Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

Matteo Bonforte

Departamento de Matemáticas,
Universidad Autónoma de Madrid,
Campus de Cantoblanco
28049 Madrid, Spain

matteo.bonforte@uam.es

http://www.uam.es/matteo.bonforte

Workshop on Aggregation-Diffusion PDEs: Variational Principles, Nonlocality and Systems

Anacapri, Italy, July 12, 2017

References:

[BV1] M. B., J. L. VÁZQUEZ, A Priori Estimates for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains.
Arch. Rat. Mech. Anal. (2015).

[BV2] M. B., J. L. VÁZQUEZ, Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part I. Existence, Uniqueness and Upper Bounds
Nonlin. Anal. TMA (2016).

[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains.
Discr. Cont. Dyn. Sys. (2015).

[BFR] M. B., A. FIGALLI, X. ROS-OTON, Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains.
To appear in Comm. Pure Appl. Math (2017).

[BFV] M. B., A. FIGALLI, J. L. VÁZQUEZ, Sharp boundary estimates and higher regularity for nonlocal porous medium-type equations in bounded domains.
Preprint (2016). https://arxiv.org/abs/1610.09881

- *A talk more focussed on the first part is available online:*
  http://www.fields.utoronto.ca/video-archive//event/2021/2016

Outline of the talk

- Introduction
  - The abstract setup of the problem
  - Some important examples
  - About Spectral Kernels
- Basic Theory
  - The Dual problem
  - Existence and uniqueness
  - First set of estimates
- Sharp Boundary Behaviour
  - Upper Boundary Estimates
  - Infinite Speed of Propagation
  - Lower Boundary Estimates
- Harnack Inequalities
- Numerics
- Regularity Estimates

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

About the operator $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e, then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

Remark. These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a remarkable variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:

- M. G. Crandall, T.M. Liggett. *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971) 265–298.
- M. Crandall, M. Pierre, *Regularizing Effects for $u_t = A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194–212

About the operator $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e , then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

Remark. These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a remarkable variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:

- M. G. Crandall, T.M. Liggett. *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971) 265–298.
- M. Crandall, M. Pierre, *Regularizing Effects for $u_t = A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194–212

About the operator $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e, then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

Remark. These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a remarkable variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:

- M. G. Crandall, T.M. Liggett. *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971) 265–298.
- M. Crandall, M. Pierre, *Regularizing Effects for $u_t = A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194–212

Assumption on the nonlinearity F

Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous and non-decreasing function, with $F(0)=0$. Moreover, it satisfies the condition:

(N1) $F\in C^1(\mathbb{R}\setminus\{0\})$ and $F/F'\in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0,\mu_1>0$ s.t.

$$\frac{1}{m_1}=1-\mu_1\le\left(\frac{F}{F'}\right)'\le 1-\mu_0=\frac{1}{m_0}$$

where F/F' is understood to vanish if $F(r)=F'(r)=0$ or $r=0$.

The main example (treated in the rest of the talk) will be

$$\boxed{F(u)=|u|^{m-1}u,\quad \text{with } m>1},\qquad \mu_0=\mu_1=\frac{m-1}{m}<1\,.$$

A simple variant is the combination of two powers:
m_1 behaviour when $u\sim\infty$ and m_0 behaviour when $u\sim 0$

Monotonicity estimates follow by (N1): the following maps

$$t\mapsto t^{\frac{1}{\mu_0}}F(u(t,x))\qquad\text{or}\qquad t\mapsto t^{\frac{1}{m-1}}u(t,x)$$

are nondecreasing in $t>0$ for a.e. $x\in\Omega$.

- P. Bénilan, M. G. Crandall. *Regularizing effects of homogeneous evolution equations*, Contributions to Analysis and Geometry, suppl. to Amer. Jour. Math., (1981). Pp. 23-39.
- M. Crandall, M. Pierre, *Regularizing Effects for $u_t=A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194–212

Assumption on the nonlinearity F

Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous and non-decreasing function, with $F(0)=0$. Moreover, it satisfies the condition:

(N1) $F\in C^1(\mathbb{R}\setminus\{0\})$ and $F/F'\in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0,\mu_1>0$ s.t.

$$\frac{1}{m_1}=1-\mu_1\le\left(\frac{F}{F'}\right)'\le 1-\mu_0=\frac{1}{m_0}$$

where F/F' is understood to vanish if $F(r)=F'(r)=0$ or $r=0$.

The main example (treated in the rest of the talk) will be

$$\boxed{F(u)=|u|^{m-1}u,\quad \text{with } m>1},\qquad \mu_0=\mu_1=\frac{m-1}{m}<1\,.$$

A simple variant is the combination of two powers:
m_1 behaviour when $u\sim\infty$ and m_0 behaviour when $u\sim 0$

Monotonicity estimates follow by (N1): the following maps

$$t\mapsto t^{\frac{1}{\mu_0}}F(u(t,x))\qquad\text{or}\qquad t\mapsto t^{\frac{1}{m-1}}u(t,x)$$

are nondecreasing in $t>0$ for a.e. $x\in\Omega$.

- P. Bénilan, M. G. Crandall. *Regularizing effects of homogeneous evolution equations*, Contributions to Analysis and Geometry, suppl. to Amer. Jour. Math., (1981). Pp. 23-39.
- M. Crandall, M. Pierre, *Regularizing Effects for $u_t=A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194–212

Assumption on the nonlinearity F

Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous and non-decreasing function, with $F(0) = 0$. Moreover, it satisfies the condition:

(N1) $F \in C^1(\mathbb{R} \setminus \{0\})$ and $F/F' \in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0, \mu_1 > 0$ s.t.

$$\frac{1}{m_1} = 1 - \mu_1 \le \left(\frac{F}{F'}\right)' \le 1 - \mu_0 = \frac{1}{m_0}$$

where F/F' is understood to vanish if $F(r) = F'(r) = 0$ or $r = 0$.

The main example (treated in the rest of the talk) will be

$$\boxed{F(u) = |u|^{m-1}u, \quad \text{with } m > 1}\,, \qquad \mu_0 = \mu_1 = \frac{m-1}{m} < 1\,.$$

A simple variant is the combination of two powers:

m_1 behaviour when $u \sim \infty$ and m_0 behaviour when $u \sim 0$

Monotonicity estimates follow by (N1): the following maps

$$t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x)) \qquad \text{or} \qquad t \mapsto t^{\frac{1}{m-1}} u(t,x)$$

are nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

- P. Bénilan, M. G. Crandall. *Regularizing effects of homogeneous evolution equations*, Contributions to Analysis and Geometry, suppl. to Amer. Jour. Math., (1981). Pp. 23-39.
- M. Crandall, M. Pierre, *Regularizing Effects for $u_t = A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194–212

Assumption on the nonlinearity F

Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous and non-decreasing function, with $F(0)=0$. Moreover, it satisfies the condition:

(N1) $F\in C^1(\mathbb{R}\setminus\{0\})$ and $F/F'\in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0,\mu_1>0$ s.t.

$$\frac{1}{m_1}=1-\mu_1\le\left(\frac{F}{F'}\right)'\le 1-\mu_0=\frac{1}{m_0}$$

where F/F' is understood to vanish if $F(r)=F'(r)=0$ or $r=0$.

The main example (treated in the rest of the talk) will be

$$\boxed{F(u)=|u|^{m-1}u,\quad \text{with } m>1},\qquad \mu_0=\mu_1=\frac{m-1}{m}<1\,.$$

A simple variant is the combination of two powers:
m_1 behaviour when $u\sim\infty$ and m_0 behaviour when $u\sim 0$

Monotonicity estimates follow by (N1): the following maps

$$t\mapsto t^{\frac{1}{\mu_0}}F(u(t,x))\qquad\text{or}\qquad t\mapsto t^{\frac{1}{m-1}}u(t,x)$$

are nondecreasing in $t>0$ for a.e. $x\in\Omega$.

- P. Bénilan, M. G. Crandall. *Regularizing effects of homogeneous evolution equations*, Contributions to Analysis and Geometry, suppl. to Amer. Jour. Math., (1981). Pp. 23-39.
- M. Crandall, M. Pierre, *Regularizing Effects for $u_t=A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194–212

Assumption on the nonlinearity F

Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous and non-decreasing function, with $F(0) = 0$. Moreover, it satisfies the condition:

(N1) $F \in C^1(\mathbb{R} \setminus \{0\})$ and $F/F' \in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0, \mu_1 > 0$ s.t.

$$\frac{1}{m_1} = 1 - \mu_1 \le \left(\frac{F}{F'}\right)' \le 1 - \mu_0 = \frac{1}{m_0}$$

where F/F' is understood to vanish if $F(r) = F'(r) = 0$ or $r = 0$.

The main example (treated in the rest of the talk) will be

$$\boxed{F(u) = |u|^{m-1}u, \quad \text{with } m > 1}, \qquad \mu_0 = \mu_1 = \frac{m-1}{m} < 1\,.$$

A simple variant is the combination of two powers:
m_1 behaviour when $u \sim \infty$ and m_0 behaviour when $u \sim 0$

Monotonicity estimates follow by (N1): the following maps

$$t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x)) \qquad \text{or} \qquad t \mapsto t^{\frac{1}{m-1}} u(t,x)$$

are nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

- P. Bénilan, M. G. Crandall. *Regularizing effects of homogeneous evolution equations*, Contributions to Analysis and Geometry, suppl. to Amer. Jour. Math., (1981). Pp. 23-39.
- M. Crandall, M. Pierre, *Regularizing Effects for $u_t = A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194–212

Assumption on the inverse operator $\mathcal{L}^{-1}$

Assumptions on the inverse of $\mathcal{L}$

We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1} : \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ with a kernel $\mathbb{K}$ such that

$$\mathcal{L}^{-1} f(x) = \int_\Omega \mathbb{K}(x,y) f(y)\, \mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$ and $c_{i,\Omega} > 0$

(K1) $$0 \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

(K2) $$c_{0,\Omega}\delta^\gamma(x)\,\delta^\gamma(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)$$

where

$$\delta^\gamma(x) := \mathrm{dist}(x, \partial\Omega)^\gamma\,.$$

When $\mathcal{L}$ has a first eigenfunction, (K1) implies $0 \le \Phi_1 \in \mathrm{L}^\infty(\Omega)$.

Moreover, (K2) implies that $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta^\gamma$ and we can rewrite (K2) as

(K3) $$c_{0,\Omega}\Phi_1(x)\Phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-x_0|^{N-2s}}\left(\frac{\Phi_1(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma}\wedge 1\right)$$

Assumption on the inverse operator $\mathcal{L}^{-1}$

Assumptions on the inverse of $\mathcal{L}$

We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1}: \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ with a kernel $\mathbb{K}$ such that

$$\mathcal{L}^{-1}f(x) = \int_\Omega \mathbb{K}(x,y) f(y)\, \mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$ and $c_{i,\Omega} > 0$

(K1) $$0 \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

(K2) $$c_{0,\Omega}\delta^\gamma(x)\,\delta^\gamma(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right)$$

where

$$\delta^\gamma(x) := \mathrm{dist}(x,\partial\Omega)^\gamma\,.$$

When $\mathcal{L}$ has a first eigenfunction, (K1) implies $0 \le \Phi_1 \in \mathrm{L}^\infty(\Omega)$.
Moreover, (K2) implies that $\Phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^\gamma = \delta^\gamma$ and we can rewrite (K2) as

(K3) $$c_{0,\Omega}\Phi_1(x)\Phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-x_0|^{N-2s}}\left(\frac{\Phi_1(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma}\wedge 1\right)$$

Examples of operators $\mathcal{L}$

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right) \frac{dt}{t^{1+s}}.$$

Examples of operators $\mathcal{L}$

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right) \frac{dt}{t^{1+s}}\,.$$

Examples of operators $\mathcal{L}$

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,
$$\widehat{((-\Delta_{\mathbb{R}^N})^s f)}(\xi) = |\xi|^{2s}\hat{f}(\xi) .$$
This formula can be used for positive and negative values of s.
2. By means of an **Hypersingular Kernel**:
if $0 < s < 1$, we can use the representation
$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}} \, \mathrm{d}z,$$
where $c_{N,s} > 0$ is a normalization constant.
3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:
$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

Examples of operators $\mathcal{L}$

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}} .$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \qquad \text{with } \gamma = 1$$

Examples of operators $\mathcal{L}$

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \quad \text{with } \gamma = 1$$

Examples of operators $\mathcal{L}$

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,, \qquad \text{with } \mathrm{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$. (J. Serra - X. Ros-Oton)

Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times (\mathbb{R}^N \setminus \Omega)\,.$$

The Green function of RFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right), \qquad \text{with } \gamma = s$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

Examples of operators $\mathcal{L}$

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The (Restricted) Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\,\text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,dz\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$. (J. Serra - X. Ros-Oton)

Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty)\times\left(\mathbb{R}^N \setminus \Omega\right).$$

The Green function of RFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}}\left(\frac{\delta^\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma}\wedge 1\right), \quad \text{with } \gamma = s$$

References. (K4) Bounds proven by Bogdan, Grzywny, Jakubowski, Kulczycki, Ryznar (1997-2010). Eigenvalues: Blumental-Getoor (1959), Chen-Song (2005)

Examples of operators $\mathcal{L}$

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \text{P.V.} \int_{\Omega} (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, dy \,, \qquad \text{with } \frac{1}{2} < s < 1 \,,$$

where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.

