrm{L}^2_V} = V/\|V\|^{(p+1)/2}_{\mathrm{L}^{p+1}}$.

All the eigenfunctions are smooth, $C^{2,\alpha}(\Omega) \cap C^\alpha(\overline{\Omega})$, and for all $x \in \Omega$:

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Linear Entropy Method and Improved Poincaré Inequalities

Consider solutions to the linearized equation: $\boxed{pV^{p-1}\partial_t f = \Delta f + \mathsf{c}pV^{p-1}f}$

Linear Entropy

$$\mathsf{E}[f] = \int_\Omega f^2 V^{p-1}\,\mathrm{d}x\,.$$

Linear Entropy-Production

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}[f(t)] = -\frac{2}{p}\left(\int_\Omega |\nabla f(t,x)|^2\,\mathrm{d}x - p\mathsf{c}\int_\Omega f^2(t,x)V^{p-1}(x)\,\mathrm{d}x\right) = -\frac{2}{p}\,\mathsf{I}[f(t)]$$

Improved Poincaré inequality $\rightsquigarrow$ Entropy-Entropy production inequality

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Linear Entropy Method and Improved Poincaré Inequalities (continued)

Entropy and Poincaré imply Exponential Decay

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{E}[f(t)] = -\frac{2}{p}\,\mathsf{I}[f(t)] \le -\frac{2}{p}\lambda_p \mathsf{E}[f(t)] \quad\Longrightarrow\quad \mathsf{E}[f(t)] \le \mathrm{e}^{-\frac{2}{p}\lambda_p t}\mathsf{E}[f(0)]$$

To prove the Improved Poincaré, we need some conditions:

(H2) There is no nontrivial solution (i.e. $\varphi \not\equiv 0$) to the homogeneous Dirichlet problem for the equation

$$-\Delta\varphi = \mathsf{c}pV^{p-1}\varphi\,.$$

Under assumption (H2), it is convenient to define the integer $k_p > 1$ as the biggest integer k for which $p\mathsf{c} > \lambda_{V,k}$, so that

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The orthogonality conditions are preserved along the linear flow.

In order to apply the improved Poincaré inequality to the solutions to the linear parabolic equation, we have to make sure that the orthogonality conditions are preserved along the evolution, namely:

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Assumption $(H2)$ is generically true

Assumption (H2) admits several equivalent statements:

- The homogeneous Dirichlet problem

$$-\Delta\varphi = \mathsf{c}pV^{p-1}\varphi \qquad \text{in } \Omega\,, \qquad \varphi = 0 \qquad \text{on } \partial\Omega\,,$$

  admits no nontrivial solution. Or, the only solution is $\varphi \equiv 0$.
- Equivalently: $\mathsf{c}p$ is not an eigenvalue for the Dirichlet Laplacian on L^2_V, i.e. $\mathsf{c}p \notin \mathrm{Spec}_{\mathrm{L}^2_V(\Omega)}(-\Delta)$.

This fact is not so easy to check in general, and it depends on the geometry of the domain. Indeed, it turns out that this result is generically true. Define

$$\mathcal{O} := \left\{\Omega \subset \mathbb{R}^N\,,\ \Omega \text{ open } : \ \overline{\Omega} \text{ is compact and } \partial\Omega \in C^{2,\alpha}\right\}$$

with the topology given by the family of neighborhoods

$$\mathcal{N}_\varepsilon(\Omega) := \left\{\Omega' \in \mathcal{O} \ : \ \exists\, \Phi \in C^{2,\alpha}(\mathbb{R}^N;\mathbb{R}^N) \text{ with } \|\Phi - \mathrm{Id}\|_{C^{2,\alpha}} < \varepsilon \text{ s.t. } \Omega' = \Phi(\Omega)\right\}$$

We define the family of (good) sets for which (H2) holds, as follows:

$$\mathcal{G} := \left\{\Omega \in \mathcal{O} \ : \ \mathsf{c}p \notin \mathrm{Spec}_{\mathrm{L}^2_V(\Omega)}(-\Delta)\right\}$$

Theorem (Good sets are Generic) Saut-Temam (CPDE 1979)

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Some examples.

