Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

Matteo Bonforte

Departamento de Matemáticas,
Universidad Autónoma de Madrid,
Campus de Cantoblanco
28049 Madrid, Spain

matteo.bonforte@uam.es

http://verso.mat.uam.es/~matteo.bonforte

Workshop in honor of Alessio Figalli's Doctor Honoris Causa at UPC
Five talks and a Round Table with Prof. Alessio Figalli
Facultat de Matematiques i Estadistica
UNIVERSITAT POLITECNICA DE CATALUNYA
Barcelona, Spain, November 21, 2019

Outline of the talk

- **Introduction to the Parabolic Problem on Domains**
- **The Classical Porous Medium Equation (PME)**
  - **A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"**
- **The Fractional PME I: Basic theory**
  - **Three Different Fractional Laplacians on Bounded Domains**
  - **Existence, Uniqueness and Boundedness**
- **The Fractional PME II: Sharp Boundary Behaviour**
  - **Positivity Estimates and Infinite Speed of Propagation**
  - **Global Harnack Principles**
  - **Asymptotic Behaviour**
  - **Anomalous Boundary Behaviour and Counterexamples**
  - **Some Numerics**

Introduction to the Parabolic Problem on Domains

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)}\qquad \begin{cases} u_t + \mathcal{L}\,F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  The classical Laplacian and *all fractional Laplacians on domains.*
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of “degenerate” nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Introduction to the Parabolic Problem on Domains

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  The classical Laplacian and *all fractional Laplacians on domains.*
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Introduction to the Parabolic Problem on Domains

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  The classical Laplacian and *all fractional Laplacians on domains.*
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of “degenerate” nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Introduction to the Parabolic Problem on Domains

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  The classical Laplacian and *all fractional Laplacians on domains.*
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

The Classical Porous Medium Equation (PME)

A Brief Summary about the Dirichlet Problem for PME in few “Blackboards”

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$\Omega \subseteq \mathbb{R}^N$ bounded domain.

$u_0 \in C_c^\infty(\Omega)$ smooth & Compactly supported.

$$\begin{cases} u_t = \Delta u^m & \text{in } (0,+\infty)\times\Omega \\ u = 0 & \text{on } (0,+\infty)\times\partial\Omega \\ u(t=0) = u_0 & \text{in } \Omega. \end{cases} \qquad m>1$$

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$\Omega \subseteq \mathbb{R}^N$ bounded domain.

$u_0 \in C_c^\infty(\Omega)$ smooth & compactly supported.

$$\begin{cases} u_t = \Delta u^m & \text{in } (0,+\infty)\times\Omega \\ u = 0 & \text{on } (0,+\infty)\times\partial\Omega \\ u(t=0) = u_0 & \text{in } \Omega. \end{cases} \qquad m>1$$

ARONSON
BENILAN
BREZIS
CAFFARELLI
DI BENEDETTO
EVANS
FRIEDMAN
KENIG
VAZQUEZ
CRANDALL
DASKALOPOULOS
PELETIER
PIERRE
GIANAZZA
VESPRI ....

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$\Omega \subseteq \mathbb{R}^N$ bounded domain.

$u_0 \in C_c^\infty(\Omega)$ smooth & compactly supported.

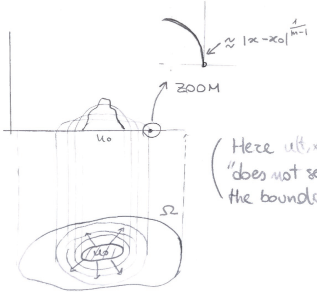
$$\begin{cases} u_t = \Delta u^m & \text{in } (0,+\infty)\times\Omega \\ u = 0 & \text{on } (0,+\infty)\times\partial\Omega \\ u(t=0) = u_0 & \text{in } \Omega. \end{cases} \qquad m>1$$

ARONSON
BENILAN
BREZIS
CAFFARELLI
DI BENEDETTO
EVANS
FRIEDMAN
KENIG
VAZQUEZ
CRANDALL
DASKALOPOULOS
PELETIER
PIERRE
GIANAZZA
VESPRI ....

(o) $0 < t < \underline{t}$ INITIAL TIMES

(Barenblatt behaviour)

$$u(t,x) \approx \left(C - \frac{|x|^2}{t^{2\beta}}\right)_+^{\frac{1}{m-1}} t^\alpha = B(t,x)$$



(Here $u(t,x)$ "does not see" the boundary)

- the support of $u(t)$ spreads from $\mathrm{supp}(u_0)$ with finite speed (close to $B(t,x)$)
- the support of $u(t)$ does NOT TOUCH the boundary $\partial\Omega$.

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$\Omega \subseteq \mathbb{R}^N$ bounded domain.

$u_0 \in C_c^\infty(\Omega)$ smooth & compactly supported.

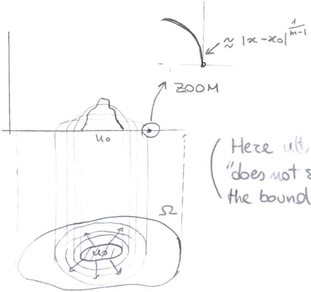
$$\begin{cases} u_t = \Delta u^m & \text{in } (0,+\infty)\times\Omega \\ u = 0 & \text{on } (0,+\infty)\times\partial\Omega \\ u(t=0) = u_0 & \text{in } \Omega. \end{cases} \qquad m>1$$

ARONSON
BENILAN
BREZIS
CAFFARELLI
DI BENEDETTO
EVANS
FRIEDMAN
KENIG
VAZQUEZ
CRANDALL
DASKALOPOULOS
PELETIER
PIERRE
GIANAZZA
VESPRI ....

(o) $0 < t < \underline{t}$ — INITIAL TIMES

(Barenblatt behaviour)

$$u(t,x) \approx \left(C - \frac{|x|^2}{t^{2\beta}}\right)_+^{\frac{1}{m-1}} t^\alpha = B(t,x)$$



(Here $u(t,x)$ "does not see" the boundary)

- the support of $u(t)$ spreads from $\mathrm{supp}(u_0)$ with finite speed (close to $B(t,x)$)
- the support of $u(t)$ does NOT TOUCH the boundary $\partial\Omega$.

$t = \underline{t}$

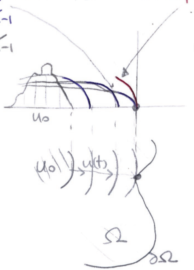
(the first "touching point")

- When the support of $u(t)$ touches $\partial\Omega$ for the first time.
- Transition of boundary behaviour:

"Barenblatt" Beh. VS "Elliptic" Beh.

$|x-x_0|^{\frac{1}{m-1}}$ | $|x-x_0|^{1/m}$

$d(x,\partial\Omega)^{\frac{1}{m-1}}$ | $d(x,\partial\Omega)^{1/m}$



(Here $u(t,x)$ "starts to see $\partial\Omega$")

the solution starts to "inflate"

from $d(x,\partial\Omega)^{\frac{1}{m-1}} \ll d(x,\partial\Omega)^{\frac{1}{m}}$ $\quad (m>1)$

TO

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$\Omega \subseteq \mathbb{R}^N$ bounded domain.

$u_0 \in C_c^\infty(\Omega)$ smooth & Compactly supported.

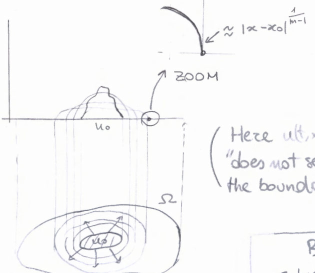
$$\begin{cases} u_t = \Delta u^m & \text{in } (0,+\infty)\times\Omega \\ u = 0 & \text{on } (0,+\infty)\times\partial\Omega \\ u(t=0) = u_0 & \text{in } \Omega \end{cases} \qquad m>1$$

ARONSON
BENILAN
BREZIS
CAFFARELLI
DiBENEDETTO
EVANS
FRIEDMAN
KENIG
VAZQUEZ
CRANDALL
DASKALOPOULOS
PELETIER
PIERRE
GIANAZZA
VESPRI ....