The Green function $\mathbb{K}(x,y)$ satisfies $(K4)$, proven by Chen, Kim and Song (2010)

$$\mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) \,, \quad \text{with } \gamma = s - \frac{1}{2} \,.$$

Remarks.

- This is a third model of Dirichlet fractional Laplacian when $\left[a(x,y) = const \right]$. This is **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

References.

- K. Bogdan, K. Burdzy, K., Z.-Q. Chen. *Censored stable processes*. Probab. Theory Relat. Fields (2003)
- Z.-Q. Chen, P. Kim, R. Song, *Two-sided heat kernel estimates for censored stable-like processes*. Probab. Theory Relat. Fields (2010)

Examples of operators $\mathcal{L}$

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \,\mathrm{d}y\,, \qquad \text{with } \frac{1}{2} < s < 1\,,$$

where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega}\times\overline{\Omega})$.

The Green function $\mathbb{K}(x,y)$ satisfies $(K4)$, proven by Chen, Kim and Song (2010)

$$\mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1\right), \quad \text{with } \gamma = s - \frac{1}{2}\,.$$

Remarks.

- This is a third model of Dirichlet fractional Laplacian when $\left[a(x,y) = const\right]$. This is **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

References.

- K. Bogdan, K. Burdzy, K., Z.-Q. Chen. *Censored stable processes*. Probab. Theory Relat. Fields (2003)
- Z.-Q. Chen, P. Kim, R. Song, *Two-sided heat kernel estimates for censored stable-like processes*. Probab. Theory Relat. Fields (2010)

Examples of operators $\mathcal{L}$

Introduced in 2003 by Bogdan, Burdzy and Chen.

Censored (Regional) Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, dy, \qquad \text{with } \frac{1}{2} < s < 1,$$

where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.

The Green function $\mathbb{K}(x,y)$ satisfies $(K4)$, proven by Chen, Kim and Song (2010)

$$\mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \qquad \text{with } \gamma = s - \frac{1}{2}.$$

Remarks.

- This is a third model of Dirichlet fractional Laplacian when $\left[a(x,y) = const \right]$. This is **not equivalent** to SFL nor to RFL.
- Roughly speaking, $s \in (0, 1/2]$ corresponds to Neumann boundary conditions.

References.

- K. Bogdan, K. Burdzy, K., Z.-Q. Chen. *Censored stable processes*. Probab. Theory Relat. Fields (2003)
- Z.-Q. Chen, P. Kim, R. Song, *Two-sided heat kernel estimates for censored stable-like processes*. Probab. Theory Relat. Fields (2010)

About the kernels of spectral nonlocal operators. Most of the examples of nonlocal operators, but the SFL, admit a representation with a kernel
A natural question is: does the SFL admit such a representation?
Let A be a uniformly elliptic linear operator. Define the s^{th} power of A:

$$\mathcal{L}g(x) = A^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{tA} g(x) - g(x)\right) \frac{\mathrm{d}t}{t^{1+s}}$$

Then it admits a representation with a Kernel plus zero order term:

$$A^s g(x) = P.V. \int_{\mathbb{R}^N} \big(g(x) - g(y) \big) K(x,y)\, \mathrm{d}y + \kappa(x) g(x) .$$

where

$$K(x,y) \asymp \frac{1}{|x-y|^{N+2s}} \left(\frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1 \right) \quad \text{and} \quad \kappa(x) \asymp \frac{1}{\mathrm{dist}(x,\partial\Omega)^{2s}} .$$

References.

- R. Song and Z. Vondracek. *Potential theory of subordinate killed Brownian motion in a domain.* Probab. Theory Relat. Fields (2003)
- N. Abatangelo, *Large solutions for fractional Laplacian operators,* PhD Thesis, 2015.

About the kernels of spectral nonlocal operators. Most of the examples of nonlocal operators, but the SFL, admit a representation with a kernel
A natural question is: does the SFL admit such a representation?
Let A be a uniformly elliptic linear operator. Define the s^{th} power of A:

$$\mathcal{L}g(x) = A^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{tA} g(x) - g(x)\right) \frac{\mathrm{d}t}{t^{1+s}}$$

Then it admits a representation with a Kernel plus zero order term:

$$A^s g(x) = P.V. \int_{\mathbb{R}^N} \left(g(x) - g(y) \right) K(x,y) \, \mathrm{d}y + \kappa(x) g(x) \, .$$

where

$$K(x,y) \asymp \frac{1}{|x-y|^{N+2s}} \left(\frac{\Phi_1(x)}{|x-y|^{\gamma}} \wedge 1 \right) \left(\frac{\Phi_1(y)}{|x-y|^{\gamma}} \wedge 1 \right) \quad \text{and} \quad \kappa(x) \asymp \frac{1}{\mathrm{dist}(x, \partial\Omega)^{2s}} \, .$$

References.

- R. Song and Z. Vondracek. *Potential theory of subordinate killed Brownian motion in a domain.* Probab. Theory Relat. Fields (2003)
- N. Abatangelo, *Large solutions for fractional Laplacian operators,* PhD Thesis, 2015.

About the kernels of spectral nonlocal operators. Most of the examples of nonlocal operators, but the SFL, admit a representation with a kernel
A natural question is: does the SFL admit such a representation?
Let A be a uniformly elliptic linear operator. Define the s^{th} power of A:

$$\mathcal{L}g(x) = A^s g(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{tA}g(x) - g(x)\right) \frac{\mathrm{d}t}{t^{1+s}}$$

Then it admits a representation with a Kernel plus zero order term:

$$A^s g(x) = P.V. \int_{\mathbb{R}^N} \big(g(x) - g(y) \big) K(x,y) \, \mathrm{d}y + \kappa(x) g(x) \,.$$

where

$$K(x,y) \asymp \frac{1}{|x-y|^{N+2s}} \left(\frac{\Phi_1(x)}{|x-y|^\gamma} \wedge 1\right) \left(\frac{\Phi_1(y)}{|x-y|^\gamma} \wedge 1\right) \quad \text{and} \quad \kappa(x) \asymp \frac{1}{\mathrm{dist}(x, \partial\Omega)^{2s}} \,.$$

References.

- R. Song and Z. Vondracek. *Potential theory of subordinate killed Brownian motion in a domain.* Probab. Theory Relat. Fields (2003)
- N. Abatangelo, *Large solutions for fractional Laplacian operators,* PhD Thesis, 2015.

Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form, with uniformly elliptic bounded measurable coefficients:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j)\,, \qquad \textit{s-power of } A \textit{ is:} \qquad \mathcal{L}f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x)$$

$\mathcal{L} = A^s$ satisfies (K3) estimates with $\gamma = 1$

$$\text{(K3)} \qquad c_{0,\Omega} \phi_1(x)\, \phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\phi_1(x)}{|x-y|} \wedge 1 \right) \left(\frac{\phi_1(y)}{|x-y|} \wedge 1 \right)$$

[General class of intrinsically ultra-contractive operators, Davies and Simon JFA 1984].

Fractional operators with "rough" kernels. Integral operators of Levy-type

$$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x+y) - f(y)) \frac{K(x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y\,.$$

where K is measurable, symmetric, bounded between two positive constants, and

$$\big| K(x,y) - K(x,x) \big| \chi_{|x-y|<1} \le c|x-y|^{\sigma}\,, \qquad \text{with } 0 < s < \sigma \le 1\,,$$

for some positive $c > 0$. We can allow even more general kernels.
The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta^\gamma(y)}{|x-y|^\gamma} \wedge 1 \right)\,, \qquad \text{with } \gamma = s$$

Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form, with uniformly elliptic bounded measurable coefficients:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j) \,, \qquad \textit{s-power of } A \textit{ is:} \qquad \mathcal{L} f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x)$$

$\mathcal{L} = A^s$ satisfies (K3) estimates with $\gamma = 1$

$$\text{(K3)} \qquad c_{0,\Omega} \phi_1(x)\, \phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\phi_1(x)}{|x-y|} \wedge 1 \right) \left(\frac{\phi_1(y)}{|x-y|} \wedge 1 \right)$$

[General class of intrinsically ultra-contractive operators, Davies and Simon JFA 1984].

Fractional operators with "rough" kernels. Integral operators of Levy-type

$$\mathcal{L} f(x) = \mathrm{P.V.} \int_{\mathbb{R}^N} (f(x+y) - f(y)) \frac{K(x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,.$$

where K is measurable, symmetric, bounded between two positive constants, and

$$\big| K(x,y) - K(x,x) \big| \, \chi_{|x-y|<1} \le c|x-y|^{\sigma} \,, \qquad \text{with } 0 < s < \sigma \le 1 \,,$$

for some positive $c > 0$. We can allow even more general kernels.
The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta^{\gamma}(x)}{|x-y|^{\gamma}} \wedge 1 \right) \left(\frac{\delta^{\gamma}(y)}{|x-y|^{\gamma}} \wedge 1 \right) \,, \qquad \text{with } \gamma = s$$

Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma , \qquad \text{with } 0 < \sigma < s \le 1 ,$$

where $(\Delta_{|\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s , \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0 ,$$

where

$$A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y| \le 1}\right) \chi_{|y| \le 1} \mathrm{d}\nu(y) ,$$

the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^N} 1 \vee |x|^2 \, \mathrm{d}\nu(y) < \infty$, together with other assumptions.

Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s , \qquad \text{with } c > 0 , \text{ and } 0 < s \le 1 .$$

The Green function $\mathbb{K}(x, y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.

Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

References. The above mentioned bounds for the Green functions have been proven by Chen, Kim, Song and Vondracek (2007, 2010, 2012, 2013).

Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma\,, \qquad \text{with } 0<\sigma<s\le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0<s<1 \quad \text{and} \quad a\ge 0\,,$$

where

$$A_s f(x) = \text{P.V.}\int_{\mathbb{R}^N}\left(f(x+y)-f(y)-\nabla f(x)\cdot y\chi_{|y|\le 1}\right)\chi_{|y|\le 1}\mathrm{d}\nu(y)\,,$$

the measure ν on $\mathbb{R}^N\setminus\{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^N} 1\vee |x|^2\,\mathrm{d}\nu(y)<\infty$, together with other assumptions.

Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s}-\Delta\right)^s\,, \qquad \text{with } c>0\,, \text{ and } 0<s\le 1\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.

Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

References. The above mentioned bounds for the Green functions have been proven by Chen, Kim, Song and Vondracek (2007, 2010, 2012, 2013).

Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma \,, \qquad \text{with } 0 < \sigma < s \le 1 \,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s \,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0 \,,$$

where

$$A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y|\le 1} \right) \chi_{|y|\le 1} \mathrm{d}\nu(y) \,,$$

the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^N} 1 \vee |x|^2 \,\mathrm{d}\nu(y) < \infty$, together with other assumptions.

Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta \right)^s \,, \qquad \text{with } c > 0 \,, \text{ and } 0 < s \le 1 \,.$$

The Green function $\mathbb{K}(x, y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.

Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

References. The above mentioned bounds for the Green functions have been proven by Chen, Kim, Song and Vondracek (2007, 2010, 2012, 2013).

Sums of two Restricted Fractional Laplacians. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma , \qquad \text{with } 0 < \sigma < s \le 1 ,$$

where $(\Delta_{|\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s , \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0 ,$$

where

$$A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y| \le 1} \right) \chi_{|y| \le 1} \mathrm{d}\nu(y) ,$$

the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^N} 1 \vee |x|^2 \, \mathrm{d}\nu(y) < \infty$, together with other assumptions.

Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta \right)^s , \qquad \text{with } c > 0 , \text{ and } 0 < s \le 1 .$$

The Green function $\mathbb{K}(x, y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.

Many other interesting examples. Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

References. The above mentioned bounds for the Green functions have been proven by Chen, Kim, Song and Vondracek (2007, 2010, 2012, 2013).

Basic Theory

- **The Dual problem**
- **Existence and uniqueness**
- **First set of estimates**

For the rest of the talk we deal with the special case:

$$\boxed{F(u) = u^m := |u|^{m-1}u}$$

The "dual" formulation of the problem.
Recall the homogeneous Cauchy-Dirichlet problem:

$$(\mathrm{CDP})\qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\partial_t U = -u^m\,,$$

where

$$U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{K}(x,y)\,\mathrm{d}y\,.$$

This formulation encodes all the possible lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.

The "dual" formulation of the problem.
Recall the homogeneous Cauchy-Dirichlet problem:

$$(\mathrm{CDP})\qquad \begin{cases} \partial_t u = -\mathcal{L}\, u^m\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\partial_t U = -u^m\,,$$

where

$$U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega u(t,y)\mathbb{K}(x,y)\,\mathrm{d}y\,.$$

This formulation encodes all the possible lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.

Existence and uniqueness of weak dual solutions

Recall that $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma$ and $\|w\|_{\mathrm{L}^1_{\Phi_1}(\Omega)} = \int_\Omega w(x)\Phi_1(x)\,\mathrm{d}x$.

Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A function u is a *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** for the equation $\partial_t u + \mathcal{L}u^m = 0$ in $Q_T = (0,T) \times \Omega$ if:

- $u \in C((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))$, $u^m \in \mathrm{L}^1\left((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega)\right)$;
- The following identity holds for every $\psi/\Phi_1 \in C^1_c((0,T) : \mathrm{L}^\infty(\Omega))$:

$$\int_0^T \int_\Omega \mathcal{L}^{-1}(u)\, \frac{\partial \psi}{\partial t}\, \mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega u^m\, \psi\, \mathrm{d}x\,\mathrm{d}t = 0.$$

- $u \in C([0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))$ and $u(0,x) = u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.

Theorem. Existence and Uniqueness (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ there *exists a unique minimal weak dual solution* to the (CDP). Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$.
In this class of solutions the standard comparison result holds.

Remarks. Mild solutions (by Crandall and Pierre) are weak dual solutions.
Weak dual solutions are very weak solutions.

Existence and uniqueness of weak dual solutions

Recall that $\Phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^\gamma$ and $\|w\|_{\mathrm{L}^1_{\Phi_1}(\Omega)} = \int_\Omega w(x)\Phi_1(x)\,\mathrm{d}x$.

Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A function u is a *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** for the equation $\partial_t u + \mathcal{L}u^m = 0$ in $Q_T = (0,T)\times\Omega$ if:

- $u \in C((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))\,,\ u^m \in \mathrm{L}^1\left((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega)\right);$
- The following identity holds for every $\psi/\Phi_1 \in C^1_c((0,T) : \mathrm{L}^\infty(\Omega))$:

$$\int_0^T\int_\Omega \mathcal{L}^{-1}(u)\,\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega u^m\,\psi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

- $u \in C([0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))$ and $u(0,x) = u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.

Theorem. Existence and Uniqueness (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ there *exists a unique minimal weak dual solution* to the (CDP). Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$.
In this class of solutions the standard comparison result holds.

Remarks. Mild solutions (by Crandall and Pierre) are weak dual solutions.
Weak dual solutions are very weak solutions.

Existence and uniqueness of weak dual solutions

Recall that $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma$ and $\|w\|_{\mathrm{L}^1_{\Phi_1}(\Omega)} = \int_\Omega w(x)\Phi_1(x)\,\mathrm{d}x$.

Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A function u is a *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** for the equation $\partial_t u + \mathcal{L}u^m = 0$ in $Q_T = (0,T)\times\Omega$ if:

- $u \in C((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))$, $u^m \in \mathrm{L}^1\left((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega)\right)$;
- The following identity holds for every $\psi/\Phi_1 \in C^1_c((0,T) : \mathrm{L}^\infty(\Omega))$:

$$\int_0^T \int_\Omega \mathcal{L}^{-1}(u)\,\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega u^m\,\psi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

- $u \in C([0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))$ and $u(0,x) = u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.

Theorem. Existence and Uniqueness (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ there *exists a unique minimal weak dual solution* to the (CDP). Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$.
In this class of solutions the standard comparison result holds.

Remarks. Mild solutions (by Crandall and Pierre) are weak dual solutions.

Weak dual solutions are very weak solutions.

Existence and uniqueness of weak dual solutions

Recall that $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma$ and $\|w\|_{\mathrm{L}^1_{\Phi_1}(\Omega)} = \int_\Omega w(x)\Phi_1(x)\,\mathrm{d}x.$

Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A function u is a *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** for the equation $\partial_t u + \mathcal{L}u^m = 0$ in $Q_T = (0,T) \times \Omega$ if:

- $u \in C((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))$, $u^m \in \mathrm{L}^1\left((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega)\right)$;
- The following identity holds for every $\psi/\Phi_1 \in C^1_c((0,T) : \mathrm{L}^\infty(\Omega))$:

$$\int_0^T \int_\Omega \mathcal{L}^{-1}(u)\,\frac{\partial\psi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega u^m\,\psi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

- $u \in C([0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))$ and $u(0,x) = u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.

Theorem. Existence and Uniqueness (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ there *exists a unique minimal weak dual solution* to the (CDP). Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$.
In this class of solutions the standard comparison result holds.

Remarks. Mild solutions (by Crandall and Pierre) are weak dual solutions.
Weak dual solutions are very weak solutions.

Basic Estimates

Theorem. First Pointwise Estimates. (M.B. and J. L. Vázquez)

Let $u \ge 0$ be a nonnegative weak dual solution to Problem (CDP).
Then, for almost every $0 \le t_0 \le t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}} u^m(t_0, x_0) \le \int_\Omega \frac{u(t_0,x) - u(t_1,x)}{t_1 - t_0} \mathbb{K}(x, x_0)\, \mathrm{d}x \le \left(\frac{t_1}{t_0}\right)^{\frac{m}{m-1}} u^m(t_1, x_0) .$$

Theorem. (Absolute upper bounds) (M.B. & J. L. Vázquez)

Let u be a weak dual solution, then there exists a constant $\overline{\kappa}_0 > 0$ depending only on N, s, m, Ω (but not on u_0 !!), such that under the minimal assumption (K1):

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}_0}{t^{\frac{1}{m-1}}}, \qquad \text{for all } \; t > 0 .$$

Theorem. (Smoothing effects) (M.B. & J. L. Vázquez)

Let $\vartheta_\gamma = 1/[2s + (N+\gamma)(m-1)]$ and assume (K2). There exists $\overline{\kappa}_1 > 0$ such that:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}_1}{t^{N\vartheta_\gamma}} \|u(t)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_\gamma} \le \frac{\overline{\kappa}_1}{t^{N\vartheta_\gamma}} \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_\gamma} \qquad \text{for all } t > 0.$$

Assuming only (K1), the above bound holds with L^1 and ϑ_0, instead of $\mathrm{L}^1_{\Phi_1}$ and ϑ_γ.

More details about this first part online: http://www.fields.utoronto.ca/video-archive//event/2021/2016

Basic Estimates

Theorem. First Pointwise Estimates. (M.B. and J. L. Vázquez)

Let $u \ge 0$ be a nonnegative weak dual solution to Problem (CDP).
Then, for almost every $0 \le t_0 \le t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}} u^m(t_0,x_0) \le \int_\Omega \frac{u(t_0,x)-u(t_1,x)}{t_1-t_0}\,\mathbb{K}(x,x_0)\,\mathrm{d}x \le \left(\frac{t_1}{t_0}\right)^{\frac{m}{m-1}} u^m(t_1,x_0)\,.$$

Theorem. (Absolute upper bounds) (M.B. & J. L. Vázquez)

Let u be a weak dual solution, then there exists a constant $\overline{\kappa}_0 > 0$ depending only on N, s, m, Ω (but not on u_0 !!), such that under the minimal assumption (K1):

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}_0}{t^{\frac{1}{m-1}}}\,, \qquad \text{for all } t>0\,.$$

Theorem. (Smoothing effects) (M.B. & J. L. Vázquez)

Let $\vartheta_\gamma = 1/[2s+(N+\gamma)(m-1)]$ and assume (K2). There exists $\overline{\kappa}_1 > 0$ such that:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}_1}{t^{N\vartheta_\gamma}}\|u(t)\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \le \frac{\overline{\kappa}_1}{t^{N\vartheta_\gamma}}\|u_0\|^{2s\vartheta_\gamma}_{\mathrm{L}^1_{\Phi_1}(\Omega)} \qquad \text{for all } t>0.$$

Assuming only (K1), the above bound holds with L^1 and ϑ_0, instead of $\mathrm{L}^1_{\Phi_1}$ and ϑ_γ.

More details about this first part online: http://www.fields.utoronto.ca/video-archive//event/2021/2016

Basic Estimates

Theorem. First Pointwise Estimates. (M.B. and J. L. Vázquez)

Let $u \geq 0$ be a nonnegative weak dual solution to Problem (CDP).
Then, for almost every $0 \leq t_0 \leq t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}} u^m(t_0, x_0) \leq \int_\Omega \frac{u(t_0,x) - u(t_1,x)}{t_1 - t_0} \mathbb{K}(x, x_0)\,\mathrm{d}x \leq \left(\frac{t_1}{t_0}\right)^{\frac{m}{m-1}} u^m(t_1, x_0)\,.$$

Theorem. (Absolute upper bounds) (M.B. & J. L. Vázquez)

Let u be a weak dual solution, then there exists a constant $\overline{\kappa}_0 > 0$ depending only on N, s, m, Ω (but not on u_0 !!), such that under the minimal assumption (K1):

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{\overline{\kappa}_0}{t^{\frac{1}{m-1}}}\,, \qquad \text{for all } t > 0\,.$$

Theorem. (Smoothing effects) (M.B. & J. L. Vázquez)

Let $\vartheta_\gamma = 1/[2s + (N+\gamma)(m-1)]$ and assume (K2). There exists $\overline{\kappa}_1 > 0$ such that:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{\overline{\kappa}_1}{t^{N\vartheta_\gamma}} \|u(t)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_\gamma} \leq \frac{\overline{\kappa}_1}{t^{N\vartheta_\gamma}} \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_\gamma} \qquad \text{for all } t > 0.$$

Assuming only (K1), the above bound holds with L^1 and ϑ_0, instead of $\mathrm{L}^1_{\Phi_1}$ and ϑ_γ.

More details about this first part online: http://www.fields.utoronto.ca/video-archive//event/2021/2016

Basic Estimates

Theorem. First Pointwise Estimates. (M.B. and J. L. Vázquez)

Let $u \ge 0$ be a nonnegative weak dual solution to Problem (CDP).
Then, for almost every $0 \le t_0 \le t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}} u^m(t_0,x_0) \le \int_\Omega \frac{u(t_0,x)-u(t_1,x)}{t_1-t_0}\,\mathbb{K}(x,x_0)\,\mathrm{d}x \le \left(\frac{t_1}{t_0}\right)^{\frac{m}{m-1}} u^m(t_1,x_0)\,.$$

Theorem. (Absolute upper bounds) (M.B. & J. L. Vázquez)

Let u be a weak dual solution, then there exists a constant $\overline{\kappa}_0 > 0$ depending only on N, s, m, Ω (but not on u_0 !!), such that under the minimal assumption (K1):

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}_0}{t^{\frac{1}{m-1}}}\,, \qquad \text{for all } t > 0\,.$$

Theorem. (Smoothing effects) (M.B. & J. L. Vázquez)

Let $\vartheta_\gamma = 1/[2s + (N+\gamma)(m-1)]$ and assume (K2). There exists $\overline{\kappa}_1 > 0$ such that:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{\overline{\kappa}_1}{t^{N\vartheta_\gamma}} \|u(t)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_\gamma} \le \frac{\overline{\kappa}_1}{t^{N\vartheta_\gamma}} \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_\gamma} \qquad \text{for all } t > 0.$$

Assuming only (K1), the above bound holds with L^1 and ϑ_0, instead of $\mathrm{L}^1_{\Phi_1}$ and ϑ_γ.

More details about this first part online: http://www.fields.utoronto.ca/video-archive//event/2021/2016

Sharp Boundary Behaviour

- Upper Boundary Estimates
- Infinite Speed of Propagation
- Lower Boundary Estimates

Sharp Upper boundary estimates

Theorem. (Upper boundary behaviour) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold. Let $u \ge 0$ be a weak dual solution to the (CDP). Let $\sigma \in (0,1]$ be

$$\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$$

Then, there exists a computable constant $\overline{\kappa} > 0$, depending only on N, s, m, and Ω, (but not on u_0 !!) such that for all $t \ge 0$ and all $x \in \Omega$:

$$u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x)^{\frac{\sigma}{m}}}{t^{\frac{1}{m-1}}} \lesssim \frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}$$

- **When $\sigma = 1$ we have sharp boundary estimates**: we will show lower bounds with matching powers.
- **When $\sigma < 1$ the estimates are not sharp in all cases**:
  - The solution by separation of variables $\mathcal{U}(t,x) = S(x)t^{-1/(m-1)}$ (asymptotic behaviour) behaves like $\Phi_1^{\sigma/m} t^{-1/(m-1)}$.
  - We will show that for small data, the boundary behaviour is different.
  - In examples, $\sigma < 1$ only happens for SFL-type, where $\gamma = 1$, and s can be small, $0 < s < 1/2 - 1/(2m)$.

Theorem. (Upper boundary behaviour) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold. Let $u \ge 0$ be a weak dual solution to the (CDP). Let $\sigma \in (0,1]$ be

$$\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$$

Then, there exists a computable constant $\overline{\kappa} > 0$, depending only on N, s, m, and Ω, (but not on u_0 !!) such that for all $t \ge 0$ and all $x \in \Omega$:

$$u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x)^{\frac{\sigma}{m}}}{t^{\frac{1}{m-1}}} \lesssim \frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}$$

- **When $\sigma = 1$ we have sharp boundary estimates**: we will show lower bounds with matching powers.
- **When $\sigma < 1$ the estimates are not sharp in all cases**:
  - The solution by separation of variables $\mathcal{U}(t,x) = S(x)t^{-1/(m-1)}$ (asymptotic behaviour) behaves like $\Phi_1^{\sigma/m} t^{-1/(m-1)}$.
  - We will show that for small data, the boundary behaviour is different.
  - In examples, $\sigma < 1$ only happens for SFL-type, where $\gamma = 1$, and s can be small, $0 < s < 1/2 - 1/(2m)$.

