Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

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Analysis and Computation
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References:

[BV1] M. B., J. L. VÁZQUEZ, A Priori Estimates for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains. *Arch. Rat. Mech. Anal. (2015).*

[BV2] M. B., J. L. VÁZQUEZ, Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part I. Existence, Uniqueness and Upper Bounds *Nonlin. Anal. TMA (2016).*

[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains. *Discr. Cont. Dyn. Sys. (2015).*

[BFR] M. B., A. FIGALLI, X. ROS-OTON, Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains. *Comm. Pure Appl. Math (2017).*

[BFV1] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary estimates and higher regularity for nonlocal porous medium-type equations in bounded domains. *Analysis & PDE (2018)*

[BFV2] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary behaviour of solutions to semilinear nonlocal elliptic equations. *Calc. Var. PDE (2018).*

- *A talk more focussed on the first three papers is available online:*

  `http://www.fields.utoronto.ca/video-archive//event/2021/2016`

Outline of the talk

- **Introduction**
  - The Parabolic problem
  - Assumptions on the (inverse) operator
  - Boundary behaviour Linear Elliptic problem
  - Some important examples
- **Semilinear Elliptic Equations**
  - Sharp boundary behaviour for Semilinear Elliptic equations
  - Parabolic solutions by separation of variables
- **Back to the Parabolic problem**
  - (More) Assumptions on the operator
  - Basic theory: existence, uniqueness and boundedness
  - Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$
- **Sharp Boundary Behaviour**
  - Harnack-type Inequalities
  - Infinite Speed of Propagation
  - Asymptotic Behaviour
  - Anomalous Boundary Behaviour and Counterexamples
  - Some Numerics

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\,F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

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About the operator $\mathcal{L}$ and $\mathcal{L}^{-1}$

Assumptions on the inverse of $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$.

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We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1} : \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ with a kernel $\mathbb{G}$ - the Green function - such that

$$\mathcal{L}^{-1} f(x) = \int_\Omega \mathbb{G}(x,y) f(y)\,\mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$

$$\text{(K1)} \qquad 0 \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

Assumption (K1) implies that $\mathcal{L}^{-1}$ is compact on $\mathrm{L}^2(\Omega)$ and has discrete spectrum.

$$\text{(K2)} \quad c_{0,\Omega}\delta^\gamma(x)\,\delta^\gamma(y) \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right)$$

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$$\delta^\gamma(x) := \mathrm{dist}(x, \partial\Omega)^\gamma\,.$$

(K2) is needed in the study of the sharp boundary behaviour.

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Examples of operators $\mathcal{L}$

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{}(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,dz,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right) \frac{dt}{t^{1+s}}.$$

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The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
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$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega) \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega)\,,$$

and ϕ_j are as smooth as $\partial\Omega$ allows: $\partial\Omega \in C^k \;\Rightarrow\; \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \quad \text{with } \boxed{\gamma = 1}$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

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Definition via the hypersingular kernel in $\mathbb{R}^N$, “restricted” to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\,\text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,dz\,, \qquad \text{with } \operatorname{supp}(g)\subseteq\overline{\Omega}\,.$$

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  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j\le\lambda_j^s$ for all $j\in\mathbb{N}$.
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Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega \frac{f(x)-f(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

- It is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum (λ_j, ϕ_j)
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Examples of operators $\mathcal{L}$

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega \frac{f(x) - f(y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with} \qquad \frac{1}{2} < s < 1\,,$$

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Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form, with uniformly elliptic bounded measurable coefficients:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j) \,, \qquad \text{s-power of A is:} \qquad \mathcal{L} f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x)$$

$\mathcal{L} = A^s$ satisfies (K3) estimates with $\gamma = 1$

$$\text{(K3)} \qquad c_{0,\Omega} \phi_1(x)\, \phi_1(y) \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\phi_1(x)}{|x-y|} \wedge 1 \right) \left(\frac{\phi_1(y)}{|x-y|} \wedge 1 \right)$$

[General class of intrinsically ultra-contractive operators, Davies and Simon JFA 1984].