(o) $0 < t < \underline{t}$ INITIAL TIMES

(Barenblatt behaviour)

$$u(t,x) \approx \left(C - \frac{|x|^2}{t^{2\beta}}\right)_+^{\frac{1}{m-1}} t^\alpha = B(t,x)$$



(Here $u(t,x)$ "does not see" the boundary)

- the support of $u(t)$ spreads from $\mathrm{supp}(u_0)$ with finite speed (close to $B(t,x)$)
- the support of $u(t)$ does NOT TOUCH the boundary $\partial\Omega$.

$t = \underline{t}$

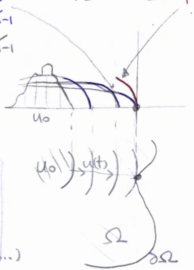
(the first "touching point")

- When the support of $u(t)$ touches $\partial\Omega$ for the first time.
- Transition of boundary behaviour:

"Barenblatt" Beh. VS "Elliptic" Beh.

$|x-x_0|^{\frac{1}{m-1}}$ — $|x-x_0|^{1/m}$

$d(x,\partial\Omega)^{\frac{1}{m-1}}$ — $d(x,\partial\Omega)^{1/m}$



(Here $u(t,x)$ "starts to see $\partial\Omega$")

REGULARITY

- solutions are smooth when positive & bounded.
- Free boundary: delicate issue (CAFFARELLI, VAZQUEZ, WOLANSKI, KOCH, ...)

the solution starts to "inflate"

from $d(x,\partial\Omega)^{\frac{1}{m-1}} \ll d(x,\partial\Omega)^{\frac{1}{m}}$ $(m>1)$

TO

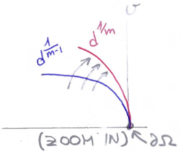
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the supp(u(t)) touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the elliptic one:

$$u(t,x) \approx \frac{d(x,\partial\Omega)^{1/m}}{t^{1/(m-1)}}$$



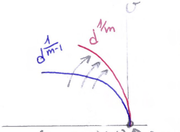
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the supp$(u(t))$ touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the elliptic one:

$$u(t,x) \approx \frac{d(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$



(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES, & LARGE)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\text{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1 \frac{\text{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

$$u(t,x) \asymp \frac{S(x)}{t^{\frac{1}{m-1}}} = \mathcal{U}(t,x)$$

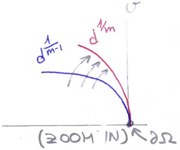
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the supp(u(t)) touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the elliptic one:

$$u(t,x) \approx \frac{d(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$



(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES & LARGE)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

⟷ (STATIONARY for RESCALED FLOW) ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \dfrac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

$$u(t,x) \asymp \frac{S(x)}{t^{\frac{1}{m-1}}} = \mathcal{U}(t,x)$$

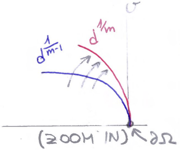
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

∘ Once the supp(u(t)) touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the elliptic one:

$$u(t,x) \approx \frac{d(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$



(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES & LARGE)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{dist(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1 \frac{dist(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

⟷ (STATIONARY for RESCALED FLOW) ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \dfrac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

↓ rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

↓ Semilinear Structure: $S^m = V$, $p = 1/m < 1$

(ELLIPTIC THEORY)

$V \asymp dist(\cdot,\partial\Omega)$

$S \asymp dist(\cdot,\partial\Omega)^{1/m}$

$$\begin{cases} -\Delta V = \dfrac{p}{1-p} V^p & \text{in } \Omega \\ V = 0 & \text{on } \partial\Omega \end{cases}$$

$$u(t,x) \asymp \frac{S(x)}{t^{\frac{1}{m-1}}} = \mathcal{U}(t,x)$$

← SEPARATE VARIABLE SOLUTION.

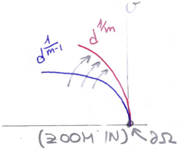
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the supp($u(t)$) touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the elliptic one:

$$u(t,x) \approx \frac{d(x,\partial\Omega)^{1/m}}{t^{1/m-1}}$$



SLOW MOTION DYNAMICS: (LOGARITHMIC TIME RESCALING)

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \rightsquigarrow \text{(SAME LATERAL B-C)} \begin{cases} v_t = \Delta v^m + \frac{v}{m-1} \\ v(t=0) = u_0. \end{cases}$$

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$

(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES & LARGE)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{1/m-1}} \le u(t,x) \le C_1 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{1/m-1}}$$

(STATIONARY for RESCALED FLOW)
ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \frac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

Semilinear Structure: $S^m = V$, $p = 1/m < 1$

(ELLIPTIC THEORY)

$$\begin{cases} -\Delta V = \frac{p}{1-p} V^p & \text{in } \Omega \\ V = 0 & \text{on } \partial\Omega \end{cases}$$

$V \asymp \mathrm{dist}(\cdot,\partial\Omega)$

$S \asymp \mathrm{dist}(\cdot,\partial\Omega)^{1/m}$

rewritten

$$C_0' \frac{S(x)}{t^{1/m-1}} \le u(t,x) \le C_1' \frac{S(x)}{t^{1/m-1}}$$

$$u(t,x) \asymp \frac{S(x)}{t^{1/m-1}} = \mathcal{U}(t,x)$$

← SEPARATE VARIABLE SOLUTION.

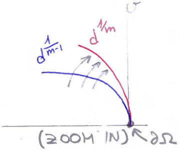
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the supp(u(t)) touches the boundary of Ω, the solution starts to inflate. The behaviour at $\partial\Omega$ becomes the elliptic one:

$$u(t,x) \approx \frac{d(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$



SLOW MOTION DYNAMICS: (LOGARITHMIC TIME RESCALING)

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \quad \overset{\text{(SAME LATERAL B.C.)}}{\rightsquigarrow} \quad \begin{cases} v_t = \Delta v^m + \dfrac{v}{m-1} \\ v(t=0) = u_0. \end{cases}$$

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau, x), \quad t = \log(\tau+1)$$

(•) $t \to +\infty$ ASYMPTOTIC BEHAVIOUR

$$\tau^{\frac{1}{m-1}} |u(\tau,x) - S(x)| \xrightarrow[\tau\to+\infty]{\text{UNIF}} 0 \quad \Big\} \quad v(t,x) \xrightarrow[t\to+\infty]{\text{UNIF}} S(x)$$

(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES & LARGE)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq C_1 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

$\longleftrightarrow$ (STATIONARY for RESCALED FLOW) ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \dfrac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

Semilinear Structure: $S^m = V$, $p = 1/m < 1$

$$\begin{cases} -\Delta V = \dfrac{p}{1-p} V^p & \text{in } \Omega \\ V = 0 & \text{on } \partial\Omega \end{cases}$$

(ELLIPTIC THEORY)

$V \asymp \mathrm{dist}(\cdot,\partial\Omega)$

$S \asymp \mathrm{dist}(\cdot,\partial\Omega)^{1/m}$

rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

$$u(t,x) \asymp \frac{S(x)}{t^{\frac{1}{m-1}}} = \mathcal{U}(t,x)$$

$\leftarrow$ SEPARATE VARIABLE SOLUTION.

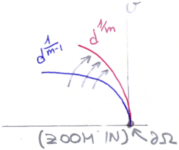
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

(•) $\underline{t} < t < t_*$ (TRANSITION OF BOUNDARY BEHAVIOUR)

REACHING THE BOUNDARY. ("forgetting u_0")

- Once the $\mathrm{supp}(u(t))$ touches the boundary of Ω, the solution starts to inflate. the behaviour at $\partial\Omega$ becomes the elliptic one:

$$u(t,x) \approx \frac{d(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$



SLOW MOTION DYNAMICS: (LOGARITHMIC TIME RESCALING)

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \rightsquigarrow \text{(SAME LATERAL B.C.)} \quad \begin{cases} v_t = \Delta v^m + \frac{v}{m-1} \\ v(t=0) = u_0. \end{cases}$$

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$

(•) $t \to +\infty$ ASYMPTOTIC BEHAVIOUR

$$\tau^{\frac{1}{m-1}} |u(\tau,x) - S(x)| \xrightarrow[\tau\to+\infty]{\text{UNIF}} 0 \qquad v(t,x) \xrightarrow[t\to+\infty]{\text{UNIF}} S(x)$$

SHARP

$$\left|\frac{u(\tau,x)}{\mathcal{U}(t,x)} - 1\right| \le \frac{C}{1+\tau} \qquad \left|\frac{v(t,x)}{S(x)} - 1\right| \le C e^{-t}$$

(•) $t \geq t_*$: POSITIVITY in all Ω (INTERMEDIATE TIMES & LARGE)

GLOBAL HARNACK PRINCIPLE:

$$C_0 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1 \frac{\mathrm{dist}(x,\partial\Omega)^{1/m}}{t^{\frac{1}{m-1}}}$$

rewritten

$$C_0' \frac{S(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le C_1' \frac{S(x)}{t^{\frac{1}{m-1}}}$$

$$u(t,x) \asymp \frac{S(x)}{t^{\frac{1}{m-1}}} = \mathcal{U}(t,x)$$

← SEPARATE VARIABLE SOLUTION.

