Fractional nonlinear degenerate diffusion equations on bounded domains

Matteo Bonforte

Departamento de Matemáticas,
Universidad Autónoma de Madrid,
Campus de Cantoblanco
28049 Madrid, Spain

matteo.bonforte@uam.es

http://www.uam.es/matteo.bonforte

Workshop on
"Nonlocal Nonlinear Partial Differential Equations and Applications"
Anacapri, Italy, September 14-18, 2015

References:

[BV1] M. B., J. L. VÁZQUEZ, A Priori Estimates for Fractional Nonlinear Degenerate Diffusion Equations on bounded domains. *Arch. Rat. Mech. Anal. (2015).*

[BV2] M. B., J. L. VÁZQUEZ, Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part I. Existence, Uniqueness and Upper Bounds *To appear in Nonlin. Anal. TMA (2015).*

[BSV] M. B., Y. SIRE, J. L. VÁZQUEZ, Existence, Uniqueness and Asymptotic behaviour for fractional porous medium equations on bounded domains. *Discr. Cont. Dyn. Sys. (2015).*

[BV3] M. B., J. L. VÁZQUEZ, Fractional Nonlinear Degenerate Diffusion Equations on Bounded Domains Part II. Positivity, Boundary behaviour and Harnack inequalities. *In preparation (2015).*

Outline of the talk

- The setup of the problem
- Existence and uniqueness
- First pointwise estimates
- Upper Estimates
- Harnack Inequalities
- Asymptotic behaviour of nonnegative solutions

The setup of the problem

- Assumption on the operator $\mathcal{L}$ and on the nonlinearity F
- Mild Solutions and Monotonicity Estimates
- Assumption on the inverse operator $\mathcal{L}^{-1}$
- Examples of operators
- The "dual" formulation of the problem
- Existence and uniqueness of weak dual solutions

Homogeneous Dirichlet Problem for
Fractional Nonlinear Degenerate Diffusion Equations

$$(\mathrm{HDP})\qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $L^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of “degenerate” nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Introduction

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \ge 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $L^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $L^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

Introduction

Homogeneous Dirichlet Problem for Fractional Nonlinear Degenerate Diffusion Equations

$$\text{(HDP)} \qquad \begin{cases} u_t + \mathcal{L}\, F(u) = 0\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

where:

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and $N \geq 1$.
- The linear operator $\mathcal{L}$ will be:
  - sub-Markovian operator
  - densely defined in $\mathrm{L}^1(\Omega)$.

  A wide class of linear operators fall in this class:
  all fractional Laplacians on domains.
- The most studied nonlinearity is $F(u) = |u|^{m-1}u\,,$ with $m > 1$.
  We deal with Degenerate diffusion of Porous Medium type.
  More general classes of "degenerate" nonlinearities F are allowed.
- The homogeneous boundary condition is posed on the lateral boundary, which may take different forms, depending on the particular choice of the operator $\mathcal{L}$.

About the operator $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e, then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

Remark. These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:

- M. G. Crandall, T.M. Liggett. *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971) 265–298.
- M. Crandall, M. Pierre, *Regularizing Effectd for $u_t = A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194-212

About the operator $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e , then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

Remark. These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:

- M. G. Crandall, T.M. Liggett. *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971) 265–298.
- M. Crandall, M. Pierre, *Regularizing Effectd for $u_t = A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194-212

About the operator $\mathcal{L}$

The linear operator $\mathcal{L} : \mathrm{dom}(A) \subseteq \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ is assumed to be densely defined and *sub-Markovian*, more precisely satisfying $(A1)$ and $(A2)$ below:

(A1) $\mathcal{L}$ is m-accretive on $\mathrm{L}^1(\Omega)$,

(A2) If $0 \le f \le 1$ then $0 \le \mathrm{e}^{-t\mathcal{L}} f \le 1$, or equivalently,

(A2') If β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ with $0 \in \beta(0)$, $u \in \mathrm{dom}(\mathcal{L})$, $\mathcal{L}u \in \mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, $v \in \mathrm{L}^{p/(p-1)}(\Omega)$, $v(x) \in \beta(u(x))$ a.e , then

$$\int_\Omega v(x)\mathcal{L}u(x)\,\mathrm{d}x \ge 0$$

Remark. These assumptions are needed for existence (and uniqueness) of semigroup (mild) solutions for the nonlinear equation $u_t = \mathcal{L}F(u)$, through a variant of the celebrated Crandall-Liggett theorem, as done by Benilan, Crandall and Pierre:

- M. G. Crandall, T.M. Liggett. *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. **93** (1971) 265–298.
- M. Crandall, M. Pierre, *Regularizing Effectd for $u_t = A\varphi(u)$ in* L^1, J. Funct. Anal. **45**, (1982), 194-212

Assumption on the nonlinearity F

Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous and non-decreasing function, with $F(0) = 0$. Moreover, it satisfies the condition:

(N1) $F \in C^1(\mathbb{R} \setminus \{0\})$ and $F/F' \in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0, \mu_1 > 0$ s.t.

$$\frac{1}{m_1} = 1 - \mu_1 \le \left(\frac{F}{F'}\right)' \le 1 - \mu_0 = \frac{1}{m_0}$$

where F/F' is understood to vanish if $F(r) = F'(r) = 0$ or $r = 0$.

The main example will be

$$F(u) = |u|^{m-1}u, \qquad \text{with } m > 1, \quad \text{and} \qquad \mu_0 = \mu_1 = \frac{m-1}{m} < 1.$$

which corresponds to the nonlocal porous medium equation studied in $[BV1]$. A simple variant is the combination of two powers:

- m_0 gives the behaviour near $u = 0$
- m_1 gives the behaviour near $u = \infty$.

Assumption on the nonlinearity F

Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous and non-decreasing function, with $F(0) = 0$. Moreover, it satisfies the condition:

(N1) $F \in C^1(\mathbb{R} \setminus \{0\})$ and $F/F' \in \mathrm{Lip}(\mathbb{R})$ and there exists $\mu_0, \mu_1 > 0$ s.t.

$$\boxed{\frac{1}{m_1} = 1 - \mu_1 \le \left(\frac{F}{F'}\right)' \le 1 - \mu_0 = \frac{1}{m_0}}$$

where F/F' is understood to vanish if $F(r) = F'(r) = 0$ or $r = 0$.

The main example will be

$$F(u) = |u|^{m-1}u, \qquad \text{with } m > 1, \quad \text{and} \qquad \mu_0 = \mu_1 = \frac{m-1}{m} < 1.$$

which corresponds to the nonlocal porous medium equation studied in $[BV1]$.
A simple variant is the combination of two powers:

- m_0 gives the behaviour near $u = 0$
- m_1 gives the behaviour near $u = \infty$.

Existence of Mild Solutions and Monotonicity Estimates

Theorem (M. Crandall and M. Pierre, JFA 1982)

Let $\mathcal{L}$ satisfy (A1) and (A2) and let F as satisfy (N1). Then for all nonnegative $u_0 \in \mathrm{L}^1(\Omega)$, there exists a unique mild solution u to equation $u_t + \mathcal{L}F(u) = 0$, and the function

(1) $\quad t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x)) \qquad$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

Moreover, the semigroup is contractive on $\mathrm{L}^1(\Omega)$ and $u \in C([0,\infty) : \mathrm{L}^1(\Omega))$.

We notice that (1) is a weak formulation of the monotonicity inequality:

$$\partial_t u \geq -\frac{1}{\mu_0\, t}\frac{F(u)}{F'(u)}\,, \qquad \text{which implies} \qquad \partial_t u \geq -\frac{1-\mu_0}{\mu_0}\frac{u}{t}$$

or equivalently, that the function

(2) $\quad t \mapsto t^{\frac{1-\mu_0}{\mu_0}}\, u(t,x) \qquad$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

- P. Bénilan, M. Crandall. *Regularizing effects of homogeneous evolution equations.* Contr. to Anal. and Geom. Johns Hopkins Univ. Press, Baltimore, Md., 1981. 23-39.
- M. Crandall, M. Pierre, *Regularizing Effect for $u_t = A\varphi(u)$ in* L^1. JFA **45**, (1982).

Existence of Mild Solutions and Monotonicity Estimates

Theorem (M. Crandall and M. Pierre, JFA 1982)

Let $\mathcal{L}$ satisfy (A1) and (A2) and let F as satisfy (N1). Then for all nonnegative $u_0 \in \mathrm{L}^1(\Omega)$, there exists a unique mild solution u to equation $u_t + \mathcal{L}F(u) = 0$, and the function

(1) $\quad t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x)) \qquad$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

Moreover, the semigroup is contractive on $\mathrm{L}^1(\Omega)$ and $u \in C([0,\infty) : \mathrm{L}^1(\Omega))$.

We notice that (1) is a weak formulation of the monotonicity inequality:

$$\partial_t u \ge -\frac{1}{\mu_0\, t}\frac{F(u)}{F'(u)}, \qquad \text{which implies} \qquad \partial_t u \ge -\frac{1-\mu_0}{\mu_0}\,\frac{u}{t}$$

or equivalently, that the function

(2) $\quad t \mapsto t^{\frac{1-\mu_0}{\mu_0}} u(t,x) \qquad$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

- P. Bénilan, M. Crandall. *Regularizing effects of homogeneous evolution equations.* Contr. to Anal. and Geom. Johns Hopkins Univ. Press, Baltimore, Md., 1981. 23-39.
- M. Crandall, M. Pierre, *Regularizing Effect for $u_t = A\varphi(u)$ in* L^1. JFA **45**, (1982).

Existence of Mild Solutions and Monotonicity Estimates

Theorem (M. Crandall and M. Pierre, JFA 1982)

Let $\mathcal{L}$ satisfy (A1) and (A2) and let F as satisfy (N1). Then for all nonnegative $u_0 \in \mathrm{L}^1(\Omega)$, there exists a unique mild solution u to equation $u_t + \mathcal{L}F(u) = 0$, and the function

(1) $\quad t \mapsto t^{\frac{1}{\mu_0}} F(u(t,x)) \qquad$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

Moreover, the semigroup is contractive on $\mathrm{L}^1(\Omega)$ and $u \in C([0,\infty) : \mathrm{L}^1(\Omega))$.

