Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

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MIT PDE/Analysis Seminar

Department of Mathematics, Massachusetts Institute of Technology
Cambridge, MA, USA, December 5, 2017

References:

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Nonlin. Anal. TMA (2016).

[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains.
Discr. Cont. Dyn. Sys. (2015).

[BFR] M. B., A. FIGALLI, X. ROS-OTON, Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains.
Comm. Pure Appl. Math (2017).

[BFV1] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary estimates and higher regularity for nonlocal porous medium-type equations in bounded domains.
To Appear in Analysis & PDE. https://arxiv.org/abs/1610.09881

[BFV2] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary behaviour of solutions to semilinear nonlocal elliptic equations.
Preprint (2017). https://arxiv.org/abs/1710.02731

- *A talk more focussed on the first three papers is available online:*
  http://www.fields.utoronto.ca/video-archive//event/2021/2016

Outline of the talk

- **Introduction**
  - The Parabolic problem
  - Assumptions on the (inverse) operator
  - Boundary behaviour Linear Elliptic problem
  - Some important examples
- **Semilinear Elliptic Equations**
  - Sharp boundary behaviour for Semilinear Elliptic equations
  - Parabolic solutions by separation of variables
- **Back to the Parabolic problem**
  - (More) Assumptions on the operator
  - Basic theory: existence, uniqueness and boundedness
  - Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$
- **Sharp Boundary Behaviour**
  - Upper Boundary Estimates
  - Infinite Speed of Propagation
  - Lower Boundary Estimates
  - Harnack-type Inequalities
  - Numerics
- **Regularity Estimates**

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

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About the operator $\mathcal{L}$ and $\mathcal{L}^{-1}$

Assumptions on the inverse of $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$.

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We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1} : \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ with a kernel $\mathbb{G}$ - the Green function - such that

$$\mathcal{L}^{-1} f(x) = \int_\Omega \mathbb{G}(x,y) f(y)\, \mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$

(K1) $$0 \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

Assumption (K1) implies that $\mathcal{L}^{-1}$ is compact on $\mathrm{L}^2(\Omega)$ and has discrete spectrum.

(K2) $$c_{0,\Omega}\delta^\gamma(x)\,\delta^\gamma(y) \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right)$$

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(K2) is needed in the study of the sharp boundary behaviour.

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Boundary behaviour for Elliptic equations

We always assume that $\mathcal{L}$ satisfies (A1), (A2) and zero Dirichlet boundary conditions.

The Linear Problem $\mathcal{L}v = f$ with $f \in \mathrm{L}^{q'}(\Omega)$

Let $\mathbb{G}$ be the kernel of $\mathcal{L}^{-1}$, and assume $(K2)$ and that $0 \le f \in \mathrm{L}^{q'}$ with $q' > N/2s$.
Then $q = \frac{q'}{q'-1} \in \left(0, \frac{N}{N-2s}\right)$ and the (weak dual) solution $v \ge 0$ satisfies $\forall x \in \Omega$

$$\|f\|_{\mathrm{L}^1_{\delta^\gamma}} \delta(x)^\gamma \lesssim v(x) \lesssim \|f\|_{\mathrm{L}^{q'}} \begin{cases} \delta(x)^\gamma\,, & 0 < q \in \left(0, \frac{N}{N-2s+\gamma}\right), \\ \delta(x)^\gamma \left(1 + \left|\log \delta(x)\right|\right)^{\frac{1}{q}}, & q = \frac{N}{N-2s+\gamma}, \\ \delta(x)^{\frac{N-q(N-2s)}{q}}\,, & q \in \left(\frac{N}{N-2s+\gamma}, \frac{N}{N-2s}\right). \end{cases}$$

The Eigenvalue Problem $\mathcal{L}\Phi_k = \lambda_k \Phi_k$

Assumption (K1) implies that $\mathcal{L}^{-1}$ is compact in $\mathrm{L}^2(\Omega)$.
Hence the operator $\mathcal{L}$ has a discrete spectrum (λ_k, Φ_k) and $\Phi_k \in \mathrm{L}^\infty(\Omega)$.
If we assume moreover that $\mathcal{L}^{-1}$ satisfies (K2) we have that

$$\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma \qquad \text{and} \qquad |\Phi_k| \lesssim \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$$

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The boundary behaviour may change depending on the relation between $2s$ and γ.

On the other hand, for eigenfunctions **we always have**

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This reveals a deep and strong difference in the boundary behaviour, typical of the different definitions of Fractional Laplacians on domains.

Many "nonlocal" results by Cabré, Caffarelli, Capella, Davila, Dupaigne, Grubb, Kassmann, Ros-Oton, Serra, Silvestre, Sire, Stinga, Torrea [...]

