Theory of fractional nonlinear diffusion equations. Estimates and traces.

Matteo Bonforte

Departamento de Matemáticas,
Universidad Autónoma de Madrid,
Campus de Cantoblanco
28049 Madrid, Spain

matteo.bonforte@uam.es

http://www.uam.es/matteo.bonforte

[Joint work with *J. L. Vázquez* (UAM)]

Analysis Seminar, Department of Mathematics
University of Texas at Austin, TX, USA
April 3, 2013

Outline

1. **The Fractional Porous Medium / Fast Diffusion Equation on $\mathbb{R}^d$**
   - **The setup of the problem**
   - **References**
   - **Several definition of solutions**
   - **Properties of weak solutions I, $m > m_c$** [DPQRV1,DPQRV2]
   - **Properties of weak solutions II, $0 < m \leq m_c$** [DPQRV1,DPQRV2]
2. **Theory of very weak solutions for FDE/PME**
   - **Weighted L^1 Estimates. The case $0 < m < 1$.**
   - **Existence and uniqueness of very weak solutions when $0 < m < 1$**
   - **Positivity estimates in the Good Fast Diffusion range**
   - **Positivity estimates in the Very Fast Diffusion range**
   - **Positivity estimates in the Porous Medium range**
3. Existence and uniqueness of initial traces
   - **Understanding the problem**
   - **Existence and uniqueness of initial trace for $0 < m < 1$.**
   - **Existence and uniqueness of initial trace for $m > 1$.**

The setup of the problem

The Cauchy Problem for The Fractional Porous Medium / Fast Diffusion Equation in $\mathbb{R}^d$

$$\text{(CP)} \quad \begin{cases} \partial_t u(t,x) + (-\Delta)^s u^m(t,x) = 0\,, & \forall (t,x) \in (0,+\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x)\,, & \forall x \in \mathbb{R}^d \end{cases}$$

where $0 < s \le 1$, $m > 0$ and $d \ge 1$.

$m > 1$ is PME, $m = 1$ is HE, $0 < m < 1$ is FDE.

Definition(s) of the fractional Laplacian

Several equivalent definitions of the nonlocal operator $(-\Delta)^{\sigma/2}$ (Laplacian of order σ):

(1) Fourier transform $\quad \widehat{((-\Delta)^{\sigma/2} f)}(x) = (2\pi|x|)^\sigma \hat{f}(x)\,,$

[can be used for positive and negative values of σ]

(2) Hypersingular Kernel $\quad (-\Delta)^{\sigma/2} g(x) = c_{d,\sigma}\ \text{P.V.} \int_{\mathbb{R}^d} \frac{g(x)-g(z)}{|x-z|^{d+\sigma}}\,dz\,,$

[can be used for $0 < \sigma < 2$, where $c_{d,\sigma} = \frac{2^{\sigma-1}\sigma\Gamma((d+\sigma)/2)}{\pi^{d/2}\Gamma(1-\sigma/2)}$ is a normalization constant.]

(3) Associated semigroup $\quad (-\Delta)^{\sigma/2} g(x) = \frac{1}{\Gamma(-\frac{\sigma}{2})}\int_0^\infty \left(e^{t\Delta} g(x) - g(x)\right)\frac{dt}{t^{1+\frac{\sigma}{2}}}\,.$

- In this talk we will always let $\boxed{\sigma = 2s\,, \quad 0 < s \le 1\,.}$

The setup of the problem

The Cauchy Problem for The Fractional Porous Medium / Fast Diffusion Equation in $\mathbb{R}^d$

$$\text{(CP)} \quad \begin{cases} \partial_t u(t,x) + (-\Delta)^s u^m(t,x) = 0\,, & \forall (t,x) \in (0,+\infty)\times\mathbb{R}^d \\ u(0,x) = u_0(x)\,, & \forall x \in \mathbb{R}^d \end{cases}$$

where $0 < s \le 1\,,\quad m > 0$ and $d \ge 1$.

$m > 1$ is PME, $\quad m = 1$ is HE, $\quad 0 < m < 1$ is FDE.

Definition(s) of the fractional Laplacian

Several equivalent definitions of the nonlocal operator $(-\Delta)^{\sigma/2}$ (Laplacian of order σ):

(1) Fourier transform $\qquad \widehat{((-\Delta)^{\sigma/2} f)}(x) = (2\pi|x|)^\sigma \hat{f}(x)\,,$

[can be used for positive and negative values of σ]

(2) Hypersingular Kernel $\qquad (-\Delta)^{\sigma/2} g(x) = c_{d,\sigma}\ \text{P.V.} \int_{\mathbb{R}^d} \frac{g(x)-g(z)}{|x-z|^{d+\sigma}}\, dz\,,$

[can be used for $0 < \sigma < 2$, where $c_{d,\sigma} = \frac{2^{\sigma-1}\sigma\Gamma((d+\sigma)/2)}{\pi^{d/2}\Gamma(1-\sigma/2)}$ is a normalization constant.]

(3) Associated semigroup $\qquad (-\Delta)^{\sigma/2} g(x) = \frac{1}{\Gamma(-\frac{\sigma}{2})} \int_0^\infty \left(e^{t\Delta} g(x) - g(x)\right) \frac{dt}{t^{1+\frac{\sigma}{2}}}\,.$

- In this talk we will always let $\boxed{\sigma = 2s\,,\quad 0 < s \le 1\,.}$

References:

[BV] M. BONFORTE, J. L. VÁZQUEZ, Quantitative Local and Global A Priori Estimates for Fractional Nonlinear Diffusion Equations, *Preprint* (2012). http://arxiv.org/abs/1210.2594

[DPQRV1] A. DE PABLO, F. QUIRÓS, A. RODRIGUEZ, J. L. VÁZQUEZ, A fractional porous medium equation *Adv. Math.* **226** (2011), no. 2, 1378 – 1409.

[DPQRV2] A. DE PABLO, F. QUIRÓS, A. RODRIGUEZ, J. L. VÁZQUEZ, A general fractional porous medium equation, *Comm. Pure Appl. Math.*, **65** (2012), 1242 – 1284.

[V] J. L. VÁZQUEZ, Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type, *Preprint,* (2012). http://arxiv.org/abs/1205.6332v2

More:

[AC] I. ATHANASOPOULOS, L. A. CAFFARELLI, Continuity of the temperature in boundary heat control problems. *Adv. Math.* **224** (2010), no. 1, 293 – 315.

[CS] L. CAFFARELLI, L. SILVESTRE, An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* **32** (2007), no. 7-9, 1245 – 1260.

[VBk] J. L. VÁZQUEZ, *The porous medium equation. Mathematical theory.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

References:

[BV] M. BONFORTE, J. L. VÁZQUEZ, Quantitative Local and Global A Priori Estimates for Fractional Nonlinear Diffusion Equations, *Preprint* (2012). http://arxiv.org/abs/1210.2594

[DPQRV1] A. DE PABLO, F. QUIRÓS, A. RODRIGUEZ, J. L. VÁZQUEZ, A fractional porous medium equation *Adv. Math.* **226** (2011), no. 2, 1378 – 1409.

[DPQRV2] A. DE PABLO, F. QUIRÓS, A. RODRIGUEZ, J. L. VÁZQUEZ, A general fractional porous medium equation, *Comm. Pure Appl. Math.,* **65** (2012), 1242 – 1284.

[V] J. L. VÁZQUEZ, Barenblatt solutions and asymptotic behaviour for a nonlinear fractional heat equation of porous medium type, *Preprint,* (2012). http://arxiv.org/abs/1205.6332v2

More:

[AC] I. ATHANASOPOULOS, L. A. CAFFARELLI, Continuity of the temperature in boundary heat control problems. *Adv. Math.* **224** (2010), no. 1, 293 – 315.

[CS] L. CAFFARELLI, L. SILVESTRE, An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations* **32** (2007), no. 7-9, 1245 – 1260.

[VBk] J. L. VÁZQUEZ, *The porous medium equation. Mathematical theory.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

Several definition of solutions

Consider the Fractional Porous Medium / Fast Diffusion Equation in $\mathbb{R}^d$:

$$\text{(EQ)} \qquad \boxed{\partial_t u(t,x) + (-\Delta)^s u^m(t,x) = 0\,.}$$

Definition of Weak and Strong solutions, [DPQRV1, DPQRV2]

- A function u is a *weak* solution to (EQ) if
  (i) $u \in C((0,\infty) : L^1(\mathbb{R}^d))$, $|u|^{m-1}u \in L^2_{\rm loc}((0,\infty) : \dot{H}^s(\mathbb{R}^d))$;
  (ii) The following identity holds for every $\varphi \in C^1_0(\mathbb{R}^d \times (0,\infty))$:

$$\int_0^\infty \int_{\mathbb{R}^d} u \frac{\partial \varphi}{\partial t}\,{\rm d}x\,{\rm d}t - \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{s/2}(|u|^{m-1}u)(-\Delta)^{s/2}\varphi\,{\rm d}x\,{\rm d}t = 0.$$

- A *weak* solution to the Cauchy Problem (CP) is a weak solution to Equation (EQ) such that moreover $u \in C([0,\infty) : L^1(\mathbb{R}^d))$ and $u(0,\cdot) = u_0 \in {\rm L}^1(\mathbb{R}^d)$.
- We say that a weak solution u to the Cauchy Problem (CP) is a *strong* solution if moreover $\partial_t u \in L^\infty((\tau,\infty) : L^1(\mathbb{R}^d))$, for every $\tau > 0$.

- The fractional Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is the completion of $C^\infty_0(\mathbb{R}^d)$ with the norm

$$\|\psi\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^\sigma |\hat{\psi}|^2\,d\xi\right)^{1/2} = \|(-\Delta)^{s/2}\psi\|_2.$$

- In [DPQRV2] there are other concepts of solutions, such as Mild solutions, Weak solutions via the so-called "Caffarelli-Silvestre extension"...

Several definition of solutions

Consider the Fractional Porous Medium / Fast Diffusion Equation in $\mathbb{R}^d$:

$$\text{(EQ)} \qquad \boxed{\partial_t u(t,x) + (-\Delta)^s u^m(t,x) = 0\,.}$$

Definition of Weak and Strong solutions, [DPQRV1, DPQRV2]

- A function u is a *weak* solution to (EQ) if
  (i) $u \in C((0,\infty) : L^1(\mathbb{R}^d))$, $|u|^{m-1}u \in L^2_{\mathrm{loc}}((0,\infty) : \dot{H}^s(\mathbb{R}^d))$;
  (ii) The following identity holds for every $\varphi \in C^1_0(\mathbb{R}^d \times (0,\infty))$:

$$\int_0^\infty \int_{\mathbb{R}^d} u\frac{\partial \varphi}{\partial t}\,\mathrm{d}x\,\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}^d} (-\Delta)^{s/2}(|u|^{m-1}u)(-\Delta)^{s/2}\varphi\,\mathrm{d}x\,\mathrm{d}t = 0.$$

- A *weak* solution to the Cauchy Problem (CP) is a weak solution to Equation (EQ) such that moreover $u \in C([0,\infty) : L^1(\mathbb{R}^d))$ and $u(0,\cdot) = u_0 \in \mathrm{L}^1(\mathbb{R}^d)$.
- We say that a weak solution u to the Cauchy Problem (CP) is a *strong* solution if moreover $\partial_t u \in L^\infty((\tau,\infty) : L^1(\mathbb{R}^d))$, for every $\tau > 0$.

