# Nonlinear and Nonlocal Degenerate Diffusions on Bounded Domains

**Matteo Bonforte**

Departamento de Matemáticas,
Universidad Autónoma de Madrid,
Campus de Cantoblanco
28049 Madrid, Spain

matteo.bonforte@uam.es

http://www.uam.es/matteo.bonforte

**Conference on Qualitative Aspects of the Theory of Nonlocal Equations**

FOCUS PROGRAM ON NONLOCAL PARTIAL DIFFERENTIAL EQUATIONS

*The Fields Institute, Toronto, Canada, May 17, 2016*

# References:

[BV1] M. B., J. L. VÁZQUEZ, A Priori Estimates for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains.
*Arch. Rat. Mech. Anal. (2015).*

[BV2] M. B., J. L. VÁZQUEZ, Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part I. Existence, Uniqueness and Upper Bounds
*Nonlin. Anal. TMA (2016).*

[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains.
*Discr. Cont. Dyn. Sys. (2015).*

[BFR] M. B., A. FIGALLI, X. ROS-OTON, Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains.
*To appear in Comm. Pure Appl. Math (2016).*

[BV3] M. B., A. FIGALLI, J. L. VÁZQUEZ, Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part II. Infinite speed of propagation, Sharp boundary behaviour, Harnack inequalities and Regularity.
*In preparation (2016).*

# Outline of the talk

- The abstract setup of the problem
- Some important examples
- Existence and uniqueness
- First pointwise estimates
- Upper and Lower Estimates
- Harnack Inequalities
- Regularity Estimates

## Introduction

**Homogeneous Dirichlet Problem for**
**Fractional Nonlinear Degenerate Diffusion Equations**

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  *all fractional Laplacians on domains.*
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities $F$ are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

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## About the operator $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is $m$-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If $\beta$ is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e, then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

**Remark.** These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:

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## Assumption on the nonlinearity $F$

Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous and non-decreasing function, with $F(0)=0$. Moreover, it satisfies the condition:

(N1) $F\in C^1(\mathbb{R}\setminus\{0\})$ and $F/F'\in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0,\mu_1>0$ s.t.

$$\boxed{\frac{1}{m_1}=1-\mu_1\le\left(\frac{F}{F'}\right)'\le 1-\mu_0=\frac{1}{m_0}}$$

where $F/F'$ is understood to vanish if $F(r)=F'(r)=0$ or $r=0$.

The main example will be

$$F(u)=|u|^{m-1}u,\qquad \text{with } m>1\,,\quad\text{and}\qquad \mu_0=\mu_1=\frac{m-1}{m}<1\,.$$

which corresponds to the nonlocal porous medium equation studied in $[BV1]$. A simple variant is the combination of two powers:

- $m_0$ gives the behaviour at zero, when $u\sim 0$
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## Existence of Mild Solutions and Monotonicity Estimates

**Theorem** (M. Crandall and M. Pierre, JFA 1982)

Let $\mathcal{L}$ satisfy (A1) and (A2) and let $F$ as satisfy (N1). Then for all $0 \le u_0 \in \mathrm{L}^1(\Omega)$, there exists a unique mild solution $u$ to equation $u_t + \mathcal{L}F(u) = 0$, and the function

$$(1) \qquad t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x)) \qquad \text{is nondecreasing in } t > 0 \text{ for a.e. } x \in \Omega\,.$$

Moreover, the semigroup is contractive on $\mathrm{L}^1(\Omega)$ and $u \in C([0,\infty) : \mathrm{L}^1(\Omega))$.

We notice that (1) is a weak formulation of the monotonicity inequality:

$$\partial_t u \ge -\frac{1}{\mu_0\, t}\frac{F(u)}{F'(u)}\,, \qquad \text{which implies} \qquad \partial_t u \ge -\frac{1-\mu_0}{\mu_0}\,\frac{u}{t}$$

or equivalently, that the function

$$(2) \qquad t \mapsto t^{\frac{1-\mu_0}{\mu_0}}\, u(t,x) \qquad \text{is nondecreasing in } t > 0 \text{ for a.e. } x \in \Omega\,.$$


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## Assumption on the inverse operator $\mathcal{L}^{-1}$

### Assumptions on the inverse of $\mathcal{L}$

We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1} : \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ with a kernel $\mathbb{K}$ such that

$$\mathcal{L}^{-1} f(x) = \int_\Omega \mathbb{K}(x,y) f(y)\, \mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$ and $c_{i,\Omega} > 0$

(K1) $$0 \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

(K2) $$c_{0,\Omega}\delta_\gamma(x)\,\delta_\gamma(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\left(\frac{\delta_\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma}\wedge 1\right)$$

where

$$\delta_\gamma(x) := \mathrm{dist}(x,\partial\Omega)^\gamma\,.$$

Indeed, $(K1)$ implies that $\mathcal{L}$ has a first eigenfunction $0 \le \Phi_1 \in \mathrm{L}^\infty(\Omega)$.
Moreover, (K2) implies that $\Phi_1 \asymp \mathrm{dist}(\cdot,\partial\Omega)^\gamma = \delta_\gamma$ and we can rewrite (K2) as

(K3) $$c_{0,\Omega}\Phi_1(x)\Phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-x_0|^{N-2s}}\left(\frac{\Phi_1(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma}\wedge 1\right)$$

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## Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$ :

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

This formula can be used for positive and negative values of $s$.

