Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

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References:

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[BV2] M. B., J. L. VÁZQUEZ, Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part I. Existence, Uniqueness and Upper Bounds *Nonlin. Anal. TMA (2016).*

[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains. *Discr. Cont. Dyn. Sys. (2015).*

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[BFV1] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary estimates and higher regularity for nonlocal porous medium-type equations in bounded domains. *Analysis & PDE (2018)*

[BFV2] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary behaviour of solutions to semilinear nonlocal elliptic equations. *Calc. Var. PDE (2018).*

Outline of the talk

- **Introduction to Parabolic Problem on Domains**
- **The Classical Porous Medium Equation (PME)**
  - **A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"**
- **The Fractional PME Part I: Basic theory**
  - **The Setup of the Problem: Nonlocal Operators and Nonlinearities**
  - **Three Different Fractional Laplacians on Bounded Domains**
  - **A Common Setup Based on Green Functions and More Examples**
- **A Detour about Elliptic Equations**
  - **Boundary behaviour Linear Elliptic problem**
  - **Regularity and Sharp boundary behaviour for Semilinear Elliptic equations**
  - **Parabolic solutions by separation of variables**
- **Back to the Parabolic problem**
  - **Basic theory: existence, uniqueness and boundedness**
    - **The Dual problem: Existence and uniqueness**
    - **An "Almost Representation" Formula**
    - **Boundedness: Absolute Bounds and Smoothing effects.**
  - **Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$**

Introduction to the Parabolic Problem on Domains

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)}\qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  The classical Laplacian and *all fractional Laplacians on domains.*
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

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The Classical Porous Medium Equation (PME)

A Brief Summary about the Dirichlet Problem for PME in few “Blackboards”

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$\Omega \subseteq \mathbb{R}^N$ bounded domain.

$u_0 \in C_c^\infty(\Omega)$ smooth & compactly supported.

$$\begin{cases} u_t = \Delta u^m & \text{in } (0,+\infty)\times\Omega \\ u = 0 & \text{on } (0,+\infty)\times\partial\Omega \\ u(t=0) = u_0 & \text{in } \Omega. \end{cases} \qquad m>1$$

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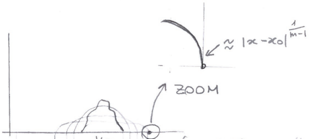
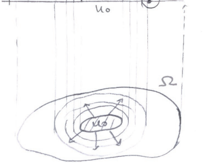
(0) $0 < t < \underline{t}$ INITIAL TIMES

(Barenblatt behaviour)

$$u(t,x) \approx \left(C - \frac{|x|^2}{t^{2\beta}}\right)_+^{\frac{1}{m-1}} t^{\alpha} = B(t,x)$$





(Here $u(t,x)$ "does not see" the boundary)

- the support of $u(t)$ spreads from $\mathrm{supp}(u_0)$ with finite speed (close to $B(t,x)$)
- the support of $u(t)$ does NOT TOUCH the boundary $\partial\Omega$.

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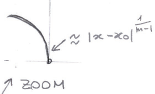

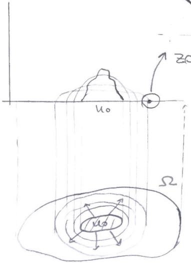

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$t = \underline{t}$

(the first "touching point")

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- Transition of boundary behaviour:

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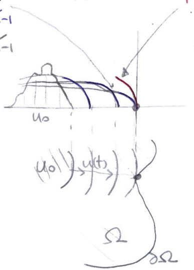
$|x-x_0|^{\frac{1}{m-1}}$ $\quad$ $|x-x_0|^{1/m}$

$d(x,\partial\Omega)^{\frac{1}{m-1}}$ $\quad$ $d(x,\partial\Omega)^{1/m}$



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the solution starts to "inflate"

from $d(x,\partial\Omega)^{\frac{1}{m-1}} \ll d(x,\partial\Omega)^{\frac{1}{m}}$ $(m>1)$

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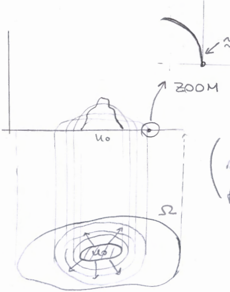

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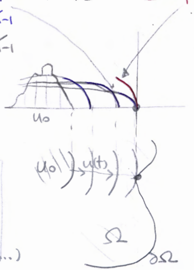

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REGULARITY

- solutions are smooth when positive & bounded.
- Free boundary: delicate issue (CAFFARELLI, VAZQUEZ, WOLANSKI, KOCH, ...)

the solution starts to "inflate"

from $d(x,\partial\Omega)^{\frac{1}{m-1}} \ll d(x,\partial\Omega)^{\frac{1}{m}}$ $\quad$ $(m>1)$

TO

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the supp$(u(t))$ touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the elliptic one:

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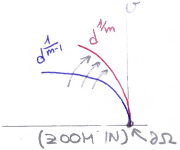

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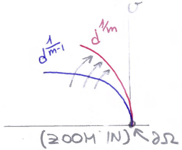

(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE & LARGE TIMES)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq C_1 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

$$u(t,x) \asymp \frac{S(x)}{t^{\frac{1}{m-1}}} = \mathcal{U}(t,x)$$

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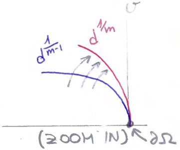

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$\longleftrightarrow$ (STATIONARY for RESCALED FLOW) ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \dfrac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

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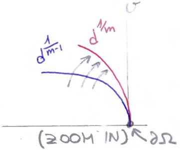

(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES & LARGE)

GLOBAL HARNACK PRINCIPLE:

$$c_0 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le c_1 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

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↓ Semilinear Structure: $S^m = V$, $p = 1/m < 1$

$$\begin{cases} -\Delta V = \dfrac{p}{1-p} V^p & \text{in } \Omega \\ V = 0 & \text{on } \partial\Omega \end{cases}$$

(ELLIPTIC THEORY)

$V \asymp \mathrm{dist}(\cdot,\partial\Omega)$

$S \asymp \mathrm{dist}(\cdot,\partial\Omega)^{1/m}$

rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

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← SEPARATE VARIABLE SOLUTION.

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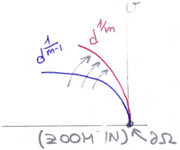

SLOW MOTION DYNAMICS: (LOGARITHMIC TIME RESCALING)

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \xrightarrow{\text{(SAME LATERAL B.C.)}} \begin{cases} v_t = \Delta v^m + \dfrac{v}{m-1} \\ v(t=0) = u_0. \end{cases}$$

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$

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Semilinear Structure: $S^m = V$, $p = 1/m < 1$

(ELLIPTIC THEORY)

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$V \asymp \operatorname{dist}(\cdot,\partial\Omega)$

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$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

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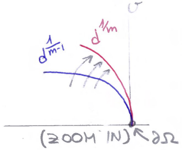

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(SAME LATERAL B.C.)