- The fractional Sobolev space $\dot{H}^s(\mathbb{R}^d)$ is the completion of $C_0^\infty(\mathbb{R}^d)$ with the norm

$$\|\psi\|_{\dot{H}^s} = \left(\int_{\mathbb{R}^d} |\xi|^\sigma |\hat{\psi}|^2\,d\xi\right)^{1/2} = \|(-\Delta)^{s/2}\psi\|_2.$$

- In [DPQRV2] there are other concepts of solutions, such as Mild solutions, Weak solutions via the so-called "Caffarelli-Silvestre extension"...

Several definition of solutions

Consider the Fractional Porous Medium / Fast Diffusion Equation in $\mathbb{R}^d$:

$$\text{(EQ)} \qquad \boxed{\partial_t u(t,x) + (-\Delta)^s u^m(t,x) = 0\,.}$$

Definition of Very Weak solutions, [BV]

- A function u is a *very weak* solution to (EQ) if
  (i) $u \in C((0,\infty) : L^1_{\rm loc}(\mathbb{R}^d))$, $|u|^{m-1}u \in L^1_{\rm loc}\left((0,\infty) : L^1\left(\mathbb{R}^d, (1+|x|)^{-(d+2s)}\,{\rm d}x\right)\right)$;

  (ii) The following identity holds for every $\varphi \in C_c^\infty([0,T]\times\mathbb{R}^d)$:

$$\int_0^\infty \int_{\mathbb{R}^d} u \frac{\partial \varphi}{\partial t}\,{\rm d}x\,{\rm d}t - \int_0^\infty \int_{\mathbb{R}^d} |u|^{m-1}u\,(-\Delta)^s\varphi\,{\rm d}x\,{\rm d}t = 0\,.$$

- A *very weak* solution to the Cauchy Problem (CP) is very weak sol. to (EQ) such that moreover $u \in C([0,\infty) : L^1_{\rm loc}(\mathbb{R}^d))$ and $u(0,\cdot) = u_0 \in {\rm L}^1_{\rm loc}(\mathbb{R}^d)$.

- Notice that $u(t,x) = $ constant is a very weak solution to (EQ).
- *At least in the Fast Diffusion case* $0 < m < 1$, there exist very weak solutions corresponding to initial data that can grow at infinity like $|x|^{\frac{2s}{m}-\varepsilon}$, for all $\varepsilon > 0$. Such solutions are moreover unique, strictly positive and they exist globally in time. Therefore, by [AC], they are regular.

Several definition of solutions

Consider the Fractional Porous Medium / Fast Diffusion Equation in $\mathbb{R}^d$:

$$\text{(EQ)} \qquad \boxed{\partial_t u(t,x) + (-\Delta)^s u^m(t,x) = 0\,.}$$

Definition of Very Weak solutions, [BV]

- A function u is a *very weak* solution to (EQ) if
  (i) $u \in C((0,\infty) : L^1_{\rm loc}(\mathbb{R}^d))$, $|u|^{m-1}u \in L^1_{\rm loc}\left((0,\infty) : L^1\left(\mathbb{R}^d, (1+|x|)^{-(d+2s)}\,{\rm d}x\right)\right)$;

  (ii) The following identity holds for every $\varphi \in C_c^\infty([0,T]\times\mathbb{R}^d)$:

$$\int_0^\infty \int_{\mathbb{R}^d} u \frac{\partial \varphi}{\partial t}\,{\rm d}x\,{\rm d}t - \int_0^\infty \int_{\mathbb{R}^d} |u|^{m-1}u\,(-\Delta)^s \varphi\,{\rm d}x\,{\rm d}t = 0\,.$$

- A *very weak* solution to the Cauchy Problem (CP) is very weak sol. to (EQ) such that moreover $u \in C([0,\infty) : L^1_{\rm loc}(\mathbb{R}^d))$ and $u(0,\cdot) = u_0 \in {\rm L}^1_{\rm loc}(\mathbb{R}^d)$.

- Notice that $u(t,x) =$ constant is a very weak solution to (EQ).
- *At least in the Fast Diffusion case* $0 < m < 1$, there exist very weak solutions corresponding to initial data that can grow at infinity like $|x|^{\frac{2s}{m}-\varepsilon}$, for all $\varepsilon > 0$. Such solutions are moreover unique, strictly positive and they exist globally in time. Therefore, by [AC], they are regular.

Properties of weak solutions I, $m > m_c$ [DPQRV1,DPQRV2]

Consider a weak solution $u(t,x)$ to the Cauchy problem (CP) and let $m_c = \dfrac{d-2s}{d}$.

The case $m > m_c$: PME + HE + Good FDE

- EXISTENCE AND UNIQUENESS FOR L^1 DATA. Let $m > m_c$, then for every $u_0 \in L^1(\mathbb{R}^d)$ there exists a *unique strong solution* of Problem (CP).
  [*The linear case $m = 1$ included*]
- WELL POSEDNESS. The solution depends continuously on the parameters $s \in (0,1)$, $m > m_c$, and $u_0 \in L^1(\mathbb{R}^d)$ in the norm of the space $C([0,\infty) : L^1(\mathbb{R}^d))$.
- SMOOTHING EFFECTS. For all $p \geq 1$,

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{L^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with} \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$

- L^1-CONTRACTION AND COMPARISON. Let $u_0, v_0 \in L^1(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} (u-v)_+(x,t)\,dx \leq \int_{\mathbb{R}^d} (u_0-v_0)_+(x)\,dx\,.$$

- MASS CONSERVATION. $\int_{\mathbb{R}^d} u(x,t)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx$ for all $t \geq 0$.
- POSITIVITY. If $u_0 \geq 0$, then $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t > 0$.
- C^α REGULARITY. If either $m \geq 1$ or $u_0 \geq 0$, then $u \in C^\alpha((0,\infty) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions with the same mass, which are positive and C^α globally in time. [Mass Conservation + Positivity + Regularity]

Properties of weak solutions I, $m > m_c$ [DPQRV1,DPQRV2]

Consider a weak solution $u(t,x)$ to the Cauchy problem (CP) and let $\boxed{m_c = \frac{d-2s}{d}}$.

The case $m > m_c$: PME + HE + Good FDE

- EXISTENCE AND UNIQUENESS FOR L^1 DATA. Let $m > m_c$, then for every $u_0 \in L^1(\mathbb{R}^d)$ there exists a *unique strong solution* of Problem (CP).
  [*The linear case $m = 1$ included*]
- WELL POSEDNESS. The solution depends continuously on the parameters $s \in (0,1)$, $m > m_c$, and $u_0 \in L^1(\mathbb{R}^d)$ in the norm of the space $C([0,\infty) : L^1(\mathbb{R}^d))$.
- SMOOTHING EFFECTS. For all $p \geq 1$,

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{L^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with} \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$

- L^1-CONTRACTION AND COMPARISON. Let $u_0, v_0 \in L^1(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} (u-v)_+(x,t)\,dx \leq \int_{\mathbb{R}^d} (u_0-v_0)_+(x)\,dx .$$

- MASS CONSERVATION. $\int_{\mathbb{R}^d} u(x,t)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx$ for all $t \geq 0$.
- POSITIVITY. If $u_0 \geq 0$, then $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t > 0$.
- C^α REGULARITY. If either $m \geq 1$ or $u_0 \geq 0$, then $u \in C^\alpha((0,\infty)\times\mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions with the same mass, which are positive and C^α globally in time. [Mass Conservation + Positivity + Regularity]

Properties of weak solutions I, $m > m_c$ [DPQRV1,DPQRV2]

Consider a weak solution $u(t,x)$ to the Cauchy problem (CP) and let $\boxed{m_c = \dfrac{d-2s}{d}\,.}$

The case $m > m_c$: PME + HE + Good FDE

- EXISTENCE AND UNIQUENESS FOR L^1 DATA. Let $m > m_c$, then for every $u_0 \in L^1(\mathbb{R}^d)$ there exists a *unique strong solution* of Problem (CP).
  [*The linear case $m = 1$ included*]
- WELL POSEDNESS. The solution depends continuously on the parameters $s \in (0,1)$, $m > m_c$, and $u_0 \in L^1(\mathbb{R}^d)$ in the norm of the space $C([0,\infty) : L^1(\mathbb{R}^d))$.
- SMOOTHING EFFECTS. For all $p \geq 1$,

$$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \leq \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{L^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with} \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$

- L^1-CONTRACTION AND COMPARISON. Let $u_0, v_0 \in L^1(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} (u-v)_+(x,t)\,dx \leq \int_{\mathbb{R}^d} (u_0 - v_0)_+(x)\,dx\,.$$

- MASS CONSERVATION. $\int_{\mathbb{R}^d} u(x,t)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx$ for all $t \geq 0$.
- POSITIVITY. If $u_0 \geq 0$, then $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t > 0$.
- C^α REGULARITY. If either $m \geq 1$ or $u_0 \geq 0$, then $u \in C^\alpha((0,\infty) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions with the same mass, which are positive and C^α globally in time. [Mass Conservation + Positivity + Regularity]

Properties of weak solutions I, $m > m_c$ [DPQRV1,DPQRV2]

Consider a weak solution $u(t,x)$ to the Cauchy problem (CP) and let $\boxed{m_c = \dfrac{d-2s}{d}}$.

The case $m > m_c$: PME + HE + Good FDE

- EXISTENCE AND UNIQUENESS FOR L^1 DATA. Let $m > m_c$, then for every $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ there exists a *unique strong solution* of Problem (CP).
  [*The linear case $m = 1$ included*]
- WELL POSEDNESS. The solution depends continuously on the parameters $s \in (0,1)$, $m > m_c$, and $u_0 \in L^1(\mathbb{R}^d)$ in the norm of the space $C([0,\infty) : L^1(\mathbb{R}^d))$.
- SMOOTHING EFFECTS. For all $p \geq 1$,

$$\|u(t,\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{\mathrm{L}^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with } \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$

- L^1-CONTRACTION AND COMPARISON. Let $u_0, v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ then

$$\int_{\mathbb{R}^d} (u-v)_+(x,t)\,dx \leq \int_{\mathbb{R}^d} (u_0 - v_0)_+(x)\,dx\,.$$

- MASS CONSERVATION. $\int_{\mathbb{R}^d} u(x,t)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx$ for all $t \geq 0$.
- POSITIVITY. If $u_0 \geq 0$, then $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t > 0$.
- C^α REGULARITY. If either $m \geq 1$ or $u_0 \geq 0$, then $u \in C^\alpha((0,\infty) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions with the same mass, which are positive and C^α globally in time. [Mass Conservation + Positivity + Regularity]

Properties of weak solutions I, $m > m_c$ [DPQRV1,DPQRV2]

Consider a weak solution $u(t,x)$ to the Cauchy problem (CP) and let $\boxed{m_c = \frac{d-2s}{d}}$.