We notice that (1) is a weak formulation of the monotonicity inequality:

$$\partial_t u \ge -\frac{1}{\mu_0\, t} \frac{F(u)}{F'(u)}, \qquad \text{which implies} \qquad \partial_t u \ge -\frac{1-\mu_0}{\mu_0} \frac{u}{t}$$

or equivalently, that the function

(2) $\quad t \mapsto t^{\frac{1-\mu_0}{\mu_0}} u(t,x) \qquad$ is nondecreasing in $t > 0$ for a.e. $x \in \Omega$.

- P. Bénilan, M. Crandall. *Regularizing effects of homogeneous evolution equations.* Contr. to Anal. and Geom. Johns Hopkins Univ. Press, Baltimore, Md., 1981. 23-39.
- M. Crandall, M. Pierre, *Regularizing Effect for $u_t = A\varphi(u)$ in* L^1. JFA **45**, (1982).

Assumption on the inverse operator $\mathcal{L}^{-1}$

Assumptions on the inverse of $\mathcal{L}$

We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1}\,:\,\mathrm{L}^1(\Omega)\to\mathrm{L}^1(\Omega)$ with a kernel $\mathbb{K}$ such that

$$\mathcal{L}^{-1}f(x)=\int_\Omega \mathbb{K}(x,y)f(y)\,\mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s\in(0,1]$ and $c_{i,\Omega}>0$

$$\text{(K1)}\qquad 0\le\mathbb{K}(x,y)\le\frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

$$\text{(K2)}\quad c_{0,\Omega}\delta_\gamma(x)\,\delta_\gamma(y)\le\mathbb{K}(x,y)\le\frac{c_{1,\Omega}}{|x-y|^{N-2s}}\left(\frac{\delta_\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma}\wedge 1\right)$$

where

$$\delta_\gamma(x):=\mathrm{dist}(x,\partial\Omega)^\gamma\,.$$

When the operator $\mathcal{L}$ has a first nonnegative eigenfunction Φ_1 , we can rewrite (K2) as

$$\text{(K3)}\quad c_{0,\Omega}\Phi_1(x)\Phi_1(y)\le\mathbb{K}(x,y)\le\frac{c_{1,\Omega}}{|x-x_0|^{N-2s}}\left(\frac{\Phi_1(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma}\wedge 1\right)$$

Indeed, (K2) implies that $\Phi_1\asymp\mathrm{dist}(\cdot,\partial\Omega)^\gamma=\delta_\gamma$.

Assumptions on the inverse of $\mathcal{L}$

We will assume that the operator $\mathcal{L}$ has an inverse $\mathcal{L}^{-1}: \mathrm{L}^1(\Omega) \to \mathrm{L}^1(\Omega)$ with a kernel $\mathbb{K}$ such that

$$\mathcal{L}^{-1} f(x) = \int_\Omega \mathbb{K}(x,y) f(y)\, \mathrm{d}y\,,$$

and that satisfies (one of) the following estimates for some $\gamma, s \in (0,1]$ and $c_{i,\Omega} > 0$

$$\text{(K1)} \qquad 0 \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}$$

$$\text{(K2)} \quad c_{0,\Omega}\delta_\gamma(x)\,\delta_\gamma(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}}\left(\frac{\delta_\gamma(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma}\wedge 1\right)$$

where

$$\delta_\gamma(x) := \mathrm{dist}(x, \partial\Omega)^\gamma\,.$$

When the operator $\mathcal{L}$ has a first nonnegative eigenfunction Φ_1, we can rewrite (K2) as

$$\text{(K3)} \quad c_{0,\Omega}\Phi_1(x)\Phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-x_0|^{N-2s}}\left(\frac{\Phi_1(x)}{|x-y|^\gamma}\wedge 1\right)\left(\frac{\Phi_1(y)}{|x-y|^\gamma}\wedge 1\right)$$

Indeed, (K2) implies that $\Phi_1 \asymp \mathrm{dist}(\cdot, \partial\Omega)^\gamma = \delta_\gamma$.

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right)\frac{dt}{t^{1+s}}.$$

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right)\frac{dt}{t^{1+s}}.$$

Reminder about the fractional Laplacian operator on $\mathbb{R}^N$

We have several equivalent definitions for $(-\Delta_{\mathbb{R}^N})^s$:

1. By means of **Fourier Transform**,

$$((-\Delta_{\mathbb{R}^N})^s f)\widehat{\ }(\xi) = |\xi|^{2s}\hat{f}(\xi)\,.$$

   This formula can be used for positive and negative values of s.

2. By means of an **Hypersingular Kernel**:
   if $0 < s < 1$, we can use the representation

$$(-\Delta_{\mathbb{R}^N})^s g(x) = c_{N,s}\ \text{P.V.}\int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z,$$

   where $c_{N,s} > 0$ is a normalization constant.

3. **Spectral definition**, in terms of the heat semigroup associated to the standard Laplacian operator:

$$(-\Delta_{\mathbb{R}^N})^s g(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_{\mathbb{R}^N}} g(x) - g(x)\right)\frac{dt}{t^{1+s}}.$$

Examples of operators $\mathcal{L}$

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\,\mathrm{d}x\,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1 \right), \qquad \text{with } \gamma = 1$$

Examples of operators $\mathcal{L}$

The Spectral Fractional Laplacian operator (SFL)

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}.$$

- Δ_Ω is the classical Dirichlet Laplacian on the domain Ω
- EIGENVALUES: $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_j \le \lambda_{j+1} \le \ldots$ and $\lambda_j \asymp j^{2/N}$.
- EIGENFUNCTIONS: ϕ_j are as smooth as the boundary of Ω allows, namely when $\partial\Omega$ is C^k, then $\phi_j \in C^\infty(\Omega) \cap C^k(\overline{\Omega})$ for all $k \in \mathbb{N}$.

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\, \mathrm{d}x \,, \qquad \text{with} \qquad \|\phi_j\|_{\mathrm{L}^2(\Omega)} = 1\,.$$

Lateral boundary conditions for the SFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times \partial\Omega\,.$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) \,, \quad \text{with } \gamma = 1$$

Examples of operators $\mathcal{L}$

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The Restricted Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s} \text{ P.V.} \int_{\mathbb{R}^N} \frac{g(x)-g(z)}{|x-z|^{N+2s}}\,\mathrm{d}z\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0,1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$.

Lateral boundary conditions for the RFL

$$u(t,x) = 0\,, \qquad \text{in } (0,\infty) \times (\mathbb{R}^N \setminus \Omega)\,.$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1\right)\left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1\right), \qquad \text{with } \gamma = s$$

Examples of operators $\mathcal{L}$

Definition via the hypersingular kernel in $\mathbb{R}^N$, "restricted" to functions that are zero outside Ω.

The Restricted Fractional Laplacian operator (RFL)

$$(-\Delta_{|\Omega})^s g(x) = c_{N,s}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{g(x) - g(z)}{|x - z|^{N+2s}}\, dz\,, \qquad \text{with } \operatorname{supp}(g) \subseteq \overline{\Omega}\,.$$

where $s \in (0, 1)$ and $c_{N,s} > 0$ is a normalization constant.

- $(-\Delta_{|\Omega})^s$ is a self-adjoint operator on $L^2(\Omega)$ with a discrete spectrum:
- EIGENVALUES: $0 < \overline{\lambda}_1 \le \overline{\lambda}_2 \le \ldots \le \overline{\lambda}_j \le \overline{\lambda}_{j+1} \le \ldots$ and $\overline{\lambda}_j \asymp j^{2s/N}$.
  Eigenvalues of the RFL are smaller than the ones of SFL: $\overline{\lambda}_j \le \lambda_j^s$ for all $j \in \mathbb{N}$.
- EIGENFUNCTIONS: $\overline{\phi}_j$ are the normalized eigenfunctions, are only Hölder continuous up to the boundary, namely $\overline{\phi}_j \in C^s(\overline{\Omega})$.

Lateral boundary conditions for the RFL

$$u(t, x) = 0\,, \qquad \text{in } (0, \infty) \times \left(\mathbb{R}^N \setminus \Omega\right).$$

The Green function of SFL satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x, y) \asymp \frac{1}{|x - y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x - y|^\gamma} \wedge 1\right) \left(\frac{\delta_\gamma(y)}{|x - y|^\gamma} \wedge 1\right), \qquad \text{with } \gamma = s$$

We can also treat more general operators of SFL and RFL type:
Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij}\partial_j) \,,$$

with bounded measurable coefficients, which are uniformly elliptic. The uniform ellipticity allows to build a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with discrete spectrum (λ_k, ϕ_k). Using the spectral theorem, we can construct the spectral power of such operator, defined as follows:

$$\mathcal{L}f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x) \qquad \text{where} \qquad \hat{f}_k = \int_\Omega f(x)\phi_k(x)\,\mathrm{d}x \,.$$

Such operators enjoy (K3) estimates with $\gamma = 1$

$$\text{(K3)} \qquad c_{0,\Omega}\phi_1(x)\,\phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\phi_1(x)}{|x-y|} \wedge 1\right)\left(\frac{\phi_1(y)}{|x-y|} \wedge 1\right)$$

We can treat the class of intrinsically ultra-contractive operators introduced by Davies and Simon.