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$

(•) $t \to +\infty$ ASYMPTOTIC BEHAVIOUR

$$\tau^{\frac{1}{m-1}} |u(\tau,x) - S(x)| \xrightarrow[\tau\to+\infty]{\text{UNIF}} 0 \quad \Big\} \quad v(t,x) \xrightarrow[t\to+\infty]{\text{UNIF}} S(x)$$

(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE & LARGE TIMES)

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$$C_0 \frac{\text{dist}(x,\partial\Omega)^{1/m}}{t^{1/m-1}} \leq u(t,x) \leq C_1 \frac{\text{dist}(x,\partial\Omega)^{1/m}}{t^{1/m-1}}$$

(STATIONARY for RESCALED FLOW) ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \frac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

↓ Semilinear Structure: $S^m = V$, $p = 1/m < 1$

$$\begin{cases} -\Delta V = \frac{p}{1-p} V^p & \text{in } \Omega \\ V = 0 & \text{on } \partial\Omega \end{cases}$$

(ELLIPTIC THEORY)

$V \asymp \text{dist}(\cdot,\partial\Omega)$

$S \asymp \text{dist}(\cdot,\partial\Omega)^{1/m}$

rewritten

$$C_0' \frac{S(x)}{t^{1/m-1}} \leq u(t,x) \leq C_1' \frac{S(x)}{t^{1/m-1}}$$

$$u(t,x) \asymp \frac{S(x)}{t^{1/m-1}} = \mathcal{U}(t,x)$$

← SEPARATE VARIABLE SOLUTION.

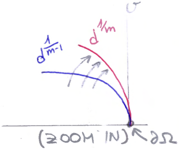
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the supp$(u(t))$ touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the elliptic one:

$$u(t,x) \approx \frac{d(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$



SLOW MOTION DYNAMICS: (LOGARITHMIC TIME RESCALING)

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \quad \overset{\text{SAME LATERAL B.C.}}{\rightsquigarrow} \quad \begin{cases} v_t = \Delta v^m + \frac{v}{m-1} \\ v(t=0) = u_0 . \end{cases}$$

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$

(•) $t \to +\infty$ ASYMPTOTIC BEHAVIOUR

$$\tau^{\frac{1}{m-1}} |u(\tau,x) - S(x)| \xrightarrow[\tau\to+\infty]{\text{UNIF}} 0 \qquad v(t,x) \xrightarrow[t\to+\infty]{\text{UNIF}} S(x)$$

SHARP

$$\left|\frac{u(\tau,x)}{\mathcal{U}(t,x)} - 1\right| \le \frac{C}{1+\tau} \qquad \left|\frac{v(t,x)}{S(x)} - 1\right| \le C e^{-t}$$

(•) $t \ge t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES & LARGE)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\text{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1 \frac{\text{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

(STATIONARY for RESCALED FLOW) ASSOCIATED ELLIPTIC PROBLEM.

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rewritten

Semilinear Structure: $S^m = V$, $p = 1/m < 1$

(ELLIPTIC THEORY)

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

$V \asymp \text{dist}(\cdot,\partial\Omega)$

$S \asymp \text{dist}(\cdot,\partial\Omega)^{1/m}$

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A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \rightsquigarrow \text{(SAME LATERAL B-C.)} \quad \begin{cases} v_t = \Delta v^m + \dfrac{v}{m-1} \\ v(t=0) = u_0. \end{cases}$$

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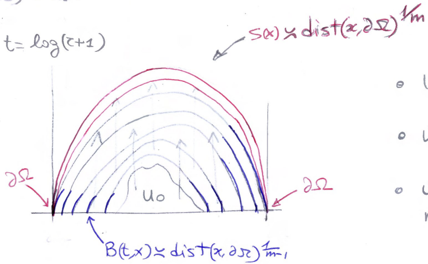

- $v_t = \Delta v^m + \dfrac{v}{m-1}$
- $v_t \geq 0 \leftarrow$ (BENILAN-CRANDALL)

  $\Downarrow$
- $v(t,x) \nearrow S(x)$

  monotonically increases to $S(x)$

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

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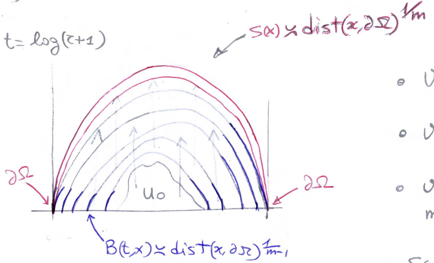

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- $S(x)$ Represents AN ABSOLUTE UPPER BOUND FOR ALL SOLUTIONS !! "FRIENDLY GIANT". (DAHLBERG-KENIG)

$$\begin{cases} u(\tau,x) = \dfrac{S(x)}{\tau^{\frac{1}{m-1}}} \\ u(0,x) = +\infty \end{cases} \quad \longleftarrow \text{RESCALING BACK.}$$

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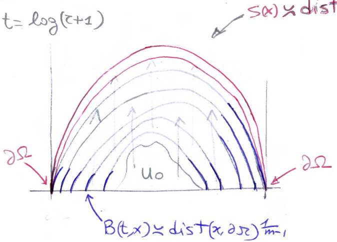

SEPARATION OF VARIABLES

$$\begin{cases} \mathcal{U}_T(\tau,x) = \dfrac{S(x)}{(T+t)^{\frac{1}{m-1}}} \\ \mathcal{U}_T(0,x) = \dfrac{S(x)}{T^{\frac{1}{m-1}}} \end{cases}$$

$T \to 0$

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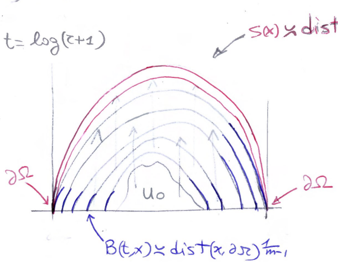

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CONVERGENCE IN RELATIVE ERROR WITH SHARP RATE:

$$\left| \frac{u(\tau,x)}{\mathcal{U}(\tau,x)} - 1 \right| \le \frac{c}{1+\tau} \quad \text{or} \quad \left| \frac{v(t,x)}{S(x)} - 1 \right| \le c\, e^{-t}$$

ARONSON-PELETIER (80s), VAZQUEZ (2000s), MB.-FIGALLI-SIRE-VAZQUEZ. (2015-18?)

The Fractional PME: Basic theory

- **The Setup of the Problem:**
  Nonlocal Operators and Nonlinearities
- **Three Different Fractional Laplacians**
  on Bounded Domains
- **A Common Setup Based on Green Functions**
- **More Examples**

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

- We have seen what happens when $\mathcal{L} = -\Delta$ is the classical Laplacian
- We now focus our attention to a particular scenario:
  - When $\mathcal{L} = (-\Delta)^s$, with $s \in (0,1)$ is a Fractional Laplacian: there are three different choices of fractional Laplacian on bounded domains.
  - When $F(u) = |u|^{m-1}u$, with $m > 1$ have the classical PME nonlinearity

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Three Different Fractional Laplacians on Bounded Domains

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**: if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \mathrm{P.V.}\int_{\mathbb{R}^N}\frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}}g(x) - g(x)\right)\frac{dt}{t^{1+s}}\,.$$

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Three Different Fractional Laplacians on Bounded Domains

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are the eigenfunctions of the classical Laplacian Δ_Ω:

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega) \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega)\,,$$

and ϕ_j are as smooth as $\partial\Omega$ allows: $\partial\Omega \in C^k \;\Rightarrow\; \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