The case $m > m_c$: PME + HE + Good FDE

- EXISTENCE AND UNIQUENESS FOR L^1 DATA. Let $m > m_c$, then for every $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ there exists a *unique strong solution* of Problem (CP).
  [*The linear case $m = 1$ included*]
- WELL POSEDNESS. The solution depends continuously on the parameters $s \in (0,1)$, $m > m_c$, and $u_0 \in L^1(\mathbb{R}^d)$ in the norm of the space $C([0,\infty) : L^1(\mathbb{R}^d))$.
- SMOOTHING EFFECTS. For all $p \geq 1$,
  $$\|u(t,\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{\mathrm{L}^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with} \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$
- L^1-CONTRACTION AND COMPARISON. Let $u_0, v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ then
  $$\int_{\mathbb{R}^d} (u-v)_+(x,t)\,dx \leq \int_{\mathbb{R}^d} (u_0 - v_0)_+(x)\,dx\,.$$
- MASS CONSERVATION. $\int_{\mathbb{R}^d} u(x,t)\,dx = \int_{\mathbb{R}^d} u_0(x)\,dx$ for all $t \geq 0$.
- POSITIVITY. If $u_0 \geq 0$, then $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t > 0$.
- C^α REGULARITY. If either $m \geq 1$ or $u_0 \geq 0$, then $u \in C^\alpha((0,\infty) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions with the same mass, which are positive and C^α globally in time. [Mass Conservation + Positivity + Regularity]

Properties of weak solutions II, $0 < m \le m_c$ [DPQRV1,DPQRV2]

Let $0 < m \le m_c$, $\boxed{m_c = \frac{d-2s}{d}.}$ and $\boxed{p_c = \frac{d(1-m)}{2s}}$

The case $0 < m \le m_c$: Very Fast FDE

- EXISTENCE AND UNIQUENESS FOR L^p DATA.
  Let $0 < m \le m_c$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p > p_c$.
  There exists a *unique strong solution* of Problem (CP).
  [*Well posedness, i.e., continuous dependence on "m, s" not known in this range*]
- SMOOTHING EFFECTS. For all $p > p_c$,
  $$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{L^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with} \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$
- L^1-CONTRACTION AND COMPARISON. As before, it holds for all $m > 0$.
- LOSS OF MASS. The mass is conserved only when $m = m_c$.
  When $0 < m < m_c$ there is a finite time $T = T(u_0) > 0$, called *Extinction Time*, such that $\boxed{u(x,t) \equiv 0 \text{ for all } t \ge T \text{ and } x \in \mathbb{R}^d}$.
- POSITIVITY. If $u_0 \ge 0$, then the solution is positive up to the extinction time T, i.e., $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t \in (0,T)$.
- C^α REGULARITY. Let $u_0 \ge 0$ and T be the extinction time.
  Then $u \in C^\alpha((0,T) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions which lose mass, are positive and C^α up to the extinction time. [Loss of Mass + Positivity + Regularity]

Properties of weak solutions II, $0 < m \le m_c$ [DPQRV1,DPQRV2]

Let $0 < m \le m_c$, $\boxed{m_c = \frac{d-2s}{d}}$. and $\boxed{p_c = \frac{d(1-m)}{2s}}$

The case $0 < m \le m_c$: Very Fast FDE

- EXISTENCE AND UNIQUENESS FOR L^p DATA.
  Let $0 < m \le m_c$ and $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^p(\mathbb{R}^d)$ with $p > p_c$.
  There exists a *unique strong solution* of Problem (CP).
  [*Well posedness, i.e., continuous dependence on "m, s" not known in this range*]
- SMOOTHING EFFECTS. For all $p > p_c$,
  $$\|u(t,\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{\mathrm{L}^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with } \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$
- L^1-CONTRACTION AND COMPARISON. As before, it holds for all $m > 0$.
- LOSS OF MASS. The mass is conserved only when $m = m_c$.
  When $0 < m < m_c$ there is a finite time $T = T(u_0) > 0$, called *Extinction Time*,
  such that $\boxed{u(x,t) \equiv 0 \text{ for all } t \ge T \text{ and } x \in \mathbb{R}^d}$.
- POSITIVITY. If $u_0 \ge 0$, then the solution is positive up to the extinction time T,
  i.e., $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t \in (0,T)$.
- C^α REGULARITY. Let $u_0 \ge 0$ and T be the extinction time.
  Then $u \in C^\alpha((0,T) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions which lose mass, are positive and C^α up to the extinction time. [Loss of Mass + Positivity + Regularity]

Properties of weak solutions II, $0 < m \le m_c$ [DPQRV1,DPQRV2]

Let $0 < m \le m_c$, $\boxed{m_c = \dfrac{d-2s}{d}}$. and $\boxed{p_c = \dfrac{d(1-m)}{2s}}$

The case $0 < m \le m_c$: Very Fast FDE

- EXISTENCE AND UNIQUENESS FOR L^p DATA.
  Let $0 < m \le m_c$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p > p_c$.
  There exists a *unique strong solution* of Problem (CP).
  [*Well posedness, i.e., continuous dependence on "m, s" not known in this range*]
- SMOOTHING EFFECTS. For all $p > p_c$,
  $$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{L^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with } \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$
- L^1-CONTRACTION AND COMPARISON. As before, it holds for all $m > 0$.
- LOSS OF MASS. The mass is conserved only when $m = m_c$.
  When $0 < m < m_c$ there is a finite time $T = T(u_0) > 0$, called *Extinction Time*, such that $\boxed{u(x,t) \equiv 0 \text{ for all } t \ge T \text{ and } x \in \mathbb{R}^d}$.
- POSITIVITY. If $u_0 \ge 0$, then the solution is positive up to the extinction time T, i.e., $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t \in (0,T)$.
- C^α REGULARITY. Let $u_0 \ge 0$ and T be the extinction time.
  Then $u \in C^\alpha((0,T) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions which lose mass, are positive and C^α up to the extinction time. [Loss of Mass + Positivity + Regularity]

Properties of weak solutions II, $0 < m \le m_c$ [DPQRV1,DPQRV2]

Let $0 < m \le m_c$, $\boxed{m_c = \frac{d-2s}{d}}$. and $\boxed{p_c = \frac{d(1-m)}{2s}}$

The case $0 < m \le m_c$: Very Fast FDE

- EXISTENCE AND UNIQUENESS FOR L^p DATA.
  Let $0 < m \le m_c$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p > p_c$.
  There exists a *unique strong solution* of Problem (CP).
  [*Well posedness, i.e., continuous dependence on "m, s" not known in this range*]
- SMOOTHING EFFECTS. For all $p > p_c$,
  $$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{L^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with } \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$
- L^1-CONTRACTION AND COMPARISON. As before, it holds for all $m > 0$.
- LOSS OF MASS. The mass is conserved only when $m = m_c$.
  When $0 < m < m_c$ there is a finite time $T = T(u_0) > 0$, called *Extinction Time*,
  such that $\boxed{u(x,t) \equiv 0 \text{ for all } t \ge T \text{ and } x \in \mathbb{R}^d}$.
- POSITIVITY. If $u_0 \ge 0$, then the solution is positive up to the extinction time T, i.e., $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t \in (0,T)$.
- C^α REGULARITY. Let $u_0 \ge 0$ and T be the extinction time.
  Then $u \in C^\alpha((0,T) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions which lose mass, are positive and C^α up to the extinction time. [Loss of Mass + Positivity + Regularity]

Properties of weak solutions II, $0 < m \le m_c$ [DPQRV1,DPQRV2]

Let $0 < m \le m_c$, $\boxed{m_c = \dfrac{d-2s}{d}}$. and $\boxed{p_c = \dfrac{d(1-m)}{2s}}$

The case $0 < m \le m_c$**:** Very Fast FDE

- EXISTENCE AND UNIQUENESS FOR L^p DATA.
  Let $0 < m \le m_c$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p > p_c$.
  There exists a *unique strong solution* of Problem (CP).
  [*Well posedness, i.e., continuous dependence on "m, s" not known in this range*]
- SMOOTHING EFFECTS. For all $p > p_c$,
  $$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{L^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with} \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$
- L^1-CONTRACTION AND COMPARISON. As before, it holds for all $m > 0$.
- LOSS OF MASS. The mass is conserved only when $m = m_c$.
  When $0 < m < m_c$ there is a finite time $T = T(u_0) > 0$, called *Extinction Time*, such that $\boxed{u(x,t) \equiv 0 \text{ for all } t \ge T \text{ and } x \in \mathbb{R}^d}$.
- POSITIVITY. If $u_0 \ge 0$, then the solution is positive up to the extinction time T, i.e., $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t \in (0,T)$.
- C^α REGULARITY. Let $u_0 \ge 0$ and T be the extinction time.
  Then $u \in C^\alpha((0,T) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions which lose mass, are positive and C^α up to the extinction time. [Loss of Mass + Positivity + Regularity]

Properties of weak solutions II, $0 < m \le m_c$ [DPQRV1,DPQRV2]

Let $0 < m \le m_c$, $\boxed{m_c = \dfrac{d-2s}{d}}$. and $\boxed{p_c = \dfrac{d(1-m)}{2s}}$

The case $0 < m \le m_c$: Very Fast FDE

- EXISTENCE AND UNIQUENESS FOR L^p DATA.
  Let $0 < m \le m_c$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p > p_c$.
  There exists a *unique strong solution* of Problem (CP).
  [*Well posedness, i.e., continuous dependence on "m, s" not known in this range*]
- SMOOTHING EFFECTS. For all $p > p_c$,
  $$\|u(t,\cdot)\|_{L^\infty(\mathbb{R}^d)} \le \frac{K_\infty}{t^{d\vartheta_p}} \|u_0\|_{L^p(\mathbb{R}^d)}^{2sp\vartheta_p}, \qquad \text{with } \begin{cases} \vartheta_p = \frac{1}{2sp+d(m-1)} > 0 \\ K_\infty = K_\infty(m,p,d,s) > 0 \end{cases}$$
- L^1-CONTRACTION AND COMPARISON. As before, it holds for all $m > 0$.
- LOSS OF MASS. The mass is conserved only when $m = m_c$.
  When $0 < m < m_c$ there is a finite time $T = T(u_0) > 0$, called *Extinction Time*, such that $\boxed{u(x,t) \equiv 0 \text{ for all } t \ge T \text{ and } x \in \mathbb{R}^d}$.
- POSITIVITY. If $u_0 \ge 0$, then the solution is positive up to the extinction time T, i.e., $u(t,x) > 0$, for all $x \in \mathbb{R}^d$ and all $t \in (0,T)$.
- C^α REGULARITY. Let $u_0 \ge 0$ and T be the extinction time.
  Then $u \in C^\alpha((0,T) \times \mathbb{R}^d)$ for some $0 < \alpha < 1$.
- REMARK: Non-negative initial data produce solutions which lose mass, are positive and C^α up to the extinction time. [Loss of Mass + Positivity + Regularity]

Weighted L^1 Estimates. The case $0 < m < 1$.