We can also treat more general operators of SFL and RFL type:
Spectral powers of uniformly elliptic operators. Consider a linear operator A in divergence form:

$$A = \sum_{i,j=1}^{N} \partial_i (a_{ij} \partial_j) \,,$$

with bounded measurable coefficients, which are uniformly elliptic. The uniform ellipticity allows to build a self-adjoint operator on $\mathrm{L}^2(\Omega)$ with discrete spectrum (λ_k, ϕ_k). Using the spectral theorem, we can construct the spectral power of such operator, defined as follows:

$$\mathcal{L} f(x) := A^s f(x) := \sum_{k=1}^{\infty} \lambda_k^s \hat{f}_k \phi_k(x) \qquad \text{where} \qquad \hat{f}_k = \int_\Omega f(x) \phi_k(x) \, \mathrm{d}x \,.$$

Such operators enjoy (K3) estimates with $\gamma = 1$

$$\text{(K3)} \qquad c_{0,\Omega} \phi_1(x)\, \phi_1(y) \le \mathbb{K}(x,y) \le \frac{c_{1,\Omega}}{|x-y|^{N-2s}} \left(\frac{\phi_1(x)}{|x-y|} \wedge 1 \right) \left(\frac{\phi_1(y)}{|x-y|} \wedge 1 \right)$$

We can treat the class of intrinsically ultra-contractive operators introduced by Davies and Simon.

Fractional operators with general kernels. Consider integral operators of the following form

$$\mathcal{L}f(x) = \text{P.V.} \int_{\mathbb{R}^N} (f(x+y) - f(y)) \frac{K(x,y)}{|x-y|^{N+2s}} \, \mathrm{d}y \, .$$

where K is a measurable symmetric function bounded between two positive constants, satisfying

$$\big|K(x,y) - K(x,x)\big| \, \chi_{|x-y|<1} \le c|x-y|^{\sigma} \, , \qquad \text{with } 0 < s < \sigma \le 1 \, ,$$

for some positive $c > 0$. We can allow even more general kernels.
The Green function satisfies a stronger assumption than (K2) or (K3), i.e.

$$\text{(K4)} \qquad \mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) \, , \quad \text{with } \gamma = s$$

Censored fractional Laplacians and operators with general kernels.
Introduced by Bogdan et al. in 2003.

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, dy \,, \qquad \text{with } \frac{1}{2} < s < 1 \,,$$

where $a(x,y)$ is a measurable symmetric function bounded between two positive constants, satisfying some further assumptions; a sufficient assumption is $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption (K_4):

$$\mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) \,, \qquad \text{with } \gamma = s - \frac{1}{2} \,.$$

This bounds has been proven by Chen, Kim and Song (2010).
Remarks.

- This is a third model of Dirichlet fractional Laplacian $\left[a(x,y) = C_{N,s} \right]$. This is **not equivalent** to SFL nor to RFL.
- Roughly speaking, when $s \in (0, 1/2]$ this corresponds to "Neumann" boundary conditions.

Censored fractional Laplacians and operators with general kernels.
Introduced by Bogdan et al. in 2003.

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}}\, \mathrm{d}y\,, \qquad \text{with } \frac{1}{2} < s < 1\,,$$

where $a(x,y)$ is a measurable symmetric function bounded between two positive constants, satisfying some further assumptions; a sufficient assumption is $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption (K_4):

$$\mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1 \right)\,, \qquad \text{with } \gamma = s - \frac{1}{2}\,.$$

This bounds has been proven by Chen, Kim and Song (2010).
Remarks.

- This is a third model of Dirichlet fractional Laplacian $\left[a(x,y) = C_{N,s}\right]$. This is **not equivalent** to SFL nor to RFL.
- Roughly speaking, when $s \in (0, 1/2]$ this corresponds to "Neumann" boundary conditions.

Censored fractional Laplacians and operators with general kernels.
Introduced by Bogdan et al. in 2003.

$$\mathcal{L}f(x) = \text{P.V.} \int_\Omega (f(x) - f(y)) \frac{a(x,y)}{|x-y|^{N+2s}} \, dy \,, \qquad \text{with } \frac{1}{2} < s < 1 \,,$$

where $a(x,y)$ is a measurable symmetric function bounded between two positive constants, satisfying some further assumptions; a sufficient assumption is $a \in C^1(\overline{\Omega} \times \overline{\Omega})$.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption (K_4):

$$\mathbb{K}(x,y) \asymp \frac{1}{|x-y|^{N-2s}} \left(\frac{\delta_\gamma(x)}{|x-y|^\gamma} \wedge 1 \right) \left(\frac{\delta_\gamma(y)}{|x-y|^\gamma} \wedge 1 \right) \,, \qquad \text{with } \gamma = s - \frac{1}{2} \,.$$

This bounds has been proven by Chen, Kim and Song (2010).
Remarks.

- This is a third model of Dirichlet fractional Laplacian $\left[a(x,y) = C_{N,s} \right]$. This is **not equivalent** to SFL nor to RFL.
- Roughly speaking, when $s \in (0, 1/2]$ this corresponds to "Neumann" boundary conditions.

Sums of two fractional operators. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma\,, \qquad \text{with } 0 < \sigma < s \le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption $(K4)$ with $\gamma = s$. The limit case $s = 1$ and $\sigma \in (0,1)$ satisfies the strongest assumption $(K4)$ with $\gamma = s = 1$.

The bounds $(K4)$ for the Green function proven by Chen, Kim, Song (2012).

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0\,,$$

where

$$A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x)\cdot y\chi_{|y|\le 1}\right)\chi_{|y|\le 1}\mathrm{d}\nu(y)\,.$$

where the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies

$$\int_{\mathbb{R}^N} 1 \vee |x|^2\, \mathrm{d}\nu(y) < \infty\,.$$

More precisely $\mathrm{d}\nu(y) = j(y)\,\mathrm{d}y$ with $j : (0,\infty) \to (0,\infty)$ is given by

$$j(r) := \int_0^\infty \frac{\mathrm{e}^{r^2/(4t)}}{(4\pi t)^{N/2}}\mathrm{d}\mu(r) \qquad \text{with} \qquad \int_0^\infty (1 \vee t)\mathrm{d}\mu(t) < \infty\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) in the form with $s = 1$ and $\gamma = 1$.
The bounds for the Green function have been proven by Chen, Kim, Song, Vondracek (2013).

Sums of two fractional operators. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma\,, \qquad \text{with } 0 < \sigma < s \le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption $(K4)$ with $\gamma = s$. The limit case $s = 1$ and $\sigma \in (0,1)$ satisfies the strongest assumption $(K4)$ with $\gamma = s = 1$.

The bounds $(K4)$ for the Green function proven by Chen, Kim, Song (2012).

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0\,,$$

where

$$A_s f(x) = \text{P.V.}\int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x)\cdot y\chi_{|y|\le 1}\right)\chi_{|y|\le 1}\mathrm{d}\nu(y)\,.$$

where the measure ν on $\mathbb{R}^N\setminus\{0\}$ is invariant under rotations around origin and satisfies

$$\int_{\mathbb{R}^N} 1\vee|x|^2\,\mathrm{d}\nu(y) < \infty\,.$$

More precisely $\mathrm{d}\nu(y) = j(y)\,\mathrm{d}y$ with $j:(0,\infty)\to(0,\infty)$ is given by

$$j(r) := \int_0^\infty \frac{\mathrm{e}^{r^2/(4t)}}{(4\pi t)^{N/2}}\mathrm{d}\mu(r) \qquad \text{with} \qquad \int_0^\infty (1\vee t)\mathrm{d}\mu(t) < \infty\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) in the form with $s=1$ and $\gamma = 1$.
The bounds for the Green function have been proven by Chen, Kim, Song, Vondracek (2013).

Sums of two fractional operators. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma\,, \qquad \text{with } 0 < \sigma < s \le 1\,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption $(K4)$ with $\gamma = s$. The limit case $s = 1$ and $\sigma \in (0,1)$ satisfies the strongest assumption $(K4)$ with $\gamma = s = 1$.
The bounds $(K4)$ for the Green function proven by Chen, Kim, Song (2012).
Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s\,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0\,,$$

where

$$A_s f(x) = \text{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x)\cdot y\chi_{|y|\le 1}\right)\chi_{|y|\le 1}\mathrm{d}\nu(y)\,.$$

where the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies

$$\int_{\mathbb{R}^N} 1 \vee |x|^2\,\mathrm{d}\nu(y) < \infty\,.$$

More precisely $\mathrm{d}\nu(y) = j(y)\,\mathrm{d}y$ with $j : (0,\infty) \to (0,\infty)$ is given by

$$j(r) := \int_0^\infty \frac{e^{r^2/(4t)}}{(4\pi t)^{N/2}}\mathrm{d}\mu(r) \qquad \text{with} \qquad \int_0^\infty (1\vee t)\mathrm{d}\mu(t) < \infty\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) in the form with $s = 1$ and $\gamma = 1$.
The bounds for the Green function have been proven by Chen, Kim, Song, Vondracek (2013).

Sums of two fractional operators. Operators of the form

$$\mathcal{L} = (\Delta_{|\Omega})^s + (\Delta_{|\Omega})^\sigma \,, \qquad \text{with } 0 < \sigma < s \le 1 \,,$$

where $(\Delta_{|\Omega})^s$ is the RFL. The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies the strongest assumption $(K4)$ with $\gamma = s$. The limit case $s = 1$ and $\sigma \in (0,1)$ satisfies the strongest assumption $(K4)$ with $\gamma = s = 1$.

The bounds $(K4)$ for the Green function proven by Chen, Kim, Song (2012).

Sum of the Laplacian and operators with general kernels. In the case

$$\mathcal{L} = a\Delta + A_s \,, \qquad \text{with } 0 < s < 1 \quad \text{and} \quad a \ge 0 \,,$$

where

$$A_s f(x) = \mathrm{P.V.} \int_{\mathbb{R}^N} \left(f(x+y) - f(y) - \nabla f(x) \cdot y \chi_{|y|\le 1} \right) \chi_{|y|\le 1} \mathrm{d}\nu(y) \,.$$

where the measure ν on $\mathbb{R}^N \setminus \{0\}$ is invariant under rotations around origin and satisfies

$$\int_{\mathbb{R}^N} 1 \vee |x|^2 \, \mathrm{d}\nu(y) < \infty \,.$$

More precisely $\mathrm{d}\nu(y) = j(y)\,\mathrm{d}y$ with $j : (0,\infty) \to (0,\infty)$ is given by

$$j(r) := \int_0^\infty \frac{\mathrm{e}^{r^2/(4t)}}{(4\pi t)^{N/2}} \mathrm{d}\mu(r) \qquad \text{with} \qquad \int_0^\infty (1 \vee t) \mathrm{d}\mu(t) < \infty \,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) in the form with $s = 1$ and $\gamma = 1$.