Theorem. (Upper boundary behaviour) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold. Let $u \ge 0$ be a weak dual solution to the (CDP). Let $\sigma \in (0,1]$ be

$$\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$$

Then, there exists a computable constant $\overline{\kappa} > 0$, depending only on N, s, m, and Ω, (but not on u_0 !!) such that for all $t \ge 0$ and all $x \in \Omega$:

$$u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x)^{\frac{\sigma}{m}}}{t^{\frac{1}{m-1}}} \lesssim \frac{\mathrm{dist}(x,\partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}$$

- **When $\sigma = 1$ we have sharp boundary estimates**: we will show lower bounds with matching powers.
- **When $\sigma < 1$ the estimates are not sharp in all cases**:
  - The solution by separation of variables $\mathcal{U}(t,x) = S(x)t^{-1/(m-1)}$ (asymptotic behaviour) behaves like $\Phi_1^{\sigma/m} t^{-1/(m-1)}$.
  - We will show that for small data, the boundary behaviour is different.
  - In examples, $\sigma < 1$ only happens for SFL-type, where $\gamma = 1$, and s can be small, $0 < s < 1/2 - 1/(2m)$.

Theorem. (Upper boundary behaviour) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold. Let $u \ge 0$ be a weak dual solution to the (CDP). Let $\sigma \in (0,1]$ be

$$\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$$

Then, there exists a computable constant $\overline{\kappa} > 0$, depending only on N, s, m, and Ω, (but not on u_0 !!) such that for all $t \ge 0$ and all $x \in \Omega$:

$$u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x)^{\frac{\sigma}{m}}}{t^{\frac{1}{m-1}}} \lesssim \frac{\operatorname{dist}(x,\partial\Omega)^{\frac{\sigma\gamma}{m}}}{t^{\frac{1}{m-1}}}$$

- **When $\sigma = 1$ we have sharp boundary estimates**: we will show lower bounds with matching powers.
- **When $\sigma < 1$ the estimates are not sharp in all cases**:
  - The solution by separation of variables $\mathcal{U}(t,x) = S(x)t^{-1/(m-1)}$ (asymptotic behaviour) behaves like $\Phi_1^{\sigma/m} t^{-1/(m-1)}$.
  - We will show that for small data, the boundary behaviour is different.
  - In examples, $\sigma < 1$ only happens for SFL-type, where $\gamma = 1$, and s can be small, $0 < s < 1/2 - 1/(2m)$.

Infinite Speed of Propagation

and

Universal Lower Bounds

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1) and (A2), and assume that

$$\mathcal{L}w(x) \ge P.V.\int_{\mathbb{R}^N} \big(w(x)-w(y)\big)K(x,y)\,\mathrm{d}y\,, \quad \text{with } K(x,y) \ge c_0\Phi_1(x)\Phi_1(y) \quad \forall\, x,y\in\Omega\,.$$

Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
Then there exists a constant $\underline{\kappa}_0 > 0$, so that the following inequality holds:

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

Here $t_* = \kappa_*\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

• Note that, for $t \ge t_*$, the dependence on the initial data disappears

$$u(t) \ge \underline{\kappa}_0\Phi_1 t^{-\frac{1}{m-1}} \qquad \forall\, t \ge t_*\,.$$

• The assumption on the kernel K of $\mathcal{L}$ holds for all examples and represent somehow the "worst case scenario" for lower estimates.
• In many cases (RFL, CFL), K satisfies a stronger property: $K \ge \underline{\kappa}_\Omega > 0$ in $\overline{\Omega}\times\overline{\Omega}$.

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1) and (A2), and assume that

$$\mathcal{L}w(x) \ge P.V. \int_{\mathbb{R}^N} \big(w(x) - w(y)\big) K(x,y)\,\mathrm{d}y\,, \quad \text{with } K(x,y) \ge c_0 \Phi_1(x)\Phi_1(y) \ \ \forall x,y \in \Omega\,.$$

Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
Then there exists a constant $\underline{\kappa}_0 > 0$, so that the following inequality holds:

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

• Note that, for $t \ge t_*$, the dependence on the initial data disappears

$$u(t) \ge \underline{\kappa}_0 \Phi_1 t^{-\frac{1}{m-1}} \qquad \forall\, t \ge t_*\,.$$

• The assumption on the kernel K of $\mathcal{L}$ holds for all examples and represent somehow the "worst case scenario" for lower estimates.

• In many cases (RFL, CFL), K satisfies a stronger property: $K \ge \underline{\kappa}_\Omega > 0$ in $\overline{\Omega} \times \overline{\Omega}$.

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1) and (A2), and assume that

$$\mathcal{L}w(x) \ge P.V. \int_{\mathbb{R}^N} \big(w(x) - w(y)\big) K(x,y)\,\mathrm{d}y\,, \quad \text{with } K(x,y) \ge c_0 \Phi_1(x)\Phi_1(y) \;\; \forall\, x,y \in \Omega\,.$$

Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
Then there exists a constant $\underline{\kappa}_0 > 0$, so that the following inequality holds:

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

• Note that, for $t \ge t_*$, the dependence on the initial data disappears

$$u(t) \ge \underline{\kappa}_0 \Phi_1 t^{-\frac{1}{m-1}} \qquad \forall\, t \ge t_*\,.$$

• The assumption on the kernel K of $\mathcal{L}$ holds for all examples and represent somehow the "worst case scenario" for lower estimates.
• In many cases (RFL, CFL), K satisfies a stronger property: $K \ge \underline{\kappa}_\Omega > 0$ in $\overline{\Omega} \times \overline{\Omega}$.

Theorem. (Universal lower bounds) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1) and (A2), and assume that

$$\mathcal{L}w(x) \ge P.V. \int_{\mathbb{R}^N} \big(w(x) - w(y)\big) K(x,y)\,\mathrm{d}y\,, \quad \text{with } K(x,y) \ge c_0 \Phi_1(x)\Phi_1(y) \quad \forall\, x,y \in \Omega\,.$$

Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
Then there exists a constant $\underline{\kappa}_0 > 0$, so that the following inequality holds:

$$\boxed{u(t,x) \ge \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

Here $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$ and $\underline{\kappa}_0, \kappa_*$ depend only on N, s, γ, m, c_0, and Ω.

• Note that, for $t \ge t_*$, the dependence on the initial data disappears

$$u(t) \ge \underline{\kappa}_0 \Phi_1 t^{-\frac{1}{m-1}} \qquad \forall\, t \ge t_*\,.$$

• The assumption on the kernel K of $\mathcal{L}$ holds for all examples and represent somehow the “worst case scenario” for lower estimates.

• In many cases (RFL, CFL), K satisfies a stronger property: $K \ge \underline{\kappa}_\Omega > 0$ in $\overline{\Omega} \times \overline{\Omega}$.

Infinite speed of propagation.

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s < 1$, contrary to the "local" case $s = 1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015]

Infinite speed of propagation.

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s < 1$, contrary to the "local" case $s = 1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015]

Infinite speed of propagation.

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s < 1$, contrary to the "local" case $s = 1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015]

Infinite speed of propagation.

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

- As a consequence, of the above universal bounds for all times, we have proven that all nonnegative solutions have **infinite speed of propagation**.
- No free boundaries when $s < 1$, contrary to the "local" case $s = 1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$, by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015]

Sharp Lower boundary estimates

Sharp lower boundary estimates I: the non-spectral case.

Let $\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$. Let $\mathcal{L}$ satisfy (A1) and (A2), and assume moreover that

$$\mathcal{L}f(x) = \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\,\mathrm{d}y\,, \qquad \text{with } \inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,.$$

Assume moreover that $\mathcal{L}$ has a first eigenfunction $\Phi_1 \asymp \operatorname{dist}(x,\partial\Omega)^\gamma$ and that

- either $\sigma = 1$;
- or $\sigma < 1$, $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$, and $\Phi_1 \in C^\gamma(\overline{\Omega})$.

Theorem. (Sharp lower bounds for all times) (M.B., A. Figalli and J. L. Vázquez)

Under the above assumptions, let $u \ge 0$ be a weak dual solution to the (CDP) with $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_1 > 0$ such that

$$\boxed{u(t,x_0) \ge \underline{\kappa}_1 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and a.e. } x \in \Omega\,.$$

where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$. The constants $\kappa_*, \underline{\kappa}_1$ depend only on $N, s, \gamma, m, \underline{\kappa}_\Omega, c_1, \Omega$.

- The *boundary behavior is sharp for all times* in view of the upper bounds.
- Within examples, this applies to RFL and CFL type, but not to SFL-type.
- For RFL, this result was obtained first by MB, A. Figalli and X. Ros-Oton.

Sharp lower boundary estimates I: the non-spectral case.

Let $\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$. Let $\mathcal{L}$ satisfy (A1) and (A2), and assume moreover that

$$\mathcal{L}f(x) = \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\,\mathrm{d}y\,, \qquad \text{with } \inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,.$$

Assume moreover that $\mathcal{L}$ has a first eigenfunction $\Phi_1 \asymp \mathrm{dist}(x,\partial\Omega)^\gamma$ and that

- either $\sigma = 1$;
- or $\sigma < 1$, $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x,y \in \mathbb{R}^N$, and $\Phi_1 \in C^\gamma(\overline{\Omega})$.

Theorem. (Sharp lower bounds for all times) (M.B., A. Figalli and J. L. Vázquez)

Under the above assumptions, let $u \ge 0$ be a weak dual solution to the (CDP) with $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_1 > 0$ such that

$$\boxed{u(t,x_0) \ge \underline{\kappa}_1 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and a.e. } x \in \Omega\,.$$

where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$. The constants $\kappa_*, \underline{\kappa}_1$ depend only on $N, s, \gamma, m, \underline{\kappa}_\Omega, c_1, \Omega$.

- The *boundary behavior is sharp for all times* in view of the upper bounds.
- Within examples, this applies to RFL and CFL type, but not to SFL-type.
- For RFL, this result was obtained first by MB, A. Figalli and X. Ros-Oton.

Sharp lower boundary estimates I: the non-spectral case.

Let $\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$. Let $\mathcal{L}$ satisfy (A1) and (A2), and assume moreover that

$$\mathcal{L}f(x) = \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\,\mathrm{d}y\,, \qquad \text{with } \inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,.$$

Assume moreover that $\mathcal{L}$ has a first eigenfunction $\Phi_1 \asymp \mathrm{dist}(x,\partial\Omega)^\gamma$ and that

- either $\sigma = 1$;
- or $\sigma < 1$, $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x,y \in \mathbb{R}^N$, and $\Phi_1 \in C^\gamma(\overline{\Omega})$.

Theorem. (Sharp lower bounds for all times) (M.B., A. Figalli and J. L. Vázquez)

Under the above assumptions, let $u \ge 0$ be a weak dual solution to the (CDP) with $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_1 > 0$ such that

$$\boxed{u(t,x_0) \ge \underline{\kappa}_1 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and a.e. } x\in\Omega\,.$$

where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$. The constants $\kappa_*, \underline{\kappa}_1$ depend only on $N, s, \gamma, m, \underline{\kappa}_\Omega, c_1, \Omega$.

- The *boundary behavior is sharp for all times* in view of the upper bounds.
- Within examples, this applies to RFL and CFL type, but not to SFL-type.
- For RFL, this result was obtained first by MB, A. Figalli and X. Ros-Oton.

Sharp lower boundary estimates I: the non-spectral case.

Let $\sigma = \frac{2sm}{\gamma(m-1)} \wedge 1$. Let $\mathcal{L}$ satisfy (A1) and (A2), and assume moreover that

$$\mathcal{L}f(x) = \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\, dy\,, \qquad \text{with } \inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0\,.$$

Assume moreover that $\mathcal{L}$ has a first eigenfunction $\Phi_1 \asymp \mathrm{dist}(x,\partial\Omega)^\gamma$ and that

- either $\sigma = 1$;
- or $\sigma < 1$, $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x,y \in \mathbb{R}^N$, and $\Phi_1 \in C^\gamma(\overline{\Omega})$.

Theorem. (Sharp lower bounds for all times) (M.B., A. Figalli and J. L. Vázquez)

Under the above assumptions, let $u \ge 0$ be a weak dual solution to the (CDP) with $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Then there exists a constant $\underline{\kappa}_1 > 0$ such that

$$\boxed{u(t,x_0) \ge \underline{\kappa}_1 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t>0 \text{ and a.e. } x \in \Omega\,.$$

where $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$. The constants $\kappa_*, \underline{\kappa}_1$ depend only on $N, s, \gamma, m, \underline{\kappa}_\Omega, c_1, \Omega$.

- The *boundary behavior is sharp for all times* in view of the upper bounds.
- Within examples, this applies to RFL and CFL type, but not to SFL-type.
- For RFL, this result was obtained first by MB, A. Figalli and X. Ros-Oton.

Sharp absolute lower estimates for large times: the case $\sigma = 1$.
When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

Theorem. (Sharp lower bounds for large times) (M.B., A. Figalli and J. L. Vázquez)

Let $(A1)$, $(A2)$, and $(K2)$ hold, and let $\sigma = 1$. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. There exists a constant $\underline{\kappa}_2 > 0$ such that

$$u(t, x_0) \geq \underline{\kappa}_2 \, \frac{\Phi_1(x_0)^{1/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \geq t_* \text{ and a.e. } x \in \Omega\,.$$

Here, $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and the constants $\kappa_*, \underline{\kappa}_2$ depend only on N, s, γ, m, and Ω.