Class of weights $\mathcal{W}_\alpha$. We say that $\varphi \in C^2(\mathbb{R}^d)$ belongs to the class $\mathcal{W}_\alpha$ if:
(i) φ is positive, radially symmetric and decreasing outside $B_1(0)$, i.e. for all $|x| \geq 1$.
(ii) $\varphi(x) \leq |x|^{-\alpha}$ and $|\mathrm{D}^2\varphi(x)| \leq c_0|x|^{-\alpha-2}$, for some $c_0 > 0$ and for any $|x| >> 1$.

Example. $$\varphi(x) = \begin{cases} 1\,, & \text{if } |x| \leq 1\,, \\ \left[1 + (|x|^2 - 1)^4\right]^{-\frac{\alpha}{8}}\,, & \text{if } |x| \geq 1\,. \end{cases}$$

Lemma. Estimates for the weights $\mathcal{W}_\alpha$

Let $\varphi \in \mathcal{W}_\alpha$, for $\alpha > 0$. Then, for all $|x| \geq |x_0| >> 1$ we have

$$|(-\Delta)^s\varphi(x)| \leq \begin{cases} c_1\,|x|^{-(\alpha+2s)}\,, & \text{if } \alpha < d\,, \\ c_2\,\log|x|\,|x|^{-(d+2s)}\,, & \text{if } \alpha = d\,, \\ c_3\,|x|^{-(d+2s)}\,, & \text{if } \alpha > d\,. \end{cases}$$

Moreover, for $\alpha > d$ the reverse estimate holds from below if $\varphi \geq 0$:

$$|(-\Delta)^s\varphi(x)| \geq \frac{c_4}{|x|^{d+2s}} \qquad \text{for all } |x| \geq |x_0| >> 1\,.$$

The positive constants $c_1, c_2, c_3, c_4 > 0$ depend only on α, s, d and $\|\varphi\|_{C^2(\mathbb{R}^d)}$.

Remark. Note that if $\varphi \in \mathcal{W}_\alpha$ is compactly supported, then $|(-\Delta)^s\varphi(x)|$ is no more compactly supported, and it behaves like $|x|^{-(d+2s)}$ for any $|x| >> 1$.

Weighted L^1 Estimates. The case $0 < m < 1$.

Class of weights $\mathcal{W}_\alpha$. We say that $\varphi \in C^2(\mathbb{R}^d)$ belongs to the class $\mathcal{W}_\alpha$ if:
(i) φ is positive, radially symmetric and decreasing outside $B_1(0)$, i.e. for all $|x| \geq 1$.
(ii) $\varphi(x) \leq |x|^{-\alpha}$ and $|\mathrm{D}^2\varphi(x)| \leq c_0|x|^{-\alpha-2}$, for some $c_0 > 0$ and for any $|x| >> 1$.

Example.
$$\varphi(x) = \begin{cases} 1\,, & \text{if } |x| \leq 1\,, \\ \left[1 + (|x|^2 - 1)^4\right]^{-\frac{\alpha}{8}}\,, & \text{if } |x| \geq 1\,. \end{cases}$$

Lemma. Estimates for the weights $\mathcal{W}_\alpha$

Let $\varphi \in \mathcal{W}_\alpha$, for $\alpha > 0$. Then, for all $|x| \geq |x_0| >> 1$ we have

$$|(-\Delta)^s\varphi(x)| \leq \begin{cases} c_1\,|x|^{-(\alpha+2s)}\,, & \text{if } \alpha < d\,, \\ c_2\,\log|x|\,|x|^{-(d+2s)}\,, & \text{if } \alpha = d\,, \\ c_3\,|x|^{-(d+2s)}\,, & \text{if } \alpha > d\,. \end{cases}$$

Moreover, for $\alpha > d$ the reverse estimate holds from below if $\varphi \geq 0$:

$$|(-\Delta)^s\varphi(x)| \geq \frac{c_4}{|x|^{d+2s}} \qquad \text{for all } |x| \geq |x_0| >> 1\,.$$

The positive constants $c_1, c_2, c_3, c_4 > 0$ depend only on α, s, d and $\|\varphi\|_{C^2(\mathbb{R}^d)}$.

Remark. Note that if $\varphi \in \mathcal{W}_\alpha$ is compactly supported, then $|(-\Delta)^s\varphi(x)|$ is no more compactly supported, and it behaves like $|x|^{-(d+2s)}$ for any $|x| >> 1$.

Weighted L^1 Estimates. The case $0 < m < 1$.

Class of weights $\mathcal{W}_\alpha$. We say that $\varphi \in C^2(\mathbb{R}^d)$ belongs to the class $\mathcal{W}_\alpha$ if:
(i) φ is positive, radially symmetric and decreasing outside $B_1(0)$, i.e. for all $|x| \ge 1$.
(ii) $\varphi(x) \le |x|^{-\alpha}$ and $|\mathrm{D}^2\varphi(x)| \le c_0|x|^{-\alpha-2}$, for some $c_0 > 0$ and for any $|x| >> 1$.

Example. $$\varphi(x) = \begin{cases} 1\,, & \text{if } |x| \le 1\,, \\ \left[1 + (|x|^2 - 1)^4\right]^{-\frac{\alpha}{8}}\,, & \text{if } |x| \ge 1\,. \end{cases}$$

Lemma. Estimates for the weights $\mathcal{W}_\alpha$

Let $\varphi \in \mathcal{W}_\alpha$, for $\alpha > 0$. Then, for all $|x| \ge |x_0| >> 1$ we have

$$|(-\Delta)^s\varphi(x)| \le \begin{cases} c_1\,|x|^{-(\alpha+2s)}\,, & \text{if } \alpha < d\,, \\ c_2\,\log|x|\,|x|^{-(d+2s)}\,, & \text{if } \alpha = d\,, \\ c_3\,|x|^{-(d+2s)}\,, & \text{if } \alpha > d\,. \end{cases}$$

Moreover, for $\alpha > d$ the reverse estimate holds from below if $\varphi \ge 0$:

$$|(-\Delta)^s\varphi(x)| \ge \frac{c_4}{|x|^{d+2s}} \qquad \text{for all } |x| \ge |x_0| >> 1\,.$$

The positive constants $c_1, c_2, c_3, c_4 > 0$ depend only on α, s, d and $\|\varphi\|_{C^2(\mathbb{R}^d)}$.

Remark. Note that if $\varphi \in \mathcal{W}_\alpha$ is compactly supported, then $|(-\Delta)^s\varphi(x)|$ is no more compactly supported, and it behaves like $|x|^{-(d+2s)}$ for any $|x| >> 1$.

Weighted L^1 Estimates. The case $0 < m < 1$.

Theorem 1. Weighted L^1 estimates for $0 < m < 1$, [BV]

Let $u \ge v$ be two ordered very weak solutions to the equation (EQ), with $0 < m < 1$. Let $\varphi_R(x) = \varphi(x/R)$ where $R > 0$ and $\varphi \in \mathcal{W}_\alpha$ with

$$\boxed{d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}.}$$

Then, there exists $C_1 > 0$ that depends only on α, m, d, such that $\forall\ 0 \le \tau, t < \infty$:

$$\left[\int_{\mathbb{R}^d} (u-v)(t,x)\varphi_R(x)\,\mathrm{d}x\right]^{1-m} \le \left[\int_{\mathbb{R}^d} (u-v)(\tau,x)\varphi_R(x)\,\mathrm{d}x\right]^{1-m} + \frac{C_1\,|t-\tau|}{R^{2s-d(1-m)}}.$$

- The estimate holds for changing sign solutions and also for $s = 1$.
- The estimate holds both for $\tau < t$ and for $\tau > t$. [Quantitative loss of mass]
- When $m_c < m < 1$: Conservation of mass, by letting $R \to \infty$.
- When $0 < m < m_c$: if $u_0 \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ with $p \ge p_c = \frac{d(1-m)}{2s}$, then sols extinguish in finite time $T > 0$, see [DPQRV2].
  Lower bound for the extinction time, [let $\tau = T$ and $t = 0$]

$$\frac{1}{C_1\,R^{d(1-m)-2s}}\left(\int_{\mathbb{R}^d} u_0\,\varphi_R\,\mathrm{d}x\right)^{1-m} \le T$$

- If u_0 is such that the limit as $R \to +\infty$ of the right-hand side diverges to $+\infty$, then the corresponding solution $u(t,x)$ exists (and is positive) globally in time.

Weighted L^1 Estimates. The case $0 < m < 1$.

Theorem 1. Weighted L^1 estimates for $0 < m < 1$, [BV]

Let $u \ge v$ be two ordered very weak solutions to the equation (EQ), with $0 < m < 1$. Let $\varphi_R(x) = \varphi(x/R)$ where $R > 0$ and $\varphi \in \mathcal{W}_\alpha$ with

$$\boxed{d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}\,.}$$

Then, there exists $C_1 > 0$ that depends only on α, m, d, such that $\forall\ 0 \le \tau, t < \infty$:

$$\left[\int_{\mathbb{R}^d} (u-v)(t,x)\varphi_R(x)\,\mathrm{d}x\right]^{1-m} \le \left[\int_{\mathbb{R}^d} (u-v)(\tau,x)\varphi_R(x)\,\mathrm{d}x\right]^{1-m} + \frac{C_1\,|t-\tau|}{R^{2s-d(1-m)}}\,.$$

- The estimate holds for changing sign solutions and also for $s = 1$.
- The estimate holds both for $\tau < t$ and for $\tau > t$. [Quantitative loss of mass]
- When $m_c < m < 1$: Conservation of mass, by letting $R \to \infty$.
- When $0 < m < m_c$: if $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^p(\mathbb{R}^d)$ with $p \ge p_c = \frac{d(1-m)}{2s}$, then sols extinguish in finite time $T > 0$, see [DPQRV2]. Lower bound for the extinction time, [let $\tau = T$ and $t = 0$]

$$\frac{1}{C_1\,R^{d(1-m)-2s}}\left(\int_{\mathbb{R}^d} u_0\,\varphi_R\,\mathrm{d}x\right)^{1-m} \le T$$

- If u_0 is such that the limit as $R \to +\infty$ of the right-hand side diverges to $+\infty$, then the corresponding solution $u(t,x)$ exists (and is positive) globally in time.

Weighted L^1 Estimates. The case $0 < m < 1$.

Theorem 1. Weighted L^1 estimates for $0 < m < 1$, [BV]

Let $u \ge v$ be two ordered very weak solutions to the equation (EQ), with $0 < m < 1$. Let $\varphi_R(x) = \varphi(x/R)$ where $R > 0$ and $\varphi \in \mathcal{W}_\alpha$ with

$$\boxed{d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}.}$$

Then, there exists $C_1 > 0$ that depends only on α, m, d, such that $\forall\ 0 \le \tau, t < \infty$:

$$\left[\int_{\mathbb{R}^d} (u-v)(t,x)\varphi_R(x)\,\mathrm{d}x\right]^{1-m} \le \left[\int_{\mathbb{R}^d} (u-v)(\tau,x)\varphi_R(x)\,\mathrm{d}x\right]^{1-m} + \frac{C_1\,|t-\tau|}{R^{2s-d(1-m)}}.$$

- The estimate holds for changing sign solutions and also for $s = 1$.
- The estimate holds both for $\tau < t$ and for $\tau > t$. [Quantitative loss of mass]
- When $m_c < m < 1$: Conservation of mass, by letting $R \to \infty$.
- When $0 < m < m_c$: if $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^p(\mathbb{R}^d)$ with $p \ge p_c = \frac{d(1-m)}{2s}$, then sols extinguish in finite time $T > 0$, see [DPQRV2].
  Lower bound for the extinction time, [let $\tau = T$ and $t = 0$]
  $$\frac{1}{C_1\, R^{d(1-m)-2s}}\left(\int_{\mathbb{R}^d} u_0\,\varphi_R\,\mathrm{d}x\right)^{1-m} \le T$$
- If u_0 is such that the limit as $R \to +\infty$ of the right-hand side diverges to $+\infty$, then the corresponding solution $u(t,x)$ exists (and is positive) globally in time.