The bounds for the Green function have been proven by Chen, Kim, Song, Vondracek (2013).

Schrödinger equations for non-symmetric diffusions. In the case

$$\mathcal{L} = A + \mu \cdot \nabla + \nu\,,$$

where A is uniformly elliptic both in divergence and non-divergence form:

$$A_1 = \frac{1}{2}\sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j) \qquad \text{or} \qquad A_2 = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}\partial_{ij}\,,$$

We assume C^1 coefficient a_{ij}, uniformly elliptic.
Moreover, μ, ν are measures belonging to suitable Kato classes.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s = 1$.
Gradient perturbation of restricted fractional Laplacians. In the case

$$\mathcal{L} = (\Delta_{|\Omega})^s + b \cdot \nabla$$

where b is a vector valued function belonging to a suitable Kato class.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.
Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s, \qquad \text{with } c > 0\,, \text{ and } 0 < s \le 1\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.
The bounds for the Green function have been proven by Chen, Kim, Song (2007, 2011, 2012).

Schrödinger equations for non-symmetric diffusions. In the case

$$\mathcal{L} = A + \mu \cdot \nabla + \nu\,,$$

where A is uniformly elliptic both in divergence and non-divergence form:

$$A_1 = \frac{1}{2}\sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j) \qquad \text{or} \qquad A_2 = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}\partial_{ij}\,,$$

We assume C^1 coefficient a_{ij}, uniformly elliptic.
Moreover, μ, ν are measures belonging to suitable Kato classes.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s = 1$.
Gradient perturbation of restricted fractional Laplacians. In the case

$$\mathcal{L} = (\Delta_{|\Omega})^s + b \cdot \nabla$$

where b is a vector valued function belonging to a suitable Kato class.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.
Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s\,, \qquad \text{with } c > 0\,, \text{ and } 0 < s \le 1\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.
The bounds for the Green function have been proven by Chen, Kim, Song (2007, 2011, 2012).

Schrödinger equations for non-symmetric diffusions. In the case

$$\mathcal{L} = A + \mu \cdot \nabla + \nu \,,$$

where A is uniformly elliptic both in divergence and non-divergence form:

$$A_1 = \frac{1}{2}\sum_{i,j=1}^{N} \partial_i (a_{ij}\partial_j) \qquad \text{or} \qquad A_2 = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}\partial_{ij} \,,$$

We assume C^1 coefficient a_{ij}, uniformly elliptic.
Moreover, μ, ν are measures belonging to suitable Kato classes.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s = 1$.
Gradient perturbation of restricted fractional Laplacians. In the case

$$\mathcal{L} = (\Delta_{|\Omega})^s + b \cdot \nabla$$

where b is a vector valued function belonging to a suitable Kato class.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.
Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s \,, \qquad \text{with } c > 0\,, \text{ and } 0 < s \le 1\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.
The bounds for the Green function have been proven by Chen, Kim, Song (2007, 2011, 2012).

Schrödinger equations for non-symmetric diffusions. In the case

$$\mathcal{L} = A + \mu \cdot \nabla + \nu \,,$$

where A is uniformly elliptic both in divergence and non-divergence form:

$$A_1 = \frac{1}{2}\sum_{i,j=1}^{N} \partial_i(a_{ij}\partial_j) \qquad \text{or} \qquad A_2 = \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}\partial_{ij} \,,$$

We assume C^1 coefficient a_{ij}, uniformly elliptic.
Moreover, μ, ν are measures belonging to suitable Kato classes.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s = 1$.
Gradient perturbation of restricted fractional Laplacians. In the case

$$\mathcal{L} = (\Delta_{|\Omega})^s + b \cdot \nabla$$

where b is a vector valued function belonging to a suitable Kato class.
The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.
Relativistic stable processes. In the case

$$\mathcal{L} = c - \left(c^{1/s} - \Delta\right)^s \,, \qquad \text{with } c > 0\,, \text{ and } 0 < s \le 1\,.$$

The Green function $\mathbb{K}(x,y)$ of $\mathcal{L}$ satisfies assumption (K_4) with $\gamma = s$.
The bounds for the Green function have been proven by Chen, Kim, Song (2007, 2011, 2012).

The "dual" formulation of the problem

Recall the homogeneous Dirichlet problem:

$$
\text{(CDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, F(u)\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}
$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$
\partial_t U = -F(u)\,,
$$

where

$$
U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega \mathbb{K}(x,y)u(t,y)\,\mathrm{d}y\,.
$$

This formulation encodes the lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.

The "dual" formulation of the problem

Recall the homogeneous Dirichlet problem:

$$\text{(CDP)} \qquad \begin{cases} \partial_t u = -\mathcal{L}\, F(u)\,, & \text{in } (0,+\infty)\times\Omega \\ u(0,x) = u_0(x)\,, & \text{in } \Omega \\ u(t,x) = 0\,, & \text{on the lateral boundary.} \end{cases}$$

We can formulate a "dual problem", using the inverse $\mathcal{L}^{-1}$ as follows

$$\partial_t U = -F(u)\,,$$

where

$$U(t,x) := \mathcal{L}^{-1}[u(t,\cdot)](x) = \int_\Omega \mathbb{K}(x,y) u(t,y)\,\mathrm{d}y\,.$$

This formulation encodes the lateral boundary conditions in the inverse operator $\mathcal{L}^{-1}$.

Remark. This formulation has been used before by Pierre, Vázquez [...] to prove (in the $\mathbb{R}^N$ case) uniqueness of the "fundamental solution", i.e. the solution corresponding to $u_0 = \delta_{x_0}$, known as the Barenblatt solution.

The "dual" formulation of the problem

Recall that

$$\|f\|_{L^1_{\delta_\gamma}(\Omega)} = \int_\Omega f(x)\delta_\gamma(x)\,dx\,, \quad \text{and} \quad L^1_{\delta_\gamma}(\Omega) := \left\{ f : \Omega \to \mathbb{R} \,\middle|\, \|f\|_{L^1_{\delta_\gamma}(\Omega)} < \infty \right\}.$$

Weak Dual Solutions

A function u is a *weak dual* solution to the Dirichlet Problem for $\partial_t + \mathcal{L}^{-1}F(u) = 0$ in $Q_T = (0,T) \times \Omega$ if:

- $u \in C((0,T) : L^1_{\delta_\gamma}(\Omega))\,,\, F(u) \in L^1\left((0,T) : L^1_{\delta_\gamma}(\Omega)\right)$;
- The following identity holds for every $\psi/\delta_\gamma \in C^1_c((0,T) : L^\infty(\Omega))$:

$$\int_0^T \int_\Omega \mathcal{L}^{-1}(u)\,\frac{\partial \psi}{\partial t}\,dx\,dt - \int_0^T \int_\Omega F(u)\,\psi\,dx\,dt = 0.$$

Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** is a weak dual solution to Equation $\partial_t + \mathcal{L}^{-1}F(u) = 0$ such that moreover

$$u \in C([0,T) : L^1_{\delta_\gamma}(\Omega)) \qquad \text{and} \qquad u(0,x) = u_0 \in L^1_{\delta_\gamma}(\Omega)\,.$$

The "dual" formulation of the problem

Recall that

$$\|f\|_{L^1_{\delta_\gamma}(\Omega)} = \int_\Omega f(x)\delta_\gamma(x)\,dx\,, \quad \text{and} \quad L^1_{\delta_\gamma}(\Omega) := \left\{f:\Omega\to\mathbb{R}\;\middle|\;\|f\|_{L^1_{\delta_\gamma}(\Omega)}<\infty\right\}.$$

Weak Dual Solutions

A function u is a *weak dual* solution to the Dirichlet Problem for $\partial_t + \mathcal{L}^{-1}F(u) = 0$ in $Q_T = (0,T)\times\Omega$ if:

- $u \in C((0,T):L^1_{\delta_\gamma}(\Omega))\,,\; F(u)\in L^1\left((0,T):L^1_{\delta_\gamma}(\Omega)\right)$;
- The following identity holds for every $\psi/\delta_\gamma \in C^1_c((0,T):L^\infty(\Omega))$:

$$\int_0^T\int_\Omega \mathcal{L}^{-1}(u)\,\frac{\partial\psi}{\partial t}\,dx\,dt - \int_0^T\int_\Omega F(u)\,\psi\,dx\,dt = 0.$$

Weak Dual Solutions for the Cauchy Dirichlet Problem (CDP)

A *weak dual* solution to the Cauchy-Dirichlet problem **(CDP)** is a weak dual solution to Equation $\partial_t + \mathcal{L}^{-1}F(u) = 0$ such that moreover

$$u\in C([0,T):L^1_{\delta_\gamma}(\Omega)) \qquad \text{and} \qquad u(0,x) = u_0 \in L^1_{\delta_\gamma}(\Omega)\,.$$

The "dual" formulation of the problem

We will use a special class of weak dual solutions:

The class $\mathcal{S}_p$ of weak dual solutions

We consider a class $\mathcal{S}_p$ of nonnegative weak dual solutions u to the (HDP) with initial data in $0 \le u_0 \in \mathrm{L}^1_{\delta_\gamma}(\Omega)$, such that:

(i) the map $u_0 \mapsto u(t)$ is "almost" order preserving in $\mathrm{L}^1_{\delta_\gamma}(\Omega)$, namely $\exists\, C > 0$ s.t.

$$\|u(t)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)} \le C\,\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)} \qquad \text{for all } 0 \le t_0 \le t.$$

(ii) for all $t > 0$ we have $u(t) \in L^p(\Omega)$ for some $p \ge 1$.