Fractional operators with "rough" kernels. Integral operators of Levy-type

$$\mathcal{L} f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y) \right) \frac{a(x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,.$$

where K is measurable, symmetric, bounded between two positive constants, and

$$\left| a(x,y) - a(x,x) \right| \chi_{|x-y|<1} \le c|x-y|^{\sigma} \,, \qquad \text{with } 0 < s < \sigma \le 1 \,,$$

for some positive $c > 0$. We can allow even more general kernels.
The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^{\gamma}(x)}{|x-y|^{\gamma}} \wedge 1 \right) \left(\frac{\delta^{\gamma}(y)}{|x-y|^{\gamma}} \wedge 1 \right) \,, \qquad \text{with } \gamma = s$$

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Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma \,, \qquad \text{with } 0<\sigma<s\le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0<s<1 \quad \text{and} \quad a\ge 0\,,$$

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the measure ν on $\mathbb{R}^N\setminus\{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^N} 1\vee|x|^2\,\mathrm{d}\nu(y)<\infty$, together with other assumptions.

Relativistic stable processes. In the case

$$\mathcal{L} = c-\left(c^{1/s}-\Delta\right)^s\,, \qquad \text{with } c>0\,, \text{ and } 0<s\le 1\,.$$

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Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

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Semilinear Elliptic Equations

- **Sharp boundary behaviour for Semilinear Elliptic equations**
- **Parabolic solutions by separation of variables**

Sharp boundary behaviour for Elliptic Equations

We always assume that $\mathcal{L}$ satisfies (A1), (A2) and zero Dirichlet boundary conditions.

The Semilinear Dirichlet Problem $\mathcal{L}v = f(v) \sim v^p$ with $0 < p < 1$

Assume moreover that $\mathcal{L}^{-1}$ satisfies (K2). Let $u \geq 0$ be a (weak dual) solution to the Dirichlet Problem, where f is a nonnegative increasing function with $f(0) = 0$ such that $F = f^{-1}$ is convex and $F(a) \asymp a^{1/p}$ when $0 \leq a \leq 1$, for some $0 < p < 1$. Then, the following sharp absolute bounds hold true for all $x \in \Omega$

$$v(x) \asymp \begin{cases} \Phi_1^{\sigma}(x) & \text{when } 2s \neq \gamma(1-p) \\ \Phi_1(x)\left(1 + |\log \Phi_1(x)|\right)^{\frac{1}{1-p}} & \text{when } 2s = \gamma(1-p), \text{ assuming (K4)} \end{cases}$$

where

$$\boxed{\sigma := 1 \wedge \frac{2s}{\gamma(1-p)}} \qquad \text{and} \qquad \Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^{\gamma} = \delta^{\gamma}$$

When $2s = \gamma(1-p)$, if (K4) does not hold, then the upper bound still holds, but the lower bound holds in a non-sharp form without the extra logarithmic term.

Remarks.

- When $2s < \gamma(1-p)$, the new power σ becomes less than 1.
- Somehow σ interpolates between the two extremal cases: $p = 0$ i.e. $\mathcal{L}v = 1$ and $p = 1$, i.e. $\mathcal{L}v = \lambda v$.

Examples.

- For the RFL ($\gamma = s$) and CFL ($\gamma = s - 1/2$) we always have $\sigma = 1$ and $2s \neq \gamma(1-p)$, hence

$$v(x) \asymp \Phi_1(x) \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$$

- For the SFL we have $\gamma = 1$ hence we have three possibilities:

$$v(x) \asymp \begin{cases} \mathrm{dist}(x,\partial\Omega) & \text{when } s > \frac{1-p}{2} \\ \mathrm{dist}(x,\partial\Omega)\left(1 + |\log \mathrm{dist}(x,\partial\Omega)|\right)^{\frac{1}{1-p}} & \text{when } s = \frac{1-p}{2} \\ \mathrm{dist}(x,\partial\Omega)^{\frac{2s}{1-p}} & \text{when } s < \frac{1-p}{2} \end{cases}$$

Regularity. Under some mild assumptions on $\mathcal{L}$ and $f \in C^\beta(\mathbb{R})$ for some $\beta > 0$, with $0 \le f(a) \le c_p a^p$ when $0 \le a \le 1$ for some $0 < p \le 1$.