- We know that (H2) is true on balls of $\mathbb{R}^N$
- In dimension $N = 2$, (H2) holds for domains which are convex in the direction x_i, $i = 1, 2$ and symmetric with respect to the hyperplanes $x_i = 0$, $i = 1, 2$. (Dancer 1990, Damascelli-Grossi-Pacella 1999).
- By the results of Zou (1994), (H2) holds for C^1 perturbation of balls.
- As for Annuli: we know that this is not true, however, if we perturb a bit the annulus in the $C^{2,\alpha}$ sense above, then $(H2)$ holds true.
- Perturbation can be done only on a small part of the boundary of Ω.
- For p close to 1 the result is true:

$$\lambda_{V,k_p+1} - \lambda_{V,k_p} \sim \lambda_{V,2} - \lambda_{V,1} \xrightarrow{p \to 1^+} \lambda_2 - \lambda_1$$

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Nonlinear Entropy method.

Recall that, as consequence of convergence in relative error: $\forall\delta \in (0,1) \;\; \exists\, t_0 > 0\;:$

$$(\mathrm{H1})_\delta \qquad |f(t,x)| = |v(t,x) - V(x)| \le \delta V(x) \qquad \text{for a.e. } (t,x) \in [t_0,\infty) \times \Omega$$

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$$\mathcal{E}[v] = \int_\Omega \left[\left(v^{p+1} - V^{p+1}\right) - \frac{p+1}{p}(v^p - V^p)V \right] \mathrm{d}x\,,$$

Comparing linear and nonlinear Entropy

Assume $(\mathrm{H1})_\delta$, then for all $t \ge t_0$ we have for some $\overline{c}_p > 0$:

$$\frac{p+1}{2(1+\overline{c}_p\delta)^2}\,\mathsf{E}[f] \;\le\; \mathcal{E}[v] \;\le\; \frac{p+1}{2}(1+\overline{c}_p\delta)^2\,\mathsf{E}[f]$$

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Assume $(\mathrm{H1})_\delta$, then for all $t \ge t_0$ we have for some $\kappa_p > 0$:

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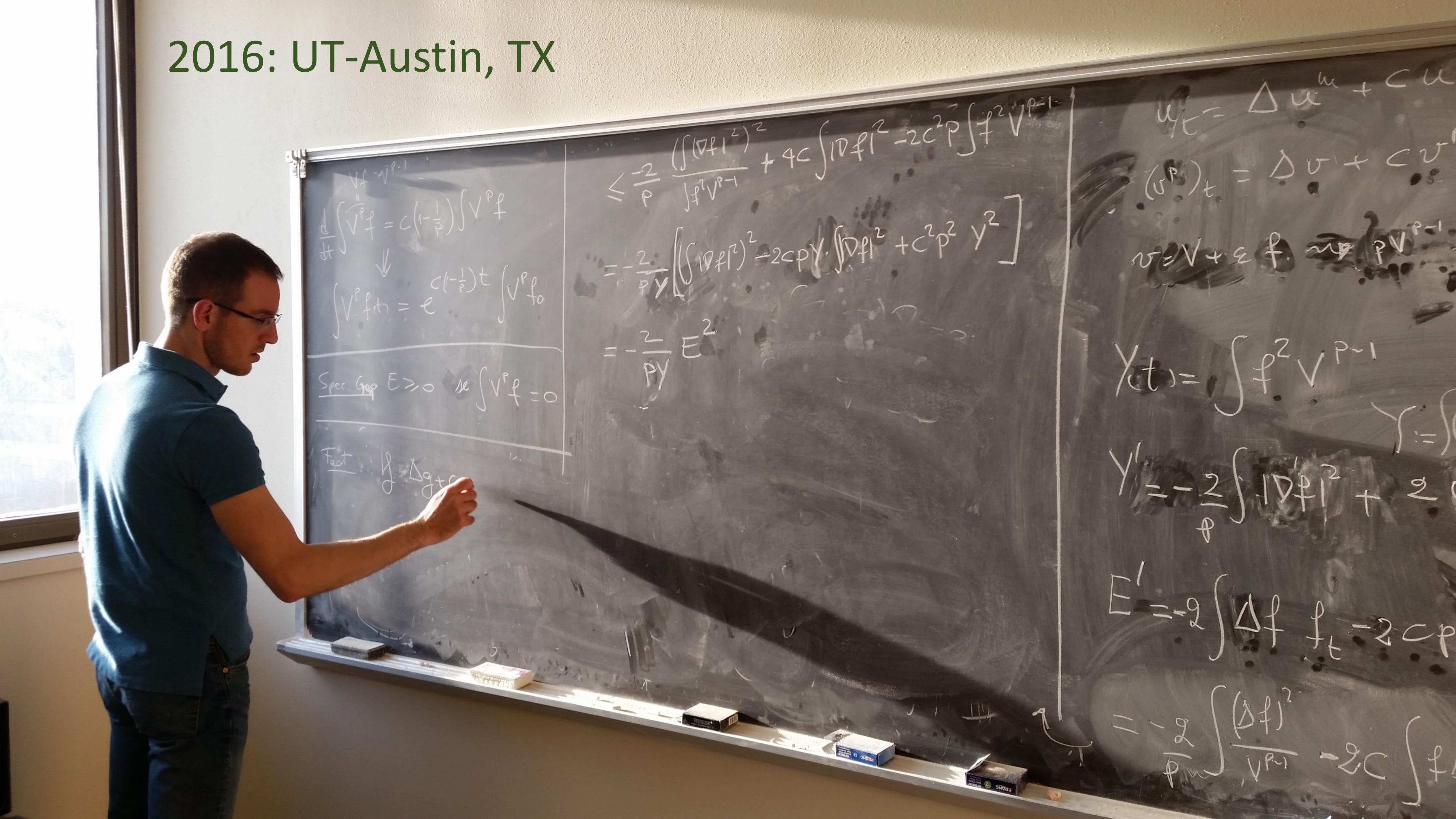