The Green function of SFL satisfies, letting $\delta^\gamma(\cdot) := \mathrm{dist}(\cdot, \partial\Omega)$,

$$\text{(K4)} \qquad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \qquad \text{with } \boxed{\gamma = 1}$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

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Three Different Fractional Laplacians on Bounded Domains

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\, dz\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
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- EIGENFUNCTIONS: $\overline{\phi}_j \in C^s(\overline{\Omega}) \cap C^\infty(\Omega)$ (J. Serra - X. Ros Oton), and

$$\phi_1 \asymp \operatorname{dist}(\cdot, \partial\Omega)^s \qquad \text{and} \qquad |\phi_j| \lesssim \operatorname{dist}(\cdot, \partial\Omega)^s\,,$$

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Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times (\mathbb{R}^N \setminus \Omega)\,.$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

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$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,, \qquad \text{with } \mathrm{supp}(g)\subseteq\overline{\Omega}\,.$$

where $s\in(0,1)$ and $c_{N,s}>0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0<\overline{\lambda}_1\le\overline{\lambda}_2\le\ldots\le\overline{\lambda}_j\le\overline{\lambda}_{j+1}\le\ldots$ and $\overline{\lambda}_j\asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j\le\lambda_j^s$ for all $j\in\mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j\in C^s(\overline{\Omega})\cap C^\infty(\Omega)$ (J. Serra - X. Ros Oton), and

$$\phi_1\asymp \mathrm{dist}(\cdot,\partial\Omega)^s \qquad \text{and} \qquad |\phi_j|\lesssim \mathrm{dist}(\cdot,\partial\Omega)^s\,,$$

The Green function of RFL satisfies, letting $\delta^\gamma(\,\cdot\,):=\mathrm{dist}(\,\cdot\,,\partial\Omega)$,

$$\text{(K4)}\qquad \mathbb{G}(x,y)\asymp\frac{1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)\,,\qquad \text{with } \boxed{\gamma=s}$$

Lateral boundary conditions for the RFL

$$u(t,x)=0\,,\qquad \text{in } (0,\infty)\times\left(\mathbb{R}^N\setminus\Omega\right).$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

Three Different Fractional Laplacians on Bounded Domains

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega \frac{f(x)-f(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

- It is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum (λ_j, ϕ_j)
- EIGENFUNCTIONS: $\overline{\phi}_j \in C^{2s-1}(\overline{\Omega}) \cap C^{2s+\alpha}(\Omega)$ (MB, A.Figalli, J. L. Vázquez)

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^{2s-1} \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega)^{2s-1}\,,$$

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Remarks.

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

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A Common Setup Based on Green Functions

Assumptions on the inverse of $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

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$$\mathcal{L}^{-1} f(x) = \int_\Omega \mathbb{G}(x,y) f(y)\, \mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$

(K1) $$0 \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

Assumption (K1) implies that $\mathcal{L}^{-1}$ is compact on $\mathrm{L}^2(\Omega)$ and has discrete spectrum.

(K2) $$c_{0,\Omega}\delta^\gamma(x)\,\delta^\gamma(y) \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right)$$

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(K2) is needed in the study of the sharp boundary behaviour.

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More Examples

Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form, with uniformly elliptic bounded measurable coefficients:

$$A = \sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j)\,, \qquad \textit{s-power of } A \textit{ is:} \qquad \mathcal{L}f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x)$$

$\mathcal{L} = A^s$ satisfies (K2) estimates with $\gamma = 1$

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[General class of intrinsically ultra-contractive operators, Davies and Simon JFA 1984].

Fractional operators with "rough" kernels. Integral operators of Levy-type

$$\mathcal{L}f(x) = \text{P.V.}\int_{\mathbb{R}^N} \left(f(x+y) - f(y)\right)\frac{a(x,y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,.$$

where K is measurable, symmetric, bounded between two positive constants, and

$$\big|a(x,y) - a(x,x)\big|\,\chi_{|x-y|<1} \le c|x-y|^{\sigma}\,, \qquad \text{with } 0 < s < \sigma \le 1\,,$$

for some positive $c > 0$. We can allow even more general kernels.
The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

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Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma\,, \qquad \text{with } 0 < \sigma < s \le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. $\mathcal{L}$ satisfies $(K4)$ with $\gamma = s$.

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0\,,$$

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the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^N} 1 \vee |x|^2\,\mathrm{d}\nu(y) < \infty$, together with other assumptions.

Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s\,, \qquad \text{with } c > 0\,, \text{ and } 0 < s \le 1\,.$$

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A Detour about Elliptic Equations

- Boundary behaviour Linear Elliptic problem
- Regularity and Sharp boundary behaviour for Semilinear Elliptic equations
- Parabolic solutions by separation of variables

Boundary behaviour for Elliptic equations

We always assume that $\mathcal{L}$ satisfies (A1), (A2) and zero Dirichlet boundary conditions.

The Linear Problem $\boxed{\mathcal{L}v = f}$ **with** $f \in \mathrm{L}^{q'}(\Omega)$

Let $\mathbb{G}$ be the kernel of $\mathcal{L}^{-1}$, and assume $(K2)$ and that $0 \le f \in \mathrm{L}^{q'}$ with $q' > N/2s$.
Then $q = \frac{q'}{q'-1} \in \left(0, \frac{N}{N-2s}\right)$ and the (weak dual) solution $v \ge 0$ satisfies $\forall x \in \Omega$

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The Eigenvalue Problem $\boxed{\mathcal{L}\Phi_k = \lambda_k \Phi_k}$

Assumption (K1) implies that $\mathcal{L}^{-1}$ is compact in $\mathrm{L}^2(\Omega)$.
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Boundary behaviour for Elliptic equations

We always assume that $\mathcal{L}$ satisfies (A1), (A2) and zero Dirichlet boundary conditions.

The Linear Problem $\boxed{\mathcal{L}v = f}$ **with** $f \in \mathrm{L}^{q'}(\Omega)$

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Some remarks about boundary behaviour for Linear Elliptic equations

Assuming (K2), that we recall here: [recall $\mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$]

$$\text{(K2)} \quad c_{0,\Omega}\delta^\gamma(x)\,\delta^\gamma(y) \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)$$

Consider for simplicity $\mathcal{L}v = f \in \mathrm{L}^\infty(\Omega) \ge 0$, hence $q = 1$. Then we have:

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The boundary behaviour may change depending on the relation between $2s$ and γ.

On the other hand, for eigenfunctions **we always have**

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This reveals a deep and strong difference in the boundary behaviour, typical of the different definitions of Fractional Laplacians on domains.

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Sharp boundary behaviour for Elliptic Equations

We always assume that $\mathcal{L}$ satisfies (A1), (A2) and zero Dirichlet boundary conditions.