- Solutions are *Hölder continuous in the interior*, and (when the operator allows it) are *classical in the interior*, namely $C^{2s+\beta}(\Omega)$.
- Assuming moreover that $\mathcal{L}^{-1}$ satisfies (K2), solutions are *Hölder continuous up to the boundary*:

$$\|u\|_{C^\eta(\overline{\Omega})} \le C \qquad \forall \eta \in (0,\gamma] \cap (0,2s).$$

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Change of notations from Elliptic to Parabolic In order to make the elliptic results "compatible" with the parabolic, we will perform the change of notations

$$\boxed{m = \frac{1}{p} > 1} \qquad \text{and} \qquad \boxed{v = S^m} \qquad \text{or} \qquad v^p = S\,.$$

The elliptic equation transforms: (we deal only with pure powers for simplicity)

$$\mathcal{L}v = f(v) = v^p \qquad \text{becomes} \qquad \boxed{\mathcal{L}S^m = \mathcal{L}F(S) = S}$$

Parabolic solutions by separation of variables. We have the following solution for the Dirichlet problem for the equation $u_t + \mathcal{L}\,u^m = 0$

$$\mathcal{U}_T(t,x) = \frac{S(x)}{(T+t)^{\frac{1}{m-1}}}$$

where $\mathcal{L}S^m = S$, and the initial datum is $U_T(0,x) = T^{-1/(m-1)}S(x)$.
When $T = 0$ we have the so-called *Friendly Giant*, corresponding to the biggest possible initial datum (useful in the asymptotic study as $t \to \infty$.)

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$$\mathcal{U}(t,x) = \frac{S(x)}{t^{\frac{1}{m-1}}} \qquad \text{with} \qquad \mathcal{U}(0,x) = +\infty\,.$$

Back to the Parabolic problem

- **(More) Assumptions on the operator**
- **Basic theory: existence, uniqueness and boundedness**
- **Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$**

For the rest of the talk we deal with the special case:

$$\boxed{F(u) = u^m := |u|^{m-1}u, \quad m > 1}$$

Recall that the linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, and we have already explained the assumptions (K1) and (K2) on the inverse.

Assumptions on the kernel.

- Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x,y)$ via the formula

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\,\mathrm{d}y\,,$$

assumption (L1) states that there exists $\underline{\kappa}_\Omega > 0$ such that

(L1) $$\inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,.$$

- Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x,y)$ and a zero order term:

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\,\mathrm{d}y + B(x)f(x),$$

assumptions (L2) states that there exists $\underline{\kappa}_\Omega > 0$ and $\gamma \in (0,1]$

(L2) $$K(x,y) \ge \underline{\kappa}_\Omega \,\mathrm{dist}(x,\partial\Omega)^\gamma\,\mathrm{dist}(y,\partial\Omega)^\gamma, \qquad \text{and} \qquad B(x) \ge 0,$$

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About the kernels of spectral nonlocal operators. Most of the examples of nonlocal operators, but the SFL, admit a representation with a kernel
A natural question is: does the SFL admit such a representation?
Let A be a uniformly elliptic linear operator. Define the s^{th} power of A:

$$\mathcal{L}g(x) = A^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{tA} g(x) - g(x)\right) \frac{dt}{t^{1+s}}$$

Then it admits a representation with a Kernel plus zero order term:

$$A^s g(x) = P.V. \int_{\mathbb{R}^N} \big(g(x) - g(y)\big) K(x,y)\, dy + \kappa(x) g(x)\,.$$

where $K \geq 0$ is compactly supported in $\overline{\Omega} \times \overline{\Omega}$ with

$$K(x,y) \asymp \frac{1}{|x-y|^{N+2s}} \left(\frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1\right) \left(\frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1\right) \quad \text{and} \quad \kappa(x) \asymp \frac{1}{\mathrm{dist}(x, \partial\Omega)^{2s}}\,.$$

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Basic theory: existence, uniqueness and boundedness (in one page)

$$\text{(CDP)}\qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\boxed{\partial_t U = -u^m\,,} \qquad \text{where} \qquad U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{G}(x,y)\,\mathrm{d}y\,.$$

- This formulation encodes the lateral boundary conditions through $\mathcal{L}^{-1}$.
- Define the *Weak Dual Solutions* (WDS), a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
- Prove existence and uniqueness of nonnegative WDS with $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Prove a number of new pointwise estimates that provide L^∞ bounds: *Absolute bounds:* ($\overline{k}$ below does NOT depend on u_0)

$$\|u(t,\cdot)\|_{\mathrm{L}^\infty(\Omega)} \le \overline{\kappa}\, t^{-\frac{1}{m-1}}\,,$$