Weighted L^1 Estimates. The case $0 < m < 1$.

Proof. • STEP 1. *A differential inequality for the weighted* L^1*-norm.*
If ψ is a smooth and sufficiently decaying function we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\big(u(t,x)-v(t,x)\big)\psi(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^d}\big((-\Delta)^s u^m-(-\Delta)^s v^m\big)\,\psi\,\mathrm{d}x\right|$$

$$=_{(a)}\left|\int_{\mathbb{R}^d}(u^m-v^m)\,(-\Delta)^s\psi\,\mathrm{d}x\right| \le_{(b)} 2^{1-m}\int_{\mathbb{R}^d}(u-v)^m\,\left|(-\Delta)^s\psi\right|\,\mathrm{d}x$$

$$\le_{(c)} 2\left(\int_{\mathbb{R}^d}(u-v)\psi\,\mathrm{d}x\right)^m\left(\int_{\mathbb{R}^d}\frac{\left|(-\Delta)^s\psi\right|^{\frac{1}{1-m}}}{\psi^{\frac{m}{1-m}}}\,\mathrm{d}x\right)^{1-m} = C_\psi^{1-m}\left(\int_{\mathbb{R}^d}(u-v)\psi\,\mathrm{d}x\right)^m.$$

Notice that in (a) we have used the fact that $(-\Delta)^s$ is a symmetric operator, while in (b) we have used that $(u^m-v^m)\le 2^{1-m}(u-v)^m$, where $u^m=|u|^{m-1}u$ as mentioned. In (c) we have used Hölder inequality with $q=1/m>1$ and $q'=1/(1-m)$.

If C_ψ is finite, integrating the above differential inequality on (τ,t) gives:

$$\left[\int_{\mathbb{R}^d}(u-v)(t,x)\psi(x)\,\mathrm{d}x\right]^{1-m}-\left[\int_{\mathbb{R}^d}(u-v)(\tau,x)\psi(x)\,\mathrm{d}x\right]^{1-m}\le(1-m)C_\psi^{1-m}\,|t-\tau|$$

• STEP 2. *Estimating the constant* C_ψ. Choose $\psi(x)=\varphi_R(x):=\varphi(x/R)=\varphi(y)$, with $\varphi\in\mathcal{W}_\alpha$. Use estimates for $\mathcal{W}_\alpha$ weights. Finally, we get that C_ψ is finite if $d-\frac{2s}{1-m}<\alpha<d+\frac{2s}{m}$. □

Weighted L^1 Estimates. The case $0 < m < 1$.

Proof. • STEP 1. *A differential inequality for the weighted* L^1*-norm.*
If ψ is a smooth and sufficiently decaying function we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\big(u(t,x)-v(t,x)\big)\psi(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^d}\left((-\Delta)^s u^m-(-\Delta)^s v^m\right)\psi\,\mathrm{d}x\right|$$

$$=_{(a)}\left|\int_{\mathbb{R}^d}(u^m-v^m)\,(-\Delta)^s\psi\,\mathrm{d}x\right| \le_{(b)} 2^{1-m}\int_{\mathbb{R}^d}(u-v)^m\,\left|(-\Delta)^s\psi\right|\,\mathrm{d}x$$

$$\le_{(c)} 2\left(\int_{\mathbb{R}^d}(u-v)\psi\,\mathrm{d}x\right)^m\left(\int_{\mathbb{R}^d}\frac{|(-\Delta)^s\psi|^{\frac{1}{1-m}}}{\psi^{\frac{m}{1-m}}}\,\mathrm{d}x\right)^{1-m} = C_\psi^{1-m}\left(\int_{\mathbb{R}^d}(u-v)\psi\,\mathrm{d}x\right)^m.$$

Notice that in (a) we have used the fact that $(-\Delta)^s$ is a symmetric operator, while in (b) we have used that $(u^m-v^m)\le 2^{1-m}(u-v)^m$, where $u^m=|u|^{m-1}u$ as mentioned. In (c) we have used Hölder inequality with $q=1/m>1$ and $q'=1/(1-m)$.
If C_ψ is finite, integrating the above differential inequality on (τ,t) gives:

$$\left[\int_{\mathbb{R}^d}(u-v)(t,x)\psi(x)\,\mathrm{d}x\right]^{1-m}-\left[\int_{\mathbb{R}^d}(u-v)(\tau,x)\psi(x)\,\mathrm{d}x\right]^{1-m}\le(1-m)C_\psi^{1-m}\,|t-\tau|$$

• STEP 2. *Estimating the constant* C_ψ. Choose $\psi(x)=\varphi_R(x):=\varphi(x/R)=\varphi(y)$, with $\varphi\in\mathcal{W}_\alpha$. Use estimates for $\mathcal{W}_\alpha$ weights. Finally, we get that C_ψ is finite if $d-\frac{2s}{1-m}<\alpha<d+\frac{2s}{m}$. □

Weighted L^1 Estimates. The case $0 < m < 1$.

Proof. • STEP 1. *A differential inequality for the weighted* L^1*-norm.*
If ψ is a smooth and sufficiently decaying function we have

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\big(u(t,x)-v(t,x)\big)\psi(x)\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^d}\left((-\Delta)^s u^m-(-\Delta)^s v^m\right)\psi\,\mathrm{d}x\right|$$

$$=_{(a)}\left|\int_{\mathbb{R}^d}(u^m-v^m)\,(-\Delta)^s\psi\,\mathrm{d}x\right| \le_{(b)} 2^{1-m}\int_{\mathbb{R}^d}(u-v)^m\,\left|(-\Delta)^s\psi\right|\,\mathrm{d}x$$

$$\le_{(c)} 2\left(\int_{\mathbb{R}^d}(u-v)\psi\,\mathrm{d}x\right)^m\left(\int_{\mathbb{R}^d}\frac{\left|(-\Delta)^s\psi\right|^{\frac{1}{1-m}}}{\psi^{\frac{m}{1-m}}}\,\mathrm{d}x\right)^{1-m} = C_\psi^{1-m}\left(\int_{\mathbb{R}^d}(u-v)\psi\,\mathrm{d}x\right)^m.$$

Notice that in (a) we have used the fact that $(-\Delta)^s$ is a symmetric operator, while in (b) we have used that $(u^m-v^m)\le 2^{1-m}(u-v)^m$, where $u^m=|u|^{m-1}u$ as mentioned. In (c) we have used Hölder inequality with $q=1/m>1$ and $q'=1/(1-m)$.
If C_ψ is finite, integrating the above differential inequality on (τ,t) gives:

$$\left[\int_{\mathbb{R}^d}(u-v)(t,x)\psi(x)\,\mathrm{d}x\right]^{1-m}-\left[\int_{\mathbb{R}^d}(u-v)(\tau,x)\psi(x)\,\mathrm{d}x\right]^{1-m}\le(1-m)C_\psi^{1-m}\,|t-\tau|$$

• STEP 2. *Estimating the constant* C_ψ. Choose $\psi(x)=\varphi_R(x):=\varphi(x/R)=\varphi(y)$, with $\varphi\in\mathcal{W}_\alpha$. Use estimates for $\mathcal{W}_\alpha$ weights. Finally, we get that C_ψ is finite if $d-\frac{2s}{1-m}<\alpha<d+\frac{2s}{m}$. □

Existence and uniqueness of very weak solutions when $0 < m < 1$

Theorem 2. Existence of very weak solutions when $0 < m < 1$, [BV]

Let $0 < m < 1$, $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x)$ and $\varphi \in \mathcal{W}_\alpha$, with $d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}$.
Then there exists a very weak solution $u(t,\cdot) \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x)$ to equation (EQ) on $[0,T] \times \mathbb{R}^d$, in the sense that for all $\psi \in C_c^\infty([0,T] \times \mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^d} u(t,x)\psi_t(t,x)\, \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} u^m(t,x)(-\Delta)^s \psi(t,x)\, \mathrm{d}x\, \mathrm{d}t\,.$$

This solution is continuous in the weighted space, $u \in C([0,T] : \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x))$.

Remark. The solutions constructed above only need to be integrable with respect to the weight φ, which has a tail of order less than $d + 2s/m$. Therefore, we have proved existence of solutions corresponding to initial data u_0 that can grow at infinity as $|x|^{(2s/m)-\varepsilon}$ for any $\varepsilon > 0$. In particular, constants are very weak solutions.
For the linear case $m = 1$ this exponent is optimal (representation formula for sols.)

Theorem 3. Uniqueness of very weak solutions when $0 < m < 1$, [BV]

The solution constructed (by approximation from below) in Theorem 2 is unique. *We call it the minimal solution.* In this class of solutions *the standard comparison result holds*, and also the weighted L^1 estimates of Theorem 1.

Existence and uniqueness of very weak solutions when $0 < m < 1$

Theorem 2. Existence of very weak solutions when $0 < m < 1$, [BV]

Let $0 < m < 1$, $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\,\mathrm{d}x)$ and $\varphi \in \mathcal{W}_\alpha$, with $d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}$.
Then there exists a very weak solution $u(t,\cdot) \in \mathrm{L}^1(\mathbb{R}^d, \varphi\,\mathrm{d}x)$ to equation (EQ) on $[0,T] \times \mathbb{R}^d$, in the sense that for all $\psi \in C_c^\infty([0,T] \times \mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^d} u(t,x)\psi_t(t,x)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} u^m(t,x)(-\Delta)^s\psi(t,x)\,\mathrm{d}x\,\mathrm{d}t\,.$$

This solution is continuous in the weighted space, $u \in C([0,T] : \mathrm{L}^1(\mathbb{R}^d, \varphi\,\mathrm{d}x))$.

Remark. The solutions constructed above only need to be integrable with respect to the weight φ, which has a tail of order less than $d + 2s/m$. Therefore, we have proved existence of solutions corresponding to initial data u_0 that can grow at infinity as $|x|^{(2s/m)-\varepsilon}$ for any $\varepsilon > 0$. In particular, constants are very weak solutions.
For the linear case $m = 1$ this exponent is optimal (representation formula for sols.)

Theorem 3. Uniqueness of very weak solutions when $0 < m < 1$, [BV]

The solution constructed (by approximation from below) in Theorem 2 is unique. *We call it the minimal solution.* In this class of solutions *the standard comparison result holds*, and also the weighted L^1 estimates of Theorem 1 .