The Semilinear Dirichlet Problem $\boxed{\mathcal{L}v = f(v) \sim v^p}$ **with** $0 < p < 1$

Assume moreover that $\mathcal{L}^{-1}$ satisfies (K2). Let $u \geq 0$ be a (weak dual) solution to the Dirichlet Problem, where f is a nonnegative increasing function with $f(0) = 0$ such that $F = f^{-1}$ is convex and $F(a) \asymp a^{1/p}$ when $0 \leq a \leq 1$, for some $0 < p < 1$. Then, the following sharp absolute bounds hold true for all $x \in \Omega$

$$v(x) \asymp \begin{cases} \Phi_1^{\sigma}(x) & \text{when } 2s \neq \gamma(1-p) \\ \Phi_1(x)\left(1 + |\log \Phi_1(x)|\right)^{\frac{1}{1-p}} & \text{when } 2s = \gamma(1-p), \text{ assuming (K4)} \end{cases}$$

where

$$\boxed{\sigma := 1 \wedge \frac{2s}{\gamma(1-p)}} \qquad \text{and} \qquad \Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^{\gamma} = \delta^{\gamma}$$

When $2s = \gamma(1-p)$, if (K4) does not hold, then the upper bound still holds, but the lower bound holds in a non-sharp form without the extra logarithmic term.

Remarks.

- When $2s < \gamma(1-p)$, the new power σ becomes less than 1.
- Somehow σ interpolates between the two extremal cases: $p = 0$ i.e. $\mathcal{L}v = 1$ and $p = 1$, i.e. $\mathcal{L}v = \lambda v$.

Examples.

- For the RFL ($\gamma = s$) and CFL ($\gamma = 2s - 1$) we always have $\sigma = 1$ and $2s \neq \gamma(1-p)$, hence

$$v(x) \asymp \Phi_1(x) \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$$

- For the SFL we have $\gamma = 1$ hence we have three possibilities:

$$v(x) \asymp \begin{cases} \mathrm{dist}(x,\partial\Omega) & \text{when } s > \frac{1-p}{2} \\ \mathrm{dist}(x,\partial\Omega)\left(1 + |\log \mathrm{dist}(x,\partial\Omega)|\right)^{\frac{1}{1-p}} & \text{when } s = \frac{1-p}{2} \\ \mathrm{dist}(x,\partial\Omega)^{\frac{2s}{1-p}} & \text{when } s < \frac{1-p}{2} \end{cases}$$

Regularity. Under some mild assumptions on $\mathcal{L}$ and $f \in C^\beta(\mathbb{R})$ for some $\beta > 0$, with $0 \le f(a) \le c_p a^p$ when $0 \le a \le 1$ for some $0 < p \le 1$.

- Solutions are *Hölder continuous in the interior*, and (when the operator allows it) are *classical in the interior*, namely $C^{2s+\beta}(\Omega)$.
- Assuming moreover that $\mathcal{L}^{-1}$ satisfies (K2), solutions are *Hölder continuous up to the boundary*:

$$\|u\|_{C^\eta(\overline{\Omega})} \le C \qquad \forall\, \eta \in (0,\gamma] \cap (0, 2s).$$

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Change of notations from Elliptic to Parabolic In order to make the elliptic results "compatible" with the parabolic, we will perform the change of notations

$$\boxed{m = \frac{1}{p} > 1} \qquad \text{and} \qquad \boxed{v = S^m} \qquad \text{or} \qquad v^p = S\,.$$

The elliptic equation transforms: (we deal only with pure powers for simplicity)

$$\mathcal{L}v = f(v) = v^p \qquad \text{becomes} \qquad \boxed{\mathcal{L}S^m = \mathcal{L}F(S) = S}$$

Parabolic solutions by separation of variables. We have the following solution for the Dirichlet problem for the equation $u_t + \mathcal{L}\,u^m = 0$

$$\mathcal{U}_T(t,x) = \frac{S(x)}{(T+t)^{\frac{1}{m-1}}}$$

where $\mathcal{L}S^m = S$, and the initial datum is $U_T(0,x) = T^{-1/(m-1)}S(x)$.
When $T = 0$ we have the so-called *Friendly Giant*, corresponding to the biggest possible initial datum (useful in the asymptotic study as $t \to \infty$.)

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Back to the Parabolic problem

- **Basic theory: existence, uniqueness and boundedness**
  - **The Dual problem: Existence and uniqueness**
  - **Mild Solutions and Time-Monotonicity**
  - **An "Almost Representation" Formula**
  - **Boundedness: Absolute Bounds and Smoothing effects.**
- **Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$**

The "dual" formulation of the problem

Recall the homogeneous Dirichlet problem:

$$\text{(CDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, F(u)\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\partial_t U = -F(u)\,,$$

where

$$U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega \mathbb{G}(x,y)u(t,y)\,\mathrm{d}y\,.$$

This formulation encodes the lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.

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Weak Dual Solutions (WDS)

A function u is a *weak dual* solution to the Dirichlet Problem for $\partial_t u + \mathcal{L}F(u) = 0$ in $Q_T = (0,T)\times\Omega$ if:

- $u \in C((0,T):\mathrm{L}^1_{\Phi_1}(\Omega))\,,\, F(u) \in \mathrm{L}^1\left((0,T):\mathrm{L}^1_{\Phi_1}(\Omega)\right);$
- The following identity holds for every $\psi/\Phi_1 \in C^1_c((0,T):\mathrm{L}^\infty(\Omega))$:

$$\int_0^T\int_\Omega \mathcal{L}^{-1}(u)\,\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega F(u)\,\psi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** is a weak dual solution to Equation $\partial_t u + \mathcal{L}F(u) = 0$ such that moreover

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Assumption on the nonlinearity F

Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous and non-decreasing function, with $F(0) = 0$. Moreover, it satisfies the condition:

(N1) $F \in C^1(\mathbb{R} \setminus \{0\})$ and $F/F' \in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0, \mu_1 > 0$ s.t.

$$\frac{1}{m_1} = 1 - \mu_1 \le \left(\frac{F}{F'}\right)' \le 1 - \mu_0 = \frac{1}{m_0}$$

where F/F' is understood to vanish if $F(r) = F'(r) = 0$ or $r = 0$.

The main example will be

$$F(u) = |u|^{m-1}u, \qquad \text{with } m > 1\,, \quad \text{and} \qquad \mu_0 = \mu_1 = \frac{m-1}{m} < 1\,.$$

which corresponds to the nonlocal porous medium equation studied in $[BV1]$. A simple variant is the combination of two powers:

- m_0 gives the behaviour at zero, when $u \sim 0$
- m_1 gives the behaviour at infinity, when $u \sim \infty$.

Assumption on the nonlinearity F

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Existence of Mild Solutions and Monotonicity Estimates

Theorem. (M. Crandall and M. Pierre, JFA 1982)

Let $\mathcal{L}$ satisfy (A1) and (A2) and let F as satisfy (N1). Then for all $0 \le u_0 \in \mathrm{L}^1(\Omega)$, there exists a unique mild solution u to equation $u_t + \mathcal{L}F(u) = 0$, and the function

(1) $\quad t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x))$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

Moreover, the semigroup is contractive on $\mathrm{L}^1(\Omega)$ and $u \in C([0,\infty) : \mathrm{L}^1(\Omega))$.