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Linear

$$p V^{p-1} \partial_t f = \Delta f + c p V^{p-1} f$$

$$E[f] = \int_\Omega f^2 V^{p-1} dx$$

$$E'[f] = -I[f]$$

$$(1) \quad = -\left(\int_\Omega |Df|^2 - cp \int f^2 V^{p-1}\right)$$

$$\lambda_p \int f^2 V^{p-1} \le \int |Df|^2 - cp \int f^2 V^{p-1}$$

$$(2) \quad \lambda_p E[f] \le I[f]$$

Combining (1)&(2)
(ORTOG-CONDITIONS)

$$E'[f(t)] \le -\lambda_p E[f(t)]$$

$$\Downarrow$$

$$E[f(t)] \le e^{-\lambda_p t} E[f_0]$$

$$\xleftarrow[v = V + \varepsilon f]{\varepsilon \to 0}$$

$$(f \approx v - V) \qquad \boxed{(H1)_\delta: \ |v - V| \le \delta V}$$

$$E[v-V] \frac{p+1}{2(1+\bar{c}\delta)} \le \mathcal{E}[v] \le \frac{p+1}{2}(1+\bar{c}\delta) E[v-V]$$

$$\mathcal{I}(v) = \frac{p+1}{p} I[v-V] - R_p[v-V]$$

$$|R_p[f]| \le C k_p \int_\Omega |f|^3 V^{p-2} dx.$$

IMPROVED POINCARÉ

$$(pc + \lambda_p) \int \varphi^2 V^{p-1} \le \int |D\varphi|^2$$

ORTHOGONALITY COND. NEEDED

$$\int f(t,x)\, \phi_{k,j}(x) = 0 \qquad \forall k = 1 \dots k_p \quad \forall j = 1 \dots N_k$$

Almost Ortogonality: $(AOL)_\varepsilon$

$$Q_{j,k}[f] = \frac{\left|\int f \phi_{k,j} V^{p-1}\right|}{\left(\int f^2 V^{p-1}\right)^{1/2}} < \varepsilon$$

DIFFICULT PART

Nonlinear

$$\partial_t v^p = \Delta v + c v^p$$

$$\mathcal{E}[v] = \int_\Omega \left(v^{p+1} - V^{p+1} - \frac{p+1}{p}(v^p - V^p)V\right) dx.$$

$$\mathcal{E}'[v] = -\mathcal{I}[v]$$

$$(3) \quad = -\frac{p+1}{p} I[f] + R_p[f]$$

IMPROVED POINCARÉ for
ALMOST ORTHOGONAL FNS:

$$(pc + \lambda_p - \gamma_p \varepsilon^2) \int \varphi^2 V^{p-1} \le \int |D\varphi|^2$$

$$(\lambda_p - \gamma_p \varepsilon^2) E[\varphi] \le I[\varphi] \quad (4)$$

Entropy-Entropy Prod. Improved
(3)+(4) give:

$$\mathcal{E}'[v] \le -\Big[\underbrace{\frac{2\lambda_p}{p} - \gamma_p(\varepsilon^2 + \delta)}_{\Lambda}\Big] \mathcal{E}(v(t))$$