Instantaneous Smoothing Effects:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$$

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Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$

Theorem. (Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Assume that $\mathcal{L}$ satisfies (A1), (A2), and (K2), and let S be the solution to $\mathcal{L}S^m = S$. Let u be any weak dual solution to the Cauchy-Dirichlet problem. Then, unless $u \equiv 0$,

$$\left\| t^{\frac{1}{m-1}} u(t,\cdot) - S \right\|_{\mathrm{L}^\infty(\Omega)} \xrightarrow{t\to\infty} 0\,.$$

This result, gives a clear suggestion of what the boundary behaviour of parabolic solutions should be,

$$u(t,x) \asymp \mathcal{U}(t,x) = \frac{S(x)}{t^{\frac{1}{m-1}}}$$

at least for large times, as it happens in the local case $s = 1$. Hence the boundary behaviour shall be dictated by the behaviour of the solution to the elliptic equation. We shall see that this is not always the case.

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Sharp Boundary Behaviour

- **Harnack-type Inequalities**
- **Infinite Speed of Propagation**
- **Asymptotic Behaviour**
- **Anomalous Boundary Behaviour and Counterexamples**
- **Some Numerics**

Global Harnack Principle I. The non-spectral case. Matching powers.

Recall: $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma$, $\sigma = 1 \wedge \dfrac{2sm}{\gamma(m-1)}$, $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$.

Theorem. (Global Harnack Principle I. The non-spectral case.) (MB & AF & JLV)

Let (A1), (A2), (L1) and (K2). Let $u \ge 0$ be a weak dual solution to the (CDP). Also, when $\sigma < 1$, assume that $K(x,y) \le c_1 |x-y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$. Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds for all $t > 0$ and all $x \in \Omega$: (when $2sm \ne \gamma(m-1)$)

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- *Notice that this result does not apply for $s=1$, is purely nonlocal.*

Consequences of GHP with matching powers

Corollary. (Local Harnack Inequalities of Elliptic/Backward Type)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, s. t. for all $t > 0$ and $h \geq 0$

$$\sup_{x\in B_R(x_0)} u(t,x) \leq \hat{H}\left[\left(1+\frac{h}{t}\right)\left(1\wedge\frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x\in B_R(x_0)} u(t+h,x)\,.$$

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Infinite speed of propagation.

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
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Anomalous boundary behaviour when $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{\gamma}{2} - \frac{\gamma}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared 'big' solutions:

$$u_0 \ge \epsilon_0 S \qquad \text{implies} \qquad u(t) \ge \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{\mathrm{L}^\infty} = 0 \qquad \text{for all } 0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)\,.$$

But this is not happening for all solutions...

Anomalous boundary behaviour when $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{\gamma}{2} - \frac{\gamma}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

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Different boundary behaviour when $\sigma < 1$. The next result shows that, in general, we cannot hope to prove that $u(t)$ is larger than $\Phi_1^{1/m}$, but always smaller than $\Phi_1^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \leq u_0 \leq c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \leq c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

When $\sigma = 1$ and $2sm = \gamma(m-1)$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{1/m}\,\big(1 + |\log \Phi_1(x)|\big)^{1/(m-1)}} = 0 \qquad \text{for any } t > 0.$$

Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

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We next show that assuming (K4), the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa}\Phi_1^\alpha(x) \qquad \text{for a.e. } x \in \Omega\,, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}\,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \leq u_0 \leq A\,\Phi_1^{1-\frac{2s}{\gamma}} \qquad \Rightarrow \qquad u(t) \leq [A^{1-m} - \tilde{C}t]^{-(m-1)}\,\Phi_1^{1-\frac{2s}{\gamma}}$$

for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C}A^{m-1})$, for some $\tilde{C} > 0$.
Recall that we have a universal lower bound (under minimal assumptions on K)

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

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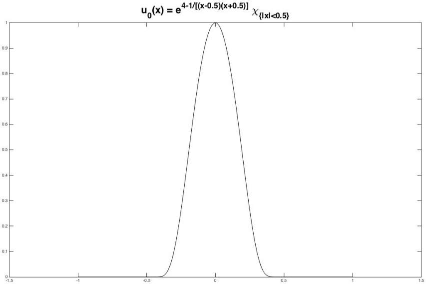
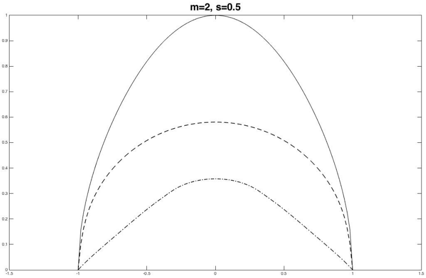
Numerical Simulations*