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Examples of operators $\mathcal{L}$

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$\widehat{((-\Delta_{\mathbb{R}^N})^s f)}(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right)\frac{dt}{t^{1+s}}.$$

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The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
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$$\phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega) \qquad \text{and} \qquad |\phi_j| \lesssim \mathrm{dist}(\cdot, \partial\Omega) \,,$$

and ϕ_j are as smooth as $\partial\Omega$ allows: $\partial\Omega \in C^k \;\Rightarrow\; \phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$

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The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \quad \mathbb{G}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) , \quad \text{with } \boxed{\gamma = 1}$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0 \,, \qquad \text{in } (0,\infty) \times \partial\Omega \,.$$

Examples of operators $\mathcal{L}$

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

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Examples of operators $\mathcal{L}$

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
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  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
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Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \left(\mathbb{R}^N \setminus \Omega\right).$$

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Remarks.

- This is a third model of Dirichlet fractional Laplacian **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.
- We can allow "coefficients", i.e. replace $K(x,y) \asymp a(x,y)|x-y|^{N-2s}$ where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, and $\big|a(x,y) - a(x,x)\big|\, \chi_{|x-y|<1} \lesssim |x-y|^\sigma$, with $0 < s < \sigma \leq 1$.

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Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form, with uniformly elliptic bounded measurable coefficients:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j) \,, \qquad \textit{s-power of A is:} \qquad \mathcal{L} f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x)$$

$\mathcal{L} = A^s$ satisfies (K3) estimates with $\gamma = 1$

$$\text{(K3)} \qquad c_{0,\Omega} \phi_1(x)\, \phi_1(y) \le \mathbb{G}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\phi_1(x)}{|x-y|} \wedge 1 \right) \left(\frac{\phi_1(y)}{|x-y|} \wedge 1 \right)$$

[General class of intrinsically ultra-contractive operators, Davies and Simon JFA 1984].

Fractional operators with "rough" kernels. Integral operators of Levy-type

$$\mathcal{L} f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x+y) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,.$$

where K is measurable, symmetric, bounded between two positive constants, and

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Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma \,, \qquad \text{with } 0 < \sigma < s \le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0\,,$$

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the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^N} 1 \vee |x|^2 \,\mathrm{d}\nu(y) < \infty$, together with other assumptions.

Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta \right)^s \,, \qquad \text{with } c > 0\,, \text{ and } 0 < s \le 1\,.$$

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Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

References. The above mentioned bounds for the Green functions have been proven by Chen, Kim, Song and Vondracek (2007, 2010, 2012, 2013).

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$$\mathcal{L} = c - \left(c^{1/s} - \Delta \right)^s , \qquad \text{with } c > 0 , \text{ and } 0 < s \leq 1 .$$

The Green function $\mathbb{G}(x,y)$ of $\mathcal{L}$ satisfies assumption $(K4)$ with $\gamma = s$.

Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

References. The above mentioned bounds for the Green functions have been proven by Chen, Kim, Song and Vondracek (2007, 2010, 2012, 2013).

Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma \,, \qquad \text{with } 0 < \sigma < s \le 1 \,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.
Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s \,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0 \,,$$

where

$$A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y| \le 1}\right) \chi_{|y|\le 1} \mathrm{d}\nu(y) \,,$$

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Semilinear Elliptic Equations

- Sharp boundary behaviour for Semilinear Elliptic equations
- Parabolic solutions by separation of variables

Sharp boundary behaviour for Elliptic Equations

We always assume that $\mathcal{L}$ satisfies (A1), (A2) and zero Dirichlet boundary conditions.

The Semilinear Dirichlet Problem $\mathcal{L}v = f(v) \sim v^p$ with $0 < p < 1$

Assume moreover that $\mathcal{L}^{-1}$ satisfies (K2). Let $u \geq 0$ be a (weak dual) solution to the Dirichlet Problem, where f is a nonnegative increasing function with $f(0) = 0$ such that $F = f^{-1}$ is convex and $F(a) \asymp a^{1/p}$ when $0 \leq a \leq 1$, for some $0 < p < 1$. Then, the following sharp absolute bounds hold true for all $x \in \Omega$

$$v(x) \asymp \begin{cases} \Phi_1^\sigma(x) & \text{when } 2s \neq \gamma(1-p) \\ \Phi_1(x)\,\left(1 + |\log \Phi_1(x)|\right)^{\frac{1}{1-p}} & \text{when } 2s = \gamma(1-p), \text{ assuming (K4)} \end{cases}$$

where

$$\boxed{\sigma := 1 \wedge \frac{2s}{\gamma(1-p)}} \qquad \text{and} \qquad \Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$$

When $2s = \gamma(1-p)$, if (K4) does not hold, then the upper bound still holds, but the lower bound holds in a non-sharp form without the extra logarithmic term.