$$\Downarrow$$

$$\mathcal{E}[v(t)] \le k\, e^{-\Lambda t}$$

then improve

Almost orthogonality conditions

We have defined k_p as the largest k such that $p\mathsf{c} > \lambda_{V,k}$ and

$$\lambda_p := \lambda_{V,k_p+1} - \mathsf{c}p > 0\,,$$

Let us define Linear Rayleigh-type quotients

$$\mathsf{Q}_{k,j}[\psi] := \frac{\left|\int_\Omega \psi\,\phi_{k,j}\,V^{p-1}\,\mathrm{d}x\right|}{\left(\int_\Omega \psi^2\,V^{p-1}\,\mathrm{d}x\right)^{\frac{1}{2}}} = \frac{\left|\langle \psi, \phi_{k,j}\rangle_{\mathrm{L}^2_V}\right|}{\|\psi\|_{\mathrm{L}^2_V}} = \frac{\left|\langle \psi, \phi_{k,j}\rangle_{\mathrm{L}^2_V}\right|}{\mathsf{E}[\psi]^{\frac{1}{2}}}\,.$$

We say that a function $f \in \mathrm{L}^2_V$ satisfies the almost-orthogonality condition if

$$(AOL)_\varepsilon \quad \exists\,\varepsilon \in (0,1) \text{ s.t. } \boxed{\mathsf{Q}_{k,j}[f] \le \varepsilon}\ \forall\, k = 1,\ldots,k_p \text{ and } j = 1,\ldots,N_k\,.$$

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and an analogous almost-orthogonality condition:

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$$\mathsf{Q}_{k,j}[\psi] := \frac{\left|\int_\Omega \psi\,\phi_{k,j}\,V^{p-1}\,\mathrm{d}x\right|}{\left(\int_\Omega \psi^2\,V^{p-1}\,\mathrm{d}x\right)^{\frac12}} = \frac{\left|\langle\psi,\phi_{k,j}\rangle_{\mathrm{L}^2_V}\right|}{\|\psi\|_{\mathrm{L}^2_V}} = \frac{\left|\langle\psi,\phi_{k,j}\rangle_{\mathrm{L}^2_V}\right|}{\mathsf{E}[\psi]^{\frac12}}\,.$$

We say that a function $f \in \mathrm{L}^2_V$ satisfies the almost-orthogonality condition if

$$(AOL)_\varepsilon \quad \exists\,\varepsilon\in(0,1)\text{ s.t. } \boxed{\mathsf{Q}_{k,j}[f] \le \varepsilon}\ \forall\,k = 1,\ldots,k_p \text{ and } j = 1,\ldots,N_k\,.$$

Let us also define Nonlinear Rayleigh-type quotients

$$\mathcal{Q}_{k,j}[v] := \frac{\left|\int_\Omega \left(v^p - V^p\right)\phi_{k,j}\,\mathrm{d}x\right|}{\left(\int_\Omega \left[\left(v^{p+1}-V^{p+1}\right) - \frac{p+1}{p}\left(v^p - V^p\right)V\right]\,\mathrm{d}x\right)^{\frac12}} := \frac{\mathcal{A}_{k,j}[v]}{\mathcal{E}[v]^{\frac12}}\,.$$

and an analogous almost-orthogonality condition:

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Almost orthogonality conditions

We have defined k_p as the largest k such that $p\mathsf{c} > \lambda_{V,k}$ and

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Comparing linear and nonlinear Rayleigh quotients

Assume $(\mathrm{H1})_\delta$, then for all $t \geq t_0$ we have

$$\frac{\sqrt{2}p}{\sqrt{p+1}(1+\overline{c}_p\delta)}\mathsf{Q}_{k,j}[f(t)] - \widetilde{c}_{k,j,p}\mathsf{E}[f]^{\frac{1}{2}} \leq \mathcal{Q}_{k,j}[v(t)]$$
$$\leq \frac{\sqrt{2}p}{\sqrt{p+1}}(1+\overline{c}_p\delta)\mathsf{Q}_{k,j}[f(t)] + \widetilde{c}_{k,j,p}\mathsf{E}[f]^{\frac{1}{2}}$$

Clearly, this implies that the Linear and Nonlinear almost orthogonality conditions are equivalent: when $\delta \ll \varepsilon$ we have

$$(AOL)_\varepsilon \quad \Longrightarrow \quad (AON)_{\kappa_1\varepsilon} \quad \Longrightarrow \quad (AOL)_{\kappa_2\varepsilon}$$