- It holds for $s = 1$, the local case, where there is finite speed of propagation.
- When $s = 1$. t_* is the time that the solution needs to be positive everywhere.
- When $\mathcal{L} = -\Delta$, proven by Aronson-Peletier ('81) and Vázquez ('04)
- Our method applies when $\mathcal{L}$ is an elliptic operator with C^1 coefficients (new result).

Sharp absolute lower estimates for large times: the case $\sigma = 1$.
When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

Theorem. (Sharp lower bounds for large times) (M.B., A. Figalli and J. L. Vázquez)

Let $(A1)$, $(A2)$, and $(K2)$ hold, and let $\sigma = 1$. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. There exists a constant $\underline{\kappa}_2 > 0$ such that

$$u(t, x_0) \geq \underline{\kappa}_2 \frac{\Phi_1(x_0)^{1/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \geq t_* \text{ and a.e. } x \in \Omega\,.$$

Here, $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and the constants κ_*, $\underline{\kappa}_2$ depend only on N, s, γ, m, and Ω.

- It holds for $s = 1$, the local case, where there is finite speed of propagation.
- When $s = 1$. t_* is the time that the solution needs to be positive everywhere.
- When $\mathcal{L} = -\Delta$, proven by Aronson-Peletier ('81) and Vázquez ('04)
- Our method applies when $\mathcal{L}$ is an elliptic operator with C^1 coefficients (new result).

Sharp absolute lower estimates for large times: the case $\sigma = 1$.
When $\sigma = 1$ we can establish a quantitative lower bound near the boundary that matches the separate-variables behavior for large times.

Theorem. (Sharp lower bounds for large times) (M.B., A. Figalli and J. L. Vázquez)

Let $(A1)$, $(A2)$, and $(K2)$ hold, and let $\sigma = 1$. Let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. There exists a constant $\underline{\kappa}_2 > 0$ such that

$$u(t, x_0) \geq \underline{\kappa}_2 \frac{\Phi_1(x_0)^{1/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \geq t_* \text{ and a.e. } x \in \Omega\,.$$

Here, $t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, and the constants κ_*, $\underline{\kappa}_2$ depend only on N, s, γ, m, and Ω.

- It holds for $s = 1$, the local case, where there is finite speed of propagation.
- When $s = 1$. t_* is the time that the solution needs to be positive everywhere.
- When $\mathcal{L} = -\Delta$, proven by Aronson-Peletier ('81) and Vázquez ('04)
- Our method applies when $\mathcal{L}$ is an elliptic operator with C^1 coefficients (new result).

Positivity for large times II: the case $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{\gamma}{2} - \frac{\gamma}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared 'big' solutions:

$$u_0 \ge \epsilon_0 S \qquad \text{implies} \qquad u(t) \ge \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{\mathrm{L}^\infty} = 0 \qquad \text{for all } 0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)\,.$$

But this is not happening for all solutions...

Positivity for large times II: the case $\sigma < 1$.

The intriguing case $\sigma < 1$ is where new and unexpected phenomena appear. Recall that

$$\sigma = \frac{2sm}{\gamma(m-1)} < 1 \qquad \text{i.e.} \qquad 0 < s < \frac{\gamma}{2} - \frac{\gamma}{2m}\,.$$

Solutions by separation of variables: the standard boundary behaviour?

Let S be a solution to the Elliptic Dirichlet problem for $\mathcal{L}S^m = c_m S$. We can define

$$\mathcal{U}(t,x) = S(x)t^{-\frac{1}{m-1}} \qquad \text{where} \qquad S \asymp \Phi_1^{\sigma/m}\,.$$

which is a solution to the (CDP), which behaves like $\Phi_1^{\sigma/m}$ at the boundary.

By comparison, we see that the same lower behaviour is shared ‘big’ solutions:

$$u_0 \ge \epsilon_0 S \qquad \text{implies} \qquad u(t) \ge \frac{S}{\left(\epsilon_0^{1-m} + t\right)^{1/(m-1)}}$$

This behaviour seems to be sharp: we have shown matching upper bounds, and also S represents the large time asymptotic behaviour:

$$\lim_{t\to\infty} \left\| t^{\frac{1}{m-1}} u(t) - S \right\|_{L^\infty} = 0 \qquad \text{for all } 0 \le u_0 \in L^1_{\Phi_1}(\Omega)\,.$$

But this is not happening for all solutions...

Different boundary behaviour when $\sigma < 1$. The next result shows that, in general, we cannot hope to prove that $u(t)$ is larger than $\Phi_1^{1/m}$, but always smaller than $\Phi_1^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \leq u_0 \leq c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \leq c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

But again, this is not happening for all solutions...

Different boundary behaviour when $\sigma < 1$. The next result shows that, in general, we cannot hope to prove that $u(t)$ is larger than $\Phi_1^{1/m}$, but always smaller than $\Phi_1^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and $u \geq 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \leq u_0 \leq c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \leq c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

But again, this is not happening for all solutions...

Different boundary behaviour when $\sigma < 1$. The next result shows that, in general, we cannot hope to prove that $u(t)$ is larger than $\Phi_1^{1/m}$, but always smaller than $\Phi_1^{\sigma/m}$.

Proposition. (Counterexample I) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and $u \ge 0$ be a weak dual solution to the (CDP). Then, there exists a constant $\hat{\kappa}$, depending only N, s, γ, m, and Ω, such that

$$0 \le u_0 \le c_0 \Phi_1 \quad \text{implies} \quad \boxed{u(t,x) \le c_0 \hat{\kappa} \frac{\Phi_1^{1/m}(x)}{t^{1/m}}} \quad \forall t > 0 \text{ and a.e. } x \in \Omega\,.$$

In particular, if $\sigma < 1$, then

$$\lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\sigma/m}} = 0 \qquad \text{for any } t > 0.$$

Idea: The proposition above could make one wonder whether or not the sharp general lower bound could be actually given by $\Phi_1^{1/m}$, as in the case $\sigma = 1$.

But again, this is not happening for all solutions...

Different boundary behaviour when $\sigma < 1$.

We next show that assuming (K4), the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa} \Phi_1^\alpha(x) \qquad \text{for a.e. } x \in \Omega \,, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma} \,.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Recall that we have a universal lower bound (under minimal assumptions on K)

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega \,.$$

Different boundary behaviour when $\sigma < 1$.

We next show that assuming (K4), the bound $u(t) \gtrsim \Phi_1^{1/m} t^{-1/(m-1)}$ is false for $\sigma < 1$.

Proposition. (Counterexample II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K4) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \leq c_0 \Phi_1$ for some $c_0 > 0$.
If there exist constants $\underline{\kappa}, T, \alpha > 0$ such that

$$u(T,x) \geq \underline{\kappa} \Phi_1^{\alpha}(x) \qquad \text{for a.e. } x \in \Omega, \quad \boxed{\text{then } \alpha \geq 1 - \frac{2s}{\gamma}.}$$

In particular, when $\sigma < 1$, we have $\alpha > \frac{1}{m} > \frac{\sigma}{m}$.

Recall that we have a universal lower bound (under minimal assumptions on K)

$$\boxed{u(t,x) \geq \underline{\kappa}_0 \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega.$$

Harnack Inequalities

- **Global Harnack Principle I. The non-spectral case.**
- **Other Harnack inequalities in the non-spectral case.**
- **Global Harnack Principle II. The remaining cases.**

Global Harnack Principle I. The non-spectral case.

Recall that

$$\Phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^\gamma\,, \qquad \sigma = 1\wedge\frac{2sm}{\gamma(m-1)}\,, \qquad t_* = \kappa_*\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}\,.$$

Theorem. (Global Harnack Principle I. The non-spectral case.)(MB & AF & JLV)

Let (A1), (A2), (K2), and $\inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0$ hold. Also, when $\sigma < 1$, assume that $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x,y\in\mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$.
Let $u \ge 0$ be a weak dual solution to the (CDP).
Then, there exist constants $\underline{\kappa},\overline{\kappa} > 0$, so that the following inequality holds:

$$\underline{\kappa}\left(1\wedge\frac{t}{t_*}\right)^{\frac{m}{m-1}}\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

The constants $\underline{\kappa},\overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$, and $\|\Phi_1\|_{C^\gamma(\Omega)}$.

- For large times $t \ge t_*$ the estimates are independent on the initial datum.
- This inequality implies local Harnack inequalities
- As a corollary we get the sharp asymptotic behaviour
- For $s=1$, $\mathcal{L}=-\Delta$, similar results by Aronson and Peletier [JDE, 1981] , Vázquez [Monatsh. Math. 2004]

Global Harnack Principle I. The non-spectral case.

Recall that

$$\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma\,, \qquad \sigma = 1 \wedge \frac{2sm}{\gamma(m-1)}, \qquad t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}.$$

Theorem. (Global Harnack Principle I. The non-spectral case.)(MB & AF & JLV)

Let (A1), (A2), (K2), and $\inf_{x,y\in\Omega} K(x,y) \geq \underline{\kappa}_\Omega > 0$ hold. Also, when $\sigma < 1$, assume that $K(x,y) \leq c_1|x-y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$.
Let $u \geq 0$ be a weak dual solution to the (CDP).
Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds:

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\, \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$, and $\|\Phi_1\|_{C^\gamma(\Omega)}$.

- For large times $t \geq t_*$ the estimates are independent on the initial datum.
- This inequality implies local Harnack inequalities
- As a corollary we get the sharp asymptotic behaviour
- For $s = 1$, $\mathcal{L} = -\Delta$, similar results by Aronson and Peletier [JDE, 1981], Vázquez [Monatsh. Math. 2004]

Global Harnack Principle I. The non-spectral case.

Recall that

$$\Phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^\gamma\,, \qquad \sigma = 1\wedge\frac{2sm}{\gamma(m-1)}, \qquad t_* = \kappa_*\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}.$$

Theorem. (Global Harnack Principle I. The non-spectral case.)(MB & AF & JLV)

Let (A1), (A2), (K2), and $\inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0$ hold. Also, when $\sigma < 1$, assume that $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x,y\in\mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$.
Let $u \ge 0$ be a weak dual solution to the (CDP).
Then, there exist constants $\underline{\kappa},\overline{\kappa} > 0$, so that the following inequality holds:

$$\underline{\kappa}\left(1\wedge\frac{t}{t_*}\right)^{\frac{m}{m-1}}\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

The constants $\underline{\kappa},\overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$, and $\|\Phi_1\|_{C^\gamma(\Omega)}$.

- For large times $t \ge t_*$ the estimates are independent on the initial datum.
- This inequality implies local Harnack inequalities
- As a corollary we get the sharp asymptotic behaviour
- For $s=1$, $\mathcal{L}=-\Delta$, similar results by Aronson and Peletier [JDE, 1981] , Vázquez [Monatsh. Math. 2004]

Global Harnack Principle I. The non-spectral case.

Recall that

$$\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma\,, \qquad \sigma = 1 \wedge \frac{2sm}{\gamma(m-1)}, \qquad t_* = \kappa_* \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}.$$

Theorem. (Global Harnack Principle I. The non-spectral case.)(MB & AF & JLV)

Let (A1), (A2), (K2), and $\inf_{x,y\in\Omega} K(x,y) \ge \underline{\kappa}_\Omega > 0$ hold. Also, when $\sigma < 1$, assume that $K(x,y) \le c_1|x-y|^{-(N+2s)}$ for a.e. $x, y \in \mathbb{R}^N$ and that $\Phi_1 \in C^\gamma(\Omega)$.
Let $u \ge 0$ be a weak dual solution to the (CDP).
Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds:

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t>0 \text{ and all } x \in \Omega\,.$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, c_1, \underline{\kappa}_\Omega, \Omega$, and $\|\Phi_1\|_{C^\gamma(\Omega)}$.

- For large times $t \ge t_*$ the estimates are independent on the initial datum.
- This inequality implies local Harnack inequalities
- As a corollary we get the sharp asymptotic behaviour
- For $s = 1$, $\mathcal{L} = -\Delta$, similar results by Aronson and Peletier [JDE, 1981] , Vázquez [Monatsh. Math. 2004]

Other Harnack inequalities in the non-spectral case.

From the Global Harnack Principle I (GHP-I) we derive local Harnack inequalities.

Theorem. (Local Harnack Inequalities of Elliptic Type) (MB & AF & JLV)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, such that

$$\sup_{x\in B_R(x_0)} u(t,x) \le \frac{\hat{H}}{\left(1\wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}}} \inf_{x\in B_R(x_0)} u(t,x) \qquad \text{for all } t>0.$$

Corollary. (Local Harnack Inequalities of Backward Type) (M.B. & A. F. & J.L.V)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, s. t. for all $t>0$ and $h\ge 0$

$$\sup_{x\in B_R(x_0)} u(t,x) \le \hat{H}\left[\left(1+\frac{h}{t}\right)\left(1\wedge\frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x\in B_R(x_0)} u(t+h,x)\,.$$

When $s=1$, backward Harnack inequalities are typical of Fast Diffusion equations (when $m<1$ there is possible extinction in finite time), and they do not happen when $m>1$ (finite speed of propagation), cf. DiBenedetto, Gianazza, Vespri and/or M.B.& J. L. Vázquez.

Other Harnack inequalities in the non-spectral case.