* Graphics obtained by numerical methods contained in: N. Cusimano, F. Del Teso, L. Gerardo-Giorda, G. Pagnini, *Discretizations of the spectral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary conditions*, SIAM Num. Anal. (2018)
Graphics and videos: courtesy of F. Del Teso (NTNU, Trondheim, Norway)

Numerical simulation for the SFL with parameters $m = 2$ and $s = 1/2$, hence $\sigma = 1$.





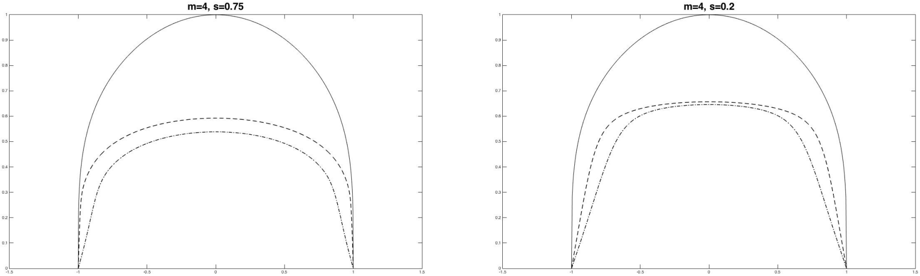
Left: the initial condition $u_0 \le C_0 \Phi_1$

Right: solid line represents $\Phi_1^{1/m}$

the dotted lines represent $\boxed{t^{\frac{1}{m-1}} u(t)}$ at time at $t = 1$ and $t = 5$

While $u(t)$ appears to behave as $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for very short times already at $t = 5$ it exhibits the matching boundary behavior $t^{\frac{1}{m-1}} u(t) \asymp \Phi_1^{1/m}$

Compare $\sigma = 1$ VS $\sigma < 1$: same $u_0 \le C_0\Phi_1$, solutions with different parameters



Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 30$ and $t = 150$; $m = 4$, $s = 3/4$, $\sigma = 1$.

Matching: $u(t)$ behaves like $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for quite some time, and only around $t = 150$ it exhibits the matching boundary behavior $u(t) \asymp \Phi_1^{1/m}$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 150$ and $t = 600$; $m = 4$, $s = 1/5$, $\sigma = 8/15 < 1$.

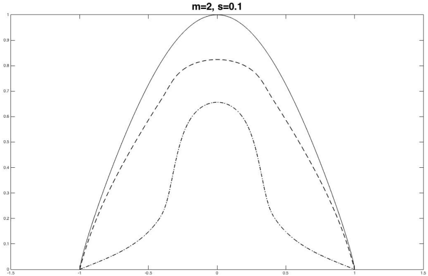
Non-matching: $u(t) \asymp \Phi_1$ even after long time.

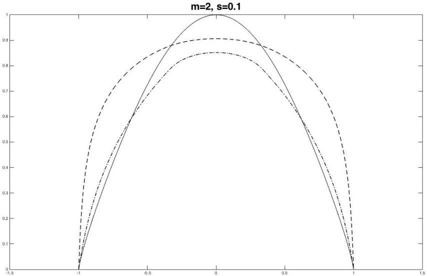
Idea: maybe when $\sigma < 1$ and $u_0 \lesssim \Phi_1$, we have $u(t) \asymp \Phi_1$ for all times...

Not True: there are cases when $u(t) \gg \Phi_1^{1-2s}$ for large times...

Non-matching when $\sigma < 1$: same data u_0, with $m = 2$ and $s = 1/10$, $\sigma = 2/5 < 1$

In both pictures, the solid line represents Φ_1^{1-2s} (anomalous behaviour)





Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 4$ and $t = 25$.

$u(t) \asymp \Phi_1$ for short times $t = 4$, then $u(t) \sim \Phi_1^{1-2s}$ for intermediate times $t = 25$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 40$ and $t = 150$. $u(t) \gg \Phi_1^{1-2s}$ for large times.

Both non-matching always different behaviour from the asymptotic profile $\Phi_1^{\sigma/m}$.