Remarks.

- When $2s < \gamma(1-p)$, the new power σ becomes less than 1.
- Somehow σ interpolates between the two extremal cases: $p = 0$ i.e. $\mathcal{L}v = 1$ and $p = 1$, i.e. $\mathcal{L}v = \lambda v$.

Examples.

- For the RFL ($\gamma = s$) and CFL ($\gamma = s - 1/2$) we always have $\sigma = 1$ and $2s \neq \gamma(1-p)$, hence

$$v(x) \asymp \Phi_1(x) \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$$

- For the SFL we have $\gamma = 1$ hence we have three possibilities:

$$v(x) \asymp \begin{cases} \mathrm{dist}(x,\partial\Omega) & \text{when } s > \frac{1-p}{2} \\ \mathrm{dist}(x,\partial\Omega)\left(1 + |\log \mathrm{dist}(x,\partial\Omega)|\right)^{\frac{1}{1-p}} & \text{when } s = \frac{1-p}{2} \\ \mathrm{dist}(x,\partial\Omega)^{\frac{2s}{1-p}} & \text{when } s < \frac{1-p}{2} \end{cases}$$

Regularity. Under some mild assumptions on $\mathcal{L}$ and $f \in C^\beta(\mathbb{R})$ for some $\beta > 0$, with $0 \le f(a) \le c_p a^p$ when $0 \le a \le 1$ for some $0 < p \le 1$.

- Solutions are *Hölder continuous in the interior*, and (when the operator allows it) are *classical in the interior*, namely $C^{2s+\beta}(\Omega)$.
- Assuming moreover that $\mathcal{L}^{-1}$ satisfies (K2), solutions are *Hölder continuous up to the boundary*:

$$\|u\|_{C^\eta(\overline{\Omega})} \le C \qquad \forall \eta \in (0,\gamma] \cap (0,2s).$$

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Change of notations from Elliptic to Parabolic In order to make the elliptic results "compatible" with the parabolic, we will perform the change of notations

$$\boxed{m = \frac{1}{p} > 1} \qquad \text{and} \qquad \boxed{v = S^m} \qquad \text{or} \qquad v^p = S\,.$$

The elliptic equation transforms: (we deal only with pure powers for simplicity)

$$\mathcal{L}v = f(v) = v^p \qquad \text{becomes} \qquad \boxed{\mathcal{L}S^m = \mathcal{L}F(S) = S}$$

Parabolic solutions by separation of variables. We have the following solution for the Dirichlet problem for the equation $u_t + \mathcal{L}\, u^m = 0$

$$\mathcal{U}_T(t,x) = \frac{S(x)}{(T+t)^{\frac{1}{m-1}}}$$

where $\mathcal{L}S^m = S$, and the initial datum is $U_T(0,x) = T^{-1/(m-1)}S(x)$.
When $T = 0$ we have the so-called *Friendly Giant*, corresponding to the biggest possible initial datum (useful in the asymptotic study as $t \to \infty$.)

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Back to the Parabolic problem

- **(More) Assumptions on the operator**
- **Basic theory: existence, uniqueness and boundedness**
- **Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$**

For the rest of the talk we deal with the special case:

$$\boxed{F(u) = u^m := |u|^{m-1}u, \quad m > 1}$$

Recall that the linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, and we have already explained the assumptions (K1) and (K2) on the inverse.

Assumptions on the kernel.

- Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x,y)$ via the formula

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big)\, K(x,y)\, \mathrm{d}y\,,$$

assumption (L1) states that there exists $\underline{\kappa}_\Omega > 0$ such that

$$\text{(L1)} \qquad \inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,.$$

- Whenever $\mathcal{L}$ is defined in terms of a kernel $K(x,y)$ and a zero order term:

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big)\, K(x,y)\, \mathrm{d}y + B(x)f(x),$$

assumptions (L2) states that there exists $\underline{\kappa}_\Omega > 0$ and $\gamma \in (0,1]$

$$\text{(L2)} \qquad K(x,y) \ge \underline{\kappa}_\Omega\, \mathrm{dist}(x,\partial\Omega)^\gamma\, \mathrm{dist}(y,\partial\Omega)^\gamma, \qquad \text{and} \qquad B(x) \ge 0,$$