From the Global Harnack Principle I (GHP-I) we derive local Harnack inequalities.

Theorem. (Local Harnack Inequalities of Elliptic Type) (MB & AF & JLV)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, such that

$$\sup_{x\in B_R(x_0)} u(t,x) \le \frac{\hat{H}}{\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}}} \inf_{x\in B_R(x_0)} u(t,x) \qquad \text{for all } t > 0.$$

Corollary. (Local Harnack Inequalities of Backward Type) (M.B. & A. F. & J.L.V)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, s. t. for all $t > 0$ and $h \ge 0$

$$\sup_{x\in B_R(x_0)} u(t,x) \le \hat{H} \left[\left(1+\frac{h}{t}\right)\left(1 \wedge \frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x\in B_R(x_0)} u(t+h,x)\,.$$

When $s = 1$, backward Harnack inequalities are typical of Fast Diffusion equations (when $m < 1$ there is possible extinction in finite time), and they do not happen when $m > 1$ (finite speed of propagation), cf. DiBenedetto, Gianazza, Vespri and/or M.B.& J. L. Vázquez.

Other Harnack inequalities in the non-spectral case.

From the Global Harnack Principle I (GHP-I) we derive local Harnack inequalities.

Theorem. (Local Harnack Inequalities of Elliptic Type) (MB & AF & JLV)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, such that

$$\sup_{x\in B_R(x_0)} u(t,x) \le \frac{\hat{H}}{\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}}} \inf_{x\in B_R(x_0)} u(t,x) \qquad \text{for all } t>0.$$

Corollary. (Local Harnack Inequalities of Backward Type) (M.B. & A. F. & J.L.V)

Assume that the (GHP-I) holds for a weak dual solution u to the (CDP). Then there exists a constant $\hat{H}$ depending only on $N, s, \gamma, m, c_1, \Omega$, s. t. for all $t>0$ and $h \ge 0$

$$\sup_{x\in B_R(x_0)} u(t,x) \le \hat{H}\left[\left(1+\frac{h}{t}\right)\left(1 \wedge \frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x\in B_R(x_0)} u(t+h,x)\,.$$

When $s=1$, backward Harnack inequalities are typical of Fast Diffusion equations (when $m<1$ there is possible extinction in finite time), and they do not happen when $m>1$ (finite speed of propagation), cf. DiBenedetto, Gianazza, Vespri and/or M.B.& J. L. Vázquez.

Global Harnack Principles II. The remaining cases.

Theorem. (Global Harnack Principle II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and let $u \geq 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Assume that:
- either $\sigma = 1$;
- or $\sigma < 1$, $u_0 \geq \underline{\kappa}_0 \Phi_1^{\sigma/m}$ for some $\underline{\kappa}_0 > 0$, and (K4) holds.

Then there exist constants $\underline{\kappa}, \overline{\kappa} > 0$ such that the following inequality holds:

$$\underline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq \overline{\kappa}\,\frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \geq t_* \text{ and all } x \in \Omega\,.$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

Corollary. Elliptic/backward local Harnack inequalities follow for large times, for all $t \geq t_*$

$$\sup_{x \in B_R(x_0)} u(t,x) \leq \hat{H}\left[\left(1+\frac{h}{t}\right)\left(1 \wedge \frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x \in B_R(x_0)} u(t+h,x)\,.$$

- For small times we can not find matching powers for a global Harnack inequality (except for special data) and such result is *actually false* for $s = 1$ (finite speed of propagation).
- Backward Harnack inequalities for the linear heat equation $s = 1$ and $m = 1$, by Fabes, Garofalo, Salsa [Ill. J. Math, 1986] and also Safonov, Yuan [Ann. of Math, 1999]
- For $s = 1$, Intrinsic (Forward) Harnack inequalities by DiBenedetto [ARMA, 1988], Daskalopoulos and Kenig [EMS Book, 2007], cf. also DiBenedetto, Gianazza, Vespri [LNM, 2011].

Global Harnack Principles II. The remaining cases.

Theorem. (Global Harnack Principle II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Assume that:
- either $\sigma = 1$;
- or $\sigma < 1$, $u_0 \ge \underline{\kappa}_0 \Phi_1^{\sigma/m}$ for some $\underline{\kappa}_0 > 0$, and (K4) holds.

Then there exist constants $\underline{\kappa}, \overline{\kappa} > 0$ such that the following inequality holds:

$$\underline{\kappa}\, \frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \ge t_* \text{ and all } x \in \Omega\,.$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

Corollary. Elliptic/backward local Harnack inequalities follow for large times, for all $t \ge t_*$

$$\sup_{x\in B_R(x_0)} u(t,x) \le \hat{H}\left[\left(1+\frac{h}{t}\right)\left(1 \wedge \frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x\in B_R(x_0)} u(t+h,x)\,.$$

- For small times we can not find matching powers for a global Harnack inequality (except for special data) and such result is *actually false* for $s = 1$ (finite speed of propagation).
- Backward Harnack inequalities for the linear heat equation $s = 1$ and $m = 1$, by Fabes, Garofalo, Salsa [Ill. J. Math, 1986] and also Safonov, Yuan [Ann. of Math, 1999]
- For $s = 1$, Intrinsic (Forward) Harnack inequalities by DiBenedetto [ARMA, 1988], Daskalopoulos and Kenig [EMS Book, 2007], cf. also DiBenedetto, Gianazza, Vespri [LNM, 2011].

Global Harnack Principles II. The remaining cases.

Theorem. (Global Harnack Principle II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Assume that:
- either $\sigma = 1$;
- or $\sigma < 1$, $u_0 \ge \underline{\kappa}_0 \Phi_1^{\sigma/m}$ for some $\underline{\kappa}_0 > 0$, and (K4) holds.

Then there exist constants $\underline{\kappa}, \overline{\kappa} > 0$ such that the following inequality holds:

$$\underline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \ge t_* \text{ and all } x \in \Omega\,.$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

Corollary. Elliptic/backward local Harnack inequalities follow for large times, for all $t \ge t_*$

$$\sup_{x \in B_R(x_0)} u(t,x) \le \hat{H}\left[\left(1 + \frac{h}{t}\right)\left(1 \wedge \frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x \in B_R(x_0)} u(t+h,x)\,.$$

• For small times we can not find matching powers for a global Harnack inequality (except for special data) and such result is *actually false* for $s = 1$ (finite speed of propagation).

• Backward Harnack inequalities for the linear heat equation $s = 1$ and $m = 1$, by Fabes, Garofalo, Salsa [Ill. J. Math, 1986] and also Safonov, Yuan [Ann. of Math, 1999]

• For $s = 1$, Intrinsic (Forward) Harnack inequalities by DiBenedetto [ARMA, 1988], Daskalopoulos and Kenig [EMS Book, 2007], cf. also DiBenedetto, Gianazza, Vespri [LNM, 2011].

Global Harnack Principles II. The remaining cases.

Theorem. (Global Harnack Principle II) (M.B., A. Figalli and J. L. Vázquez)

Let (A1), (A2), and (K2) hold, and let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Assume that:
- either $\sigma = 1$;
- or $\sigma < 1$, $u_0 \ge \underline{\kappa}_0 \Phi_1^{\sigma/m}$ for some $\underline{\kappa}_0 > 0$, and (K4) holds.

Then there exist constants $\underline{\kappa}, \overline{\kappa} > 0$ such that the following inequality holds:

$$\underline{\kappa}\,\frac{\Phi_1(x)^{\sigma/m}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t \ge t_* \text{ and all } x \in \Omega\,.$$

The constants $\underline{\kappa}, \overline{\kappa}$ depend only on $N, s, \gamma, m, \underline{\kappa}_0, \underline{\kappa}_\Omega$, and Ω.

Corollary. Elliptic/backward local Harnack inequalities follow for large times, for all $t \ge t_*$

$$\sup_{x\in B_R(x_0)} u(t,x) \le \hat{H}\left[\left(1+\frac{h}{t}\right)\left(1\wedge\frac{t}{t_*}\right)^{-m}\right]^{\frac{1}{m-1}} \inf_{x\in B_R(x_0)} u(t+h,x)\,.$$

• For small times we can not find matching powers for a global Harnack inequality (except for special data) and such result is *actually false* for $s = 1$ (finite speed of propagation).
• Backward Harnack inequalities for the linear heat equation $s = 1$ and $m = 1$, by Fabes, Garofalo, Salsa [Ill. J. Math, 1986] and also Safonov, Yuan [Ann. of Math, 1999]
• For $s = 1$, Intrinsic (Forward) Harnack inequalities by DiBenedetto [ARMA, 1988], Daskalopoulos and Kenig [EMS Book, 2007], cf. also DiBenedetto, Gianazza, Vespri [LNM, 2011].

Hence, in the remaining cases, we have only the following general result.

Theorem. (Global Harnack Principle III) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1) and (A2), and (K2). Assume moreover that

$$\mathcal{L}w(x) = P.V. \int_{\mathbb{R}^N} \big(w(x) - w(y)\big) K(x,y)\, \mathrm{d}y\,,$$

with $K(x,y) \ge c_0 \Phi_1(x)\Phi_1(y) \quad \forall\, x,y \in \Omega$.
Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds:

$$\underline{\kappa} \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

- This is sufficient to ensure interior regularity, under 'minimal' assumptions.
- This bound holds for all times and for a large class of operators.
- This is not sufficient to ensure C^α_x boundary regularity.

Hence, in the remaining cases, we have only the following general result.

Theorem. (Global Harnack Principle III) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1) and (A2), and (K2). Assume moreover that

$$\mathcal{L}w(x) = P.V. \int_{\mathbb{R}^N} \big(w(x) - w(y)\big) K(x,y)\,\mathrm{d}y\,,$$

with $K(x,y) \ge c_0 \Phi_1(x)\Phi_1(y) \quad \forall x,y \in \Omega$.
Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds:

$$\underline{\kappa} \left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t>0 \text{ and all } x \in \Omega\,.$$

- This is sufficient to ensure interior regularity, under ‘minimal’ assumptions.
- This bound holds for all times and for a large class of operators.
- This is not sufficient to ensure C^α_x boundary regularity.

Hence, in the remaining cases, we have only the following general result.

Theorem. (Global Harnack Principle III) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1) and (A2), and (K2). Assume moreover that

$$\mathcal{L}w(x) = P.V. \int_{\mathbb{R}^N} \big(w(x) - w(y)\big) K(x,y)\,\mathrm{d}y\,,$$

with $K(x,y) \ge c_0 \Phi_1(x)\Phi_1(y) \quad \forall\, x,y \in \Omega$.
Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds:

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\,\frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t>0 \text{ and all } x\in\Omega\,.$$

- This is sufficient to ensure interior regularity, under ‘minimal’ assumptions.

- This bound holds for all times and for a large class of operators.

- This is not sufficient to ensure C^α_x boundary regularity.

Hence, in the remaining cases, we have only the following general result.

Theorem. (Global Harnack Principle III) (M.B., A. Figalli and J. L. Vázquez)

Let $\mathcal{L}$ satisfy (A1) and (A2), and (K2). Assume moreover that

$$\mathcal{L}w(x) = P.V. \int_{\mathbb{R}^N} \big(w(x) - w(y)\big) K(x,y)\,\mathrm{d}y\,,$$

with $K(x,y) \ge c_0 \Phi_1(x)\Phi_1(y) \quad \forall x, y \in \Omega$.
Let $u \ge 0$ be a weak dual solution to the (CDP) corresponding to $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
Then, there exist constants $\underline{\kappa}, \overline{\kappa} > 0$, so that the following inequality holds:

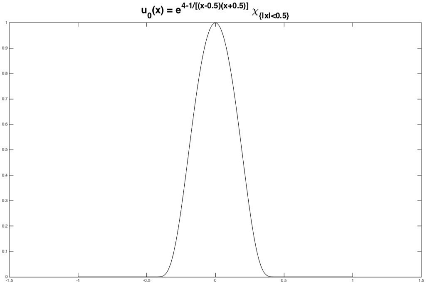
$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x_0)^{\sigma/m}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t > 0 \text{ and all } x \in \Omega\,.$$

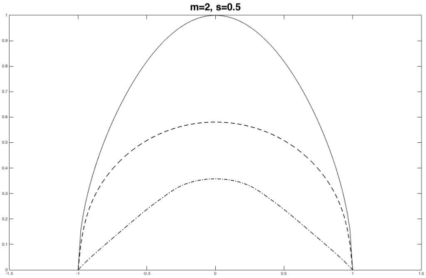
- This is sufficient to ensure interior regularity, under ‘minimal’ assumptions.
- This bound holds for all times and for a large class of operators.
- This is not sufficient to ensure C^α_x boundary regularity.

Numerical Simulations*

* Graphics obtained by numerical methods contained in: N. Cusimano, F. Del Teso, L. Gerardo-Giorda, G. Pagnini, *Discretizations of the spectral fractional Laplacian on general domains with Dirichlet, Neumann, and Robin boundary conditions*, Preprint (2017).
Graphics and videos: courtesy of F. Del Teso (NTNU, Trondheim, Norway)

Numerical simulation for the SFL with parameters $m = 2$ and $s = 1/2$, hence $\sigma = 1$.