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Almost orthogonality, improved Poincaré inequalities and Entropy-Production

Improved Poincaré inequality for almost-orthogonal functions

Under assumption (H2), let $\varphi \in \mathrm{L}^2_V$ be such that $(AOL)_\varepsilon$ holds true with $\varepsilon > 0$. Then, the following improved Poincaré inequality holds

$$(p\mathsf{C} + \lambda_p - \gamma_p \varepsilon^2) \int_\Omega \varphi^2 V^{p-1}\,\mathrm{d}x \le \int_\Omega |\nabla\varphi|^2\,\mathrm{d}x\,,$$

Equivalently:

$$(\lambda_p - \gamma_p \varepsilon^2)\,\mathsf{E}[\varphi] \le \mathsf{I}[\varphi]\,.$$

Entropy Entropy-Production inequality I

Assume $(\mathrm{H1})_\delta$ and (H2) and assume that for some $t \ge t_0$ we have that $f(t)$ satisfies $(AOL)_\varepsilon$. Then we have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[v(t)] \le -\left[\frac{2\lambda_p}{p} - \gamma_p(\varepsilon^2 + \delta)\right]\mathcal{E}[v(t)]\,.$$

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Entropy Entropy-Production inequality II

Assume $(\mathrm{H1})_\delta$ and (H2), and assume that for some $\eta > 0$ we have:

$$\left\| \frac{v(t) - V}{V} \right\|_{\mathrm{L}^\infty(\Omega)} \le \overline{\kappa}\, \mathcal{E}[v(t-1)]^\eta \quad \text{and} \quad \mathsf{Q}_{k,j}[f(t)] \le \overline{c}_{p,k,j}\, \mathcal{E}[v(t-1)]^{\frac{\eta}{2}} ,$$

for all $t \ge t_0 \ge 1$ and all $k = 1, \ldots, k_p, j = 1, \ldots, N_k$.
Then, for all $t \ge t_0 \ge 1$ we obtain

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}[v(t)] \le -\frac{2\lambda_p}{p}\, \mathcal{E}[v(t)] + \kappa_p\, \mathcal{E}[v(t-1)]^\eta\, \mathcal{E}[v(t)] \,.}$$

Super solutions to ODEs with delay

Let $\sigma > 0$ and $Y : [t_0, \infty) \to [0, \infty)$ satisfy the following ODE for all $t \ge t_0 + 1$

$$Y'(t) \le -\lambda Y(t) + Y^\sigma(t-1) Y(t) \,.$$

If $C := \lambda Y(t_0)^{-\sigma} - 1 > 0$, then for all $t \ge t_0 + 1$:

$$Y(t) \le \overline{Y}(t) := \frac{\lambda^{\frac{1}{\sigma}} \mathrm{e}^{-\lambda t}}{\left[\mathrm{e}^{-\lambda\sigma(t-1)} + C\right]^{\frac{1}{\sigma}}} \,.$$

Entropy Entropy-Production inequality II

Assume $(\mathrm{H1})_\delta$ and (H2), and assume that for some $\eta > 0$ we have:

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Possible blow up when almost orthogonality fails

In order to get the (sharp) rate of decay we have to show that almost orthogonality is true along the nonlinear flow, hence ensuring the hypotheses $(AOL)_\varepsilon$ or $(AON)_\varepsilon$ needed in the previous steps. Recall that

$$\mathcal{Q}_{k,j}[v] := \frac{\left|\int_\Omega \left(v^p - V^p\right)\phi_{k,j}\,\mathrm{d}x\right|}{\left(\int_\Omega \left[\left(v^{p+1} - V^{p+1}\right) - \frac{p+1}{p}\left(v^p - V^p\right)V\right]\,\mathrm{d}x\right)^{\frac{1}{2}}} := \frac{\mathcal{A}_{k,j}[v]}{\mathcal{E}[v]^{\frac{1}{2}}}\,.$$

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Let $v = f + V$ and assume $(\mathrm{H1})_\delta$. Fix two integers $k \in [1, k_p]$ and $j \in [1, N_k]$, and fix also $t \ge t_0 \ge 0$ and $\varepsilon_0 \in (0, 1/2)$. There exists $\underline{\kappa}_0 > 0$ such that the following holds: if

$$\delta < \underline{\kappa}_0\,\varepsilon_0 \qquad \text{and} \qquad \mathcal{Q}_{k,j}[f(t)] \ge \varepsilon_0$$

then there exists $\underline{\kappa}_1 > 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A}_{k,j}[v(t)] \ge \underline{\kappa}_1 \varepsilon_0 \mathcal{A}_{k,j}[v(t)]\,.$$

When the quotients $\mathcal{Q}_{k,j}$ are relatively big, the corresponding projections $\mathcal{A}_{k,j}$ tend to blow up exponentially in infinite time, as in the linear case.
This is the most delicate part of this method and the core of the proof.