We prove that the mild solutions of Crandall and Pierre fall into this class:

Proposition. Semigroup solutions are weak dual solutions

Let u be the unique semigroup (mild) solution to the (CDP) corresponding to the initial datum $u_0 \in \mathrm{L}^p(\Omega)$ with $p \ge 1$. Then u is a weak dual solution of (CDP) and is contained in the class $\mathcal{S}_p$.

The "dual" formulation of the problem

We will use a special class of weak dual solutions:

The class $\mathcal{S}_p$ of weak dual solutions

We consider a class $\mathcal{S}_p$ of nonnegative weak dual solutions u to the (HDP) with initial data in $0 \le u_0 \in \mathrm{L}^1_{\delta_\gamma}(\Omega)$, such that:

(i) the map $u_0 \mapsto u(t)$ is "almost" order preserving in $\mathrm{L}^1_{\delta_\gamma}(\Omega)$, namely $\exists\, C > 0$ s.t.

$$\|u(t)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)} \le C\,\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)} \qquad \text{for all } 0 \le t_0 \le t.$$

(ii) for all $t > 0$ we have $u(t) \in L^p(\Omega)$ for some $p \ge 1$.

We prove that the mild solutions of Crandall and Pierre fall into this class:

Proposition. Semigroup solutions are weak dual solutions

Let u be the unique semigroup (mild) solution to the (CDP) corresponding to the initial datum $u_0 \in \mathrm{L}^p(\Omega)$ with $p \ge 1$. Then u is a weak dual solution of (CDP) and is contained in the class $\mathcal{S}_p$.

Reminder about Mild solutions and their properties

Mild (or semigroup) solutions have been obtained by Benilan, Crandall and Pierre via Crandall-Liggett type theorems; the underlying idea is the use of an Implicit Time Discretization (ITD) method: consider the partition of $[0,T]$

$$t_k = \frac{k}{n}T\,, \quad \text{for any } 0 \le k \le n\,, \quad \text{with } t_0 = 0\,, t_n = T\,, \text{and} \quad h = t_{k+1} - t_k = \frac{T}{n}\,.$$

For any $t \in (0,T)$, the (unique) semigroup solution $u(t,\cdot)$ is obtained as the limit in $L^1(\Omega)$ of the solutions $u_{k+1}(\cdot) = u(t_{k+1},\cdot)$ which solve the following elliptic equation (u_k is the datum, is given by the previous iterative step)

$$h\mathcal{L}F(u_{k+1}) + u_{k+1} = u_k \qquad \text{or equivalently} \qquad \frac{u_{k+1} - u_k}{h} = -\mathcal{L}F(u_{k+1})\,.$$

Usually such solutions are difficult to treat since a priori they are merely very weak solutions. We can prove the following result:

Proposition. Semigroup solutions are weak dual solutions

Let u be the unique semigroup (mild) solution to the (CDP) corresponding to the initial datum $u_0 \in L^p(\Omega)$ with $p \ge 1$. Then u is a weak dual solution of (CDP) and is contained in the class $\mathcal{S}_p$.

Reminder about Mild solutions and their properties

Mild (or semigroup) solutions have been obtained by Benilan, Crandall and Pierre via Crandall-Liggett type theorems; the underlying idea is the use of an Implicit Time Discretization (ITD) method: consider the partition of $[0,T]$

$$t_k = \frac{k}{n}T\,, \quad \text{for any } 0 \le k \le n\,, \quad \text{with } t_0 = 0\,, t_n = T\,, \text{and} \quad h = t_{k+1} - t_k = \frac{T}{n}\,.$$

For any $t \in (0,T)$, the (unique) semigroup solution $u(t,\cdot)$ is obtained as the limit in $L^1(\Omega)$ of the solutions $u_{k+1}(\cdot) = u(t_{k+1},\cdot)$ which solve the following elliptic equation (u_k is the datum, is given by the previous iterative step)

$$h\mathcal{L}F(u_{k+1}) + u_{k+1} = u_k \qquad \text{or equivalently} \qquad \frac{u_{k+1} - u_k}{h} = -\mathcal{L}F(u_{k+1})\,.$$

Usually such solutions are difficult to treat since a priori they are merely very weak solutions. We can prove the following result:

Proposition. Semigroup solutions are weak dual solutions

Let u be the unique semigroup (mild) solution to the (CDP) corresponding to the initial datum $u_0 \in L^p(\Omega)$ with $p \ge 1$. Then u is a weak dual solution of (CDP) and is contained in the class $\mathcal{S}_p$.

Existence and uniqueness of weak dual solutions

Theorem. Existence of weak dual solutions (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in L^1_{\delta_\gamma}(\Omega)$ there exists a minimal weak dual solution to the (CDP). Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : L^1_{\delta_\gamma}(\Omega))$. Mild solutions are weak dual solutions and the set of such solutions has the properties needed to form a class of type $\mathcal{S}$.

Theorem. Uniqueness of weak dual solutions (M.B. and J. L. Vázquez)

The solution constructed in the above Theorem by approximation of the initial data from below is unique. We call it the *minimal solution*. In this class of solutions the standard comparison result holds, and also the weighted L^1 estimates .

Existence and uniqueness of weak dual solutions

Theorem. Existence of weak dual solutions (M.B. and J. L. Vázquez)

For every nonnegative $u_0 \in \mathrm{L}^1_{\delta_\gamma}(\Omega)$ there exists a minimal weak dual solution to the (CDP). Such a solution is obtained as the monotone limit of the semigroup (mild) solutions that exist and are unique. The minimal weak dual solution is continuous in the weighted space $u \in C([0,\infty) : \mathrm{L}^1_{\delta_\gamma}(\Omega))$. Mild solutions are weak dual solutions and the set of such solutions has the properties needed to form a class of type $\mathcal{S}$.

Theorem. Uniqueness of weak dual solutions (M.B. and J. L. Vázquez)

The solution constructed in the above Theorem by approximation of the initial data from below is unique. We call it the *minimal solution*. In this class of solutions the standard comparison result holds, and also the weighted L^1 estimates .

First Pointwise Estimates

Theorem. (M.B. and J. L. Vázquez)

Let $u \geq 0$ be a solution in the class $\mathcal{S}_p$ of very weak solutions to Problem (CDP) with $p > N/2s$. Then,

$$\int_\Omega u(t_1,x)\mathbb{K}(x,x_0)\,\mathrm{d}x \leq \int_\Omega u(t_0,x)\mathbb{K}(x,x_0)\,\mathrm{d}x\,, \qquad \text{for all } t_1 \geq t_0 \geq 0\,.$$

Moreover, for almost every $0 \leq t_0 \leq t_1$ and almost every $x_0 \in \Omega$, we have

$$\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}} (t_1 - t_0)\, F(u(t_0,x_0)) \leq \int_\Omega \left[u(t_0,x) - u(t_1,x)\right] \mathbb{K}(x,x_0)\,\mathrm{d}x$$

$$\leq (m_0 - 1)\frac{t_1^{\frac{1}{\mu_0}}}{t_0^{\frac{1-\mu_0}{\mu_0}}}\, F(u(t_1,x_0))\,.$$

Remark. As a consequence of the above inequality and Hölder inequality, we have that $\mathcal{S}_p = \mathcal{S}_\infty$, when $p > N/2s$.

Sketch of the proof of the First Pointwise Estimates

We would like to take as test function

$$\psi(t,x)=\psi_1(t)\psi_2(x)=\chi_{[t_0,t_1]}(t)\,\mathbb{K}(x_0,x)\,,$$

This is not admissible in the Definition of Weak Dual solutions.
Plugging such test function in the definition of weak dual solution gives the formula

$$\int_\Omega u(t_0,x)\mathbb{K}(x_0,x)\,\mathrm{d}x-\int_\Omega u(t_1,x)\mathbb{K}(x_0,x)\,\mathrm{d}x=\int_{t_0}^{t_1}F(u(\tau,x_0))\mathrm{d}\tau\,.$$

This formula can be proven rigorously though careful approximation.
Next, we use the monotonicity estimates,

$$t\mapsto t^{\frac{1}{\mu_0}}F(u(t,x))\qquad \text{is nondecreasing in } t>0 \text{ for a.e. } x\in\Omega\,.$$

to get for all $0\le t_0\le t_1$, recalling that $\frac{1}{\mu_0}=\frac{m_0}{m_0-1}$

$$\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}}(t_1-t_0)F(u(t_0,x_0))\le\int_{t_0}^{t_1}F(u(\tau,x_0))\mathrm{d}\tau\le\frac{m_0-1}{t_0^{\frac{1}{m_0-1}}}t_1^{\frac{1}{\mu_0}}F(u(t_1,x_0)).\ \square$$

Sketch of the proof of the First Pointwise Estimates

We would like to take as test function

$$\psi(t,x)=\psi_1(t)\psi_2(x)=\chi_{[t_0,t_1]}(t)\,\mathbb{K}(x_0,x)\,,$$

This is not admissible in the Definition of Weak Dual solutions.
Plugging such test function in the definition of weak dual solution gives the formula

$$\int_\Omega u(t_0,x)\mathbb{K}(x_0,x)\,\mathrm{d}x-\int_\Omega u(t_1,x)\mathbb{K}(x_0,x)\,\mathrm{d}x=\int_{t_0}^{t_1}F(u(\tau,x_0))\mathrm{d}\tau\,.$$