2. By means of an **Hypersingular Kernel**:
if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right)\frac{dt}{t^{1+s}}.$$

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## Examples of operators $\mathcal{L}$

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## Examples of operators $\mathcal{L}$

### The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x)\right) \frac{dt}{t^{1+s}} .$$

- $\Delta_\Omega$ is the classical Dirichlet Laplacian on the domain $\Omega$
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: $\phi_j$ are as smooth as the boundary of $\Omega$ allows, namely when $\partial\Omega$ is $C^k$, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

### Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1\right), \qquad \text{with } \gamma = 1$$

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## Examples of operators $\mathcal{L}$

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside $\Omega$.

### The Restricted Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\, dz\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$.

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### Censored Fractional Laplacians (CFL)

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}}\,\mathrm{d}y\,, \qquad \text{with } \frac{1}{2} < s < 1\,,$$

where $a(x,y)$ is a measurable, symmetric function bounded between two positive constants, satisfying some further assumptions; for instance $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.

The Green function $\mathbb{K}(x,y)$ satisfies $(K4)$ , proven by Chen, Kim and Song (2010)

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**Remarks.**

- This is a third model of Dirichlet fractional Laplacian when $\left[a(x,y) = const\right]$. This is **not equivalent** to SFL nor to RFL.
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**Spectral powers of uniformly elliptic operators.** Consider a linear operator $A$ in divergence form, with uniformly elliptic bounded measurable coefficients:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j) \,, \qquad s\text{-power of } A \text{ is:} \qquad \mathcal{L} f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x)$$

$\mathcal{L} = A^s$ satisfies (K3) estimates with $\gamma = 1$

$$\text{(K3)} \qquad c_{0,\Omega} \phi_1(x)\, \phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left( \frac{\phi_1(x)}{|x-y|} \wedge 1 \right) \left( \frac{\phi_1(y)}{|x-y|} \wedge 1 \right)$$

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**Fractional operators with "rough" kernels.** Integral operators of Levy-type

$$\mathcal{L} f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left( f(x+y) - f(y) \right) \frac{K(x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y \,.$$

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for some positive $c > 0$. We can allow even more general kernels.

The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

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$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma\,, \qquad \text{with } 0 < \sigma < s \le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. Satisfy $(K4)$ with $\gamma = s$.

**Sum of the Laplacian and operators with general kernels.** In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0\,,$$

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the measure $\nu$ on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies $\int_{\mathbb{R}^N} 1 \vee |x|^2 \,\mathrm{d}\nu(y) < \infty$, together with other assumptions.

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$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s\,, \qquad \text{with } c > 0\,, \text{ and } 0 < s \le 1\,.$$

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**Many other interesting examples.** Schrödinger equations for non-symmetric diffusions, Gradient perturbation of RFL...

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## The "dual" formulation of the problem

Recall the homogeneous Dirichlet problem:

$$(\mathrm{CDP}) \qquad \begin{cases} \partial_t u = -\mathcal{L}\, F(u)\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\partial_t U = -F(u)\,,$$

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$$U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega \mathbb{K}(x,y)u(t,y)\,\mathrm{d}y\,.$$

This formulation encodes the lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

**Remark.** This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.

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### Weak Dual Solutions

A function $u$ is a *weak dual* solution to the Dirichlet Problem for $\partial_t u + \mathcal{L}F(u) = 0$ in $Q_T = (0,T) \times \Omega$ if:

- $u \in C((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega))\,, F(u) \in \mathrm{L}^1\left((0,T) : \mathrm{L}^1_{\Phi_1}(\Omega)\right);$
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### Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** is a weak dual solution to Equation $\partial_t u + \mathcal{L}F(u) = 0$ such that moreover

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## Existence and uniqueness of weak dual solutions

**Theorem. Existence of weak dual solutions** (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ there exists a minimal weak dual solution to the $(CDP)$. Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : \mathrm{L}^1_{\Phi_1}(\Omega))$.
Mild solutions (constructed by Crandall and Pierre) are weak dual solutions and if $u_0 \in \mathrm{L}^p(\Omega)$ then $u(t) \in \mathrm{L}^p(\Omega)$ for all $t > 0$.