Existence and uniqueness of very weak solutions when $0 < m < 1$

Theorem 2. Existence of very weak solutions when $0 < m < 1$, [BV]

Let $0 < m < 1$, $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x)$ and $\varphi \in \mathcal{W}_\alpha$, with $d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}$.
Then there exists a very weak solution $u(t, \cdot) \in \mathrm{L}^1(\mathbb{R}^d, \varphi\ \mathrm{d}x)$ to equation (EQ) on $[0, T] \times \mathbb{R}^d$, in the sense that for all $\psi \in C_c^\infty([0, T] \times \mathbb{R}^d)$

$$\int_0^T \int_{\mathbb{R}^d} u(t,x)\psi_t(t,x)\ \mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}^d} u^m(t,x)(-\Delta)^s\psi(t,x)\ \mathrm{d}x\,\mathrm{d}t\,.$$

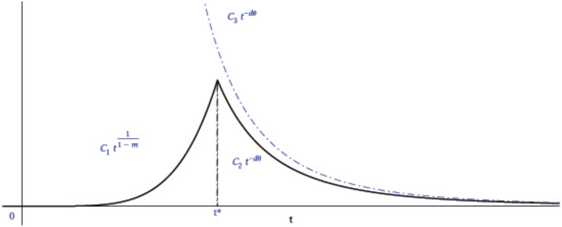
This solution is continuous in the weighted space, $u \in C([0, T] : \mathrm{L}^1(\mathbb{R}^d, \varphi\ \mathrm{d}x))$.

Remark. The solutions constructed above only need to be integrable with respect to the weight φ, which has a tail of order less than $d + 2s/m$. Therefore, we have proved existence of solutions corresponding to initial data u_0 that can grow at infinity as $|x|^{(2s/m)-\varepsilon}$ for any $\varepsilon > 0$. In particular, constants are very weak solutions.
For the linear case $m = 1$ this exponent is optimal (representation formula for sols.)

Theorem 3. Uniqueness of very weak solutions when $0 < m < 1$, [BV]

The solution constructed (by approximation from below) in Theorem 2 is unique. *We call it the minimal solution.* In this class of solutions *the standard comparison result holds*, and also the weighted L^1 estimates of Theorem 1.

Positivity estimates in the Good Fast Diffusion range



$$m_c = \frac{d-2s}{d}\,, \quad \vartheta = \frac{1}{2s - d(1-m)}$$

Figure:

Black: Lower bounds in the two time ranges.

Blue: Upper bounds (smoothing effects)

Note: when $t \geq t_*$ upper and lower bounds have the same time behaviour .

Theorem 4. Local lower bounds when $m_c < m < 1$, [BV]

Let $m_c < m < 1$, $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\,\mathrm{d}x)$ and $\varphi \in \mathcal{W}_\alpha$, with $d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}$.
Let $u(t,\cdot) \in \mathrm{L}^1(\mathbb{R}^d, \varphi\,\mathrm{d}x)$ be a very weak solution to Equation (EQ) corresponding to the initial datum u_0. Then for any $R_0 > 0$, there exists a time

$$t_* := C_*\, R_0^{2s-d(1-m)}\, \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m}$$

such that

$$\inf_{x \in B_{R_0/2}} u(t,x) \geq K_1\, R_0^{-\frac{2s}{1-m}}\, t^{\frac{1}{1-m}}\,, \qquad \text{if } 0 \leq t \leq t_*\,,$$

$$\inf_{x \in B_{R_0/2}} u(t,x) \geq K_2\, \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}^{2s\vartheta}}{t^{d\vartheta}}\,, \qquad \text{if } t \geq t_*\,.$$

The constants $C_*, K_1, K_2 > 0$ are explicit and depend on m, s and $d \geq 1$.

Positivity estimates in the Good Fast Diffusion range

Remarks.

1. The lower estimate for small times: for all $0 \le t \le t_* = C_* R_0^{2s-d(1-m)} \|u_0\|_{L^1(B_{R_0})}^{1-m}$

$$\inf_{x \in B_{R_0/2}} u(t,x) \ge K_1 R_0^{-\frac{2s}{1-m}} t^{\frac{1}{1-m}},$$

is an *absolute lower bound* , i.e., does not depend on the initial data (but t_* does).

2. When $s = 1$, we recover the lower Harnack inequality of B.-Vázquez [JFA 2006, Adv.Math. 2010] and DiBenedetto-Gianazza-Vespri [Ann.SNS Pisa 2009]. (Different proof)

3. The lower estimate for large times has the flavour of "reverse smoothing effect", indeed, for all $t \ge t_*$, $x_0 \in \mathbb{R}^d$ and $R_0 > 0$: (recall that $\vartheta = 1/[2s + d(m-1)] > 0$)

$$K_2 \frac{\|u_0\|_{L^1(B_{R_0}(x_0))}^{2s\vartheta}}{t^{d\vartheta}} \le u(t,x_0) \le K_\infty \frac{\|u_0\|_{L^1(\mathbb{R}^d)}^{2s\vartheta}}{t^{d\vartheta}}$$

The blue part is the smoothing effect proven in [DPQRV2] .

4. The lower estimate for large times, holds in the limit $m \to 1$:

Proposition. Lower estimate for Fractional Heat Equation, $m = 1$.

Let $m = 1$, and let $u(t,x)$ be the very weak solution to the Problem (CP) corresponding to $u_0 \in L^1(\mathbb{R}^d, \varphi\, dx)$ with $\varphi \in \mathcal{W}_\alpha$, with $\alpha < d + 2s$. Then, for all $R_0 > 0$ there exists C_* , K_2 depending only on α, s and d, such that

$$\inf_{x \in B_{R_0/2}} u(t,x) \ge K_2 \frac{\|u_0\|_{L^1(B_{R_0})}}{t^{d/2s}}, \qquad \forall t \ge t_* = C_* R_0^{2s}.$$

Positivity estimates in the Good Fast Diffusion range

Remarks.

1. The lower estimate for small times: for all $0 \le t \le t_* = C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m}$

$$\inf_{x\in B_{R_0/2}} u(t,x) \ge K_1\, R_0^{-\frac{2s}{1-m}}\, t^{\frac{1}{1-m}}\,,$$

   is an *absolute lower bound* , i.e., does not depend on the initial data (but t_* does).

2. When $s = 1$, we recover the lower Harnack inequality of B.-Vázquez [JFA 2006, Adv.Math. 2010] and DiBenedetto-Gianazza-Vespri [Ann.SNS Pisa 2009]. (Different proof)

3. The lower estimate for large times has the flavour of "reverse smoothing effect", indeed, for all $t \ge t_*$, $x_0 \in \mathbb{R}^d$ and $R_0 > 0$: (recall that $\vartheta = 1/[2s + d(m-1)] > 0$)

$$K_2 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0}(x_0))}^{2s\vartheta}}{t^{d\vartheta}} \le u(t,x_0) \le K_\infty \frac{\|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}^{2s\vartheta}}{t^{d\vartheta}}$$

   The blue part is the smoothing effect proven in [DPQRV2] .

4. The lower estimate for large times, holds in the limit $m \to 1$:

Proposition. Lower estimate for Fractional Heat Equation, $m = 1$.

Let $m = 1$, and let $u(t,x)$ be the very weak solution to the Problem (CP) corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x)$ with $\varphi \in \mathcal{W}_\alpha$, with $\alpha < d + 2s$. Then, for all $R_0 > 0$ there exists C_* , K_2 depending only on α, s and d, such that

$$\inf_{x\in B_{R_0/2}} u(t,x) \ge K_2 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}}{t^{d/2s}}\,, \qquad \forall t \ge t_* = C_*\, R_0^{2s}\,.$$

Positivity estimates in the Good Fast Diffusion range

Remarks.

1. The lower estimate for small times: for all $0 \le t \le t_* = C_* R_0^{2s-d(1-m)} \|u_0\|_{L^1(B_{R_0})}^{1-m}$

$$\inf_{x\in B_{R_0/2}} u(t,x) \ge K_1 \, R_0^{-\frac{2s}{1-m}} \, t^{\frac{1}{1-m}} \,,$$

is an *absolute lower bound* , i.e., does not depend on the initial data (but t_* does).

2. When $s = 1$, we recover the lower Harnack inequality of B.-Vázquez [JFA 2006, Adv.Math. 2010] and DiBenedetto-Gianazza-Vespri [Ann.SNS Pisa 2009]. (Different proof)

3. The lower estimate for large times has the flavour of "reverse smoothing effect", indeed, for all $t \ge t_*$, $x_0 \in \mathbb{R}^d$ and $R_0 > 0$: (recall that $\vartheta = 1/[2s + d(m-1)] > 0$)

$$K_2 \frac{\|u_0\|_{L^1(B_{R_0}(x_0))}^{2s\vartheta}}{t^{d\vartheta}} \le u(t,x_0) \le K_\infty \frac{\|u_0\|_{L^1(\mathbb{R}^d)}^{2s\vartheta}}{t^{d\vartheta}}$$

The blue part is the smoothing effect proven in [DPQRV2] .

4. The lower estimate for large times, holds in the limit $m \to 1$:

Proposition. Lower estimate for Fractional Heat Equation, $m = 1$.

Let $m = 1$, and let $u(t,x)$ be the very weak solution to the Problem (CP) corresponding to $u_0 \in L^1(\mathbb{R}^d, \varphi\,dx)$ with $\varphi \in \mathcal{W}_\alpha$, with $\alpha < d + 2s$. Then, for all $R_0 > 0$ there exists C_* , K_2 depending only on α, s and d, such that

$$\inf_{x\in B_{R_0/2}} u(t,x) \ge K_2 \frac{\|u_0\|_{L^1(B_{R_0})}}{t^{d/2s}} \,, \qquad \forall t \ge t_* = C_* R_0^{2s} \,.$$

Positivity estimates in the Good Fast Diffusion range

Remarks.

1. The lower estimate for small times: for all $0 \le t \le t_* = C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m}$
$$\inf_{x\in B_{R_0/2}} u(t,x) \ge K_1\, R_0^{-\frac{2s}{1-m}}\, t^{\frac{1}{1-m}} ,$$
is an *absolute lower bound* , i.e., does not depend on the initial data (but t_* does).
2. When $s = 1$, we recover the lower Harnack inequality of B.-Vázquez [JFA 2006, Adv.Math. 2010] and DiBenedetto-Gianazza-Vespri [Ann.SNS Pisa 2009]. (Different proof)
3. The lower estimate for large times has the flavour of "reverse smoothing effect", indeed, for all $t \ge t_*$, $x_0 \in \mathbb{R}^d$ and $R_0 > 0$: (recall that $\vartheta = 1/[2s + d(m-1)] > 0$)
$$K_2 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0}(x_0))}^{2s\vartheta}}{t^{d\vartheta}} \le u(t,x_0) \le K_\infty \frac{\|u_0\|_{\mathrm{L}^1(\mathbb{R}^d)}^{2s\vartheta}}{t^{d\vartheta}}$$
The blue part is the smoothing effect proven in [DPQRV2] .
4. The lower estimate for large times, holds in the limit $m \to 1$:

Proposition. Lower estimate for Fractional Heat Equation, $m = 1$.