We notice that (1) is a weak formulation of the monotonicity inequality:

$$\partial_t u \ge -\frac{1}{\mu_0\, t}\frac{F(u)}{F'(u)}\,, \qquad \text{which implies} \qquad \partial_t u \ge -\frac{1-\mu_0}{\mu_0}\frac{u}{t}$$

or equivalently, that the function

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Reminder about Mild solutions and their properties

Mild (or semigroup) solutions have been obtained by Benilan, Crandall and Pierre via Crandall-Liggett type theorems; the underlying idea is the use of an Implicit Time Discretization (ITD) method: consider the partition of $[0,T]$

$$t_k = \frac{k}{n}T\,, \quad \text{for any } 0 \le k \le n\,, \quad \text{with } t_0 = 0\,, t_n = T\,, \text{ and } \quad h = t_{k+1} - t_k = \frac{T}{n}\,.$$

For any $t \in (0,T)$, the (unique) semigroup solution $u(t,\cdot)$ is obtained as the limit in $\mathrm{L}^1(\Omega)$ of the solutions $u_{k+1}(\cdot) = u(t_{k+1},\cdot)$ which solve the following elliptic equation (u_k is the datum, is given by the previous iterative step)

$$h\mathcal{L}F(u_{k+1}) + u_{k+1} = u_k \qquad \text{or equivalently} \qquad \frac{u_{k+1} - u_k}{h} = -\mathcal{L}F(u_{k+1})\,.$$

Usually such solutions are difficult to treat since a priori they are merely very weak solutions. We can prove the following result:

Proposition. Semigroup solutions are weak dual solutions

Let u be the unique semigroup (mild) solution to the (CDP) corresponding to the initial datum $u_0 \in \mathrm{L}^p(\Omega)$ with $p \ge 1$. Then u is a weak dual solution of (CDP) and $u(t) \in \mathrm{L}^p(\Omega)$ for all $t > 0$.

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Existence and uniqueness of weak dual solutions

Theorem. Existence of Weak Dual Solutions (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ there exists a minimal weak dual solution to the (CDP). Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$.

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Moreover, the time-monotonicity holds: for a.e. $x \in \Omega$

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An "Almost" Representation Formula

For simplicity, from now on, we take $F(u) = u^m$, with $m > 1$.

Theorem. (First Pointwise Estimates) (M.B. and J. L. Vázquez)

Let $u \ge 0$ be a weak dual solution to Problem (CDP) with $u_0 \in \mathrm{L}^p(\Omega)$, $p > N/2s$. Then,

$$\int_\Omega u(t_1,x)\mathbb{G}(x,x_0)\,\mathrm{d}x \le \int_\Omega u(t_0,x)\mathbb{G}(x,x_0)\,\mathrm{d}x\,, \qquad \text{for all } t_1 \ge t_0 \ge 0\,.$$

Moreover, for almost every $0 \le t_0 \le t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}} (t_1 - t_0)\, u^m(t_0,x_0) \le \int_\Omega \left[u(t_0,x) - u(t_1,x)\right] \mathbb{G}(x,x_0)\,\mathrm{d}x$$
$$\le (m-1)\frac{t_1^{\frac{m}{m-1}}}{t_0^{\frac{1}{m-1}}}\, u^m(t_1,x_0)\,.$$

Remark. As a consequence of the above inequality and Hölder inequality, we have that $u(t) \in \mathrm{L}^\infty(\Omega)$ when $u_0 \in \mathrm{L}^p(\Omega)$, with $p > N/(2s)$.

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Sketch of the proof of the First Pointwise Estimates

We would like to take as test function

$$\psi(t,x)=\psi_1(t)\psi_2(x)=\chi_{[t_0,t_1]}(t)\,\mathbb{G}(x_0,x)\,,$$

(This is NOT an admissible test in the Definition of WDS: approximation needed)
Plugging such test function in the definition of weak dual solution gives the formula

$$\int_\Omega u(t_0,x)\mathbb{G}(x_0,x)\,\mathrm{d}x-\int_\Omega u(t_1,x)\mathbb{G}(x_0,x)\,\mathrm{d}x=\int_{t_0}^{t_1}u^m(\tau,x_0)\mathrm{d}\tau\,.$$

This formula can be proven rigorously though careful approximation.
Next, we use the monotonicity estimates,

$$t\mapsto t^{\frac{1}{m-1}}\,u(t,x)\qquad\text{is nondecreasing in } t>0 \text{ for a.e. } x\in\Omega\,.$$

to get for all $0\le t_0\le t_1$

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The single power case $F(u) = u^m$. Absolute bounds.

Theorem. (Absolute upper bounds)(M.B. & J. L. Vázquez)

Let u be a weak dual solution, then there exists constants $K_1 > 0$ depending only on N, s, m, Ω (but not on u_0 !!), such that (K1) assumption implies:

$$\boxed{\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_1}{t^{\frac{1}{m-1}}}\,,} \qquad \text{for all } \ t > 0\,.$$

- This is a very strong regularization *independent* of the initial datum u_0.
- Time decay is sharp, but only for large times, say $t \ge 1$. For small times when $0 < t < 1$ a better time decay is obtained in the form of smoothing effects.

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Sketch of the proof of Absolute Bounds

• STEP 1. *First upper estimates.* Recall the pointwise estimate:

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}} (t_1 - t_0)\, u^m(t_0, x_0) \le \int_\Omega u(t_0, x) G_\Omega(x, x_0)\, dx - \int_\Omega u(t_1, x) G_\Omega(x, x_0)\, dx\,.$$

for any $u \in \mathcal{S}_p$, all $0 \le t_0 \le t_1$ and all $x_0 \in \Omega$. Choose $t_1 = 2t_0$ to get

$$(*) \qquad \boxed{u^m(t_0, x_0) \le \frac{2^{\frac{m}{m-1}}}{t_0} \int_\Omega u(t_0, x) G_\Omega(x, x_0)\, dx\,.}$$

Recall that $u \in \mathcal{S}_p$ with $p > N/(2s)$, means $u(t) \in \mathrm{L}^p(\Omega)$ for all $t > 0$, so that:

$$u^m(t_0, x_0) \le \frac{c_0}{t_0} \int_\Omega u(t_0, x) G_\Omega(x, x_0)\, dx \le \frac{c_0}{t_0} \|u(t_0)\|_{\mathrm{L}^p(\Omega)} \|G_\Omega(\cdot, x_0)\|_{\mathrm{L}^q(\Omega)} < +\infty$$

since $G_\Omega(\cdot, x_0) \in \mathrm{L}^q(\Omega)$ for all $0 < q < N/(N-2s)$, so that $u(t_0) \in \mathrm{L}^\infty(\Omega)$ for all $t_0 > 0$.