**Theorem. Uniqueness of weak dual solutions** (M.B. and J. L. Vázquez)

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## First Pointwise Estimates

**Theorem.** (M.B. and J. L. Vázquez)

Let $u \ge 0$ be a weak dual solution to Problem $(CDP)$ with $u_0 \in \mathrm{L}^p(\Omega)$, $p > N/2s$. Then,

$$\int_\Omega u(t_1,x)\mathbb{K}(x,x_0)\,\mathrm{d}x \le \int_\Omega u(t_0,x)\mathbb{K}(x,x_0)\,\mathrm{d}x\,, \qquad \text{for all } t_1 \ge t_0 \ge 0\,.$$

Moreover, for almost every $0 \le t_0 \le t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}} (t_1 - t_0)\, F(u(t_0,x_0)) \le \int_\Omega \left[u(t_0,x) - u(t_1,x)\right] \mathbb{K}(x,x_0)\,\mathrm{d}x$$

$$\le (m_0-1)\frac{t_1^{\frac{1}{\mu_0}}}{t_0^{\frac{1-\mu_0}{\mu_0}}}\, F(u(t_1,x_0))\,.$$

**Remark.** As a consequence of the above inequality and Hölder inequality, we have that $u(t) \in \mathrm{L}^\infty(\Omega)$ when $u_0 \in \mathrm{L}^p(\Omega)$, with $p > N/(2s)$.

## First Pointwise Estimates

**Theorem.** (M.B. and J. L. Vázquez)

Let $u \ge 0$ be a weak dual solution to Problem $(CDP)$ with $u_0 \in \mathrm{L}^p(\Omega)$, $p > N/2s$. Then,

$$\int_\Omega u(t_1,x)\mathbb{K}(x,x_0)\,\mathrm{d}x \le \int_\Omega u(t_0,x)\mathbb{K}(x,x_0)\,\mathrm{d}x\,, \qquad \text{for all } t_1 \ge t_0 \ge 0\,.$$

Moreover, for almost every $0 \le t_0 \le t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}} (t_1 - t_0)\,F(u(t_0,x_0)) \le \int_\Omega \left[u(t_0,x) - u(t_1,x)\right]\mathbb{K}(x,x_0)\,\mathrm{d}x$$

$$\le (m_0-1)\,\frac{t_1^{\frac{1}{\mu_0}}}{t_0^{\frac{1-\mu_0}{\mu_0}}}\,F(u(t_1,x_0))\,.$$

**Remark.** As a consequence of the above inequality and Hölder inequality, we have that $u(t) \in \mathrm{L}^\infty(\Omega)$ when $u_0 \in \mathrm{L}^p(\Omega)$, with $p > N/(2s)$.

## Sketch of the proof of the First Pointwise Estimates

We would like to take as test function

$$\psi(t,x)=\psi_1(t)\psi_2(x)=\chi_{[t_0,t_1]}(t)\,\mathbb{K}(x_0,x)\,,$$

(This is not an admissible test in the Definition of Weak Dual solutions)
Plugging such test function in the definition of weak dual solution gives the formula

$$\int_\Omega u(t_0,x)\mathbb{K}(x_0,x)\,\mathrm{d}x-\int_\Omega u(t_1,x)\mathbb{K}(x_0,x)\,\mathrm{d}x=\int_{t_0}^{t_1}F(u(\tau,x_0))\mathrm{d}\tau\,.$$

This formula can be proven rigorously though careful approximation.
Next, we use the monotonicity estimates,

$$t\mapsto t^{\frac{1}{\mu_0}}F(u(t,x))\qquad\text{is nondecreasing in } t>0 \text{ for a.e. } x\in\Omega\,.$$

to get for all $0\le t_0\le t_1$, recalling that $\frac{1}{\mu_0}=\frac{m_0}{m_0-1}$

$$\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}}(t_1-t_0)F(u(t_0,x_0))\le\int_{t_0}^{t_1}F(u(\tau,x_0))\mathrm{d}\tau\le\frac{m_0-1}{t_0^{\frac{1}{m_0-1}}}t_1^{\frac{1}{\mu_0}}F(u(t_1,x_0)).\ \square$$

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# Upper Bounds

## For the rest of the talk we deal with the special case:

$$\boxed{F(u) = u^m := |u|^{m-1}u}$$

## The power case. Absolute bounds and boundary behaviour

**Theorem. (Absolute upper bounds and boundary behaviour)**(M.B. & J. L. Vázquez)

Let $u$ be a weak dual solution, then there exists constants $K_1, K_2 > 0$ depending only on $N, s, m, \Omega$ (but not on $u_0$ !!) , such that
(K1) assumption implies:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_1}{t^{\frac{1}{m-1}}}\,, \qquad \text{for all } \ t>0\,.$$