(STATIONARY for RESCALED FLOW)

ASSOCIATED ELLIPTIC PROBLEM.

$$\begin{cases} -\Delta S^m = \frac{S}{m-1} & \text{in } \Omega \\ S = 0, & \text{on } \partial\Omega \end{cases}$$

Semilinear Structure: $S^m = V$, $p = 1/m < 1$

$$\begin{cases} -\Delta V = \frac{p}{1-p} V^p & \text{in } \Omega \\ V = 0 & \text{on } \partial\Omega \end{cases}$$

(ELLIPTIC THEORY)

$V \asymp \mathrm{dist}(\cdot,\partial\Omega)$

$S \asymp \mathrm{dist}(\cdot,\partial\Omega)^{1/m}$

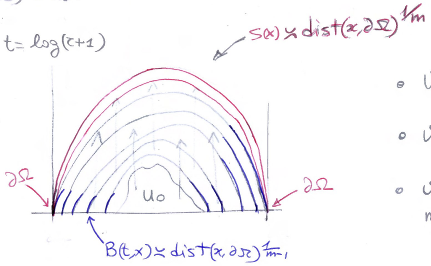
A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \quad \rightsquigarrow \quad \begin{cases} v_t = \Delta v^m + \frac{v}{m-1} \\ v(t=0) = u_0 . \end{cases}$$

(SAME LATERAL B.C.)

SLOW MOTION DYNAMICS

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$



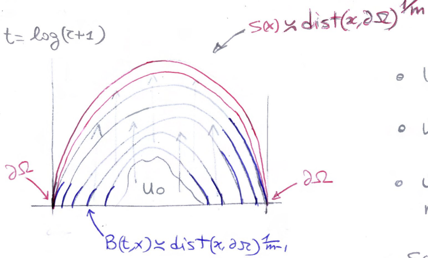
- $v_t = \Delta v^m + \frac{v}{m-1}$
- $v_t \geq 0 \leftarrow$ (BENILAN-CRANDALL)
  $\Downarrow$
- $v(t,x) \nearrow S(x)$
  monotonically increases to $S(x)$

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \quad \rightsquigarrow \text{(SAME LATERAL B.C.)} \quad \begin{cases} v_t = \Delta v^m + \dfrac{v}{m-1} \\ v(t=0) = u_0 . \end{cases}$$

SLOW MOTION DYNAMICS

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau, x), \quad t = \log(\tau+1)$$



- $v_t = \Delta v^m + \dfrac{v}{m-1}$
- $v_t \geq 0 \leftarrow$ (BENILAN–CRANDALL)
  $\Downarrow$
- $v(t,x) \nearrow S(x)$ monotonically increases to $S(x)$
- $S(x)$ Represents AN ABSOLUTE UPPER BOUND FOR ALL SOLUTIONS !! "FRIENDLY GIANT". (DAHLBERG–KENIG)

$$\begin{cases} u(\tau,x) = \dfrac{S(x)}{\tau^{\frac{1}{m-1}}} \\ u(0,x) = +\infty \end{cases} \quad \longleftarrow \text{RESCALING BACK.}$$

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \rightsquigarrow \begin{cases} v_t = \Delta v^m + \frac{v}{m-1} \\ v(t=0) = u_0 . \end{cases}$$

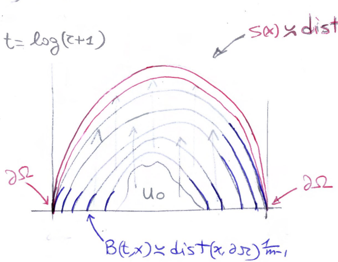
(SAME LATERAL B.C.)

SLOW MOTION DYNAMICS

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau, x), \quad t = \log(\tau+1)$$

SEPARATION OF VARIABLES

$$\begin{cases} \mathcal{U}_T(\tau,x) = \dfrac{S(x)}{(T+t)^{\frac{1}{m-1}}} \\ \mathcal{U}_T(0,x) = \dfrac{S(x)}{T^{\frac{1}{m-1}}} \end{cases}$$



- $v_t = \Delta v^m + \frac{v}{m-1}$
- $v_t \geq 0 \leftarrow$ (BENILAN–CRANDALL) $\Downarrow$
- $v(t,x) \nearrow S(x)$ monotonically increases to $S(x)$
- $S(x)$ Represents AN ABSOLUTE UPPER BOUND FOR ALL SOLUTIONS !! "FRIENDLY GIANT". (DAHLBERG–KENIG)

$T \to 0$

$$\begin{cases} \mathcal{U}(\tau,x) = \dfrac{S(x)}{\tau^{\frac{1}{m-1}}} \\ \mathcal{U}(0,x) = +\infty \end{cases}$$

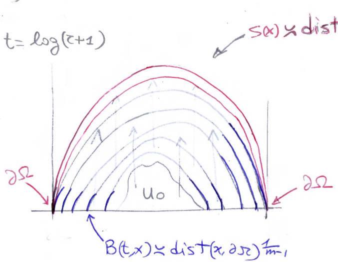
$\leftarrow$ RESCALING BACK.

A Brief Summary about the Dirichlet Problem for PME in few "Blackboards"

$$\begin{cases} u_\tau = \Delta u^m \\ u(\tau=0) = u_0 \end{cases} \quad \xrightarrow{\text{(SAME LATERAL B.C.)}} \quad \begin{cases} v_t = \Delta v^m + \dfrac{v}{m-1} \\ v(t=0) = u_0 . \end{cases}$$

SLOW MOTION DYNAMICS

$$v(t,x) = \tau^{\frac{1}{m-1}} u(\tau,x), \quad t = \log(\tau+1)$$



- $v_t = \Delta v^m + \dfrac{v}{m-1}$
- $v_t \geq 0 \leftarrow$ (BENILAN–CRANDALL) $\Downarrow$
- $v(t,x) \nearrow S(x)$ monotonically increases to $S(x)$
- $S(x)$ Represents AN ABSOLUTE UPPER BOUND FOR ALL SOLUTIONS !! "FRIENDLY GIANT". (DAHLBERG–KENIG)

SEPARATION OF VARIABLES

$$\begin{cases} \mathcal{U}_T(\tau,x) = \dfrac{S(x)}{(T+t)^{\frac{1}{m-1}}} \\ \mathcal{U}_T(0,x) = \dfrac{S(x)}{T^{\frac{1}{m-1}}} \end{cases}$$

$T \to 0$

$$\begin{cases} \mathcal{U}(\tau,x) = \dfrac{S(x)}{\tau^{\frac{1}{m-1}}} \\ \mathcal{U}(0,x) = +\infty \end{cases} \quad \xleftarrow{\text{RESCALING BACK.}}$$

CONVERGENCE IN RELATIVE ERROR WITH SHARP RATE:

$$\left| \frac{u(\tau,x)}{\mathcal{U}(\tau,x)} - 1 \right| \leq \frac{c}{1+\tau} \quad \text{or} \quad \left| \frac{v(t,x)}{S(x)} - 1 \right| \leq c\, e^{-t}$$

ARONSON–PELETIER (80s), VAZQUEZ (2000s), MB.–FIGALLI–SIRE–VAZQUEZ (2015–18?)