• STEP 2. Let us estimate the r.h.s. of $(*)$ as follows:

$$u^m(t_0, x_0) \le \frac{c_0}{t_0} \int_\Omega u(t_0, x) G_\Omega(x, x_0)\, dx \le \|u(t_0)\|_{\mathrm{L}^\infty(\Omega)} \frac{c_0}{t_0} \int_\Omega G_\Omega(x, x_0)\, dx\,.$$

Taking the supremum over $x_0 \in \Omega$ of both sides, we get:

$$\boxed{\|u(t_0)\|^{m-1}_{\mathrm{L}^\infty(\Omega)} \le \frac{c_0}{t_0} \sup_{x_0 \in \Omega} \int_\Omega G_\Omega(x, x_0)\, dx \le \frac{K_1^{m-1}}{t_0}} \qquad \square$$

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The single power case $F(u) = u^m$. Smoothing Effects

Define the exponents:

$$\vartheta_{1,\gamma} = \frac{1}{2s + (N+\gamma)(m-1)} \qquad \text{and} \qquad \vartheta_1 = \vartheta_{1,0} = \frac{1}{2s + N(m-1)}$$

Theorem. (Smoothing effects) (M.B. & J. L. Vázquez)

There exist universal constants $K_3, K_4 > 0$ such that:

L^1-L^∞ SMOOTHING EFFECT: (K1) assumption implies for all $t > 0$:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_3}{t^{N\vartheta_1}} \|u(t)\|_{\mathrm{L}^1(\Omega)}^{2s\vartheta_1} \le \frac{K_3}{t^{N\vartheta_1}} \|u_0\|_{\mathrm{L}^1(\Omega)}^{2s\vartheta_1}$$

$\mathrm{L}^1_{\Phi_1}$-L^∞ SMOOTHING EFFECT: (K2) assumption implies for all $t > 0$:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u(t)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}}.$$

- A novelty is that we get instantaneous smoothing effects.
- Also the weighted smoothing effect is new (as far as we know).
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

$$(N+\gamma)\vartheta_{1,\gamma} = \frac{N+\gamma}{2 + (N+\gamma)(m-1)} < \frac{1}{m-1}.$$

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$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u(t)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} .$$

- A novelty is that we get instantaneous smoothing effects.
- Also the weighted smoothing effect is new (as far as we know).
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

$$(N+\gamma)\vartheta_{1,\gamma} = \frac{N+\gamma}{2 + (N+\gamma)(m-1)} < \frac{1}{m-1} .$$

The single power case $F(u) = u^m$. Smoothing Effects

Theorem. (Backward Smoothing effects) (M.B. & J. L. Vázquez)

There exists a universal constant $K_4 > 0$ such that for all $t, h > 0$

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_4}{t^{(d+\gamma)\vartheta_{1,\gamma}}} \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{1,\gamma}}{m-1}} \|u(t+h)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} .$$

Proof. By the monotonicity estimates , the function $u(x,t)t^{1/(m-1)}$ is non-decreasing in time for fixed x, therefore using the smoothing effect, we get for all $t_1 \ge t$:

$$\begin{aligned}\|u(t)\|_{\mathrm{L}^\infty(\Omega)} &\le \frac{K_4}{t^{(N+1)\vartheta_{1,\gamma}}} \left(\int_\Omega u(t,x)\Phi_1(x)\,\mathrm{d}x\right)^{2s\vartheta_{1,\gamma}} \\ &\le \frac{K_4}{t^{(N+1)\vartheta_{1,\gamma}}} \left(\frac{t_1^{\frac{1}{m-1}}}{t^{\frac{1}{m-1}}} \int_\Omega u(t_1,x)\Phi_1(x)\,\mathrm{d}x\right)^{2s\vartheta_{1,\gamma}}\end{aligned}$$

where K_4 is as in the smoothing effects. Finally, let $t_1 = t + h$. □

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Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$

Let S be the unique solution to the Elliptic Dirichlet Problem for $\mathcal{L}S^m = S$.

Theorem. (Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Let $u \geq 0$ be any nonnegative WDS to the Cauchy-Dirichlet problem. Then, unless $u \equiv 0$,

$$\sup_{x\in\Omega} \left| t^{\frac{1}{m-1}} u(t,x) - S(x) \right| \xrightarrow[t\to\infty]{} 0\,.$$

This result, gives a clear suggestion of what the boundary behaviour of parabolic solutions should be,

$$\boxed{u(t,x) \asymp \mathcal{U}(t,x) = \frac{S(x)}{t^{\frac{1}{m-1}}}}$$

at least for large times, as it happens in the local case $s = 1$.

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Sharp Boundary Behaviour

- **(More) Assumptions on the operator & Spectral Kernels**
- **Upper Boundary Estimates**
- **Infinite Speed of Propagation**
- **Lower Boundary Estimates**
- **Harnack-type Inequalities**
- **Numerics**

Recall that the linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, and we have already explained the assumptions (K1) and (K2) on the inverse.

Assumptions on the kernel.

- Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x,y)$ via the formula

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\, \mathrm{d}y\,,$$

assumption (L1) states that there exists $\underline{\kappa}_\Omega > 0$ such that

$$\text{(L1)} \qquad \inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,.$$

- Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x,y)$ and a zero order term:

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\, \mathrm{d}y + B(x) f(x),$$

assumptions (L2) states that there exists $\underline{\kappa}_\Omega > 0$ and $\gamma \in (0,1]$

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About the kernels of spectral nonlocal operators. Most of the examples of nonlocal operators, but the SFL, admit a representation with a kernel
A natural question is: does the SFL admit such a representation?
Let A be a uniformly elliptic linear operator. Define the s^{th} power of A:

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Then it admits a representation with a Kernel plus zero order term:

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Let (A1), (A2), and (K2) hold. Let $u \geq 0$ be a weak dual solution to the (CDP). Let $\sigma \in (0, 1]$ be

$$\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$$

Then, there exists a computable constant $\overline{\kappa} > 0$, depending only on N, s, m, and Ω, (but not on u_0 !!) such that for all $t \geq 0$ and all $x \in \Omega$:

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Infinite Speed of Propagation

and

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Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

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$$\mathcal{L}f(x) = P.V.\int_{\mathbb{R}^N} \big(f(x)-f(y)\big)\, K(x,y)\,\mathrm{d}y + B(x) f(x), \quad \text{with} \quad \begin{cases} K(x,y) \gtrsim \delta^\gamma(x)\,\delta^\gamma(y), \\ B(x) \ge 0, \end{cases}$$

• In many cases (RFL, CFL), K satisfies a stronger property: $K \ge \underline{\kappa}_\Omega > 0$ in $\overline{\Omega} \times \overline{\Omega}$.

Infinite speed of propagation.

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and all } x \in \Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s < 1$, contrary to the "local" case $s = 1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
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Sharp Lower boundary estimates

Sharp lower boundary estimates I: the non-spectral case.

Let $\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$. Let $\mathcal{L}$ satisfy (A1) and (A2), and assume moreover that

$$\mathcal{L}f(x) = \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\,dy\,, \qquad \text{with } \inf_{x,y\in\Omega} K(x,y) \geq \underline{\kappa}_\Omega > 0\,.$$

Assume moreover that $\mathcal{L}$ has a first eigenfunction $\Phi_1 \asymp \mathrm{dist}(x,\partial\Omega)^\gamma$ and that
- either $\sigma = 1$;
- or $\sigma < 1$, $K(x,y) \leq c_1|x-y|^{-(N+2s)}$ for a.e. $x,y \in \mathbb{R}^N$, and $\Phi_1 \in C^\gamma(\overline{\Omega})$.

Theorem. (Sharp lower bounds for all times) (M.B., A. Figalli and J. L. Vázquez)

Under the above assumptions, let $u \geq 0$ be a weak dual solution to the (CDP) with $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_1 > 0$ such that

$$\boxed{u(t,x_0) \geq \underline{\kappa}_1 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and a.e. } x \in \Omega\,.$$

where $t_* = \kappa_* \|u_0\|^{-(m-1)}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$. The constants $\kappa_*, \underline{\kappa}_1$ depend only on $N, s, \gamma, m, \underline{\kappa}_\Omega, c_1, \Omega$.