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About the kernels of spectral nonlocal operators. Most of the examples of nonlocal operators, but the SFL, admit a representation with a kernel
A natural question is: does the SFL admit such a representation?
Let A be a uniformly elliptic linear operator. Define the s^{th} power of A:

$$\mathcal{L}g(x) = A^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{tA} g(x) - g(x)\right) \frac{\mathrm{d}t}{t^{1+s}}$$

Then it admits a representation with a Kernel plus zero order term:

$$A^s g(x) = P.V. \int_{\mathbb{R}^N} \big(g(x) - g(y) \big) K(x,y)\, \mathrm{d}y + \kappa(x) g(x)\,.$$

where $K \geq 0$ is compactly supported in $\overline{\Omega} \times \overline{\Omega}$ with

$$K(x,y) \asymp \frac{1}{|x-y|^{N+2s}} \left(\frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1 \right) \quad \text{and} \quad \kappa(x) \asymp \frac{1}{\mathrm{dist}(x, \partial\Omega)^{2s}}\,.$$

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Basic theory: existence, uniqueness and boundedness (in one page)

$$\text{(CDP)}\qquad\begin{cases}\partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega\\ u(0,x)=u_0(x)\,, & \text{in } \Omega\\ u(t,x)=0\,, & \text{on the lateral boundary.}\end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\boxed{\partial_t U = -u^m\,,}\qquad \text{where}\qquad U(t,x):=\mathcal{L}^{-1}[u(t,\cdot)](x)=\int_\Omega u(t,y)\mathbb{G}(x,y)\,\mathrm{d}y\,.$$

- This formulation encodes the lateral boundary conditions through $\mathcal{L}^{-1}$.
- Define the *Weak Dual Solutions* (WDS), a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
- Prove existence and uniqueness of nonnegative WDS with $0\le u_0\in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Prove a number of new pointwise estimates that provide L^∞ bounds:
  Absolute bounds: ($\overline{k}$ below does NOT depend on u_0)

$$\|u(t,\cdot)\|_{\mathrm{L}^\infty(\Omega)}\le \overline{\kappa}\, t^{-\frac{1}{m-1}}\,,$$

  Instantaneous Smoothing Effects:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)}\le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}\le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$$

- For more details on this part "old slides": http://verso.mat.uam.es/~matteo.bonforte
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Basic theory: existence, uniqueness and boundedness (in one page)

$$\text{(CDP)}\qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega\\ u(0,x) = u_0(x)\,, & \text{in } \Omega\\ u(t,x) = 0\,, & \text{on the lateral boundary.}\end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\boxed{\partial_t U = -u^m\,,}\qquad \text{where}\qquad U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{G}(x,y)\,\mathrm{d}y\,.$$

- This formulation encodes the lateral boundary conditions through $\mathcal{L}^{-1}$.
- Define the *Weak Dual Solutions* (WDS), a new concept compatible with more standard solutions: very weak, weak (energy), mild, strong [...]
- Prove existence and uniqueness of nonnegative WDS with $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Prove a number of new pointwise estimates that provide L^∞ bounds:
  Absolute bounds: ($\overline{k}$ below does NOT depend on u_0)

$$\|u(t,\cdot)\|_{\mathrm{L}^\infty(\Omega)} \le \overline{\kappa}\, t^{-\frac{1}{m-1}}\,,$$

  Instantaneous Smoothing Effects:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \le \frac{\overline{\kappa}}{t^{N\vartheta_\gamma}}\|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$$

- For more details on this part "old slides": `http://verso.mat.uam.es/~matteo.bonforte`
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Elliptic VS Parabolic: Asymptotic Behaviour as $t \to \infty$

Theorem. (Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Assume that $\mathcal{L}$ satisfies (A1), (A2), and (K2), and let S be the solution to $\mathcal{L}S^m = S$. Let u be any weak dual solution to the Cauchy-Dirichlet problem. Then, unless $u \equiv 0$,

$$\left\| t^{\frac{1}{m-1}} u(t,\cdot) - S \right\|_{\mathrm{L}^\infty(\Omega)} \xrightarrow{t\to\infty} 0\,.$$

This result, gives a clear suggestion of what the boundary behaviour of parabolic solutions should be,

$$u(t,x) \asymp \mathcal{U}(t,x) = \frac{S(x)}{t^{\frac{1}{m-1}}}$$

at least for large times, as it happens in the local case $s = 1$. Hence the boundary behaviour shall be dictated by the behaviour of the solution to the elliptic equation. We shall see that this is not always the case.