- **The Fractional PME I: Basic theory**
  - **Three Different Fractional Laplacians on Bounded Domains**
  - **Existence, Uniqueness and Boundedness of solutions**

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

- We have seen what happens when $\mathcal{L} = -\Delta$ is the classical Laplacian
- We now focus our attention to a particular scenario:
  - When $\mathcal{L} = (-\Delta)^s$, with $s \in (0,1)$ is a Fractional Laplacian: there are three different choices of fractional Laplacian on bounded domains.
  - When $F(u) = |u|^{m-1}u$, with $m > 1$ have the classical PME nonlinearity

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)}\qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

- We have seen what happens when $\mathcal{L} = -\Delta$ is the classical Laplacian
- We now focus our attention to a particular scenario:
  - When $\mathcal{L} = (-\Delta)^s$, with $s \in (0,1)$ is a Fractional Laplacian: there are three different choices of fractional Laplacian on bounded domains.
  - When $F(u) = |u|^{m-1}u$, with $m > 1$ have the classical PME nonlinearity

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)}\qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

- We have seen what happens when $\mathcal{L} = -\Delta$ is the classical Laplacian
- We now focus our attention to a particular scenario:
  - When $\mathcal{L} = (-\Delta)^s$, with $s \in (0,1)$ is a Fractional Laplacian: there are three different choices of fractional Laplacian on bounded domains.
  - When $F(u) = |u|^{m-1}u$, with $m > 1$ have the classical PME nonlinearity

Three Different Fractional Laplacians on Bounded Domains

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s} \hat{f}(\xi) \,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x-z|^{N+2s}} \, dz,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x) \right) \frac{dt}{t^{1+s}} \,.$$

Three Different Fractional Laplacians on Bounded Domains

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{}(\xi) = |\xi|^{2s} \hat{f}(\xi) \, .$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}} \, \mathrm{d}z,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

Three Different Fractional Laplacians on Bounded Domains

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right) \frac{dt}{t^{1+s}}\,.$$

Three Different Fractional Laplacians on Bounded Domains

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are the eigenfunctions of the classical Laplacian Δ_Ω:

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega) \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega) \,,$$

and ϕ_j are as smooth as $\partial\Omega$ allows: $\partial\Omega \in C^k \;\Rightarrow\; \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x \,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1 \,.$$

The Green function of SFL satisfies, letting $\delta^\gamma(\cdot) := \mathrm{dist}(\cdot, \partial\Omega)$,

$$\text{(K4)} \qquad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) , \qquad \text{with } \boxed{\gamma = 1}$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0 \,, \qquad \text{in } (0,\infty) \times \partial\Omega \,.$$

Three Different Fractional Laplacians on Bounded Domains

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are the eigenfunctions of the classical Laplacian Δ_Ω:

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega) \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega)\,,$$

and ϕ_j are as smooth as $\partial\Omega$ allows: $\partial\Omega \in C^k \ \Rightarrow \ \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

The Green function of SFL satisfies, letting $\delta^\gamma(\cdot) := \mathrm{dist}(\cdot, \partial\Omega)$,

$$\text{(K4)} \quad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \quad \text{with } \boxed{\gamma = 1}$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

Three Different Fractional Laplacians on Bounded Domains

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are the eigenfunctions of the classical Laplacian Δ_Ω:

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega) \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega) \,,$$

and ϕ_j are as smooth as $\partial\Omega$ allows: $\partial\Omega \in C^k \;\Rightarrow\; \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\, dx \,, \qquad \text{with} \qquad \|\phi_j\|_{L^2(\Omega)} = 1 \,.$$

The Green function of SFL satisfies, letting $\delta^\gamma(\,\cdot\,) := \mathrm{dist}(\,\cdot\,, \partial\Omega)$,

$$\text{(K4)} \qquad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) , \quad \text{with } \boxed{\gamma = 1}$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0 \,, \qquad \text{in } (0,\infty) \times \partial\Omega \,.$$

Three Different Fractional Laplacians on Bounded Domains

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\ \mathrm{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\, \mathrm{d}z\,, \qquad \text{with } \mathrm{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j \in C^s(\overline{\Omega}) \cap C^\infty(\Omega)$ (J. Serra - X. Ros Oton), and

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^s \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega)^s\,,$$

The Green function of RFL satisfies, letting $\delta^\gamma(\cdot) := \mathrm{dist}(\cdot, \partial\Omega)$,

$$\text{(K4)} \qquad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \quad \text{with } \boxed{\gamma = s}$$

Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \left(\mathbb{R}^N \setminus \Omega\right).$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

Three Different Fractional Laplacians on Bounded Domains

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,dz\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j \in C^s(\overline{\Omega}) \cap C^\infty(\Omega)$ (J. Serra - X. Ros Oton), and

$$\phi_1 \asymp \operatorname{dist}(\cdot,\partial\Omega)^s \qquad \text{and} \qquad |\phi_j| \lesssim \operatorname{dist}(\cdot,\partial\Omega)^s\,,$$

The Green function of RFL satisfies, letting $\delta^\gamma(\,\cdot\,) := \operatorname{dist}(\,\cdot\,,\partial\Omega)$,

$$\text{(K4)} \qquad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right), \qquad \text{with } \boxed{\gamma = s}$$

Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \left(\mathbb{R}^N \setminus \Omega\right).$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

Three Different Fractional Laplacians on Bounded Domains

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\ \text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,dz\,, \qquad \text{with } \operatorname{supp}(g)\subseteq \overline{\Omega}\,.$$

where $s\in(0,1)$ and $c_{N,s}>0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0<\overline{\lambda}_1\le\overline{\lambda}_2\le\ldots\le\overline{\lambda}_j\le\overline{\lambda}_{j+1}\le\ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j\in\mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j \in C^s(\overline{\Omega})\cap C^\infty(\Omega)$ (J. Serra - X. Ros Oton), and

$$\phi_1 \asymp \operatorname{dist}(\cdot,\partial\Omega)^s \qquad \text{and} \qquad |\phi_j| \lesssim \operatorname{dist}(\cdot,\partial\Omega)^s\,,$$

The Green function of RFL satisfies, letting $\delta^\gamma(\,\cdot\,):=\operatorname{dist}(\,\cdot\,,\partial\Omega)$,

$$\text{(K4)}\quad \mathbb{G}(x,y)\asymp \frac{1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)\,,\quad \text{with } \boxed{\gamma = s}$$

Lateral boundary conditions for the RFL

$$u(t,x)=0\,,\qquad \text{in } (0,\infty)\times\left(\mathbb{R}^N\setminus\Omega\right).$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

Three Different Fractional Laplacians on Bounded Domains

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \mathrm{P.V.}\int_\Omega \frac{f(x)-f(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

- It is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum (λ_j, ϕ_j)
- EIGENFUNCTIONS: $\overline{\phi}_j \in C^{2s-1}(\overline{\Omega}) \cap C^{2s+\alpha}(\Omega)$ (MB, A.Figalli, J. L. Vázquez)

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^{2s-1} \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega)^{2s-1}\,,$$

The Green function $\mathbb{G}(x,y)$ satisfies, letting $\delta^\gamma(\,\cdot\,) := \mathrm{dist}(\,\cdot\,, \partial\Omega)$,

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)\,, \quad \text{with} \quad \boxed{\gamma = 2s-1}$$

Remarks.

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

Three Different Fractional Laplacians on Bounded Domains

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \mathrm{P.V.} \int_\Omega \frac{f(x)-f(y)}{|x-y|^{N+2s}}\,dy\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

- It is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum (λ_j, ϕ_j)
- EIGENFUNCTIONS: $\overline{\phi_j} \in C^{2s-1}(\overline{\Omega}) \cap C^{2s+\alpha}(\Omega)$ (MB, A.Figalli, J. L. Vázquez)

$$\phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^{2s-1} \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot,\partial\Omega)^{2s-1}\,,$$

The Green function $\mathbb{G}(x,y)$ satisfies, letting $\delta^\gamma(\,\cdot\,) := \mathrm{dist}(\,\cdot\,,\partial\Omega)$,

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)\,, \text{ with } \boxed{\gamma = 2s-1}$$

Remarks.

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

Three Different Fractional Laplacians on Bounded Domains

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \mathrm{P.V.}\int_\Omega \frac{f(x)-f(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

- It is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum (λ_j, ϕ_j)
- EIGENFUNCTIONS: $\overline{\phi_j} \in C^{2s-1}(\overline{\Omega}) \cap C^{2s+\alpha}(\Omega)$ (MB, A.Figalli, J. L. Vázquez)

$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^{2s-1} \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega)^{2s-1}\,,$$

The Green function $\mathbb{G}(x,y)$ satisfies, letting $\delta^\gamma(\,\cdot\,) := \mathrm{dist}(\,\cdot\,, \partial\Omega)$,

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)\,, \text{ with } \boxed{\gamma = 2s-1}$$

Remarks.