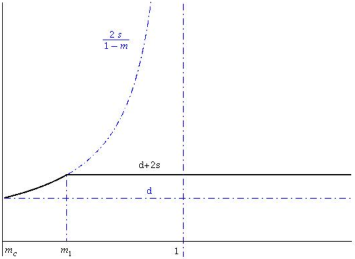
Let $m = 1$, and let $u(t,x)$ be the very weak solution to the Problem (CP) corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, dx)$ with $\varphi \in \mathcal{W}_\alpha$, with $\alpha < d + 2s$. Then, for all $R_0 > 0$ there exists C_* , K_2 depending only on α, s and d, such that
$$\inf_{x\in B_{R_0/2}} u(t,x) \ge K_2 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}}{t^{d/2s}} , \qquad \forall t \ge t_* = C_*\, R_0^{2s} .$$

Positivity estimates in the Good Fast Diffusion range

Theorem 5. Spatial lower bounds when $m_c < m < m_1$, [BV]

Let $m_c < m < 1$, $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x)$ and $\varphi \in \mathcal{W}_\alpha$, with $d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}$. Then, for all $x >> 1 \in \mathbb{R}^d$, and for all $t \geq 0$ we have the sharp lower bounds

$$u(t,x) \geq K_3(u_0,t)\, |x|^{-\frac{2s}{1-m}} .$$



$$\frac{d-2s}{d} = m_c < m_1 = \frac{d}{d+2s}$$

Figure: Barenblatt solutions' tails, [V]. According to the results of [V], the Barenblatt solutions $\mathcal{B}(t,x)$ have the precise spatial behaviour for $|x|$ large enough:

$$\mathcal{B}(t,x) = \frac{1}{t^{d\vartheta}} F\left(\frac{|x|}{t^{\vartheta}}\right) \asymp \frac{1}{|x|^{\beta}} \,, \quad \text{for } |x| >> 1$$

$$\text{with } \beta = \begin{cases} 2s/(1-m)\,, & \text{if } m_c < m < m_1 \\ d+2s\,, & \text{if } m_1 < m < 1 \end{cases}$$

Theorem 6. Spatial Lower Bounds when $m_1 < m < 1$, [BV]

Under the assumptions of Theorem 5, in the range $m_1 < m < 1$ we have

$$u(t,x) \geq K_4(t,u_0)|x|^{-(d+2s)} \qquad \text{when } |x| >> 1.$$

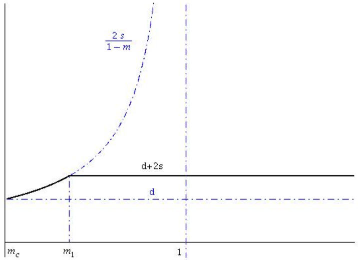
for all $0 < t < T$ with some bounded function $K_4 > 0$ that depends on t, T and u_0.

Positivity estimates in the Good Fast Diffusion range

Theorem 5. Spatial lower bounds when $m_c < m < m_1$, [BV]

Let $m_c < m < 1$, $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, dx)$ and $\varphi \in \mathcal{W}_\alpha$, with $d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}$.
Then, for all $x >> 1 \in \mathbb{R}^d$, and for all $t \geq 0$ we have the sharp lower bounds

$$u(t,x) \geq K_3(u_0, t)\, |x|^{-\frac{2s}{1-m}} .$$



$$\frac{d-2s}{d} = m_c < m_1 = \frac{d}{d+2s}$$

Figure: Barenblatt solutions' tails, [V]. According to the results of [V], the Barenblatt solutions $\mathcal{B}(t,x)$ have the precise spatial behaviour for $|x|$ large enough:

$$\mathcal{B}(t,x) = \frac{1}{t^{d\vartheta}} F\left(\frac{|x|}{t^{\vartheta}}\right) \asymp \frac{1}{|x|^{\beta}} , \quad \text{for } |x| >> 1$$

$$\text{with } \beta = \begin{cases} 2s/(1-m) , & \text{if } m_c < m < m_1 \\ d + 2s , & \text{if } m_1 < m < 1 \end{cases}$$

Theorem 6. Spatial Lower Bounds when $m_1 < m < 1$, [BV]

Under the assumptions of Theorem 5, in the range $m_1 < m < 1$ we have

$$u(t,x) \geq K_4(t, u_0) |x|^{-(d+2s)} \qquad \text{when } |x| >> 1.$$

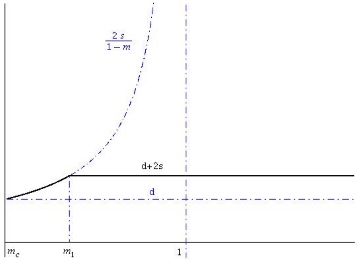
for all $0 < t < T$ with some bounded function $K_4 > 0$ that depends on t, T and u_0.

Positivity estimates in the Good Fast Diffusion range

Theorem 5. Spatial lower bounds when $m_c < m < m_1$, [BV]

Let $m_c < m < 1$, $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x)$ and $\varphi \in \mathcal{W}_\alpha$, with $d - \frac{2s}{1-m} < \alpha < d + \frac{2s}{m}$. Then, for all $x >> 1 \in \mathbb{R}^d$, and for all $t \geq 0$ we have the sharp lower bounds

$$u(t,x) \geq K_3(u_0, t)\, |x|^{-\frac{2s}{1-m}} .$$



$$\frac{d-2s}{d} = m_c < m_1 = \frac{d}{d+2s}$$

Figure: Barenblatt solutions' tails, [V]. According to the results of [V], the Barenblatt solutions $\mathcal{B}(t,x)$ have the precise spatial behaviour for $|x|$ large enough:

$$\mathcal{B}(t,x) = \frac{1}{t^{d\vartheta}} F\left(\frac{|x|}{t^\vartheta}\right) \asymp \frac{1}{|x|^\beta}, \quad \text{for } |x| >> 1$$

$$\text{with } \beta = \begin{cases} 2s/(1-m), & \text{if } m_c < m < m_1 \\ d+2s, & \text{if } m_1 < m < 1 \end{cases}$$

Theorem 6. Spatial Lower Bounds when $m_1 < m < 1$, [BV]

Under the assumptions of Theorem 5, in the range $m_1 < m < 1$ we have

$$u(t,x) \geq K_4(t, u_0)|x|^{-(d+2s)} \qquad \text{when } |x| >> 1.$$

for all $0 < t < T$ with some bounded function $K_4 > 0$ that depends on t, T and u_0.

Positivity estimates in the Very Fast Diffusion range

Theorem 7. Local lower bounds I, when $0 < m < m_c$, [BV]

Let u be a weak solution to the equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^{p_c}(\mathbb{R}^d)$ with $0 < m < m_c = d/(d-2s)$, $0 < s < 1$ and let $p_c = d(1-m)/(2s)$. Let also $T = T(u_0)$ be the finite extinction time for u. Then, for every $R_0 > 0$, there exists

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0),$$

such that

$$\inf_{x \in B_{R_0/2}} u(t,x) \ge K_5 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{m(1-m)}}} \quad \text{if } 0 \le t \le t_*,$$

where C_* and K_5 are explicit positive universal constants, that depend only on m, s, d.

Remarks. (i) This result can be re-written alternatively: Under the above assumptions, there exists $K_6 > 0$ such that for any $0 \le t \le T$ and $R > 0$ we have

$$\frac{\|u_0\|_{\mathrm{L}^1(B_R)}}{R^d} \le K_6 \left[\frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{T^{\frac{1}{1-m}}}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x \in B_{R/2}} u^m(t,x) \right].$$

(ii) When $s \to 1$, we recover the lower bounds of B.-Vázquez [Adv. Math. 2010], with different proof. Such estimates (valid for local weak solutions) have been called Aronson-Caffarelli there, since they are quite similar to the one obtained by Aronson-Caffarelli for $m > 1$, [TAMS 1983].

(iii) We need upper bounds for $T(u_0)$ needed to eliminate T from the above estimates.

Positivity estimates in the Very Fast Diffusion range

Theorem 7. Local lower bounds I, when $0 < m < m_c$, [BV]

Let u be a weak solution to the equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^{p_c}(\mathbb{R}^d)$ with $0 < m < m_c = d/(d-2s)$, $0 < s < 1$ and let $p_c = d(1-m)/(2s)$. Let also $T = T(u_0)$ be the finite extinction time for u. Then, for every $R_0 > 0$, there exists

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0),$$

such that

$$\inf_{x \in B_{R_0/2}} u(t,x) \ge K_5 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{m(1-m)}}} \quad \text{if } 0 \le t \le t_*,$$

where C_* and K_5 are explicit positive universal constants, that depend only on m, s, d.

Remarks. (i) This result can be re-written alternatively: Under the above assumptions, there exists $K_6 > 0$ such that for any $0 \le t \le T$ and $R > 0$ we have

$$\frac{\|u_0\|_{\mathrm{L}^1(B_R)}}{R^d} \le K_6 \left[\frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{T^{\frac{1}{1-m}}}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x \in B_{R/2}} u^m(t,x) \right].$$

(ii) When $s \to 1$, we recover the lower bounds of B.-Vázquez [Adv. Math. 2010], with different proof. Such estimates (valid for local weak solutions) have been called Aronson-Caffarelli there, since they are quite similar to the one obtained by Aronson-Caffarelli for $m > 1$, [TAMS 1983].

(iii) We need upper bounds for $T(u_0)$ needed to eliminate T from the above estimates .

Positivity estimates in the Very Fast Diffusion range

Theorem 7. Local lower bounds I, when $0 < m < m_c$, [BV]

Let u be a weak solution to the equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^{p_c}(\mathbb{R}^d)$ with $0 < m < m_c = d/(d-2s)$, $0 < s < 1$ and let $p_c = d(1-m)/(2s)$. Let also $T = T(u_0)$ be the finite extinction time for u. Then, for every $R_0 > 0$, there exists

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0)\,,$$

such that

$$\inf_{x\in B_{R_0/2}} u(t,x) \ge K_5 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{m(1-m)}}} \qquad \text{if } 0 \le t \le t_*\,,$$

where C_* and K_5 are explicit positive universal constants, that depend only on m, s, d.

Remarks. (i) This result can be re-written alternatively: Under the above assumptions, there exists $K_6 > 0$ such that for any $0 \le t \le T$ and $R > 0$ we have

$$\frac{\|u_0\|_{\mathrm{L}^1(B_R)}}{R^d} \le K_6 \left[\frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{T^{\frac{1}{1-m}}}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x\in B_{R/2}} u^m(t,x) \right].$$

(ii) When $s \to 1$, we recover the lower bounds of B.-Vázquez [Adv. Math. 2010], with different proof. Such estimates (valid for local weak solutions) have been called Aronson-Caffarelli there, since they are quite similar to the one obtained by Aronson-Caffarelli for $m > 1$, [TAMS 1983].

(iii) We need upper bounds for $T(u_0)$ needed to eliminate T from the above estimates .