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Sharp Boundary Behaviour

- Upper Boundary Estimates
- Infinite Speed of Propagation
- Lower Boundary Estimates
- Harnack-type Inequalities
- Numerics

Theorem. (Upper boundary behaviour) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold. Let $u \ge 0$ be a weak dual solution to the (CDP). Let $\sigma \in (0,1]$ be

$$\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$$

Then, there exists a computable constant $\overline{\kappa} > 0$, depending only on N, s, m, and Ω, (but not on u_0 !!) such that for all $t \ge 0$ and all $x \in \Omega$:

$$u(t,x) \le \frac{\overline{\kappa}}{t^{\frac{1}{m-1}}} \begin{cases} \Phi_1(x)^{\frac{\sigma}{m}} & \text{if } \gamma \ne 2sm/(m-1), \\ \Phi_1(x)^{\frac{1}{m}} \left(1 + |\log \Phi_1(x)|\right)^{\frac{1}{m-1}} & \text{if } \gamma = 2sm/(m-1). \end{cases}$$

- **When $\sigma = 1$ and $\gamma \ne 2sm/(m-1)$ we have sharp boundary estimates**: we will show lower bounds with matching powers.
- **When $\sigma < 1$ the estimates are not sharp in all cases**:
  - The solution by separation of variables $\mathcal{U}(t,x) = S(x)t^{-1/(m-1)}$ (asymptotic behaviour) behaves like $\Phi_1^{\sigma/m} t^{-1/(m-1)}$.
  - We will show that for small data, the boundary behaviour is different.
  - In examples, $\sigma < 1$ only happens for SFL-type, where $\gamma = 1$, and s can be small, $0 < s < 1/2 - 1/(2m)$.

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Infinite Speed of Propagation

and

Universal Lower Bounds

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1), (A2) and (L2). Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_0 > 0$, so that the following inequality holds:

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and all } x \in \Omega\,.$$

Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

• Note that, for $t \geq t_*$, the dependence on the initial data disappears

$$u(t) \geq \underline{\kappa}_0 \Phi_1 t^{-\frac{1}{m-1}} \qquad \forall\, t \geq t_*\,.$$

• The assumption on the kernel K of $\mathcal{L}$ holds for all examples and represent somehow the "worst case scenario" for lower estimates:

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x)-f(y)\big)\, K(x,y)\,\mathrm{d}y + B(x) f(x), \quad \text{with} \quad \begin{cases} K(x,y) \gtrsim \delta^\gamma(x)\,\delta^\gamma(y), \\ B(x) \geq 0, \end{cases}$$

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Infinite speed of propagation.

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- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s < 1$, contrary to the "local" case $s = 1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
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Sharp Lower boundary estimates

Sharp lower boundary estimates I: the non-spectral case.

Let $\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$. Let $\mathcal{L}$ satisfy (A1) and (A2), and assume moreover that

$$\mathcal{L}f(x) = \int_{\mathbb{R}^N} \big(f(x)-f(y)\big)K(x,y)\,\mathrm{d}y\,, \qquad \text{with } \inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,.$$

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- either $\sigma = 1$;
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$$\boxed{u(t,x_0) \ge \underline{\kappa}_1 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and a.e. } x \in \Omega\,.$$

where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$. The constants $\kappa_*, \underline{\kappa}_1$ depend only on $N, s, \gamma, m, \underline{\kappa}_\Omega, c_1, \Omega$.

- The *boundary behavior is sharp for all times* in view of the upper bounds.
- Within examples, this applies to RFL and CFL type, but not to SFL-type.
- For RFL, this result was obtained first by MB, A. Figalli and X. Ros-Oton.

Sharp absolute lower estimates for large times: the case $\sigma = 1$.
When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

Theorem. (Sharp lower bounds for large times) (M.B., A. Figalli and J. L. Vázquez)

Let $(A1)$, $(A2)$, and $(K2)$ hold, and let $\sigma = 1$ and $2sm \neq \gamma(m-1)$. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. There exists a constant $\underline{\kappa}_2 > 0$ such that

$$u(t, x_0) \geq \underline{\kappa}_2 \, \frac{\Phi_1(x_0)^{1/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \geq t_* \text{ and a.e. } x \in \Omega\,.$$

Here, $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and the constants $\kappa_*, \underline{\kappa}_2$ depend only on N, s, γ, m, and Ω.