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

Three Different Fractional Laplacians on Bounded Domains

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \mathrm{P.V.}\int_\Omega \frac{f(x)-f(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

- It is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum (λ_j, ϕ_j)
- EIGENFUNCTIONS: $\overline{\phi}_j \in C^{2s-1}(\overline{\Omega}) \cap C^{2s+\alpha}(\Omega)$ (MB, A.Figalli, J. L. Vázquez)

$$\phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^{2s-1} \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot,\partial\Omega)^{2s-1}\,,$$

The Green function $\mathbb{G}(x,y)$ satisfies, letting $\delta^\gamma(\,\cdot\,) := \mathrm{dist}(\,\cdot\,,\partial\Omega)$,

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)\,, \text{ with } \boxed{\gamma = 2s-1}$$

Remarks.

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

Three Different Fractional Laplacians on Bounded Domains

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega \frac{f(x) - f(y)}{|x-y|^{N+2s}}\, dy\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

- It is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum (λ_j, ϕ_j)
- EIGENFUNCTIONS: $\overline{\phi}_j \in C^{2s-1}(\overline{\Omega}) \cap C^{2s+\alpha}(\Omega)$ (MB, A.Figalli, J. L. Vázquez)

$$\phi_1 \asymp \text{dist}(\cdot, \partial\Omega)^{2s-1} \qquad \text{and} \qquad |\phi_j| \lesssim \text{dist}(\cdot, \partial\Omega)^{2s-1}\,,$$

The Green function $\mathbb{G}(x,y)$ satisfies, letting $\delta^\gamma(\,\cdot\,) := \text{dist}(\,\cdot\,, \partial\Omega)$,

$$\mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right)\,, \text{ with } \boxed{\gamma = 2s-1}$$

Remarks.

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

Basic theory: existence, uniqueness and boundedness (in one page)

$$\text{(CDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m \,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\boxed{\partial_t U = -u^m\,,} \qquad \text{where} \qquad U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{G}(x,y)\,{\rm d}y\,.$$

- This formulation encodes the lateral boundary conditions through $\mathcal{L}^{-1}$.
- Define the *Weak Dual Solutions* (WDS), a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
- Prove *Existence and Uniqueness* of nonnegative WDS with $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Prove a number of new pointwise estimates that provide L^∞ bounds:
  Absolute bounds: ($\overline{\kappa}$ below does NOT depend on u_0)

$$|u(t,x)| \le \|u(t,\cdot)\|_{\mathrm{L}^\infty(\Omega)} \le \overline{\kappa}\, t^{-\frac{1}{m-1}}\,,$$

  Instantaneous Smoothing Effects:

$$|u(t,x)| \le \|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}} \|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}} \|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$$

Basic theory: existence, uniqueness and boundedness (in one page)

$$\text{(CDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\boxed{\partial_t U = -u^m\,,} \qquad \text{where} \qquad U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{G}(x,y)\,\mathrm{d}y\,.$$

- This formulation encodes the lateral boundary conditions through $\mathcal{L}^{-1}$.
- Define the *Weak Dual Solutions* (WDS), a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
- Prove *Existence and Uniqueness* of nonnegative WDS with $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Prove a number of new pointwise estimates that provide L^∞ bounds:
  Absolute bounds: ($\overline{\kappa}$ below does NOT depend on u_0)

$$|u(t,x)| \le \|u(t,\cdot)\|_{\mathrm{L}^\infty(\Omega)} \le \overline{\kappa}\, t^{-\frac{1}{m-1}}\,,$$

  Instantaneous Smoothing Effects:

$$|u(t,x)| \le \|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$$

Basic theory: existence, uniqueness and boundedness (in one page)

$$\text{(CDP)}\qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\boxed{\partial_t U = -u^m\,,}\qquad \text{where}\qquad U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{G}(x,y)\,dy\,.$$

- This formulation encodes the lateral boundary conditions through $\mathcal{L}^{-1}$.
- Define the *Weak Dual Solutions (WDS)*, a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
- Prove *Existence and Uniqueness* of nonnegative WDS with $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Prove a number of new pointwise estimates that provide L^∞ bounds:
  Absolute bounds: ($\overline{\kappa}$ below does NOT depend on u_0)

$$|u(t,x)| \le \|u(t,\cdot)\|_{\mathrm{L}^\infty(\Omega)} \le \overline{\kappa}\, t^{-\frac{1}{m-1}}\,,$$

Instantaneous Smoothing Effects:

$$|u(t,x)| \le \|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$$

Basic theory: existence, uniqueness and boundedness (in one page)

$$(\mathrm{CDP})\qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\boxed{\partial_t U = -u^m\,,}\qquad \text{where}\qquad U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{G}(x,y)\,\mathrm{d}y\,.$$

- This formulation encodes the lateral boundary conditions through $\mathcal{L}^{-1}$.
- Define the *Weak Dual Solutions (WDS)*, a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
- Prove *Existence and Uniqueness of nonnegative WDS* with $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Prove a number of new pointwise estimates that provide L^∞ bounds: *Absolute bounds:* ($\overline{\kappa}$ below does NOT depend on u_0)

$$|u(t,x)| \le \|u(t,\cdot)\|_{\mathrm{L}^\infty(\Omega)} \le \overline{\kappa}\, t^{-\frac{1}{m-1}}\,,$$

Instantaneous Smoothing Effects:

$$|u(t,x)| \le \|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$$

Basic theory: existence, uniqueness and boundedness (in one page)

$$\text{(CDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\boxed{\partial_t U = -u^m\,,} \qquad \text{where} \qquad U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{G}(x,y)\,\mathrm{d}y\,.$$

- This formulation encodes the lateral boundary conditions through $\mathcal{L}^{-1}$.
- Define the *Weak Dual Solutions (WDS)*, a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
- Prove *Existence and Uniqueness* of nonnegative WDS with $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Prove a number of new pointwise estimates that provide L^∞ bounds:
  Absolute bounds: ($\overline{\kappa}$ below does NOT depend on u_0)

$$|u(t,x)| \le \|u(t,\cdot)\|_{\mathrm{L}^\infty(\Omega)} \le \overline{\kappa}\, t^{-\frac{1}{m-1}}\,,$$

Instantaneous Smoothing Effects:

$$|u(t,x)| \le \|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$$

Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$

Let S be the unique solution to the Elliptic Dirichlet Problem for $\mathcal{L}S^m = S$.

Theorem. (Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Let $u \ge 0$ be any nonnegative WDS to the Cauchy-Dirichlet problem. Then, unless $u \equiv 0$,

$$\sup_{x\in\Omega}\left|t^{\frac{1}{m-1}}u(t,x) - S(x)\right| \xrightarrow[t\to\infty]{} 0\,.$$

This result, gives a clear suggestion of what the boundary behaviour of parabolic solutions should be,

$$\boxed{u(t,x) \asymp \mathcal{U}(t,x) = \frac{S(x)}{t^{\frac{1}{m-1}}}}$$

at least for large times, as it happens in the local case $s = 1$. Hence the boundary behaviour shall be dictated by the behaviour of the solution to the elliptic equation.

We shall see that this is not always the case.

Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$

Let S be the unique solution to the Elliptic Dirichlet Problem for $\mathcal{L}S^m = S$.

Theorem. (Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Let $u \ge 0$ be any nonnegative WDS to the Cauchy-Dirichlet problem. Then, unless $u \equiv 0$,

$$\sup_{x\in\Omega}\left|t^{\frac{1}{m-1}}u(t,x) - S(x)\right| \xrightarrow[t\to\infty]{} 0\,.$$

This result, gives a clear suggestion of what the boundary behaviour of parabolic solutions should be,

$$\boxed{u(t,x) \asymp \mathcal{U}(t,x) = \frac{S(x)}{t^{\frac{1}{m-1}}}}$$

at least for large times, as it happens in the local case $s = 1$. Hence the boundary behaviour shall be dictated by the behaviour of the solution to the elliptic equation.

We shall see that this is not always the case.