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Almost orthogonality improves along the nonlinear flow

Qualitative almost orthogonality along the nonlinear flow

Assume $(\mathrm{H1})_\delta$ with $0 < \delta < 1/p$ and (H2). Then, for every $\varepsilon > 0$ there exists $t_\varepsilon \geq t_0 \geq 0$ such that for all $1 \leq k \leq k_p$ and $1 \leq j \leq N_k$

$$\mathcal{Q}_{k,j}[v(t)] \leq \varepsilon \qquad \text{for all } t \geq t_\varepsilon \, .$$

The nonlinear flow turns out to be "more stable" than the linearized one: it is surprising that the almost orthogonality is eventually true along the nonlinear flow (for any initial data), while it is false along the linear flow, unless we impose precise orthogonality conditions on the data.

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Assume $(\mathrm{H1})_\delta$ and (H2). Assume moreover that $\mathcal{E}[v(t_0-1)] \leq 1$ and

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Then, there exist $T_0 \geq t_0$ and $\overline{\kappa}_p > 0$ such that for all $1 \leq k \leq k_p$ and $1 \leq j \leq N_k$

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Smoothing effects for the relative error

The last assumption that we have to check is a uniform control of the relative error in terms of the entropy (delicate).

Weighted smoothing effects

Assume $(\mathrm{H1})_\delta$ and $t_0 \geq 1 \vee T \log 2$. Then the following estimates hold true for any $t \geq t_0$

$$\left\| \frac{v(t) - V}{V} \right\|_{\mathrm{L}^\infty(\Omega)} \leq \overline{\kappa}_\infty \frac{\mathrm{e}^{2\mathsf{c}m(t-t_0)}}{t - t_0} \sup_{\tau \in [t_0, t]} \mathcal{E}[v(\tau)]^{\frac{1}{4N}} + 2\mathsf{c}m(t - t_0)\mathrm{e}^{2\mathsf{c}m(t-t_0)} .$$

As a consequence:

Entropy controls the L^∞ norm of the relative error

Under the above assumptions, the following estimates hold true for any $t \geq t_0 + 1 \geq 0$:

$$\left\| \frac{v(t) - V}{V} \right\|_{\mathrm{L}^\infty(\Omega)} \leq \overline{\kappa}_\infty \mathcal{E}[v(t-1)]^{\frac{1}{8N}} .$$

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The End

Thank You!!!

Grazie Mille!!!

Muchas Gracias!!!

Congratulazioni Alessio!!!!!

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Some References: (in inverse chronological order)

(This talk is based on [BF20] where a more complete list of References can be found)

[BF20] M. B., A. FIGALLI, Extinction Rates for Fast Diffusion Equations on Generic Bounded Domains. *Comm. Pure Appl. Math (2020). Online First* http://dx.doi.org/10.1002/cpa.21887

[BGV12] M. B., G. GRILLO, J. L. VÁZQUEZ, Behaviour near extinction for the Fast Diffusion Equation on bounded domains, *J. Math. Pures Appl.* **97**, (2012), n.1, 1–38. http://dx.doi.org/10.1016/j.matpur.2011.03.002

[FS00] E. FEIREISL, F. SIMONDON, Convergence for semilinear degenerate parabolic equations in several space dimensions. *J. Dynam. Differential Equations* **12** (2000), no. 3, 647–673. doi:10.1023/A:1026467729263

[DKV91] E. DIBENEDETTO, Y. KWONG, V. VESPRI, Local space-analyticity of solutions of certain singular parabolic equations. *Indiana Univ. Math. J.* **40** (1991), no. 2, 741–765. doi:10.1512/iumj.1991.40.40033

[BH80] J. G. BERRYMAN, C. J. HOLLAND, Stability of the separable solution for fast diffusion. *Arch. Rational Mech. Anal.* **74** (1980), no. 4, 379–388. doi:10.1007/BF0024968