- It holds for $s = 1$, the local case, where there is finite speed of propagation.
- When $s = 1$, t_* is the time that the solution needs to be positive everywhere.
- When $\mathcal{L} = -\Delta$, proven by Aronson-Peletier ('81) and Vázquez ('04)
- Our method applies when $\mathcal{L}$ is an elliptic operator with C^1 coefficients (new result).
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Positivity for large times II: the case $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{\gamma}{2} - \frac{\gamma}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared 'big' solutions:

$$u_0 \ge \epsilon_0 S \qquad \text{implies} \qquad u(t) \ge \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{\mathrm{L}^\infty} = 0 \qquad \text{for all } 0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)\,.$$

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Different boundary behaviour when $\sigma < 1$. The next result shows that, in general, we cannot hope to prove that $u(t)$ is larger than $\Phi_1^{1/m}$, but always smaller than $\Phi_1^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \leq u_0 \leq c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \leq c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

When $\sigma = 1$ and $2sm = \gamma(m-1)$, then

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Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

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We next show that assuming (K4), the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \le c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \ge \underline{\kappa}\Phi_1^\alpha(x) \qquad \text{for a.e. } x \in \Omega\,, \quad \boxed{\text{then } \alpha \ge 1 - \frac{2s}{\gamma}\,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Under mild assumptions on the operator (for example SFL-type), we can prove:

$$0 \le u_0 \le A\,\Phi_1^{1-\frac{2s}{\gamma}} \qquad \Rightarrow \qquad u(t) \le [A^{1-m} - \tilde{C}t]^{-(m-1)}\Phi_1^{1-\frac{2s}{\gamma}}$$

for small times $t \in [0, T_A]$, where $T_A := 1/(\tilde{C}A^{m-1})$, for some $\tilde{C} > 0$.
Recall that we have a universal lower bound (under minimal assumptions on K)

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

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Harnack-type Inequalities

- Global Harnack Principle I. The non-spectral case.
- Consequences of GHP.
- Global Harnack Principle II. The remaining cases.

Global Harnack Principle I. The non-spectral case.

Recall that

$$\Phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^\gamma\,, \qquad \sigma = 1\wedge\frac{2sm}{\gamma(m-1)}, \qquad t_* = \kappa_*\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}.$$

Theorem. (Global Harnack Principle I. The non-spectral case.)(MB & AF & JLV)

Let (A1), (A2), (L1) and (K2). Let $u\ge 0$ be a weak dual solution to the (CDP). Also, when $\sigma<1$, assume that $K(x,y)\le c_1|x-y|^{-(N+2s)}$ for a.e. $x,y\in\mathbb{R}^N$ and that $\Phi_1\in C^\gamma(\Omega)$.
Then, there exist constants $\underline{\kappa},\overline{\kappa}>0$, so that the following inequality holds:

$$\underline{\kappa}\left(1\wedge\frac{t}{t_*}\right)^{\frac{m}{m-1}}\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}\le u(t,x)\le\overline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}\qquad\text{for all } t>0 \text{ and all } x\in\Omega\,.$$

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- For large times $t\ge t_*$ the estimates are independent on the initial datum.
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Consequences of GHP with matching powers

Corollary. (Local Harnack Inequalities of Elliptic/Backward Type)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, s. t. for all $t > 0$ and $h \geq 0$

$$\sup_{x \in B_R(x_0)} u(t,x) \leq \hat{H} \left[\left(1 + \frac{h}{t}\right) \left(1 \wedge \frac{t}{t_*}\right)^{-m} \right]^{\frac{1}{m-1}} \inf_{x \in B_R(x_0)} u(t+h,x) \, .$$

When $s = 1$, backward Harnack inequalities are typical of Fast Diffusion eq. ($m < 1$, possible extinction in finite time), and they do not happen when $m > 1$ (finite speed of propagation)

Theorem. (Sharp Asymptotic behaviour) (M.B., A. Figalli, Y. Sire, J. L. Vázquez)

Assume that a GHP with matching powers hold. Set $\mathcal{U}(t,x) := t^{-\frac{1}{m-1}} S(x)$. Then there exists $c_0 > 0$ such that, for all $t \geq t_0 := c_0 \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have

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This asymptotic result is sharp: check by considering $u(t,x) = \mathcal{U}(t+1,x)$. For the classical case $\mathcal{L} = \Delta$, we recover the results of Aronson-Peletier and Vazquez with a different proof.

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This asymptotic result is sharp: check by considering $u(t,x) = \mathcal{U}(t+1,x)$. For the classical case $\mathcal{L} = \Delta$, we recover the results of Aronson-Peletier and Vazquez with a different proof.

Global Harnack Principles II. The remaining cases.

Theorem. (Global Harnack Principle II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Assume that:
- either $\sigma = 1$ and $2sm \ne \gamma(m-1)$;
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Then there exist constants $\underline{\kappa}, \overline{\kappa} > 0$ such that the following inequality holds:

$$\underline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \ge t_* \text{ and all } x \in \Omega\,.$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

- For large times, we can prove as before Local Harnack inequalities of Elliptic/Backward type.
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Hence, in the remaining cases, we have only the following general result.