Moreover, (K2) assumption implies, for $0<\gamma\le 2sm/(m-1)$

$$u(t,x) \le K_2\,\frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}\,, \qquad \text{for all } t>0 \text{ and } x\in\Omega\,.$$

When $\gamma > 2sm/(m-1)$ the power of $\Phi_1$ becomes $\frac{\sigma}{m} := \frac{2s}{(m-1)\gamma} < \frac{1}{m}$

**Remarks.**

- This is a very strong regularization *independent* of the initial datum $u_0$.
- Sharp boundary estimates: we will show lower bounds with matching powers. The power decay of $u^m$ is $\sigma = 1 \wedge 2sm/[(m-1)\gamma]$
  In examples, only for SFL-type, $\gamma = 1$, and $s$ small, $0 < s < 1/2 - 1/(2m)$
- Time decay is sharp, but only for large times, say $t \ge 1$. For small times when $0 < t < 1$ a better time decay is obtained in the form of smoothing effects

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## Sketch of the proof of Absolute Bounds

• STEP 1. *First upper estimates.* Recall the pointwise estimate:

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}}(t_1-t_0)\,u^m(t_0,x_0)\le\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,{\rm d}x-\int_\Omega u(t_1,x)G_\Omega(x,x_0)\,{\rm d}x\,.$$

for any $u\in\mathcal{S}_p$, all $0\le t_0\le t_1$ and all $x_0\in\Omega$. Choose $t_1=2t_0$ to get

$$(*)\qquad \boxed{u^m(t_0,x_0)\le\frac{2^{\frac{m}{m-1}}}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,{\rm d}x\,.}$$

Recall that $u\in\mathcal{S}_p$ with $p>N/(2s)$, means $u(t)\in{\rm L}^p(\Omega)$ for all $t>0$, so that:

$$u^m(t_0,x_0)\le\frac{c_0}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,{\rm d}x\le\frac{c_0}{t_0}\|u(t_0)\|_{{\rm L}^p(\Omega)}\,\|G_\Omega(\cdot,x_0)\|_{{\rm L}^q(\Omega)}<+\infty$$

since $G_\Omega(\cdot,x_0)\in{\rm L}^q(\Omega)$ for all $0<q<N/(N-2s)$, so that $u(t_0)\in{\rm L}^\infty(\Omega)$ for all $t_0>0$.

• STEP 2. Let us estimate the r.h.s. of $(*)$ as follows:

$$u^m(t_0,x_0)\le\frac{c_0}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,{\rm d}x\le\|u(t_0)\|_{{\rm L}^\infty(\Omega)}\frac{c_0}{t_0}\int_\Omega G_\Omega(x,x_0)\,{\rm d}x\,.$$

Taking the supremum over $x_0\in\Omega$ of both sides, we get:

$$\boxed{\|u(t_0)\|^{m-1}_{{\rm L}^\infty(\Omega)}\le\frac{c_0}{t_0}\sup_{x_0\in\Omega}\int_\Omega G_\Omega(x,x_0)\,{\rm d}x\le\frac{K_1^{m-1}}{t_0}}\qquad\square$$

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for any $u \in \mathcal{S}_p$, all $0 \le t_0 \le t_1$ and all $x_0 \in \Omega$. Choose $t_1 = 2t_0$ to get

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Define the exponents:

$$\vartheta_{1,\gamma} = \frac{1}{2s + (N+\gamma)(m-1)} \qquad \text{and} \qquad \vartheta_1 = \vartheta_{1,0} = \frac{1}{2s + N(m-1)}$$

**Theorem. (Smoothing effects)** (M.B. & J. L. Vázquez)

There exist universal constants $K_3, K_4 > 0$ such that:

$L^1$-$L^\infty$ SMOOTHING EFFECT: (K1) assumption implies for all $t > 0$:

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_3}{t^{N\vartheta_1}} \|u(t)\|_{L^1(\Omega)}^{2s\vartheta_1} \le \frac{K_3}{t^{N\vartheta_1}} \|u_0\|_{L^1(\Omega)}^{2s\vartheta_1}$$

$L^1_{\Phi_1}$-$L^\infty$ SMOOTHING EFFECT: (K2) assumption implies for all $t > 0$:

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u(t)\|_{L^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} \le \frac{K_4}{t^{(N+\gamma)\vartheta_{1,\gamma}}} \|u_0\|_{L^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} .$$

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**Theorem. (Backward Smoothing effects)** (M.B. & J. L. Vázquez)

There exists a universal constant $K_4 > 0$ such that for all $t, h > 0$

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_4}{t^{(d+\gamma)\vartheta_{1,\gamma}}} \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{1,\gamma}}{m-1}} \|u(t+h)\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{2s\vartheta_{1,\gamma}} .$$