The Fractional PME II

Sharp Boundary Behaviour

- **Positivity Estimates and Infinite Speed of Propagation**
- **Global Harnack Principles**
- **Asymptotic Behaviour**
- **Anomalous Boundary Behaviour and Counterexamples**
- **Some Numerics**

Positivity Estimates and Infinite Speed of Propagation

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $0 < s < 1$ and $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_0 > 0$, such that

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^\gamma}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

(recall that $\gamma = 1$ for SFL, $\gamma = s$ for the RFL and $\gamma = 2s - 1$ for the CFL)

• Note that, for $t \geq t_*$, the dependence on the initial data disappears

$$u(t,x) \geq \underline{\kappa}_0 \mathrm{dist}(x,\partial\Omega)^\gamma t^{-\frac{1}{m-1}} \qquad \forall\, t \geq t_*\,.$$

(like in the local case $s = 1$)

• But also note that these estimates can not hold for small times when $s = 1$, by the finite speed of propagation that holds in the local case...

Positivity Estimates and Infinite Speed of Propagation

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $0 < s < 1$ and $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_0 > 0$, such that

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^{\gamma}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

(recall that $\gamma = 1$ for SFL, $\gamma = s$ for the RFL and $\gamma = 2s - 1$ for the CFL)

- Note that, for $t \geq t_*$, the dependence on the initial data disappears

$$u(t,x) \geq \underline{\kappa}_0 \mathrm{dist}(x,\partial\Omega)^{\gamma} t^{-\frac{1}{m-1}} \qquad \forall\, t \geq t_*\,.$$

(like in the local case $s = 1$)

- But also note that these estimates can not hold for small times when $s = 1$, by the finite speed of propagation that holds in the local case...

Positivity Estimates and Infinite Speed of Propagation

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $0 < s < 1$ and $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_0 > 0$, such that

$$u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^\gamma}{t^{\frac{1}{m-1}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

(recall that $\gamma = 1$ for SFL, $\gamma = s$ for the RFL and $\gamma = 2s - 1$ for the CFL)

• Note that, for $t \ge t_*$, the dependence on the initial data disappears

$$u(t,x) \ge \underline{\kappa}_0 \mathrm{dist}(x,\partial\Omega)^\gamma t^{-\frac{1}{m-1}} \qquad \forall\, t \ge t_*\,.$$

(like in the local case $s = 1$)

• But also note that these estimates can not hold for small times when $s = 1$, by the finite speed of propagation that holds in the local case...

Universal lower bounds and Infinite speed of propagation.

Recall that $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^\gamma}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s<1$, contrary to the "local" case $s=1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015, ARMA 2019]
- **Question:** Is this estimate sharp?
More precisely, is the power γ of the distance to the boundary the better one?

Universal lower bounds and Infinite speed of propagation.

Recall that $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^\gamma}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s<1$, contrary to the "local" case $s=1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015, ARMA 2019]
- **Question:** Is this estimate sharp?
More precisely, is the power γ of the distance to the boundary the better one?

Universal lower bounds and Infinite speed of propagation.

Recall that $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\operatorname{dist}(x,\partial\Omega)^\gamma}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s<1$, contrary to the "local" case $s=1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015, ARMA 2019]
- **Question:** Is this estimate sharp?
  More precisely, is the power γ of the distance to the boundary the better one?

Universal lower bounds and Infinite speed of propagation.

Recall that $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^{\gamma}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s<1$, contrary to the "local" case $s=1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015, ARMA 2019]
- **Question:** Is this estimate sharp?
  More precisely, is the power γ of the distance to the boundary the better one?

Universal lower bounds and Infinite speed of propagation.

Recall that $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\operatorname{dist}(x,\partial\Omega)^\gamma}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s<1$, contrary to the “local” case $s=1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015, ARMA 2019]
- **Question:** Is this estimate sharp?
  More precisely, is the power γ of the distance to the boundary the better one?

Global Harnack Principle I. The non-spectral case. Matching powers.

Theorem. (GHP I) (M.B., A. Figall, X. Ros Oton & J. L. Vázquez)

Let $\mathcal{L}$ be either the RFL ($\gamma = s$) or the CFL ($\gamma = 2s - 1$). Let $u \geq 0$ be a weak dual solution to the (CDP). Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds for all $t > 0$ and all $x \in \Omega$:

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x, \partial\Omega)^{\frac{\gamma}{m}}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\mathrm{dist}(x, \partial\Omega)^{\frac{\gamma}{m}}}{t^{\frac{1}{m-1}}}\,.$$

Where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$.

- For large times $t \geq t_*$ the estimates are independent on the initial datum.
- *Notice that this result* ***does not apply for*** $s = 1$*, is purely nonlocal.*
- In the local case $s = 1$ the above result holds only for $t \geq t_*$ (finite speed of propagation)

Global Harnack Principle I. The non-spectral case. Matching powers.

Theorem. (GHP I) (M.B., A. Figall, X. Ros Oton & J. L. Vázquez)

Let $\mathcal{L}$ be either the RFL ($\gamma = s$) or the CFL ($\gamma = 2s-1$). Let $u \geq 0$ be a weak dual solution to the (CDP). Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds for all $t > 0$ and all $x \in \Omega$:

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\gamma}{m}}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\,\frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\gamma}{m}}}{t^{\frac{1}{m-1}}}\,.$$

Where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$.

- For large times $t \geq t_*$ the estimates are independent on the initial datum.
- *Notice that this result* ***does not apply for*** *$s = 1$, is purely nonlocal.*
- In the local case $s = 1$ the above result holds only for $t \geq t_*$ (finite speed of propagation)

Global Harnack Principle I. The non-spectral case. Matching powers.

Theorem. (GHP I) (M.B., A. Figall, X. Ros Oton & J. L. Vázquez)

Let $\mathcal{L}$ be either the RFL ($\gamma = s$) or the CFL ($\gamma = 2s - 1$). Let $u \geq 0$ be a weak dual solution to the (CDP). Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds for all $t > 0$ and all $x \in \Omega$:

$$\boxed{\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\gamma}{m}}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\,\frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\gamma}{m}}}{t^{\frac{1}{m-1}}}}\,.$$

Where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$.

- For large times $t \geq t_*$ the estimates are independent on the initial datum.
- *Notice that this result* ***does not apply for*** $s = 1$, *is purely nonlocal.*
- In the local case $s = 1$ the above result holds only for $t \geq t_*$ (finite speed of propagation)

Global Harnack Principle I. The non-spectral case. Matching powers.

Theorem. (GHP I) (M.B., A. Figall, X. Ros Oton & J. L. Vázquez)

Let $\mathcal{L}$ be either the RFL ($\gamma = s$) or the CFL ($\gamma = 2s - 1$). Let $u \geq 0$ be a weak dual solution to the (CDP). Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds for all $t > 0$ and all $x \in \Omega$:

$$\boxed{\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x, \partial\Omega)^{\frac{\gamma}{m}}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\mathrm{dist}(x, \partial\Omega)^{\frac{\gamma}{m}}}{t^{\frac{1}{m-1}}}\,.}$$

Where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$.

- For large times $t \geq t_*$ the estimates are independent on the initial datum.
- *Notice that this result* ***does not apply for*** $s = 1$, *is purely nonlocal.*
- In the local case $s = 1$ the above result holds only for $t \geq t_*$ (finite speed of propagation)

Consequences of GHP with matching powers

As a consequence of GHP with matching powers we get:

Theorem. (Sharp Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Assume that a GHP with matching powers hold. Set $\mathcal{U}(t,x) := t^{-\frac{1}{m-1}} S(x)$. Then there exists $c_0 > 0$ such that, for all $t \geq t_0 := c_0 \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have

$$\sup_{x\in\Omega} \left| \frac{u(t,x)}{\mathcal{U}(t,)} - 1 \right| \leq \frac{2}{m-1} \, \frac{t_0}{t_0+t} \,.$$

This asymptotic result is sharp: check by considering $u(t,x) = \mathcal{U}(t+1,x)$. For the classical case $\mathcal{L} = \Delta$, we recover the results of Aronson-Peletier (1981) and Vázquez (2004) with a different proof.

Consequences of GHP with matching powers

As a consequence of GHP with matching powers we get:

Theorem. (Sharp Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Assume that a GHP with matching powers hold. Set $\mathcal{U}(t,x) := t^{-\frac{1}{m-1}} S(x)$. Then there exists $c_0 > 0$ such that, for all $t \geq t_0 := c_0 \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have

$$\sup_{x\in\Omega} \left| \frac{u(t,x)}{\mathcal{U}(t,)} - 1 \right| \leq \frac{2}{m-1} \, \frac{t_0}{t_0+t} \,.$$

This asymptotic result is sharp: check by considering $u(t,x) = \mathcal{U}(t+1,x)$. For the classical case $\mathcal{L} = \Delta$, we recover the results of Aronson-Peletier (1981) and Vázquez (2004) with a different proof.