In this case we show that if $u_0(x) \le C_0 \Phi_1(x)$ then for all $t > 0$

$$u(t,x) \le C_1 \left[\frac{\Phi_1(x)}{t}\right]^{\frac{1}{m}} \qquad \text{and} \qquad \lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\frac{\sigma}{m}}} = 0 \qquad \text{for any } t > 0.$$

The End

Thank You!!!

Grazie Mille!!!

Muchas Gracias!!!

Regularity Estimates

- Interior Regularity
- Hölder continuity up to the boundary
- Higher interior regularity for RFL

Interior Regularity

The regularity results, require the validity of a Global Harnack Principle.

(R) The operator $\mathcal{L}$ satisfies (A1) and (A2), and $\mathcal{L}^{-1}$ satisfies (K2). Moreover, we consider

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big)K(x,y)\,\mathrm{d}y + B(x)f(x)\,, \qquad \text{with}$$

$$K(x,y) \asymp |x-y|^{-(N+2s)} \quad \text{in } B_{2r}(x_0) \subset \Omega, \qquad K(x,y) \lesssim |x-y|^{-(N+2s)} \quad \text{in } \mathbb{R}^N \setminus B_{2r}(x_0).$$

As a consequence, for any ball $B_{2r}(x_0) \subset\subset \Omega$ and $0 < t_0 < T_1$, there exist $\delta, M > 0$ such that

$$\begin{aligned} &0 < \delta \le u(t,x) && \text{for a.e. } (t,x) \in (T_0,T_1) \times B_{2r}(x_0),\\ &0 \le u(t,x) \le M && \text{for a.e. } (t,x) \in (T_0,T_1) \times \Omega. \end{aligned}$$

The constants in the regularity estimates will depend on the solution only through δ, M.

Theorem. (Interior Regularity) (M.B., A. Figalli and J. L. Vázquez)

Assume (R) and let u be a nonnegative bounded weak dual solution to problem (CDP).

1. Then u is **Hölder continuous in the interior**. More precisely, there exists $\alpha > 0$ such that, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{\alpha/2s,\alpha}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

2. Assume in addition $|K(x,y) - K(x',y)| \le c|x-x'|^\beta\,|y|^{-(N+2s)}$ for some $\beta \in (0, 1\wedge 2s)$ such that $\beta + 2s \notin \mathbb{N}$. Then u **is a classical solution in the interior**. More precisely, for all $0 < T_0 < T_2 < T_1$,

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Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
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- Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spacial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)
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• Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spacial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)

• Previous regularity results: (I apologize if I forgot someone)

- *C^α regularity:*
  Athanasopoulos and Caffarelli [Adv. Math, 2010], (RFL domains)
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Hölder continuity up to the boundary

Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
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Higher Interior Regularity for RFL.

Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)**, then $u \in C_x^\infty((0,\infty)\times\Omega)$.
More precisely, let $k \geq 1$ be any positive integer, and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then, for any $t \geq t_0 > 0$ we have

$$\left|D_x^k u(t,x)\right| \leq C\,[d(x)]^{\frac{s}{m}-k},$$

where C depends only on N, s, m, k, Ω, t_0, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in t. To our knowledge also open for the local case $s = 1$.
- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C_x^\infty$

  $$\|u\|_{C_x^{k,\alpha}((\frac{1}{2},1)\times B_{1/2})} \leq C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \geq 0.$$

  Analogous estimates in time do not hold for $k \geq 1$ and $\alpha \in (0,1)$.
  Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not C^1 in t in $(\frac{1}{2},1)\times B_{1/2}$. [Chang-Lara, Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains , and also for operator with more general kernels.
- Also the "classical/local" case $s = 1$ works after the waiting time t_*:
  $u \in C_{x,t}^{\frac{1}{m},\frac{1}{2m}}\left(\overline{\Omega}\times[t_*,T]\right)$, $C_x^\infty((0,\infty)\times\Omega)$ and $C_t^{1,\alpha}([t_0,T]\times K)$.

Higher Interior Regularity for RFL.

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- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C_x^\infty$
  $$\|u\|_{C_x^{k,\alpha}((\frac12,1)\times B_{1/2})} \leq C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \geq 0.$$
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Under the running assumptions **(R)**, then $u \in C_x^\infty((0,\infty)\times\Omega)$.
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