*Proof.* By the monotonicity estimates , the function $u(x,t)t^{1/(m-1)}$ is non-decreasing in time for fixed $x$, therefore using the smoothing effect, we get for all $t_1 \ge t$:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_4}{t^{(N+1)\vartheta_{1,\gamma}}} \left(\int_\Omega u(t,x)\Phi_1(x)\,\mathrm{d}x\right)^{2s\vartheta_{1,\gamma}}$$

$$\le \frac{K_4}{t^{(N+1)\vartheta_{1,\gamma}}} \left(\frac{t_1^{\frac{1}{m-1}}}{t^{\frac{1}{m-1}}} \int_\Omega u(t_1,x)\Phi_1(x)\,\mathrm{d}x\right)^{2s\vartheta_{1,\gamma}}$$

where $K_4$ is as in the smoothing effects. Finally, let $t_1 = t + h$ . □

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## Absolute upper bounds. $F$ case

# Upper Bounds for general $F$

## Absolute upper bounds. $F$ case

**Theorem. (Absolute upper estimate)** (M.B. & J. L. Vázquez)

Let $u$ be a nonnegative weak dual solution corresponding to $u_0 \in \mathrm{L}^1_{\delta_\gamma}(\Omega)$. Then, there exists universal constants $K_0, K_1, K_2 > 0$ such that the following estimates hold true for all $t > 0$:

$$F\left(\|u(t)\|_{\mathrm{L}^\infty(\Omega)}\right) \le F^*\left(\frac{K_1}{t}\right).$$

Moreover, there exists a time $\tau_1(u_0)$ with $0 \le \tau_1(u_0) \le K_0$ such that

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le 1 \qquad \text{for all } t \ge \tau_1\,,$$

so that

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le \frac{K_2}{t^{\frac{1}{m_i-1}}} \qquad \text{with} \qquad \begin{cases} i=0 & \text{if } t \le K_0 \\ i=1 & \text{if } t \ge K_0 \end{cases}$$

The Legendre transform of $F$ is defined as a function $F^* : \mathbb{R} \to \mathbb{R}$ with

$$F^*(z) = \sup_{r\in\mathbb{R}} \left(zr - F(r)\right) = z\,(F')^{-1}(z) - F\left((F')^{-1}(z)\right) = F'(r)\,r + F(r)\,,$$

with the choice $r = (F')^{-1}(z)$.

## Smoothing Effects

Let $\gamma, s \in [0,1]$ be the exponents appearing in assumption $(K2)$. Define

$$\vartheta_{i,\gamma} = \frac{1}{2s + (N+\gamma)(m_i - 1)} \qquad \text{with} \qquad m_i = \frac{1}{1-\mu_i} > 1$$

**Theorem. (Weighted $\mathrm{L}^1 - \mathrm{L}^\infty$ smoothing effect)** (M.B. & J. L. Vázquez)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_6 > 0$ s.t.

$$F\left(\|u(t)\|_{\mathrm{L}^\infty(\Omega)}\right) \le K_6 \, \frac{\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2sm_i\vartheta_{i,\gamma}}}{t^{m_i(N+\gamma)\vartheta_{i,\gamma}}}\,, \qquad \text{for all } 0 \le t_0 \le t\,,$$

with $i = 1$ if $t \ge \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ and $i = 0$ if $t \le \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ .

- A novelty is that we get instantaneous smoothing effects, new even when $s = 1$.
- The weighted smoothing effect is new even for $s = 1$.
- **Corollary.** Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$ .
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

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with $i = 1$ if $t \ge \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ and $i = 0$ if $t \le \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$.

- A novelty is that we get instantaneous smoothing effects, new even when $s = 1$.
- The weighted smoothing effect is new even for $s = 1$.
- **Corollary.** Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$.
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

$$(N+\gamma)\vartheta_{i,\gamma} = \frac{N+\gamma}{2s+(N+\gamma)(m_i-1)} < \frac{1}{m_i - 1}.$$

## Smoothing Effects

Let $\gamma, s \in [0,1]$ be the exponents appearing in assumption $(K2)$. Define

$$\vartheta_{i,\gamma} = \frac{1}{2s + (N+\gamma)(m_i - 1)} \qquad \text{with} \qquad m_i = \frac{1}{1-\mu_i} > 1$$

**Theorem. (Weighted $\mathrm{L}^1 - \mathrm{L}^\infty$ smoothing effect)** (M.B. & J. L. Vázquez)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_6 > 0$ s.t.

$$F\left(\|u(t)\|_{\mathrm{L}^\infty(\Omega)}\right) \le K_6 \, \frac{\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2sm_i\vartheta_{i,\gamma}}}{t^{m_i(N+\gamma)\vartheta_{i,\gamma}}}\,, \qquad \text{for all } 0 \le t_0 \le t\,,$$

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# Lower bounds and speed of propagation