Global Harnack Principles II. The Spectral case. Non-Matching powers.

In the case of the SFL, $\gamma = 1$, and a new exponent enters the game:

$$\sigma = \min\left\{1, \frac{2sm}{\gamma(m-1)}\right\}$$

Theorem. (GHP II) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist $\underline{\kappa}, \overline{\kappa} > 0$, such that for all $t > 0$ and $x \in \Omega$

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x, \partial\Omega)^{\gamma}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\mathrm{dist}(x, \partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}\,.$$

- This is a universal bound: it holds for all nonlocal operators that we consider $s < 1$ and shows *infinite speed of propagation* in a quantitative way.
- This is sufficient to ensure interior regularity, under 'minimal' assumptions.
- This bound holds for all times and for a large class of operators.
- This is not sufficient to ensure C^α_x boundary regularity.
- **Question:** Can the estimate be improved to get matching powers also in this case?

Global Harnack Principles II. The Spectral case. Non-Matching powers.

In the case of the SFL, $\gamma = 1$, and a new exponent enters the game:

$$\sigma = \min\left\{1, \frac{2sm}{\gamma(m-1)}\right\}$$

Theorem. (GHP II) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist $\underline{\kappa}, \overline{\kappa} > 0$, such that for all $t > 0$ and $x \in \Omega$

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^{\gamma}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}.$$

• This is a universal bound: it holds for all nonlocal operators that we consider $s < 1$ and shows *infinite speed of propagation* in a quantitative way.

• This is sufficient to ensure interior regularity, under 'minimal' assumptions.

• This bound holds for all times and for a large class of operators.

• This is not sufficient to ensure C^{α}_{x} boundary regularity.

• **Question:** Can the estimate be improved to get matching powers also in this case?

Global Harnack Principles II. The Spectral case. Non-Matching powers.

In the case of the SFL, $\gamma = 1$, and a new exponent enters the game:

$$\sigma = \min\left\{1, \frac{2sm}{\gamma(m-1)}\right\}$$

Theorem. (GHP II) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist $\underline{\kappa}, \overline{\kappa} > 0$, such that for all $t > 0$ and $x \in \Omega$

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x, \partial\Omega)^{\gamma}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\mathrm{dist}(x, \partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}\,.$$

• This is a universal bound: it holds for all nonlocal operators that we consider $s < 1$ and shows *infinite speed of propagation* in a quantitative way.

• This is sufficient to ensure interior regularity, under 'minimal' assumptions.

• This bound holds for all times and for a large class of operators.

• This is not sufficient to ensure C^{α}_x boundary regularity.

• **Question:** Can the estimate be improved to get matching powers also in this case?

Global Harnack Principles II. The Spectral case. Non-Matching powers.

In the case of the SFL, $\gamma = 1$, and a new exponent enters the game:

$$\sigma = \min\left\{1, \frac{2sm}{\gamma(m-1)}\right\}$$

Theorem. (GHP II) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist $\underline{\kappa}, \overline{\kappa} > 0$, such that for all $t > 0$ and $x \in \Omega$

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^{\gamma}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}\,.$$

- This is a universal bound: it holds for all nonlocal operators that we consider $s < 1$ and shows *infinite speed of propagation* in a quantitative way.
- This is sufficient to ensure interior regularity, under 'minimal' assumptions.
- This bound holds for all times and for a large class of operators.
- This is not sufficient to ensure C_x^α boundary regularity.
- **Question:** Can the estimate be improved to get matching powers also in this case?

Global Harnack Principles II. The Spectral case. Non-Matching powers.

In the case of the SFL, $\gamma = 1$, and a new exponent enters the game:

$$\sigma = \min\left\{1, \frac{2sm}{\gamma(m-1)}\right\}$$

Theorem. (GHP II) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist $\underline{\kappa}, \overline{\kappa} > 0$, such that for all $t > 0$ and $x \in \Omega$

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x,\partial\Omega)^{\gamma}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}\,.$$

- This is a universal bound: it holds for all nonlocal operators that we consider $s < 1$ and shows *infinite speed of propagation* in a quantitative way.
- This is sufficient to ensure interior regularity, under 'minimal' assumptions.
- This bound holds for all times and for a large class of operators.
- This is not sufficient to ensure C^{α}_{x} boundary regularity.
- **Question:** Can the estimate be improved to get matching powers also in this case?

Global Harnack Principle II. Non-Matching powers.

Global Harnack Principles II. The Spectral case. Non-Matching powers.

In the case of the SFL, $\gamma = 1$, and a new exponent enters the game:

$$\sigma = \min\left\{1, \frac{2sm}{\gamma(m-1)}\right\}$$

Theorem. (GHP II) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL, and let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then, there exist $\underline{\kappa}, \overline{\kappa} > 0$, such that for all $t > 0$ and $x \in \Omega$

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\mathrm{dist}(x, \partial\Omega)^{\gamma}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\, \frac{\mathrm{dist}(x, \partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}\,.$$

- This is a universal bound: it holds for all nonlocal operators that we consider $s < 1$ and shows *infinite speed of propagation* in a quantitative way.
- This is sufficient to ensure interior regularity, under 'minimal' assumptions.
- This bound holds for all times and for a large class of operators.
- This is not sufficient to ensure C^α_x boundary regularity.
- **Question:** Can the estimate be improved to get matching powers also in this case?

Anomalous boundary behaviour when $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear.

We consider the SFL, hence $\gamma = 1$ from now on. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} = \frac{2sm}{m-1} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{1}{2} - \frac{1}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared 'big' solutions:

$$u_0 \ge \epsilon_0 S \qquad \text{implies} \qquad u(t) \ge \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{L^\infty} = 0 \qquad \text{for all } 0 \le u_0 \in L^1_{\Phi_1}(\Omega)\,.$$

But this is not happening for all solutions...

Anomalous Boundary Behaviour and Counterexamples

Anomalous boundary behaviour when $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear.

We consider the SFL, hence $\gamma = 1$ from now on. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} = \frac{2sm}{m-1} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{1}{2} - \frac{1}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared 'big' solutions:

$$u_0 \ge \epsilon_0 S \qquad \text{implies} \qquad u(t) \ge \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{L^\infty} = 0 \qquad \text{for all } 0 \le u_0 \in L^1_{\Phi_1}(\Omega)\,.$$

But this is not happening for all solutions...

Anomalous Boundary Behaviour and Counterexamples

Anomalous boundary behaviour when $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear.

We consider the SFL, hence $\gamma = 1$ from now on. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} = \frac{2sm}{m-1} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{1}{2} - \frac{1}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared 'big' solutions:

$$u_0 \ge \epsilon_0 S \qquad \text{implies} \qquad u(t) \ge \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{\mathrm{L}^\infty} = 0 \qquad \text{for all } 0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)\,.$$

But this is not happening for all solutions...

Anomalous Boundary Behaviour and Counterexamples

Different boundary behaviour when $\sigma < 1$. We now show that, in general, we cannot hope to prove that $u(t)$ is larger than $\mathrm{dist}^{1/m}$, but always smaller than $\mathrm{dist}^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL ($\gamma = 1$) and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \le u_0 \le c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \le c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

When $\sigma = 1$ and $2sm = \gamma(m-1)$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{1/m} \left(1 + |\log \Phi_1(x)|\right)^{1/(m-1)}} = 0 \qquad \text{for any } t > 0.$$

Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

But again, this is not happening for all solutions...

Anomalous Boundary Behaviour and Counterexamples

Different boundary behaviour when $\sigma < 1$. We now show that, in general, we cannot hope to prove that $u(t)$ is larger than $\mathrm{dist}^{1/m}$, but always smaller than $\mathrm{dist}^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL ($\gamma = 1$) and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \leq u_0 \leq c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \leq c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

When $\sigma = 1$ and $2sm = \gamma(m-1)$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{1/m} \left(1 + |\log \Phi_1(x)|\right)^{1/(m-1)}} = 0 \qquad \text{for any } t > 0.$$

Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

But again, this is not happening for all solutions...