- The *boundary behavior is sharp for all times* in view of the upper bounds.
- Within examples, this applies to RFL and CFL type, but not to SFL-type.
- For RFL, this result was obtained first by MB, A. Figalli and X. Ros-Oton.

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Sharp absolute lower estimates for large times: the case $\sigma = 1$.
When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

Theorem. (Sharp lower bounds for large times) (M.B., A. Figalli and J. L. Vázquez)

Let $(A1)$, $(A2)$, and $(K2)$ hold, and let $\sigma = 1$ and $2sm \neq \gamma(m-1)$. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. There exists a constant $\underline{\kappa}_2 > 0$ such that

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Here, $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and the constants $\kappa_*, \underline{\kappa}_2$ depend only on N, s, γ, m, and Ω.

- It holds for $s = 1$, the local case, where there is finite speed of propagation.
- When $s = 1$, t_* is the time that the solution needs to be positive everywhere.
- When $\mathcal{L} = -\Delta$, proven by Aronson-Peletier ('81) and Vázquez ('04)
- Our method applies when $\mathcal{L}$ is an elliptic operator with C^1 coefficients (new result).
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Positivity for large times II: the case $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{\gamma}{2} - \frac{\gamma}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared 'big' solutions:

$$u_0 \ge \epsilon_0 S \qquad \text{implies} \qquad u(t) \ge \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{L^\infty} = 0 \qquad \text{for all } 0 \le u_0 \in L^1_{\Phi_1}(\Omega)\,.$$

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Different boundary behaviour when $\sigma < 1$. The next result shows that, in general, we cannot hope to prove that $u(t)$ is larger than $\Phi_1^{1/m}$, but always smaller than $\Phi_1^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \leq u_0 \leq c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \leq c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

When $\sigma = 1$ and $2sm = \gamma(m-1)$, then

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Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

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Different boundary behaviour when $\sigma < 1$.

We next show that assuming (K4), the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa} \Phi_1^{\alpha}(x) \qquad \text{for a.e. } x \in \Omega\,, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}\,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \leq u_0 \leq A\,\Phi_1^{1-\frac{2s}{\gamma}} \qquad \Rightarrow \qquad u(t) \leq \left[A^{1-m} - \tilde{C}t\right]^{-(m-1)} \Phi_1^{1-\frac{2s}{\gamma}}$$

for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C}A^{m-1})$, for some $\tilde{C} > 0$.
Recall that we have a universal lower bound (under minimal assumptions on K)

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

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Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa}\Phi_1^{\alpha}(x) \qquad \text{for a.e. } x \in \Omega\,, \qquad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}\,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \leq u_0 \leq A\,\Phi_1^{1-\frac{2s}{\gamma}} \qquad \Rightarrow \qquad u(t) \leq [A^{1-m} - \tilde{C}t]^{-(m-1)}\,\Phi_1^{1-\frac{2s}{\gamma}}$$

for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C}A^{m-1})$, for some $\tilde{C} > 0$.
Recall that we have a universal lower bound (under minimal assumptions on K)

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Harnack-type Inequalities

- Global Harnack Principle I. The non-spectral case.
- Consequences of GHP.
- Global Harnack Principle II. The remaining cases.

Global Harnack Principle I. The non-spectral case.

Recall that

$$\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma, \quad \sigma = 1 \wedge \frac{2sm}{\gamma(m-1)}, \quad t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}.$$

Theorem. (Global Harnack Principle I. The non-spectral case.)(MB & AF & JLV)

Let (A1), (A2), (L1) and (K2). Let $u \geq 0$ be a weak dual solution to the (CDP). Also, when $\sigma < 1$, assume that $K(x,y) \leq c_1|x-y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$.

Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds:

$$\boxed{\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$, and $\|\Phi_1\|_{C^\gamma(\Omega)}$.

- For large times $t \geq t_*$ the estimates are independent on the initial datum.
- For $s = 1$, $\mathcal{L} = -\Delta$, similar results by Aronson and Peletier [JDE, 1981], Vázquez [Monatsh. Math. 2004]

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Consequences of GHP with matching powers

Corollary. (Local Harnack Inequalities of Elliptic/Backward Type)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, s. t. for all $t > 0$ and $h \geq 0$

$$\sup_{x\in B_R(x_0)} u(t,x) \leq \hat{H}\left[\left(1+\frac{h}{t}\right)\left(1\wedge\frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x\in B_R(x_0)} u(t+h,x)\,.$$

When $s = 1$, backward Harnack inequalities are typical of Fast Diffusion eq. ($m < 1$, possible extinction in finite time), and they do not happen when $m > 1$ (finite speed of propagation)

Theorem. (Sharp Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Assume that a GHP with matching powers hold. Set $\mathcal{U}(t,x) := t^{-\frac{1}{m-1}} S(x)$. Then there exists $c_0 > 0$ such that, for all $t \geq t_0 := c_0 \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have

$$\left\|\frac{u(t,\cdot)}{\mathcal{U}(t,\cdot)} - 1\right\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{2}{m-1}\,\frac{t_0}{t_0+t}\,.$$

This asymptotic result is sharp: check by considering $u(t,x) = \mathcal{U}(t+1,x)$. For the classical case $\mathcal{L} = \Delta$, we recover the results of Aronson-Peletier and Vazquez with a different proof.

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Global Harnack Principles II. The remaining cases.

Theorem. (Global Harnack Principle II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Assume that:
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The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

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Hence, in the remaining cases, we have only the following general result.

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- This is sufficient to ensure interior regularity, under ‘minimal’ assumptions.
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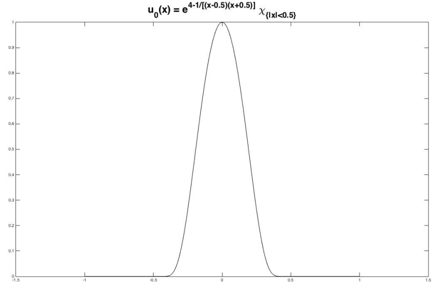
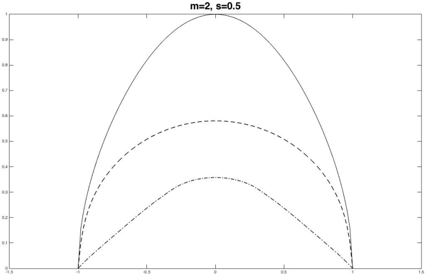
Numerical Simulations*

* Graphics obtained by numerical methods contained in: N. Cusimano, F. Del Teso, L. Gerardo-Giorda, G. Pagnini, *Discretizations of the spectral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary conditions*, SIAM Num. Anal. (2018).
Graphics and videos: courtesy of F. Del Teso (NTNU, Trondheim, Norway)

Numerics I. Matching

Numerical simulation for the SFL with parameters $m = 2$ and $s = 1/2$, hence $\sigma = 1$.