This formula can be proven rigorously though careful approximation.
Next, we use the monotonicity estimates,

$$t\mapsto t^{\frac{1}{\mu_0}}F(u(t,x))\qquad\text{is nondecreasing in } t>0 \text{ for a.e. } x\in\Omega\,.$$

to get for all $0\le t_0\le t_1$, recalling that $\frac{1}{\mu_0}=\frac{m_0}{m_0-1}$

$$\left(\frac{t_0}{t_1}\right)^{\frac{1}{\mu_0}}(t_1-t_0)F(u(t_0,x_0))\le\int_{t_0}^{t_1}F(u(\tau,x_0))\mathrm{d}\tau\le\frac{m_0-1}{t_0^{\frac{1}{m_0-1}}}t_1^{\frac{1}{\mu_0}}F(u(t_1,x_0)).\;\square$$

Upper Estimates

- **Absolute upper bounds**
  - **Absolute bounds**
  - **The power case. Absolute bounds and boundary behaviour**
- **Smoothing Effects**
  - L^1-L^∞ **Smoothing Effects**
  - $L^1_{\delta_\gamma}$-L^∞ **Smoothing Effects**
  - **Backward in time Smoothing effects**

Absolute upper bounds

Theorem. (Absolute upper estimate) (M.B. & J. L. Vázquez)

Let u be a nonnegative weak dual solution corresponding to $u_0 \in L^1_{\delta_\gamma}(\Omega)$. Then, there exists universal constants $K_0, K_1, K_2 > 0$ such that the following estimates hold true for all $t > 0$:

$$F\left(\|u(t)\|_{L^\infty(\Omega)}\right) \le F^*\left(\frac{K_1}{t}\right) .$$

Moreover, there exists a time $\tau_1(u_0)$ with $0 \le \tau_1(u_0) \le K_0$ such that

$$\|u(t)\|_{L^\infty(\Omega)} \le 1 \qquad \text{for all } t \ge \tau_1 ,$$

so that

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_2}{t^{\frac{1}{m_i - 1}}} \qquad \text{with} \qquad \begin{cases} i = 0 & \text{if } t \le K_0 \\ i = 1 & \text{if } t \ge K_0 \end{cases}$$

The Legendre transform of F is defined as a function $F^* : \mathbb{R} \to \mathbb{R}$ with

$$F^*(z) = \sup_{r \in \mathbb{R}} \left(zr - F(r)\right) = z\,(F')^{-1}(z) - F\left((F')^{-1}(z)\right) = F'(r)\,r + F(r) ,$$

with the choice $r = (F')^{-1}(z)$.

The power case. Absolute bounds and boundary behaviour

Theorem. (Absolute upper estimate and boundary behaviour) (M.B. & J. L. Vázquez)

Let u be a weak dual solution. Then, there exists universal constants $K_1, K_2 > 0$ such that the following estimates hold true: (K1) assumption implies:

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_1}{t^{\frac{1}{m-1}}}\,, \qquad \text{for all } \ t > 0\,.$$

Moreover, (K2) assumption implies:

$$u(t,x) \le K_2\,\frac{\delta_\gamma(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t > 0 \text{ and } x \in \Omega\,.$$

Remark.

- This is a very strong regularization *independent* of the initial datum u_0.
- The boundary estimates are sharp, since we will obtain lower bounds with matching powers.
- This bounds give a sharp time decay for the solution, but only for large times, say $t \ge 1$. For small times we will obtain a better time decay when $0 < t < 1$, in the form of smoothing effects

The power case. Absolute bounds and boundary behaviour

Theorem. (Absolute upper estimate and boundary behaviour) (M.B. & J. L. Vázquez)

Let u be a weak dual solution. Then, there exists universal constants $K_1, K_2 > 0$ such that the following estimates hold true: (K1) assumption implies:

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_1}{t^{\frac{1}{m-1}}}, \qquad \text{for all } \ t > 0\,.$$

Moreover, (K2) assumption implies:

$$u(t,x) \le K_2 \frac{\delta_\gamma(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \qquad \text{for all } t > 0 \text{ and } x \in \Omega\,.$$

Remark.

- This is a very strong regularization *independent* of the initial datum u_0.
- The boundary estimates are sharp, since we will obtain lower bounds with matching powers.
- This bounds give a sharp time decay for the solution, but only for large times, say $t \ge 1$. For small times we will obtain a better time decay when $0 < t < 1$, in the form of smoothing effects

The power case. Absolute bounds and boundary behaviour

Sketch of the proof of Absolute Bounds

• STEP 1. *First upper estimates.* Recall the pointwise estimate:

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}}(t_1-t_0)\,u^m(t_0,x_0)\le\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,dx-\int_\Omega u(t_1,x)G_\Omega(x,x_0)\,dx\,.$$

for any $u\in\mathcal{S}_p$, all $0\le t_0\le t_1$ and all $x_0\in\Omega$. Choose $t_1=2t_0$ to get

$$(*)\qquad \boxed{u^m(t_0,x_0)\le\frac{2^{\frac{m}{m-1}}}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,dx\,.}$$

Recall that $u\in\mathcal{S}_p$ with $p>N/(2s)$, means $u(t)\in L^p(\Omega)$ for all $t>0$, so that:

$$u^m(t_0,x_0)\le\frac{c_0}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,dx\le\frac{c_0}{t_0}\|u(t_0)\|_{L^p(\Omega)}\,\|G_\Omega(\cdot,x_0)\|_{L^q(\Omega)}<+\infty$$

since $G_\Omega(\cdot,x_0)\in L^q(\Omega)$ for all $0<q<N/(N-2s)$, so that $u(t_0)\in L^\infty(\Omega)$ for all $t_0>0$.

• STEP 2. Let us estimate the r.h.s. of $(*)$ as follows:

$$u^m(t_0,x_0)\le\frac{c_0}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,dx\le\|u(t_0)\|_{L^\infty(\Omega)}\frac{c_0}{t_0}\int_\Omega G_\Omega(x,x_0)\,dx\,.$$

Taking the supremum over $x_0\in\Omega$ of both sides, we get:

$$\boxed{\|u(t_0)\|^{m-1}_{L^\infty(\Omega)}\le\frac{c_0}{t_0}\sup_{x_0\in\Omega}\int_\Omega G_\Omega(x,x_0)\,dx\le\frac{K_1^{m-1}}{t_0}}\qquad\square$$

Sketch of the proof of Absolute Bounds

• STEP 1. *First upper estimates.* Recall the pointwise estimate:

$$\left(\frac{t_0}{t_1}\right)^{\frac{m}{m-1}}(t_1-t_0)\,u^m(t_0,x_0)\le\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,dx-\int_\Omega u(t_1,x)G_\Omega(x,x_0)\,dx\,.$$

for any $u\in\mathcal{S}_p$, all $0\le t_0\le t_1$ and all $x_0\in\Omega$. Choose $t_1=2t_0$ to get

$$(*)\qquad u^m(t_0,x_0)\le\frac{2^{\frac{m}{m-1}}}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,dx\,.$$

Recall that $u\in\mathcal{S}_p$ with $p>N/(2s)$, means $u(t)\in L^p(\Omega)$ for all $t>0$, so that:

$$u^m(t_0,x_0)\le\frac{c_0}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,dx\le\frac{c_0}{t_0}\|u(t_0)\|_{L^p(\Omega)}\|G_\Omega(\cdot,x_0)\|_{L^q(\Omega)}<+\infty$$

since $G_\Omega(\cdot,x_0)\in L^q(\Omega)$ for all $0<q<N/(N-2s)$, so that $u(t_0)\in L^\infty(\Omega)$ for all $t_0>0$.

• STEP 2. Let us estimate the r.h.s. of $(*)$ as follows:

$$u^m(t_0,x_0)\le\frac{c_0}{t_0}\int_\Omega u(t_0,x)G_\Omega(x,x_0)\,dx\le\|u(t_0)\|_{L^\infty(\Omega)}\frac{c_0}{t_0}\int_\Omega G_\Omega(x,x_0)\,dx\,.$$

Taking the supremum over $x_0\in\Omega$ of both sides, we get:

$$\|u(t_0)\|_{L^\infty(\Omega)}^{m-1}\le\frac{c_0}{t_0}\sup_{x_0\in\Omega}\int_\Omega G_\Omega(x,x_0)\,dx\le\frac{K_1^{m-1}}{t_0}\qquad\square$$

Smoothing Effects

Let $\gamma, s \in [0,1]$ be the exponents appearing in assumption $(K2)$. Define

$$\vartheta_{i,\gamma} = \frac{1}{2s + (N+\gamma)(m_i - 1)} \qquad \text{with} \qquad m_i = \frac{1}{1-\mu_i} > 1$$

Theorem. (Weighted $\mathrm{L}^1 - \mathrm{L}^\infty$ smoothing effect) (M.B. & J. L. Vázquez)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_6 > 0$ s.t.

$$F\left(\|u(t)\|_{\mathrm{L}^\infty(\Omega)}\right) \le K_6 \frac{\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2sm_i\vartheta_{i,\gamma}}}{t^{m_i(N+\gamma)\vartheta_{i,\gamma}}}, \qquad \text{for all } 0 \le t_0 \le t,$$

with $i = 1$ if $t \ge \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ and $i = 0$ if $t \le \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$.

- A novelty is that we get instantaneous smoothing effects, new even when $s = 1$.
- The weighted smoothing effect is new even for $s = 1$.
- **Corollary.** Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$.
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

$$(N+\gamma)\vartheta_{i,\gamma} = \frac{N+\gamma}{2s + (N+\gamma)(m_i - 1)} < \frac{1}{m_i - 1}.$$

Smoothing Effects

Let $\gamma, s \in [0,1]$ be the exponents appearing in assumption $(K2)$. Define

$$\vartheta_{i,\gamma} = \frac{1}{2s + (N+\gamma)(m_i - 1)} \qquad \text{with} \qquad m_i = \frac{1}{1-\mu_i} > 1$$

Theorem. (Weighted $\mathrm{L}^1 - \mathrm{L}^\infty$ smoothing effect) (M.B. & J. L. Vázquez)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_6 > 0$ s.t.

$$F\left(\|u(t)\|_{\mathrm{L}^\infty(\Omega)}\right) \le K_6 \frac{\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2sm_i\vartheta_{i,\gamma}}}{t^{m_i(N+\gamma)\vartheta_{i,\gamma}}}, \qquad \text{for all } 0 \le t_0 \le t,$$

with $i = 1$ if $t \ge \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ and $i = 0$ if $t \le \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$.