**Theorem. (Lower absolute and boundary estimates)** (M.B. & J. L. Vázquez)

Let let $m>1$ and let $u\ge 0$ be a weak dual solution to the (CDP), corresponding to the initial datum $0\le u_0\in \mathrm{L}^1_{\Phi_1}(\Omega)$ . Then, there exist constants $l_0(\Omega), l_1(\Omega)>0$ , so that, setting

$$t_*=\frac{l_0(\Omega)}{\left(\int_\Omega u_0\Phi_1\,\mathrm{d}x\right)^{m-1}}\,,$$

we have that for all $t\ge t_*$ and all $x_0\in\Omega$, the following inequality holds when $0<\gamma\le 2sm/(m-1)$

$$u(t,x_0)\ge l_1(\Omega)\,\frac{\Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}\,.$$

When $\gamma>2sm/(m-1)$ the power of $\Phi_1$ changes to $2s/[(m-1)\gamma]<1/m$

The constants $l_0(\Omega), l_1(\Omega)>0$ , depend on $N,m,s$ and on $\Omega$ , but not on $u$ (or any norm of $u$); they have an explicit form. Recall that $\Phi_1\asymp\delta_\gamma$

## Remarks.

- This boundary behaviour is sharp because we have upper bounds with matching powers of $\Phi_1$.
- $t_*$ is an estimate the time that it takes "to fill the hole": if $u_0$ is concentrated close to the border (leaves an hole in the middle of $\Omega$), then $\int_\Omega u_0\Phi_1\,dx$ is small, therefore $t_*$ becomes very large, therefore it takes a lot of time to fill the hole.
  When $s = 1$ it is known that the PME has finite speed of propagation.
  **Question:** Is the speed of propagation finite when $s < 1$ ?
- These estimates can also be rewritten "á la" Aronson-Caffarelli:

$$\text{either} \quad t \le t_* = \frac{l_0}{\left(\int_\Omega u_0\Phi_1\,dx\right)^{m-1}}, \quad \text{or} \quad u(t,x_0) \ge l_1 \frac{\Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \quad \forall t \ge t_*,$$

  which gives, for all $t \ge 0$ and all $x_0 \in \Omega$:

$$u(t,x_0) \ge \frac{l_1\Phi_1(x_0)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}\left[1 - \left(\frac{t_*}{t}\right)^{\frac{1}{m-1}}\right].$$

- *Open problem:* find precise lower bounds for small times, $0 < t < t_*$.
- *Solved for RFL, with $s < 1$:* precise lower bounds for small times proven for Restricted-type Fractional Laplacians (on any domain), by MB, A. Figalli and X. Ros-Oton.

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# Harnack Inequalities

Joining our upper and lower bounds we obtain

**Theorem. (Global Harnack Principle)** (M.B. & J. L. Vázquez)

There exist universal constants $H_0, H_1, l_0 > 0$ such that setting

$$t_* = l_0 \left( \int_\Omega u_0 \Phi_1 \, \mathrm{d}x \right)^{-(m-1)} ,$$

we have that for all $t \geq t_*$ and all $x \in \Omega$, when $0 < \gamma \leq 2sm/(m-1)$

$$H_0 \, \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq H_1 \, \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}$$

When $\gamma > 2sm/(m-1)$ the power of $\Phi_1$ changes to $2s/[(m-1)\gamma] < 1/m$

Recall that $\Phi_1 \asymp \mathrm{dist}(\cdot\,, \partial\Omega)^\gamma$ , is the first eigenfunction of $\mathcal{L}$.

**Remarks.**

- This inequality implies local Harnack inequalities of elliptic type
- As a corollary we get the sharp asymptotic behaviour
- For $s = 1$ similar results by Aronson and Peletier [JDE, 1981] , Vázquez [Monatsh. Math. 2004]

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Solutions $u$ to the parabolic problem inherit the Harnack inequality for $\Phi_1$:

$$\sup_{x\in B_R(x_0)} \Phi_1(x) \le \mathcal{H} \inf_{x\in B_R(x_0)} \Phi_1(x) \qquad \forall B_R(x_0) \in \Omega.$$

**Theorem. (Local Harnack Inequalities of Elliptic Type)** (M.B. & J. L. Vázquez)

There exist universal constants $H_0, H_1, l_0 > 0$ such that setting $t_* = l_0 \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have that for all $t \ge t_*$ and all $B_R(x_0) \in \Omega$:

$$\sup_{x\in B_R(x_0)} u(t,x) \le \frac{H_1\,\mathcal{H}^{\frac{1}{m}}}{H_0} \inf_{x\in B_R(x_0)} u(t,x)$$