Left: the initial condition $u_0 \le C_0 \Phi_1$

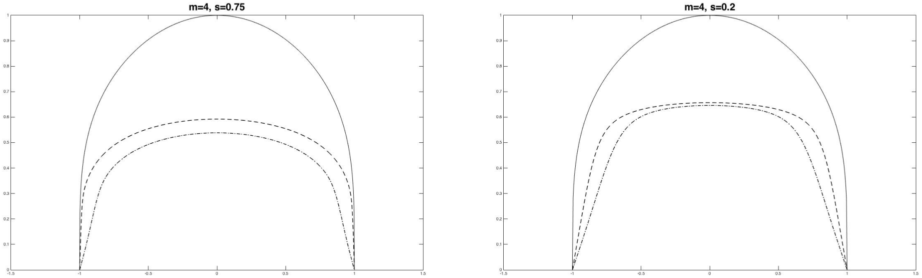
Right: solid line represents $\Phi_1^{1/m}$

the dotted lines represent $\boxed{t^{\frac{1}{m-1}} u(t)}$ at time at $t = 1$ and $t = 5$

While $u(t)$ appears to behave as $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for very short times
already at $t = 5$ it exhibits the matching boundary behavior $t^{\frac{1}{m-1}} u(t) \asymp \Phi_1^{1/m}$

Numerics II. Matching VS Non-Matching

Compare $\sigma = 1$ VS $\sigma < 1$: same $u_0 \le C_0\Phi_1$, solutions with different parameters



Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 30$ and $t = 150$; $m = 4$, $s = 3/4$, $\sigma = 1$.

Matching: $u(t)$ behaves like $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for quite some time,
and only around $t = 150$ it exhibits the matching boundary behavior $u(t) \asymp \Phi_1^{1/m}$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 150$ and $t = 600$; $m = 4$, $s = 1/5$, $\sigma = 8/15 < 1$.

Non-matching: $u(t) \asymp \Phi_1$ even after long time.

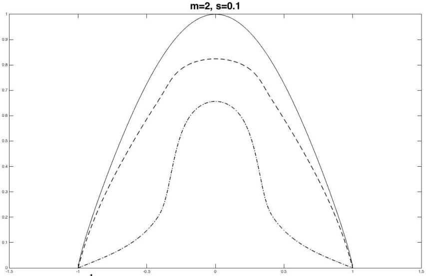
Idea: maybe when $\sigma < 1$ and $u_0 \lesssim \Phi_1$, we have $u(t) \asymp \Phi_1$ for all times...

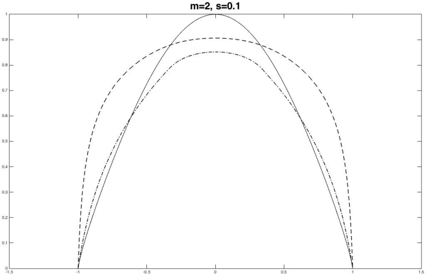
Not True: there are cases when $u(t) \gg \Phi_1^{1-2s}$ for large times...

Numerics III. Non-Matching

Non-matching when $\sigma < 1$: same data u_0, with $m = 2$ and $s = 1/10$, $\sigma = 2/5 < 1$

In both pictures, the solid line represents Φ_1^{1-2s} (anomalous behaviour)





Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 4$ and $t = 25$.

$u(t) \asymp \Phi_1$ for short times $t = 4$, then $u(t) \sim \Phi_1^{1-2s}$ for intermediate times $t = 25$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 40$ and $t = 150$. $u(t) \gg \Phi_1^{1-2s}$ for large times.

Both non-matching always different behaviour from the asymptotic profile $\Phi_1^{\sigma/m}$.

In this case we show that if $u_0(x) \le C_0 \Phi_1(x)$ then for all $t > 0$

$$u(t,x) \le C_1 \left[\frac{\Phi_1(x)}{t}\right]^{\frac{1}{m}} \qquad \text{and} \qquad \lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\frac{\sigma}{m}}} = 0 \qquad \text{for any } t > 0.$$

Regularity Estimates

- Interior Regularity
- Hölder continuity up to the boundary
- Higher interior regularity for RFL

Interior Regularity

The regularity results, require the validity of a Global Harnack Principle.

(R) The operator $\mathcal{L}$ satisfies (A1) and (A2), and $\mathcal{L}^{-1}$ satisfies (K2). Moreover, we consider

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\,\mathrm{d}y + B(x)f(x)\,, \qquad \text{with}$$

$$K(x,y) \asymp |x-y|^{-(N+2s)} \quad \text{in } B_{2r}(x_0) \subset \Omega, \qquad K(x,y) \lesssim |x-y|^{-(N+2s)} \quad \text{in } \mathbb{R}^N \setminus B_{2r}(x_0).$$

As a consequence, for any ball $B_{2r}(x_0) \subset\subset \Omega$ and $0 < t_0 < T_1$, there exist $\delta, M > 0$ such that

$$0 < \delta \le u(t,x) \qquad \text{for a.e. } (t,x) \in (T_0, T_1) \times B_{2r}(x_0),$$
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The constants in the regularity estimates will depend on the solution only through δ, M.

Theorem. (Interior Regularity) (M.B., A. Figalli and J. L. Vázquez)

Assume (R) and let u be a nonnegative bounded weak dual solution to problem (CDP).

1. Then u is **Hölder continuous in the interior**. More precisely, there exists $\alpha > 0$ such that, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{\alpha/2s,\alpha}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

2. Assume in addition $|K(x,y) - K(x',y)| \le c|x-x'|^{\beta}\,|y|^{-(N+2s)}$ for some $\beta \in (0, 1\wedge 2s)$ such that $\beta + 2s \notin \mathbb{N}$. Then u **is a classical solution in the interior**. More precisely, for all $0 < T_0 < T_2 < T_1$,

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Interior Regularity

The regularity results, require the validity of a Global Harnack Principle.
(R) The operator $\mathcal{L}$ satisfies (A1) and (A2), and $\mathcal{L}^{-1}$ satisfies (K2). Moreover, we consider

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Hölder continuity up to the boundary

Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

$$\|u\|_{C^{\frac{\gamma}{m\vartheta},\frac{\gamma}{m}}_{t,x}((T_2,T_1)\times\Omega)} \le C \qquad \text{with} \qquad \vartheta := 2s - \gamma\left(1 - \frac{1}{m}\right).$$

- Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spacial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)
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Higher Interior Regularity for RFL.

Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)** , then $u \in C_x^\infty((0,\infty)\times\Omega)$.
More precisely, let $k \geq 1$ be any positive integer, and $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, then, for any $t \geq t_0 > 0$ we have

$$\left|D_x^k u(t,x)\right| \leq C\,\delta(x)^{\frac{s}{m}-k},$$

where C depends only on N, s, m, k, Ω, t_0, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in t. To our knowledge also open for the local case $s = 1$.
- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C_x^\infty$
  $$\|u\|_{C_x^{k,\alpha}((\frac{1}{2},1)\times B_{1/2})} \leq C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \geq 0.$$
  Analogous estimates in time do not hold for $k \geq 1$ and $\alpha \in (0,1)$.
  Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not C^1 in t in $(\frac{1}{2},1)\times B_{1/2}$. [Chang-Lara, Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains , and also for operator with more general kernels.
- Also the "classical/local" case $s = 1$ works after the waiting time t_*:
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The End

Thank You!!!

Arigatō!!!

Grazie Mille!!!