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Let $\mathcal{L}$ satisfy (A1),(A2), (L2) and (K2).Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
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- This is sufficient to ensure interior regularity, under 'minimal' assumptions.
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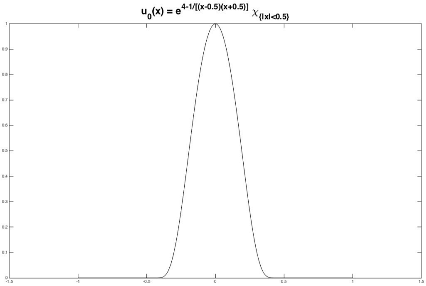
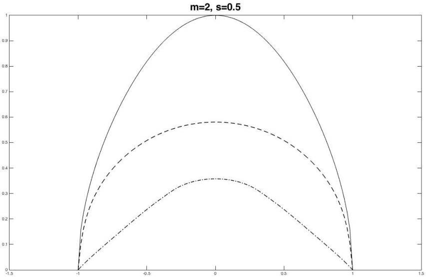
Numerical Simulations*

* Graphics obtained by numerical methods contained in: N. Cusimano, F. Del Teso, L. Gerardo-Giorda, G. Pagnini, *Discretizations of the spectral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary conditions*, Preprint (2017).
Graphics and videos: courtesy of F. Del Teso (NTNU, Trondheim, Norway)

Numerical simulation for the SFL with parameters $m = 2$ and $s = 1/2$, hence $\sigma = 1$.





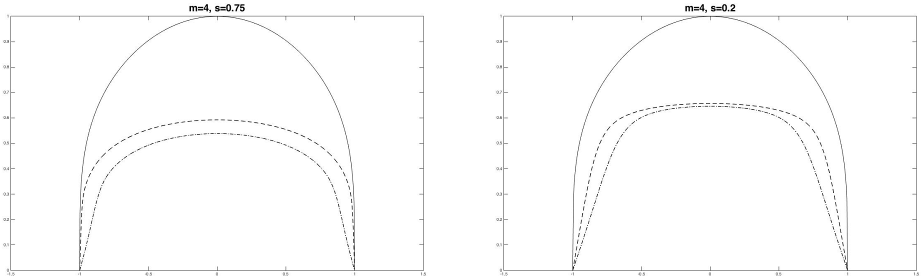
Left: the initial condition $u_0 \le C_0 \Phi_1$

Right: solid line represents $\Phi_1^{1/m}$

the dotted lines represent $\boxed{t^{\frac{1}{m-1}} u(t)}$ at time at $t = 1$ and $t = 5$

While $u(t)$ appears to behave as $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for very short times already at $t = 5$ it exhibits the matching boundary behavior $t^{\frac{1}{m-1}} u(t) \asymp \Phi_1^{1/m}$

Compare $\sigma = 1$ VS $\sigma < 1$: same $u_0 \le C_0 \Phi_1$, solutions with different parameters



Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 30$ and $t = 150$; $m = 4$, $s = 3/4$, $\sigma = 1$.

Matching: $u(t)$ behaves like $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for quite some time,
and only around $t = 150$ it exhibits the matching boundary behavior $u(t) \asymp \Phi_1^{1/m}$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 150$ and $t = 600$; $m = 4$, $s = 1/5$, $\sigma = 8/15 < 1$.

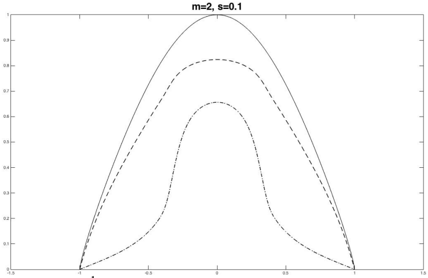
Non-matching: $u(t) \asymp \Phi_1$ even after long time.

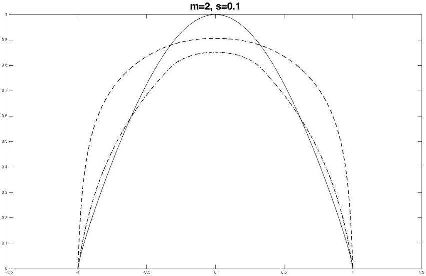
Idea: maybe when $\sigma < 1$ and $u_0 \lesssim \Phi_1$, we have $u(t) \asymp \Phi_1$ for all times...

Not True: there are cases when $u(t) \gg \Phi_1^{1-2s}$ for large times...

Non-matching when $\sigma < 1$: same data u_0, with $m = 2$ and $s = 1/10$, $\sigma = 2/5 < 1$

In both pictures, the solid line represents Φ_1^{1-2s} (anomalous behaviour)





Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 4$ and $t = 25$.