**Corollary. (Local Harnack Inequalities of Backward Type)**

Under the runninig assumptions, for all $t \ge t_*$ and all $B_R(x_0) \in \Omega$, we have:

$$\sup_{x\in B_R(x_0)} u(t,x) \le 2\frac{H_1\,\mathcal{H}^{\frac{1}{m}}}{H_0} \inf_{x\in B_R(x_0)} u(t+h,x) \qquad \text{for all } 0 \le h \le t_* .$$

- Backward Harnack inequalities for the linear heat equation $s = 1$ and $m = 1$, by Fabes, Garofalo, Salsa [Ill. J. Math, 1986]
- For $s = 1$, Intrinsic (Forward) Harnack inequalities by DiBenedetto [ARMA, 1988], Daskalopoulos and Kenig [EMS Book, 2007], cf. also DiBenedetto, Gianazza, Vespri [Monograph, Springer, 2011].

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# Harnack Inequalities
# and
# Higher Regularity for RFL

## For the rest of the talk we deal with the special case:

$$\mathcal{L}(u)(x) = (-\Delta_{|\Omega})^s u(x) = C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{n+2s}} \, \mathrm{d}y$$

For the RFL we solve the problem of sharp lower bounds for small times. Recall that here $\gamma = s$ and $\Phi_1 \asymp \delta_\gamma = \mathrm{dist}(\cdot, \partial\Omega)^s$ .

**Theorem. (Global quantitative positivity)** (M.B., A. Figalli, X. Ros-Oton)

Let $m > 1$, $0 < s < 1$, and $N > 2s$. Let $\Omega$ be a bounded domain of class $C^{1,1}$, and let $u$ be a weak dual solution to the (CDP) corresponding to $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$ . Then the following bound holds true:

$$u(t,x) \ge \kappa \|u_0\|^m_{\mathrm{L}^1_{\Phi_1}(\Omega)}\, t\, \Phi_1(x)^{\frac{1}{m}}\,, \qquad \text{for all } 0 \le t \le t_* \text{ and all } x \in \overline{\Omega}\,,$$

where $t_* = l_0 \|u_0\|^{-(m-1)}_{\mathrm{L}^1_{\Phi_1}(\Omega)}$ and $l_0, \kappa > 0$ depend only on $N, s, m, \Omega$.

As a consequence, solutions to the (CDP) corresponding to nonnegative and nontrivial initial data, have *infinite speed of propagation.*

- No free boundaries when $s < 1$, contrary to the "local" case $s = 1$, cf. Barenblatt, Aronson, Caffarelli, Vázquez, Wolansky [...]
- Qualitative version of infinite speed of propagation for the Cauchy problem on $\mathbb{R}^N$ , by De Pablo, Quíros, Rodriguez, Vázquez [Adv. Math. 2011, CPAM 2012]
- Different from the so-called Caffarelli-Vázquez model (on $\mathbb{R}^N$) that has *finite speed of propagation* [ARMA 2011, DCDS 2011] and also Stan, del Teso Vázquez [CRAS 2014, NLTMA 2015, JDE 2015], cf. also Coxeter lecture by Caffarelli yesterday :)

**Theorem. (Global Harnack Principle for all times)**(M.B., A. Figalli, X. Ros-Oton)

Let $m > 1$, $0 < s < 1$, and $N > 2s$. Let $\Omega$ be a bounded domain of class $C^{1,1}$, and let $u$ be a weak dual solution to the (CDP) corresponding to $0 \le u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$. Let $t_*$ be as above. Then for all $t > 0$ and all $x \in \overline{\Omega}$

$$\underline{\kappa}\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}} \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \le u(t,x) \le \overline{\kappa}\, \frac{\Phi_1(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}},$$

where $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^s$, and $\overline{\kappa}, \underline{\kappa} > 0$ depend only on $N, s, m, \Omega$.

**Theorem. (Local Harnack inequalities for all times)**(M.B., A. Figalli, X. Ros-Oton)

Under the above assumptions, for all balls $B_R(x_0) \subset\subset \Omega$, we have

$$\sup_{x \in B_R(x_0)} u(t,x) \le \frac{\mathcal{H}}{\left(1 \wedge \frac{t}{t_*}\right)^{\frac{m}{m-1}}} \inf_{x \in B_R(x_0)} u(t,x), \qquad \text{for all } t > 0,$$

where $\mathcal{H} > 0$ depend only on $N, s, m, \Omega, \mathrm{dist}(B_R(x_0), \partial\Omega)$.

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## Hölder Regularity up to the boundary.

The following regularity results hold true under the running assumptions:

**(R)** Let $m > 1$, $0 < s < 1$, and $N > 2s$. Let $\Omega$ be a bounded domain of class $C^{1,1}$, and let $u$ be a solution to the (CDP) corresponding to a nonnegative initial datum $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.