Anomalous Boundary Behaviour and Counterexamples

Different boundary behaviour when $\sigma < 1$. We now show that, in general, we cannot hope to prove that $u(t)$ is larger than $\mathrm{dist}^{1/m}$, but always smaller than $\mathrm{dist}^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ be the SFL ($\gamma = 1$) and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \leq u_0 \leq c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \leq c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

When $\sigma = 1$ and $2sm = \gamma(m-1)$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{1/m} \left(1 + |\log \Phi_1(x)|\right)^{1/(m-1)}} = 0 \qquad \text{for any } t > 0.$$

Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

But again, this is not happening for all solutions...

Anomalous Boundary Behaviour and Counterexamples

Different boundary behaviour when $\sigma < 1$.

We next show that the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa} \Phi_1^{\alpha}(x) \qquad \text{for a.e. } x \in \Omega\,, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}\,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \leq u_0 \leq A\,\Phi_1^{1-\frac{2s}{\gamma}} \qquad \Rightarrow \qquad u(t) \leq \left[A^{1-m} - \tilde{C}t\right]^{-(m-1)} \Phi_1^{1-\frac{2s}{\gamma}}$$

for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C}A^{m-1})$, for some $\tilde{C} > 0$.
Recall that we have a universal lower bound

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Different boundary behaviour when $\sigma < 1$.

We next show that the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa} \Phi_1^{\alpha}(x) \qquad \text{for a.e. } x \in \Omega\,, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}\,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \leq u_0 \leq A\, \Phi_1^{1-\frac{2s}{\gamma}} \qquad \Rightarrow \qquad u(t) \leq [A^{1-m} - \tilde{C} t]^{-(m-1)} \Phi_1^{1-\frac{2s}{\gamma}}$$

for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C} A^{m-1})$, for some $\tilde{C} > 0$.

Recall that we have a universal lower bound

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Different boundary behaviour when $\sigma < 1$.

We next show that the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa} \Phi_1^{\alpha}(x) \qquad \text{for a.e. } x \in \Omega\,, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}\,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \leq u_0 \leq A\, \Phi_1^{1-\frac{2s}{\gamma}} \qquad \Rightarrow \qquad u(t) \leq [A^{1-m} - \tilde{C}t]^{-(m-1)} \Phi_1^{1-\frac{2s}{\gamma}}$$

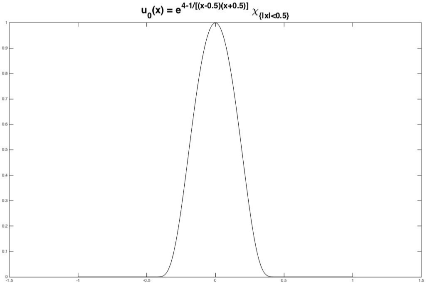
for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C}A^{m-1})$, for some $\tilde{C} > 0$.
Recall that we have a universal lower bound

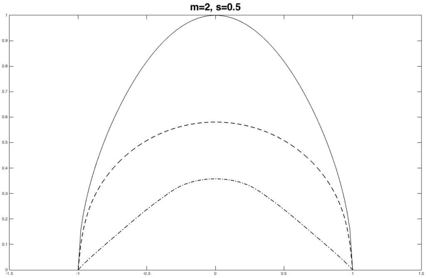
$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Numerical Simulations*

* Graphics obtained by numerical methods contained in: N. Cusimano, F. Del Teso, L. Gerardo-Giorda, G. Pagnini, *Discretizations of the spectral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary conditions*, SIAM Num. Anal. (2018)
Graphics and videos: courtesy of F. Del Teso (BCAM, Bilbao, ES)

Numerical simulation for the SFL with parameters $m = 2$ and $s = 1/2$, hence $\sigma = 1$.





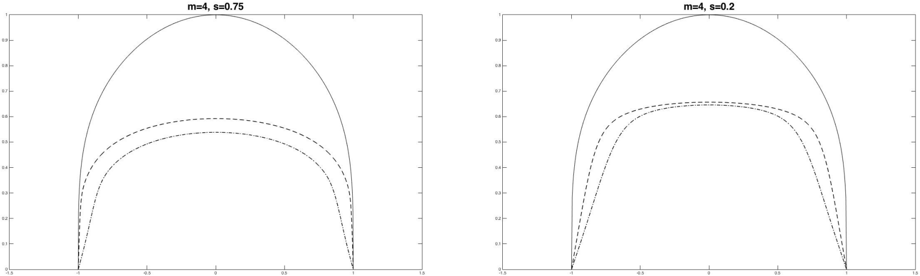
Left: the initial condition $u_0 \le C_0 \Phi_1$

Right: solid line represents $\Phi_1^{1/m}$

the dotted lines represent $\boxed{t^{\frac{1}{m-1}} u(t)}$ at time at $t = 1$ and $t = 5$

While $u(t)$ appears to behave as $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for very short times already at $t = 5$ it exhibits the matching boundary behavior $t^{\frac{1}{m-1}} u(t) \asymp \Phi_1^{1/m}$

Compare $\sigma = 1$ VS $\sigma < 1$: same $u_0 \le C_0 \Phi_1$, solutions with different parameters



Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 30$ and $t = 150$; $m = 4$, $s = 3/4$, $\sigma = 1$.

Matching: $u(t)$ behaves like $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for quite some time,
and only around $t = 150$ it exhibits the matching boundary behavior $u(t) \asymp \Phi_1^{1/m}$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 150$ and $t = 600$; $m = 4$, $s = 1/5$, $\sigma = 8/15 < 1$.

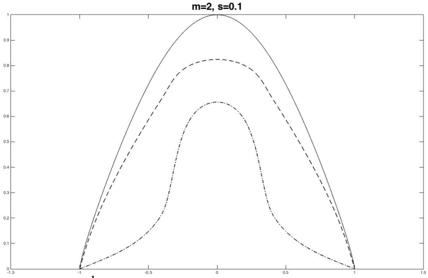
Non-matching: $u(t) \asymp \Phi_1$ even after long time.

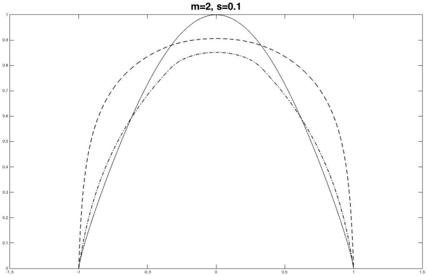
Idea: maybe when $\sigma < 1$ and $u_0 \lesssim \Phi_1$, we have $u(t) \asymp \Phi_1$ for all times...

Not True: there are cases when $u(t) \gg \Phi_1^{1-2s}$ for large times...

Non-matching when $\sigma < 1$: same data u_0, with $m = 2$ and $s = 1/10$, $\sigma = 2/5 < 1$

In both pictures, the solid line represents Φ_1^{1-2s} (anomalous behaviour)





Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 4$ and $t = 25$.

$u(t) \asymp \Phi_1$ for short times $t = 4$, then $u(t) \sim \Phi_1^{1-2s}$ for intermediate times $t = 25$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 40$ and $t = 150$. $u(t) \gg \Phi_1^{1-2s}$ for large times.

Both non-matching always different behaviour from the asymptotic profile $\Phi_1^{\sigma/m}$.

In this case we show that if $u_0(x) \le C_0 \Phi_1(x)$ then for all $t > 0$

$$u(t,x) \le C_1 \left[\frac{\Phi_1(x)}{t}\right]^{\frac{1}{m}} \qquad \text{and} \qquad \lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\frac{\sigma}{m}}} = 0 \qquad \text{for any } t > 0.$$

The End

Muchas Gracias!!!

Moltes Grácies!!!

Thank You!!!

Congratulazioni Alessio!!!!!

The End

Muchas Gracias!!!

Moltes Grácies!!!

Thank You!!!

Congratulazioni Alessio!!!!!

References:

[BV1] M. B., J. L. VÁZQUEZ, A Priori Estimates for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains. *Arch. Rat. Mech. Anal. (2015).*

[BV2] M. B., J. L. VÁZQUEZ, Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part I. Existence, Uniqueness and Upper Bounds *Nonlin. Anal. TMA (2016).*

[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains. *Discr. Cont. Dyn. Sys. (2015).*

[BFR] M. B., A. FIGALLI, X. ROS-OTON, Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains. *Comm. Pure Appl. Math (2017).*

[BFV1] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary estimates and higher regularity for nonlocal porous medium-type equations in bounded domains. *Analysis & PDE (2018)*

[BFV2] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary behaviour of solutions to semilinear nonlocal elliptic equations. *Calc. Var. PDE (2018).*