$u(t) \asymp \Phi_1$ for short times $t = 4$, then $u(t) \sim \Phi_1^{1-2s}$ for intermediate times $t = 25$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 40$ and $t = 150$. $u(t) \gg \Phi_1^{1-2s}$ for large times.

Both non-matching always different behaviour from the asymptotic profile $\Phi_1^{\sigma/m}$.

In this case we show that if $u_0(x) \le C_0 \Phi_1(x)$ then for all $t > 0$

$$u(t,x) \le C_1 \left[\frac{\Phi_1(x)}{t}\right]^{\frac{1}{m}} \qquad \text{and} \qquad \lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\frac{\sigma}{m}}} = 0 \qquad \text{for any } t > 0.$$

Regularity Estimates

- **Interior Regularity**
- **Hölder continuity up to the boundary**
- **Higher interior regularity for RFL**

Interior Regularity

The regularity results, require the validity of a Global Harnack Principle.

(R) The operator $\mathcal{L}$ satisfies (A1) and (A2), and $\mathcal{L}^{-1}$ satisfies (K2). Moreover, we consider

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big)K(x,y)\,\mathrm{d}y + B(x)f(x)\,, \qquad \text{with}$$

$$K(x,y) \asymp |x-y|^{-(N+2s)} \quad \text{in } B_{2r}(x_0) \subset \Omega, \qquad K(x,y) \lesssim |x-y|^{-(N+2s)} \quad \text{in } \mathbb{R}^N \setminus B_{2r}(x_0).$$

As a consequence, for any ball $B_{2r}(x_0) \subset\subset \Omega$ and $0 < t_0 < T_1$, there exist $\delta, M > 0$ such that

$$0 < \delta \le u(t,x) \qquad \text{for a.e. } (t,x) \in (T_0,T_1) \times B_{2r}(x_0),$$
$$0 \le u(t,x) \le M \qquad \text{for a.e. } (t,x) \in (T_0,T_1) \times \Omega.$$

The constants in the regularity estimates will depend on the solution only through δ, M.

Theorem. (Interior Regularity) (M.B., A. Figalli and J. L. Vázquez)

Assume (R) and let u be a nonnegative bounded weak dual solution to problem (CDP).

1. Then u is **Hölder continuous in the interior**. More precisely, there exists $\alpha > 0$ such that, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{\alpha/2s,\alpha}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

2. Assume in addition $|K(x,y) - K(x',y)| \le c|x-x'|^\beta\,|y|^{-(N+2s)}$ for some $\beta \in (0, 1 \wedge 2s)$ such that $\beta + 2s \notin \mathbb{N}$. Then u **is a classical solution in the interior**. More precisely, for all $0 < T_0 < T_2 < T_1$,

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Hölder continuity up to the boundary

Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
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- Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spacial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)
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    Athanasopoulos and Caffarelli [Adv. Math, 2010], (RFL domains)
    De Pablo, Quirós, Rodriguez, Vázquez [CPAM 2012] (RFL on $\mathbb{R}^N$, SFL-Dirichlet)
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Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

$$\|u\|_{C^{\frac{\gamma}{m\vartheta},\frac{\gamma}{m}}_{t,x}((T_2,T_1)\times\Omega)} \le C \qquad \text{with} \qquad \vartheta := 2s - \gamma\left(1 - \frac{1}{m}\right).$$

• Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \operatorname{dist}(x,\partial\Omega)^{\gamma/m}$, **the spacial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)

• Previous regularity results: (I apologize if I forgot someone)

- C^α *regularity:*
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Higher Interior Regularity for RFL.

Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)** , then $u \in C_x^\infty((0,\infty)\times\Omega)$.
More precisely, let $k \ge 1$ be any positive integer, and $d(x) = \mathrm{dist}(x,\partial\Omega)$, then, for any $t \ge t_0 > 0$ we have

$$\left|D_x^k u(t,x)\right| \le C\,[d(x)]^{\frac{s}{m}-k},$$

where C depends only on N, s, m, k, Ω, t_0, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in t. To our knowledge also open for the local case $s = 1$.
- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C_x^\infty$

  $$\|u\|_{C_x^{k,\alpha}((\frac12,1)\times B_{1/2})} \le C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \ge 0.$$

  Analogous estimates in time do not hold for $k \ge 1$ and $\alpha \in (0,1)$.
  Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not C^1 in t in $(\frac12, 1)\times B_{1/2}$. [Chang-Lara, Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains , and also for operator with more general kernels.
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The End

Thank You!!!

Grazie Mille!!!

Muchas Gracias!!!