**Theorem. (Hölder regularity up to the boundary)**(M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)**, then, for each $0 < t_0 < T$ we have

$$\|u\|_{C^{\frac{s}{m},\,\frac{1}{2m}}_{x,t}\left(\overline{\Omega}\times[t_0,T]\right)} \le C,$$

where $C$ depends only on $N, s, m, \Omega, t_0$, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

**Remarks.**

- Notice that the $C^{s/m}_x$ regularity up to the boundary is optimal, since we have that $u(t,x) \ge c(u_0,t)\mathrm{dist}(x,\partial\Omega)^{s/m}$, with $c(u_0,t) > 0$ for all $t > 0$, and therefore $u(t,\cdot) \notin C_x^{\frac{s}{m}+\epsilon}(\overline{\Omega})$ for any $\epsilon > 0$.
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- Previous result on $C^\alpha$ regularity by Athanasopoulos and Caffarelli [Adv. Math, 2010].

**Higher Regularity.** Under the running assumptions **(R)**, we prove interior $C^\infty$ regularity in the $x$-variable and interior $C^{1,\alpha}$ regularity in the $t$-variable

**Theorem. (Higher interior regularity in space)** (M.B., A. Figalli, X. Ros-Oton)

Under the running assumptions **(R)** , then $u \in C_x^\infty((0,\infty)\times\Omega)$.
More precisely, let $k \geq 1$ be any positive integer, and $d(x) = \mathrm{dist}(x, \partial\Omega)$, then, for any $t \geq t_0 > 0$ we have

$$\left|D_x^k u(t,x)\right| \leq C\,[d(x)]^{\frac{s}{m}-k},$$

where $C$ depends only on $N, s, m, k, \Omega, t_0$, and $\|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}$.

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Under the running assumptions **(R)** , then $u \in C_t^{1,\alpha}((0,\infty)\times\Omega)$ for some $\alpha > 0$ that depends only on $s$ and $m$. Moreover, for any compact set $K \subset\subset \Omega$, and any $0 < t_0 < T$, we have

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**Remarks.**

- A possible value for the exponent $\alpha$ in the previous theorem on time regularity is $\alpha = \min\left\{\frac{1}{2m}, 1-s\right\}$.
- Notice that the above regularity results imply that *solutions to (CDP) are classical* for any nonnegative initial datum $u_0 \in \mathrm{L}^1_{\Phi_1}(\Omega)$.
- Higher regularity in time is a difficult open problem. It is connected to higher order boundary regularity in $t$. To our knowledge also open for the local case $s = 1$.
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# The End

Thank You!!!

Merci Beaucoup!!!

Muchas Gracias!!!

## Asymptotic behaviour of nonnegative solutions

- **Convergence to the stationary profile**
- **Convergence with optimal rate**

## Convergence to the stationary profile

In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$ .

**Theorem. (Asymptotic behaviour)** (M.B. , Y. Sire, J. L. Vázquez)

There exists a unique nonnegative selfsimilar solution of the above Dirichlet Problem

$$U(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}},$$

for some bounded function $S : \Omega \to \mathbb{R}$. Let $u$ be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$)

$$\lim_{\tau\to\infty} \tau^{\frac{1}{m-1}} \|u(\tau, \cdot) - U(\tau, \cdot)\|_{L^\infty(\Omega)} = 0 .$$

The previous theorem admits the following corollary.

**Theorem. (Elliptic problem)** (M.B. , Y. Sire, J. L. Vázquez)

Let $m > 1$. There exists a unique weak dual solution to the elliptic problem

$$\begin{cases} \mathcal{L}(S^m) = \dfrac{S}{m-1} & \text{in } \Omega, \\ S(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Notice that the previous theorem is obtained in the present paper through a parabolic technique.

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## Convergence with optimal rate

**Theorem. (Sharp asymptotic with rates)** (M.B. , Y. Sire, J. L. Vázquez)

Let $u$ be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$) that there exist $t_0 > 0$ of the form

$$t_0 = \overline{k} \left[ \frac{\int_\Omega \Phi_1 \, \mathrm{d}x}{\int_\Omega u_0 \Phi_1 \, \mathrm{d}x} \right]^{m-1}$$

such that for all $t \geq t_0$ we have

$$\left\| \frac{u(t,\cdot)}{U(t,\cdot)} - 1 \right\|_{\mathrm{L}^\infty(\Omega)} \leq \frac{2}{m-1} \, \frac{t_0}{t_0 + t} \, .$$

The constant $\overline{k} > 0$ only depends on $m, N, s,$ and $|\Omega|$.

**Remarks.**

- We provide two different proofs of the above result.
- One proof is based on the construction of the so-called Friendly-Giant solution, namely the solution with initial data $u_0 = +\infty$ , and is based on the Global Harnack Principle of Part 4
- The second proof is based on a new Entropy method, which is based on a parabolic version of the Caffarelli-Silvestre extension.

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