- A novelty is that we get instantaneous smoothing effects, new even when $s = 1$.
- The weighted smoothing effect is new even for $s = 1$.
- **Corollary.** Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$.
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

$$(N+\gamma)\vartheta_{i,\gamma} = \frac{N+\gamma}{2s + (N+\gamma)(m_i - 1)} < \frac{1}{m_i - 1}.$$

Smoothing Effects

Let $\gamma, s \in [0,1]$ be the exponents appearing in assumption $(K2)$. Define

$$\vartheta_{i,\gamma} = \frac{1}{2s + (N+\gamma)(m_i - 1)} \qquad \text{with} \qquad m_i = \frac{1}{1-\mu_i} > 1$$

Theorem. (Weighted $L^1 - L^\infty$ smoothing effect) (M.B. & J. L. Vázquez)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_6 > 0$ s.t.

$$F\left(\|u(t)\|_{L^\infty(\Omega)}\right) \le K_6 \frac{\|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2sm_i\vartheta_{i,\gamma}}}{t^{m_i(N+\gamma)\vartheta_{i,\gamma}}}, \qquad \text{for all } 0 \le t_0 \le t,$$

with $i = 1$ if $t \ge \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ and $i = 0$ if $t \le \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$.

- A novelty is that we get instantaneous smoothing effects, new even when $s = 1$.
- The weighted smoothing effect is new even for $s = 1$.
- **Corollary.** Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{L^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{L^1(\Omega)}$.
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

$$(N+\gamma)\vartheta_{i,\gamma} = \frac{N+\gamma}{2s + (N+\gamma)(m_i - 1)} < \frac{1}{m_i - 1}.$$

Smoothing Effects

Let $\gamma, s \in [0,1]$ be the exponents appearing in assumption $(K2)$. Define

$$\vartheta_{i,\gamma} = \frac{1}{2s + (N+\gamma)(m_i - 1)} \qquad \text{with} \qquad m_i = \frac{1}{1-\mu_i} > 1$$

Theorem. (Weighted $\mathrm{L}^1 - \mathrm{L}^\infty$ smoothing effect) (M.B. & J. L. Vázquez)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_6 > 0$ s.t.

$$F\left(\|u(t)\|_{\mathrm{L}^\infty(\Omega)}\right) \le K_6 \, \frac{\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2sm_i\vartheta_{i,\gamma}}}{t^{m_i(N+\gamma)\vartheta_{i,\gamma}}}\,, \qquad \text{for all } 0 \le t_0 \le t\,,$$

with $i = 1$ if $t \ge \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$ and $i = 0$ if $t \le \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}$.

- A novelty is that we get instantaneous smoothing effects, new even when $s = 1$.
- The weighted smoothing effect is new even for $s = 1$.
- **Corollary.** Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$.
- The time decay is better for small times $0 < t < 1$ than the one given by absolute bounds:

$$(N+\gamma)\vartheta_{i,\gamma} = \frac{N+\gamma}{2s + (N+\gamma)(m_i - 1)} < \frac{1}{m_i - 1}\,.$$

Smoothing Effects

Corollary.

As a consequence of $(K2)$ hypothesis, there exists a constant $K_7 > 0$ s.t.
WEIGHTED $L^1 - L^\infty$ SMOOTHING EFFECT FOR SMALL TIMES:

$$\|u(t)\|_{L^\infty(\Omega)} \le K_7 \frac{\|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{0,\gamma}}}{t^{(N+\gamma)\vartheta_{0,\gamma}}}, \qquad \text{for all } 0 \le t_0 \le t \le \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}.$$

WEIGHTED $L^1 - L^\infty$ SMOOTHING EFFECT FOR LARGE TIMES:

$$\|u(t)\|_{L^\infty(\Omega)} \le K_7 \frac{\|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{1,\gamma}}}{t^{(d+\gamma)\vartheta_{1,\gamma}}}, \qquad \text{for all } t \ge \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{d+\gamma}}.$$

Moreover, the condition $t \ge \|u(t_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{d+\gamma}}$, is implied by $t \ge \left(K_1 \|\delta_\gamma\|_{L^1(\Omega)}\right)^{\vartheta_{1,\gamma}(m_1-1)}$.

Corollary. Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{L^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{L^1(\Omega)}$.

Smoothing Effects

Corollary.

As a consequence of $(K2)$ hypothesis, there exists a constant $K_7 > 0$ s.t.

WEIGHTED $\mathrm{L}^1 - \mathrm{L}^\infty$ SMOOTHING EFFECT FOR SMALL TIMES:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le K_7 \frac{\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{0,\gamma}}}{t^{(N+\gamma)\vartheta_{0,\gamma}}}, \qquad \text{for all } 0 \le t_0 \le t \le \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{N+\gamma}}.$$

WEIGHTED $\mathrm{L}^1 - \mathrm{L}^\infty$ SMOOTHING EFFECT FOR LARGE TIMES:

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le K_7 \frac{\|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{1,\gamma}}}{t^{(d+\gamma)\vartheta_{1,\gamma}}}, \qquad \text{for all } t \ge \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{d+\gamma}}.$$

Moreover, the condition $t \ge \|u(t_0)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{\frac{2s}{d+\gamma}}$, is implied by $t \ge \left(K_1 \|\delta_\gamma\|_{\mathrm{L}^1(\Omega)}\right)^{\vartheta_{1,\gamma}(m_1-1)}$.

Corollary. Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$.

Corollary. (Backward weighted $\mathrm{L}^1 - \mathrm{L}^\infty$ smoothing effects)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_7 > 0$ s.t.

For small times: for all $t, h > 0$ and for all $0 \le t \le \|u(t)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$,

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le 2K_7 \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{0,\gamma}}{m_0-1}} \frac{\|u(t+h)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{0,\gamma}}}{t^{(N+\gamma)\vartheta_{0,\gamma}}} .$$

For large times: for all $t, h > 0$ and for all $t \ge \|u(t)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$,

$$\|u(t)\|_{\mathrm{L}^\infty(\Omega)} \le 2K_7 \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{1,\gamma}}{m_1-1}} \frac{\|u(t+h)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{1,\gamma}}}{t^{(N+\gamma)\vartheta_{1,\gamma}}} .$$

Moreover, the condition $t \ge \|u(t)\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$, is implied by $t \ge \left(K_1 \|\delta_\gamma\|_{\mathrm{L}^1(\Omega)}\right)^{\vartheta_{1,\gamma}(m_1-1)}$.

Corollary. Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{\mathrm{L}^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{\mathrm{L}^1(\Omega)}$.

Corollary. (Backward weighted $L^1 - L^\infty$ smoothing effects)

As a consequence of $(K2)$ hypothesis, there exists a constant $K_7 > 0$ s.t.

For small times: for all $t, h > 0$ and for all $0 \le t \le \|u(t)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$,

$$\|u(t)\|_{L^\infty(\Omega)} \le 2K_7 \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{0,\gamma}}{m_0-1}} \frac{\|u(t+h)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{0,\gamma}}}{t^{(N+\gamma)\vartheta_{0,\gamma}}} .$$

For large times: for all $t, h > 0$ and for all $t \ge \|u(t)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$,

$$\|u(t)\|_{L^\infty(\Omega)} \le 2K_7 \left(1 \vee \frac{h}{t}\right)^{\frac{2s\vartheta_{1,\gamma}}{m_1-1}} \frac{\|u(t+h)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s\vartheta_{1,\gamma}}}{t^{(N+\gamma)\vartheta_{1,\gamma}}} .$$

Moreover, the condition $t \ge \|u(t)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$, is implied by $t \ge \left(K_1 \|\delta_\gamma\|_{L^1(\Omega)}\right)^{\vartheta_{1,\gamma}(m_1-1)}$.

Corollary. Under the weaker assumption $(K1)$ instead of $(K2)$, the above result holds true with $\gamma = 0$ and replacing $\|\cdot\|_{L^1_{\delta_\gamma}(\Omega)}$ with $\|\cdot\|_{L^1(\Omega)}$.

Harnack inequalities

- **Global Harnack Principle**
- **Local Harnack inequalities**

In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$.

Theorem. (Global Harnack Principle) (M.B. & J. L. Vázquez)

There exist universal constants $H_0, H_1, L_0 > 0$ such that setting

$$t_* = \frac{L_0}{\left(\int_\Omega u_0 \delta_\gamma \,\mathrm{d}x\right)^{m-1}} ,$$

we have that for all $t \geq t_*$ and all $x \in \Omega$, the following inequality holds:

$$H_0 \frac{\delta_\gamma(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}} \leq u(t,x) \leq H_1 \frac{\delta_\gamma(x)^{\frac{1}{m}}}{t^{\frac{1}{m-1}}}$$

Remarks.

- This inequality implies local Harnack inequalities of elliptic type
- Useful to study the sharp asymptotic behaviour

Local Harnack inequalities

Theorem. (Local Harnack Inequalities of Elliptic Type) (M.B. & J. L. Vázquez)

There exist constants $H_R, L_0 > 0$ such that setting $t_* = L_0 \|u_0\|_{\mathrm{L}^1_{\Phi_1}(\Omega)}^{-(m-1)}$, we have that for all $t \ge t_*$ and all $B_R(x_0) \in \Omega$, the following inequality holds:

$$\sup_{x \in B_R(x_0)} u(t,x) \le H_R \inf_{x \in B_R(x_0)} u(t,x)$$

The constant H_R depends on $\mathrm{dist}(B_R(x_0), \partial\Omega)$.