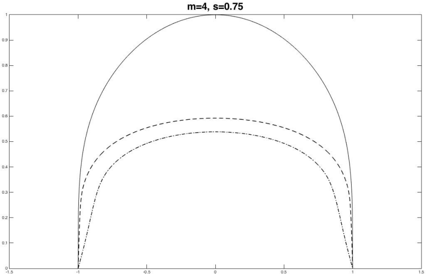
Left: the initial condition $u_0 \le C_0 \Phi_1$

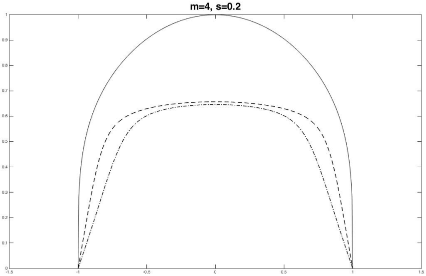
Right: solid line represents $\Phi_1^{1/m}$

the dotted lines represent $\boxed{t^{\frac{1}{m-1}} u(t)}$ at time at $t = 1$ and $t = 5$

While $u(t)$ appears to behave as $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for very short times already at $t = 5$ it exhibits the matching boundary behavior $t^{\frac{1}{m-1}} u(t) \asymp \Phi_1^{1/m}$

Compare $\sigma = 1$ VS $\sigma < 1$: same $u_0 \le C_0 \Phi_1$, solutions with different parameters





Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 30$ and $t = 150$; $m = 4$, $s = 3/4$, $\sigma = 1$.

Matching: $u(t)$ behaves like $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)$ for quite some time,
and only around $t = 150$ it exhibits the matching boundary behavior $u(t) \asymp \Phi_1^{1/m}$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 150$ and $t = 600$; $m = 4$, $s = 1/5$, $\sigma = 8/15 < 1$.

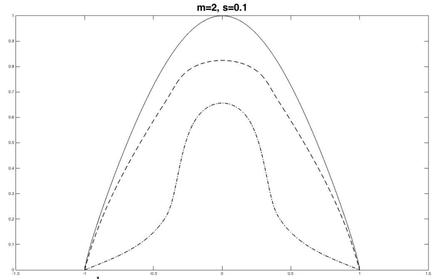
Non-matching: $u(t) \asymp \Phi_1$ even after long time.

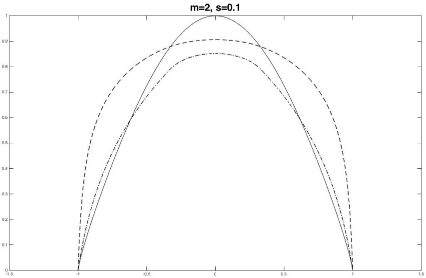
Idea: maybe when $\sigma < 1$ and $u_0 \lesssim \Phi_1$, we have $u(t) \asymp \Phi_1$ for all times...

Not True: there are cases when $u(t) \gg \Phi_1^{1-2s}$ for large times...

Non-matching when $\sigma < 1$: same data u_0, with $m = 2$ and $s = 1/10$, $\sigma = 2/5 < 1$

In both pictures, the solid line represents Φ_1^{1-2s} (anomalous behaviour)





Left: $t^{\frac{1}{m-1}} u(t)$ at time $t = 4$ and $t = 25$.

$u(t) \asymp \Phi_1$ for short times $t = 4$, then $u(t) \sim \Phi_1^{1-2s}$ for intermediate times $t = 25$

Right: $t^{\frac{1}{m-1}} u(t)$ at time $t = 40$ and $t = 150$. $u(t) \gg \Phi_1^{1-2s}$ for large times.

Both non-matching always different behaviour from the asymptotic profile $\Phi_1^{1/m}$.

In this case we show that if $u_0(x) \le C_0 \Phi_1(x)$ then for all $t > 0$

$$u(t,x) \le C_1 \left[\frac{\Phi_1(x)}{t}\right]^{\frac{1}{m}} \qquad \text{and} \qquad \lim_{x \to \partial\Omega} \frac{u(t,x)}{\Phi_1(x)^{\frac{1}{m}}} = 0 \qquad \text{for any } t > 0.$$

Regularity Estimates

- Interior Regularity
- Hölder continuity up to the boundary
- Higher interior regularity for RFL

Interior Regularity

The regularity results, require the validity of a Global Harnack Principle.
(R) The operator $\mathcal{L}$ satisfies (A1) and (A2), and $\mathcal{L}^{-1}$ satisfies (K2). Moreover, we consider

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x)-f(y)\big)K(x,y)\,\mathrm{d}y\,, \qquad \text{with}$$

$$K(x,y) \asymp |x-y|^{-(N+2s)} \quad \text{in } B_{2r}(x_0)\subset\Omega, \qquad K(x,y) \lesssim |x-y|^{-(N+2s)} \quad \text{in } \mathbb{R}^N\setminus B_{2r}(x_0).$$

As a consequence, for any ball $B_{2r}(x_0) \subset\subset \Omega$ and $0<t_0<T_1$, there exist $\delta, M>0$ such that

$$0<\delta\le u(t,x) \qquad \text{for a.e. } (t,x)\in(T_0,T_1)\times B_{2r}(x_0),$$
$$0\le u(t,x)\le M \qquad \text{for a.e. } (t,x)\in(T_0,T_1)\times\Omega.$$

The constants in the regularity estimates will depend on the solution only through δ, M.

Theorem. (Interior Regularity) (M.B., A. Figalli and J. L. Vázquez)

Assume (R) and let u be a nonnegative bounded weak dual solution to problem (CDP).
1. Then u is **Hölder continuous in the interior**. More precisely, there exists $\alpha>0$ such that, for all $0<T_0<T_2<T_1$,

$$\|u\|_{C^{\alpha/2s,\alpha}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

2. Assume in addition $|K(x,y)-K(x',y)| \le c|x-x'|^\beta\,|y|^{-(N+2s)}$ for some $\beta\in(0,1\wedge 2s)$ such that $\beta+2s\notin\mathbb{N}$. Then u **is a classical solution in the interior**. More precisely, for all $0<T_0<T_2<T_1$,

$$\|u\|_{C^{1+\beta/2s,2s+\beta}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

Interior Regularity

The regularity results, require the validity of a Global Harnack Principle.

(R) The operator $\mathcal{L}$ satisfies (A1) and (A2), and $\mathcal{L}^{-1}$ satisfies (K2). Moreover, we consider

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x)-f(y)\big)K(x,y)\,\mathrm{d}y\,, \qquad \text{with}$$

$$K(x,y) \asymp |x-y|^{-(N+2s)} \quad \text{in } B_{2r}(x_0)\subset\Omega, \qquad K(x,y) \lesssim |x-y|^{-(N+2s)} \quad \text{in } \mathbb{R}^N\setminus B_{2r}(x_0).$$

As a consequence, for any ball $B_{2r}(x_0) \subset\subset \Omega$ and $0 < t_0 < T_1$, there exist $\delta, M > 0$ such that

$$0<\delta\le u(t,x) \qquad \text{for a.e. } (t,x)\in(T_0,T_1)\times B_{2r}(x_0),$$
$$0\le u(t,x)\le M \qquad \text{for a.e. } (t,x)\in(T_0,T_1)\times\Omega.$$

The constants in the regularity estimates will depend on the solution only through δ, M.

Theorem. (Interior Regularity) (M.B., A. Figalli and J. L. Vázquez)

Assume (R) and let u be a nonnegative bounded weak dual solution to problem (CDP).

1. Then u is **Hölder continuous in the interior**. More precisely, there exists $\alpha > 0$ such that, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{\alpha/2s,\alpha}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

2. Assume in addition $|K(x,y)-K(x',y)| \le c|x-x'|^\beta\,|y|^{-(N+2s)}$ for some $\beta\in(0,1\wedge 2s)$ such that $\beta+2s\notin\mathbb{N}$. Then u **is a classical solution in the interior**. More precisely, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{1+\beta/2s,2s+\beta}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

Interior Regularity

The regularity results, require the validity of a Global Harnack Principle.
(R) The operator $\mathcal{L}$ satisfies (A1) and (A2), and $\mathcal{L}^{-1}$ satisfies (K2). Moreover, we consider

$$\mathcal{L}f(x) = P.V. \int_{\mathbb{R}^N} \big(f(x) - f(y)\big) K(x,y)\, dy\,, \qquad \text{with}$$

$$K(x,y) \asymp |x-y|^{-(N+2s)} \quad \text{in } B_{2r}(x_0) \subset \Omega, \qquad K(x,y) \lesssim |x-y|^{-(N+2s)} \quad \text{in } \mathbb{R}^N \setminus B_{2r}(x_0).$$

As a consequence, for any ball $B_{2r}(x_0) \subset\subset \Omega$ and $0 < t_0 < T_1$, there exist $\delta, M > 0$ such that

$$0 < \delta \le u(t,x) \qquad \text{for a.e. } (t,x) \in (T_0, T_1) \times B_{2r}(x_0),$$
$$0 \le u(t,x) \le M \qquad \text{for a.e. } (t,x) \in (T_0, T_1) \times \Omega.$$

The constants in the regularity estimates will depend on the solution only through δ, M.

Theorem. (Interior Regularity) (M.B., A. Figalli and J. L. Vázquez)

Assume (R) and let u be a nonnegative bounded weak dual solution to problem (CDP).
1. Then u is **Hölder continuous in the interior**. More precisely, there exists $\alpha > 0$ such that, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{\alpha/2s,\alpha}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

2. Assume in addition $|K(x,y) - K(x',y)| \le c|x-x'|^\beta\, |y|^{-(N+2s)}$ for some $\beta \in (0, 1\wedge 2s)$ such that $\beta + 2s \notin \mathbb{N}$. Then u **is a classical solution in the interior**. More precisely, for all $0 < T_0 < T_2 < T_1$,

$$\|u\|_{C^{1+\beta/2s,2s+\beta}_{t,x}((T_2,T_1)\times B_r(x_0))} \le C.$$

Hölder continuity up to the boundary

Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

$$\|u\|_{C^{\frac{\gamma}{m\vartheta},\frac{\gamma}{m}}_{t,x}((T_2,T_1)\times\Omega)} \le C \qquad \text{with} \qquad \vartheta := 2s - \gamma\left(1 - \frac{1}{m}\right).$$

- Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spatial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)
- Previous regularity results: (I apologize if I forgot someone)
  - *C^α regularity:*
    Athanasopoulos and Caffarelli [Adv. Math, 2010], (RFL domains)
    De Pablo, Quirós, Rodriguez, Vázquez [CPAM 2012] (RFL on $\mathbb{R}^N$, SFL-Dirichlet)
    De Pablo, Quirós, Rodriguez [NLTMA 2016]. (RFL-rough kernels $\mathbb{R}^N$)
  - *Classical Solutions:*
    Vázquez, De Pablo, Quirós, Rodriguez [JEMS 2016] (RFL on $\mathbb{R}^N$)
    M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)
  - *Higher regularity: C^∞_x* and C^α up to the boundary:
    M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)

Hölder continuity up to the boundary

Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

$$\|u\|_{C^{\frac{\gamma}{m\vartheta},\frac{\gamma}{m}}_{t,x}((T_2,T_1)\times\Omega)} \le C \qquad \text{with} \qquad \vartheta := 2s - \gamma\left(1 - \frac{1}{m}\right).$$

• Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spatial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)

• Previous regularity results: (I apologize if I forgot someone)

- *C^α regularity:*
  Athanasopoulos and Caffarelli [Adv. Math, 2010], (RFL domains)
  De Pablo, Quirós, Rodriguez, Vázquez [CPAM 2012] (RFL on $\mathbb{R}^N$, SFL-Dirichlet)
  De Pablo, Quirós, Rodriguez [NLTMA 2016]. (RFL-rough kernels $\mathbb{R}^N$)
- *Classical Solutions:*
  Vázquez, De Pablo, Quirós, Rodriguez [JEMS 2016] (RFL on $\mathbb{R}^N$)
  M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)
- *Higher regularity:* C^∞_x and C^α up to the boundary:
  M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)

Hölder continuity up to the boundary

Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

$$\|u\|_{C^{\frac{\gamma}{m\vartheta},\frac{\gamma}{m}}_{t,x}((T_2,T_1)\times\Omega)} \le C \qquad \text{with} \qquad \vartheta := 2s - \gamma\left(1 - \frac{1}{m}\right).$$

• Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spatial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)

• Previous regularity results: (I apologize if I forgot someone)

- *C^α regularity:*
  Athanasopoulos and Caffarelli [Adv. Math, 2010], (RFL domains)
  De Pablo, Quirós, Rodriguez, Vázquez [CPAM 2012] (RFL on $\mathbb{R}^N$, SFL-Dirichlet)
  De Pablo, Quirós, Rodriguez [NLTMA 2016]. (RFL-rough kernels $\mathbb{R}^N$)
- *Classical Solutions:*
  Vázquez, De Pablo, Quirós, Rodriguez [JEMS 2016] (RFL on $\mathbb{R}^N$)
  M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)
- *Higher regularity:* C^∞_x and C^α up to the boundary:
  M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)

Hölder continuity up to the boundary

Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

$$\|u\|_{C^{\frac{\gamma}{m\vartheta},\frac{\gamma}{m}}_{t,x}((T_2,T_1)\times\Omega)} \le C \qquad \text{with} \qquad \vartheta := 2s - \gamma\left(1 - \frac{1}{m}\right).$$

• Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spatial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)

• Previous regularity results: (I apologize if I forgot someone)

- *C^α regularity:*
  Athanasopoulos and Caffarelli [Adv. Math, 2010], (RFL domains)
  De Pablo, Quirós, Rodriguez, Vázquez [CPAM 2012] (RFL on $\mathbb{R}^N$, SFL-Dirichlet)
  De Pablo, Quirós, Rodriguez [NLTMA 2016]. (RFL-rough kernels $\mathbb{R}^N$)
- *Classical Solutions:*
  Vázquez, De Pablo, Quirós, Rodriguez [JEMS 2016] (RFL on $\mathbb{R}^N$)
  M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)
- *Higher regularity: C^∞_x* and C^α up to the boundary:
  M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)

Hölder continuity up to the boundary

Theorem. (Hölder continuity up to the boundary) (M.B., A. Figalli and J. L. Vázquez)

Assume (R), hypothesis **2** of the interior regularity and in addition that $2s > \gamma$.
Then u **is Hölder continuous up to the boundary**.
More precisely, for all $0 < T_0 < T_2 < T_1$ there exists a constant $C > 0$ such that

$$\|u\|_{C^{\frac{\gamma}{m\vartheta},\frac{\gamma}{m}}_{t,x}((T_2,T_1)\times\Omega)} \le C \qquad \text{with} \qquad \vartheta := 2s - \gamma\left(1 - \frac{1}{m}\right).$$

• Since $u(t,x) \asymp \Phi_1(x)^{1/m} \asymp \mathrm{dist}(x,\partial\Omega)^{\gamma/m}$, **the spatial Hölder exponent is sharp,** while the Hölder exponent in time is the natural one by scaling. ($2s > \gamma$ implies $\sigma = 1$)

• Previous regularity results: (I apologize if I forgot someone)

- *C^α regularity:*
  Athanasopoulos and Caffarelli [Adv. Math, 2010], (RFL domains)
  De Pablo, Quirós, Rodriguez, Vázquez [CPAM 2012] (RFL on $\mathbb{R}^N$, SFL-Dirichlet)
  De Pablo, Quirós, Rodriguez [NLTMA 2016]. (RFL-rough kernels $\mathbb{R}^N$)
- *Classical Solutions:*
  Vázquez, De Pablo, Quirós, Rodriguez [JEMS 2016] (RFL on $\mathbb{R}^N$)
  M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)
- *Higher regularity:* C^∞_x and C^α up to the boundary:
  M.B., Figalli, Ros-Oton [CPAM2016] (RFL Dirichlet, even unbounded domains)

Higher Interior Regularity for RFL.

Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)**, then $u \in C_x^\infty((0,\infty)\times\Omega)$.
More precisely, let $k \geq 1$ be any positive integer, and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then, for any $t \geq t_0 > 0$ we have

$$\left|D_x^k u(t,x)\right| \leq C\,[d(x)]^{\frac{s}{m}-k},$$

where C depends only on N, s, m, k, Ω, t_0, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in t. To our knowledge also open for the local case $s = 1$.
- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C_x^\infty$

  $$\|u\|_{C_x^{k,\alpha}((\frac{1}{2},1)\times B_{1/2})} \leq C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \geq 0.$$

  Analogous estimates in time do not hold for $k \geq 1$ and $\alpha \in (0,1)$.
  Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not C^1 in t in $(\frac{1}{2},1)\times B_{1/2}$. [Chang-Lara, Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains , and also for operator with more general kernels.
- Also the "classical/local" case $s = 1$ works after the waiting time t_*:
  $u \in C_{x,t}^{\frac{1}{m},\frac{1}{2m}}(\overline{\Omega}\times[t_*,T])$, $C_x^\infty((0,\infty)\times\Omega)$ and $C_t^{1,\alpha}([t_0,T]\times K)$.

Higher Interior Regularity for RFL.

Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)**, then $u \in C_x^\infty((0,\infty)\times\Omega)$.
More precisely, let $k \ge 1$ be any positive integer, and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then, for any $t \ge t_0 > 0$ we have

$$\left|D_x^k u(t,x)\right| \le C\,[d(x)]^{\frac{s}{m}-k},$$

where C depends only on N, s, m, k, Ω, t_0, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in t. To our knowledge also open for the local case $s = 1$.
- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C_x^\infty$
$$\|u\|_{C_x^{k,\alpha}((\frac{1}{2},1)\times B_{1/2})} \le C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \ge 0.$$
Analogous estimates in time do not hold for $k \ge 1$ and $\alpha \in (0,1)$.
Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not C^1 in t in $(\frac{1}{2},1)\times B_{1/2}$. [Chang-Lara, Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains , and also for operator with more general kernels.
- Also the “classical/local” case $s = 1$ works after the waiting time t_*:
$u \in C_{x,t}^{\frac{1}{m},\frac{1}{2m}}(\overline{\Omega}\times[t_*,T])$, $C_x^\infty((0,\infty)\times\Omega)$ and $C_t^{1,\alpha}([t_0,T]\times K)$.

Higher Interior Regularity for RFL.

Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)** , then $u \in C_x^\infty((0,\infty)\times\Omega)$.
More precisely, let $k \ge 1$ be any positive integer, and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then, for any $t \ge t_0 > 0$ we have

$$\left|D_x^k u(t,x)\right| \le C\,[d(x)]^{\frac{s}{m}-k},$$

where C depends only on N, s, m, k, Ω, t_0, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in t. To our knowledge also open for the local case $s = 1$.
- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C_x^\infty$

$$\|u\|_{C_x^{k,\alpha}((\frac{1}{2},1)\times B_{1/2})} \le C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \ge 0.$$

Analogous estimates in time do not hold for $k \ge 1$ and $\alpha \in (0,1)$.
Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not C^1 in t in $(\frac{1}{2},1)\times B_{1/2}$. [Chang-Lara, Davila, JDE (2014)]

- Our techniques allow to prove regularity also in unbounded domains , and also for operator with more general kernels.
- Also the "classical/local" case $s = 1$ works after the waiting time t_*:
$u \in C_{x,t}^{\frac{1}{m},\frac{1}{2m}}\left(\overline{\Omega}\times[t_*,T]\right)$, $C_x^\infty((0,\infty)\times\Omega)$ and $C_t^{1,\alpha}([t_0,T]\times K)$.

Higher Interior Regularity for RFL.

Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)** , then $u \in C_x^\infty((0,\infty)\times\Omega)$.
More precisely, let $k \geq 1$ be any positive integer, and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then, for any $t \geq t_0 > 0$ we have

$$\left|D_x^k u(t,x)\right| \leq C\,[d(x)]^{\frac{s}{m}-k},$$

where C depends only on N, s, m, k, Ω, t_0, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in t. To our knowledge also open for the local case $s = 1$.
- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C_x^\infty$

  $$\|u\|_{C_x^{k,\alpha}((\frac{1}{2},1)\times B_{1/2})} \leq C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \geq 0.$$

  Analogous estimates in time do not hold for $k \geq 1$ and $\alpha \in (0,1)$.
  Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not C^1 in t in $(\frac{1}{2},1)\times B_{1/2}$. [Chang-Lara, Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains , and also for operator with more general kernels.
- Also the "classical/local" case $s = 1$ works after the waiting time t_*:
  $u \in C_{x,t}^{\frac{1}{m},\frac{1}{2m}}(\overline{\Omega}\times[t_*,T])$, $C_x^\infty((0,\infty)\times\Omega)$ and $C_t^{1,\alpha}([t_0,T]\times K)$.

Higher Interior Regularity for RFL.

Theorem. (Higher interior regularity in space) (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)**, then $u \in C^\infty_x((0,\infty)\times\Omega)$.
More precisely, let $k \geq 1$ be any positive integer, and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then, for any $t \geq t_0 > 0$ we have

$$\left|D^k_x u(t,x)\right| \leq C\,[d(x)]^{\frac{s}{m}-k},$$

where C depends only on N, s, m, k, Ω, t_0, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in t. To our knowledge also open for the local case $s = 1$.
- When $m = 1$ (FHE) $u_t + (-\Delta_{|\Omega})^s u = 0$ on $(0,1)\times B_1$ we have $u \in C^\infty_x$
$$\|u\|_{C^{k,\alpha}_x((\frac{1}{2},1)\times B_{1/2})} \leq C\|u\|_{L^\infty((0,1)\times\mathbb{R}^N)}, \qquad \text{for all } k \geq 0.$$
Analogous estimates in time do not hold for $k \geq 1$ and $\alpha \in (0,1)$.
Indeed, one can construct a solution to the (FHE) which is bounded in all of $\mathbb{R}^N$, but which is not C^1 in t in $(\frac{1}{2},1)\times B_{1/2}$. [Chang-Lara, Davila, JDE (2014)]
- Our techniques allow to prove regularity also in unbounded domains , and also for operator with more general kernels.
- Also the "classical/local" case $s = 1$ works after the waiting time t_*:
$u \in C^{\frac{1}{m},\frac{1}{2m}}_{x,t}\left(\overline{\Omega}\times[t_*,T]\right)$, $C^\infty_x((0,\infty)\times\Omega)$ and $C^{1,\alpha}_t([t_0,T]\times K)$.

The End

Thank You!!!

Grazie Mille!!!

Muchas Gracias!!!

Summary

Asymptotic behaviour of nonnegative solutions

- Convergence to the stationary profile
- Convergence with optimal rate

Convergence to the stationary profile

In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$.

Theorem. (Asymptotic behaviour) (M.B. , Y. Sire, J. L. Vázquez)

There exists a unique nonnegative selfsimilar solution of the above Dirichlet Problem

$$U(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}} ,$$

for some bounded function $S : \Omega \to \mathbb{R}$. Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$)

$$\lim_{\tau \to \infty} \tau^{\frac{1}{m-1}} \|u(\tau, \cdot) - U(\tau, \cdot)\|_{L^\infty(\Omega)} = 0 .$$

The previous theorem admits the following corollary.

Theorem. (Elliptic problem) (M.B. , Y. Sire, J. L. Vázquez)

Let $m > 1$. There exists a unique weak dual solution to the elliptic problem

$$\begin{cases} \mathcal{L}(S^m) = \dfrac{S}{m-1} & \text{in } \Omega, \\ S(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Notice that the previous theorem is obtained in the present paper through a parabolic technique.

Convergence to the stationary profile

In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$.

Theorem. (Asymptotic behaviour) (M.B. , Y. Sire, J. L. Vázquez)

There exists a unique nonnegative selfsimilar solution of the above Dirichlet Problem

$$U(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}},$$

for some bounded function $S : \Omega \to \mathbb{R}$. Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$)

$$\lim_{\tau \to \infty} \tau^{\frac{1}{m-1}} \|u(\tau, \cdot) - U(\tau, \cdot)\|_{L^\infty(\Omega)} = 0\,.$$

The previous theorem admits the following corollary.

Theorem. (Elliptic problem) (M.B. , Y. Sire, J. L. Vázquez)

Let $m > 1$. There exists a unique weak dual solution to the elliptic problem

$$\begin{cases} \mathcal{L}(S^m) = \dfrac{S}{m-1} & \text{in } \Omega, \\ S(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Notice that the previous theorem is obtained in the present paper through a parabolic technique.

Convergence with optimal rate

Theorem. (Sharp asymptotic with rates) (M.B. , Y. Sire, J. L. Vázquez)

Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$) that there exist $t_0 > 0$ of the form

$$t_0 = \overline{k} \left[\frac{\int_\Omega \Phi_1 \, dx}{\int_\Omega u_0 \Phi_1 \, dx} \right]^{m-1}$$

such that for all $t \geq t_0$ we have

$$\left\| \frac{u(t,\cdot)}{U(t,\cdot)} - 1 \right\|_{L^\infty(\Omega)} \leq \frac{2}{m-1} \, \frac{t_0}{t_0 + t} \, .$$

The constant $\overline{k} > 0$ only depends on m, N, s, and $|\Omega|$.

Remarks.

- We provide two different proofs of the above result.
- One proof is based on the construction of the so-called Friendly-Giant solution, namely the solution with initial data $u_0 = +\infty$, and is based on the Global Harnack Principle of Part 4
- The second proof is based on a new Entropy method, which is based on a parabolic version of the Caffarelli-Silvestre extension.

Convergence with optimal rate

Theorem. (Sharp asymptotic with rates) (M.B. , Y. Sire, J. L. Vázquez)

Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$) that there exist $t_0 > 0$ of the form

$$t_0 = \overline{k}\left[\frac{\int_\Omega \Phi_1\,\mathrm{d}x}{\int_\Omega u_0 \Phi_1\,\mathrm{d}x}\right]^{m-1}$$

such that for all $t \ge t_0$ we have

$$\left\|\frac{u(t,\cdot)}{U(t,\cdot)} - 1\right\|_{\mathrm{L}^\infty(\Omega)} \le \frac{2}{m-1}\,\frac{t_0}{t_0+t}\,.$$

The constant $\overline{k} > 0$ only depends on m, N, s, and $|\Omega|$.

Remarks.

- We provide two different proofs of the above result.
- One proof is based on the construction of the so-called Friendly-Giant solution, namely the solution with initial data $u_0 = +\infty$, and is based on the Global Harnack Principle of Part 4
- The second proof is based on a new Entropy method, which is based on a parabolic version of the Caffarelli-Silvestre extension.