Corollary. (Local Harnack Inequalities of Backward Type)

Under the runninig assumptions, for all $t \ge t_*$ and all $B_R(x_0) \in \Omega$, we have:

$$\sup_{x \in B_R(x_0)} u(t,x) \le 2H_R \inf_{x \in B_R(x_0)} u(t+h,x) \qquad \text{for all } 0 \le h \le t_* .$$

Asymptotic behaviour of nonnegative solutions

- **Convergence to the stationary profile**
- **Convergence with optimal rate**

Convergence to the stationary profile

In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$.

Theorem. (Asymptotic behaviour) (M.B. , Y. Sire, J. L. Vázquez)

There exists a unique nonnegative selfsimilar solution of the above Dirichlet Problem

$$U(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}} \, ,$$

for some bounded function $S : \Omega \to \mathbb{R}$. Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$)

$$\lim_{\tau \to \infty} \tau^{\frac{1}{m-1}} \left\| u(\tau, \cdot) - U(\tau, \cdot) \right\|_{L^\infty(\Omega)} = 0 \, .$$

The previous theorem admits the following corollary.

Theorem. (Elliptic problem) (M.B. , Y. Sire, J. L. Vázquez)

Let $m > 1$. There exists a unique weak dual solution to the elliptic problem

$$\begin{cases} \mathcal{L}(S^m) = \dfrac{S}{m-1} & \text{in } \Omega, \\ S(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Convergence to the stationary profile

In the rest of the talk we consider the nonlinearity $F(u) = |u|^{m-1}u$ with $m > 1$.

Theorem. (Asymptotic behaviour) (M.B. , Y. Sire, J. L. Vázquez)

There exists a unique nonnegative selfsimilar solution of the above Dirichlet Problem

$$U(\tau, x) = \frac{S(x)}{\tau^{\frac{1}{m-1}}} ,$$

for some bounded function $S : \Omega \to \mathbb{R}$. Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$)

$$\lim_{\tau\to\infty} \tau^{\frac{1}{m-1}} \|u(\tau,\cdot) - U(\tau,\cdot)\|_{L^\infty(\Omega)} = 0 .$$

The previous theorem admits the following corollary.

Theorem. (Elliptic problem) (M.B. , Y. Sire, J. L. Vázquez)

Let $m > 1$. There exists a unique weak dual solution to the elliptic problem

$$\begin{cases} \mathcal{L}(S^m) = \dfrac{S}{m-1} & \text{in } \Omega, \\ S(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Convergence with optimal rate

Theorem. (Sharp asymptotic with rates) (M.B. , Y. Sire, J. L. Vázquez)

Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$) that there exist $t_0 > 0$ of the form

$$t_0 = \overline{k} \left[\frac{\int_\Omega \Phi_1 \, \mathrm{d}x}{\int_\Omega u_0 \Phi_1 \, \mathrm{d}x} \right]^{m-1}$$

such that for all $t \ge t_0$ we have

$$\left\| \frac{u(t,\cdot)}{U(t,\cdot)} - 1 \right\|_{\mathrm{L}^\infty(\Omega)} \le \frac{2}{m-1} \, \frac{t_0}{t_0 + t} \, .$$

The constant $\overline{k} > 0$ only depends on m, N, s, and $|\Omega|$.

Remarks.

- We provide two different proofs of the above result.
- One proof is based on the construction of the so-called Friendly-Giant solution, namely the solution with initial data $u_0 = +\infty$, and is based on the Global Harnack Principle of Part 3
- The second proof is based on a new Entropy method, which is based on a parabolic version of the Caffarelli-Silvestre extension.

Convergence with optimal rate

Theorem. (Sharp asymptotic with rates) (M.B. , Y. Sire, J. L. Vázquez)

Let u be any nonnegative weak dual solution to the (CDP) , then we have (unless $u \equiv 0$) that there exist $t_0 > 0$ of the form

$$t_0 = \overline{k} \left[\frac{\int_\Omega \Phi_1 \, \mathrm{d}x}{\int_\Omega u_0 \Phi_1 \, \mathrm{d}x} \right]^{m-1}$$

such that for all $t \geq t_0$ we have

$$\left\| \frac{u(t,\cdot)}{U(t,\cdot)} - 1 \right\|_{L^\infty(\Omega)} \leq \frac{2}{m-1} \, \frac{t_0}{t_0 + t} \,.$$

The constant $\overline{k} > 0$ only depends on m, N, s, and $|\Omega|$.

Remarks.

- We provide two different proofs of the above result.
- One proof is based on the construction of the so-called Friendly-Giant solution, namely the solution with initial data $u_0 = +\infty$, and is based on the Global Harnack Principle of Part 3
- The second proof is based on a new Entropy method, which is based on a parabolic version of the Caffarelli-Silvestre extension.

Convergence with optimal rate

The End

Muchas Gracias!!!

Thank You!!!

Grazie Mille!!!

Weighted L^1 estimates

- L^1 **estimates with** Φ_1 **weight**
- L^1 **estimates with** δ_γ **weight**

Weighted $L^1_{\Phi_1}$ estimates

To simplify the presentation, we first treat the case in which $\mathcal{L}$ has a first nonnegative eigenfunction Φ_1; we recall that $\Phi_1 \asymp \delta_\gamma$ on $\overline{\Omega}$, by hyp. (K2).

Proposition. (Weighted L^1 estimates for ordered solutions)

Let $u \ge v$ be two ordered weak dual solutions to the Problem (CDP) corresponding to the initial data $0 \le u_0, v_0 \in L^1_{\Phi_1}(\Omega)$. Then for all $t_1 \ge t_0 \ge 0$

$$\int_\Omega \left[u(t_1,x) - v(t_1,x)\right]\Phi_1(x)\,dx \le \int_\Omega \left[u(t_0,x) - v(t_0,x)\right]\Phi_1(x)\,dx\,.$$

Moreover, for all $0 \le \tau_0 \le \tau, t \; < +\infty$
such that either $t, \tau \le K_0$ or $\tau_0 \ge K_0$, we have

$$\int_\Omega \left[u(\tau,x) - v(\tau,x)\right]\Phi_1(x)\,dx \le \int_\Omega \left[u(t,x) - v(t,x)\right]\Phi_1(x)\,dx$$
$$+ K_8\,\|u(\tau_0)\|_{L^1_{\Phi_1}(\Omega)}^{2s(m_i-1)\vartheta_{i,\gamma}}\,|t-\tau|^{2s\vartheta_{i,\gamma}}\int_\Omega \left[u(\tau_0,x) - v(\tau_0,x)\right]\Phi_1\,dx$$

where $i = 0$ if $t, \tau \le \|u(\tau_0)\|_{L^1_{\Phi_1}(\Omega)}^{\frac{2s}{d+\gamma}}$ and $i = 1$ if $t, \tau \ge \|u(\tau_0)\|_{L^1_{\Phi_1}(\Omega)}^{\frac{2s}{d+\gamma}}$.

Weighted $\mathrm{L}^1_{\delta_\gamma}$ estimates

Taking any nonnegative function $\psi \in \mathrm{L}^\infty(\Omega)$, using assumption (K2) gives

$$\mathcal{L}^{-1}\psi(x) \asymp \delta_\gamma(x) \qquad \text{for a.e. } x \in \Omega\,.$$

This will imply the monotonicity of some L^1-weighted norm.

Proposition. (Weighted L^1 estimates for ordered solutions)

Let $u \geq v$ be two ordered weak dual solutions to the Problem (CDP) corresponding to $0 \leq u_0, v_0 \in \mathrm{L}^1_{\delta_\gamma}(\Omega)$. Then for all $0 \leq \psi \in \mathrm{L}^\infty(\Omega)$ and all $0 \leq \tau \leq t$

$$\int_\Omega \big[u(t,x) - v(t,x)\big]\mathcal{L}^{-1}\psi(x)\,\mathrm{d}x \leq \int_\Omega \big[u(\tau,x) - v(\tau,x)\big]\mathcal{L}^{-1}\psi(x)\,\mathrm{d}x\,.$$

As a consequence, there exists a constant $C_{\Omega,\gamma} > 0$ such that for all $0 \leq \tau \leq t$

$$\int_\Omega \big[u(t,x) - v(t,x)\big]\,\delta_\gamma(x)\,\mathrm{d}x \leq C_{\Omega,\gamma}\int_\Omega \big[u(\tau,x) - v(\tau,x)\big]\,\delta_\gamma(x)\,\mathrm{d}x\,.$$

Moreover...

Weighted $L^1_{\delta_\gamma}$ estimates

Let's put $\boxed{\Psi_1 = \mathcal{L}^{-1}\delta_\gamma}$, in analogy with the formula $\Phi_1 = \lambda_1^{-1}\mathcal{L}^{-1}\Phi_1$.

Proposition. (Weighted L^1 estimates for ordered solutions) Continued

Moreover, for all $0 \le \tau_0 \le \tau, t \; < +\infty$ such that either $t, \tau \le K_0$ or $\tau_0 \ge K_0$, we have

$$\int_\Omega \big[u(\tau,x) - v(\tau,x)\big]\Psi_1(x)\,\mathrm{d}x \le \int_\Omega \big[u(t,x) - v(t,x)\big]\Psi_1(x)\,\mathrm{d}x$$
$$+ K_8\,\|u(\tau_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s(m_i-1)\vartheta_{i,\gamma}}\;|t-\tau|^{2s\vartheta_{i,\gamma}}\int_\Omega \big[u(\tau_0,x) - v(\tau_0,x)\big]\delta_\gamma(x)\,\mathrm{d}x$$

where $i = 0$ if $t, \tau \le \|u(\tau_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$ and $i = 1$ if $t, \tau \ge \|u(\tau_0)\|_{L^1_{\delta_\gamma}(\Omega)}^{2s/(N+\gamma)}$.

Remark. The above inequality, together with monotonicity, allows to prove that weak dual solutions constructed by approximation from below by mild solutions belong to the space

$$u \in C([0,\infty) \;:\; L^1_{\delta_\gamma}(\Omega))\,.$$