Positivity estimates in the Very Fast Diffusion range

Theorem 7. Local lower bounds I, when $0 < m < m_c$, [BV]

Let u be a weak solution to the equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^{p_c}(\mathbb{R}^d)$ with $0 < m < m_c = d/(d-2s)$, $0 < s < 1$ and let $p_c = d(1-m)/(2s)$. Let also $T = T(u_0)$ be the finite extinction time for u. Then, for every $R_0 > 0$, there exists

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0) ,$$

such that

$$\inf_{x \in B_{R_0/2}} u(t,x) \ge K_5 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{T^{\frac{1}{m(1-m)}}} \qquad \text{if } 0 \le t \le t_* ,$$

where C_* and K_5 are explicit positive universal constants, that depend only on m, s, d.

Remarks. (i) This result can be re-written alternatively: Under the above assumptions, there exists $K_6 > 0$ such that for any $0 \le t \le T$ and $R > 0$ we have

$$\frac{\|u_0\|_{\mathrm{L}^1(B_R)}}{R^d} \le K_6 \left[\frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{T^{\frac{1}{1-m}}}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x \in B_{R/2}} u^m(t,x) \right] .$$

(ii) When $s \to 1$, we recover the lower bounds of B.-Vázquez [Adv. Math. 2010], with different proof. Such estimates (valid for local weak solutions) have been called Aronson-Caffarelli there, since they are quite similar to the one obtained by Aronson-Caffarelli for $m > 1$, [TAMS 1983].

(iii) We need upper bounds for $T(u_0)$ needed to eliminate T from the above estimates .

Positivity estimates in the Very Fast Diffusion range

Minimal life time and solutions that do not extinguish.

By comparison it is easy to prove that this estimates hold for the class of very weak solutions to the Cauchy Problem (CP) constructed in Theorem 2. The quantitative lower bounds of Theorem 7 apply to solutions u that may not extinguish in finite time. Hence, we can interpret T as the *minimal life time* for the solution $u(t,\cdot)$, concept introduced in B.-Vázquez [Adv. Math. 2010]. Theorem 7 also gives a lower bound for the minimal life time T

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0)\,.$$

Corollary. Solutions that do not extinguish in finite time, [BV]

Let $0 < m < m_c$ and consider an initial datum $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x)$, where φ is as in Theorem 2, or simply $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$. Assume moreover that

$$\liminf_{R\to+\infty} R^{\frac{2s}{1-m}-d}\|u_0\|_{\mathrm{L}^1(B_R)} = +\infty\,.$$

Then the corresponding solution $u(t,x)$ exists and is positive globally in space and time, hence does not extinguish in finite time. Moreover the quantitative lower bounds of Theorem 7 hold for any $0 \le t \le t_*$ with $t_* \le T = T(u_0\chi_{B_{R_0}}) < +\infty$, where T is the extinction time of a reduced problem.

Remark. A practical assumption on the initial datum u_0: $\displaystyle \liminf_{|x|\to+\infty} |x|^{\frac{2s}{1-m}} u_0(x) = +\infty$.

Positivity estimates in the Very Fast Diffusion range

Minimal life time and solutions that do not extinguish.

By comparison it is easy to prove that this estimates hold for the class of very weak solutions to the Cauchy Problem (CP) constructed in Theorem 2. The quantitative lower bounds of Theorem 7 apply to solutions u that may not extinguish in finite time. Hence, we can interpret T as the *minimal life time* for the solution $u(t,\cdot)$, concept introduced in B.-Vázquez [Adv. Math. 2010]. Theorem 7 also gives a lower bound for the minimal life time T

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0)\,.$$

Corollary. Solutions that do not extinguish in finite time, [BV]

Let $0 < m < m_c$ and consider an initial datum $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\, \mathrm{d}x)$, where φ is as in Theorem 2, or simply $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$. Assume moreover that

$$\liminf_{R\to+\infty} R^{\frac{2s}{1-m}-d} \|u_0\|_{\mathrm{L}^1(B_R)} = +\infty\,.$$

Then the corresponding solution $u(t,x)$ exists and is positive globally in space and time, hence does not extinguish in finite time. Moreover the quantitative lower bounds of Theorem 7 hold for any $0 \le t \le t_*$ with $t_* \le T = T(u_0\chi_{B_{R_0}}) < +\infty$, where T is the extinction time of a reduced problem.

Remark. A practical assumption on the initial datum u_0: $\liminf_{|x|\to+\infty} |x|^{\frac{2s}{1-m}} u_0(x) = +\infty$.

Positivity estimates in the Very Fast Diffusion range

Minimal life time and solutions that do not extinguish.

By comparison it is easy to prove that this estimates hold for the class of very weak solutions to the Cauchy Problem (CP) constructed in Theorem 2. The quantitative lower bounds of Theorem 7 apply to solutions u that may not extinguish in finite time. Hence, we can interpret T as the *minimal life time* for the solution $u(t,\cdot)$, concept introduced in B.-Vázquez [Adv. Math. 2010]. Theorem 7 also gives a lower bound for the minimal life time T

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0)\,.$$

Corollary. Solutions that do not extinguish in finite time, [BV]

Let $0<m<m_c$ and consider an initial datum $0 \le u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi\,\mathrm{d}x)$, where φ is as in Theorem 2, or simply $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$. Assume moreover that

$$\liminf_{R\to+\infty} R^{\frac{2s}{1-m}-d}\|u_0\|_{\mathrm{L}^1(B_R)} = +\infty\,.$$

Then the corresponding solution $u(t,x)$ exists and is positive globally in space and time, hence does not extinguish in finite time. Moreover the quantitative lower bounds of Theorem 7 hold for any $0\le t\le t_*$ with $t_*\le T = T(u_0\chi_{B_{R_0}}) < +\infty$, where T is the extinction time of a reduced problem.

Remark. A practical assumption on the initial datum u_0: $\boxed{\liminf_{|x|\to+\infty} |x|^{\frac{2s}{1-m}} u_0(x) = +\infty}$.

Positivity estimates in the Very Fast Diffusion range

Estimating the extinction time.

This result extends a nowadays classical result of Benilan and Crandall [Indiana 1981].

Proposition. Upper bounds for the extinction time, [DPQRV2, BV]

Let u be a weak solution to the equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^{p_c}(\mathbb{R}^d)$ with $0 < m < m_c = d/(d-2s)$, $0 < s < 1$ and let $p_c = d(1-m)/(2s)$. Then for all $0 \le \tau \le t$ the following estimate holds true

$$\left[\int_{\mathbb{R}^d} |u(t,x)|^{p_c}\, \mathrm{d}x\right]^{\frac{2s}{d}} \le \left[\int_{\mathbb{R}^d} |u(\tau,x)|^{p_c}\, \mathrm{d}x\right]^{\frac{2s}{d}} - K_7(t-\tau)$$

where $K_7 > 0$ is explicit and only depends on d, s, m. Moreover, there exists a finite extinction time $T \ge 0$ which can be bounded above as follows

$$T = T(u_0) \le K_7^{-1} \|u_0\|_{\mathrm{L}^{p_c}(\mathbb{R}^d)}^{1-m} .$$

Remarks. (i) The above proposition allows to get rid of the extinction time T in the lower estimates of Theorem 7.

(ii) In view of this proposition, the exponent in the condition for solutions that do not extinguish in finite time is sharp:

$$\liminf_{R\to+\infty} R^{\frac{2s}{1-m}-d} \|u_0\|_{\mathrm{L}^1(B_R)} = +\infty .$$

Estimating the extinction time.

This result extends a nowadays classical result of Benilan and Crandall [Indiana 1981].

Proposition. Upper bounds for the extinction time, [DPQRV2, BV]

Let u be a weak solution to the equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^{p_c}(\mathbb{R}^d)$ with $0 < m < m_c = d/(d-2s)$, $0 < s < 1$ and let $p_c = d(1-m)/(2s)$. Then for all $0 \le \tau \le t$ the following estimate holds true

$$\left[\int_{\mathbb{R}^d} |u(t,x)|^{p_c}\,\mathrm{d}x\right]^{\frac{2s}{d}} \le \left[\int_{\mathbb{R}^d} |u(\tau,x)|^{p_c}\,\mathrm{d}x\right]^{\frac{2s}{d}} - K_7(t-\tau)$$

where $K_7 > 0$ is explicit and only depends on d, s, m. Moreover, there exists a finite extinction time $T \ge 0$ which can be bounded above as follows

$$T = T(u_0) \le K_7^{-1} \|u_0\|_{\mathrm{L}^{p_c}(\mathbb{R}^d)}^{1-m} .$$

Remarks. (i) The above proposition allows to get rid of the extinction time T in the lower estimates of Theorem 7.

(ii) In view of this proposition, the exponent in the condition for solutions that do not extinguish in finite time is sharp:

$$\liminf_{R \to +\infty} R^{\frac{2s}{1-m} - d} \|u_0\|_{\mathrm{L}^1(B_R)} = +\infty .$$

Positivity estimates in the Very Fast Diffusion range

Estimating the extinction time.

This result extends a nowadays classical result of Benilan and Crandall [Indiana 1981].

Proposition. Upper bounds for the extinction time, [DPQRV2, BV]

Let u be a weak solution to the equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^{p_c}(\mathbb{R}^d)$ with $0 < m < m_c = d/(d-2s)$, $0 < s < 1$ and let $p_c = d(1-m)/(2s)$. Then for all $0 \le \tau \le t$ the following estimate holds true

$$\left[\int_{\mathbb{R}^d} |u(t,x)|^{p_c}\,\mathrm{d}x\right]^{\frac{2s}{d}} \le \left[\int_{\mathbb{R}^d} |u(\tau,x)|^{p_c}\,\mathrm{d}x\right]^{\frac{2s}{d}} - K_7(t-\tau)$$

where $K_7 > 0$ is explicit and only depends on d, s, m. Moreover, there exists a finite extinction time $T \ge 0$ which can be bounded above as follows

$$T = T(u_0) \le K_7^{-1} \|u_0\|_{\mathrm{L}^{p_c}(\mathbb{R}^d)}^{1-m}\,.$$

Remarks. (i) The above proposition allows to get rid of the extinction time T in the lower estimates of Theorem 7.

(ii) In view of this proposition, the exponent in the condition for solutions that do not extinguish in finite time is sharp:

$$\liminf_{R\to+\infty} R^{\frac{2s}{1-m}-d} \|u_0\|_{\mathrm{L}^1(B_R)} = +\infty\,.$$

Positivity estimates in the Very Fast Diffusion range

Theorem 8. Local lower bounds II, when $0 < m < m_c$, [BV]

Under the assumptions of the above proposition, let $T(u_0)$ be the finite extinction time for u. Then, for every ball $B_{2R_0} \subset \Omega$, there exists explicit positive universal constants C_* , $K_8 > 0$ (that depend only on m, s, d) and a time t_* such that

$$t_* := C_* R_0^{2s-d(1-m)} \|u_0\|_{L^1(B_{R_0})}^{1-m} \le T(u_0) \le K_7^{-1} \|u_0\|_{L^{p_c}(\mathbb{R}^d)}^{1-m} ,$$

and for all $0 \le t \le t_*$

$$\inf_{x \in B_{R_0/2}} u(t,x) \ge K_8 \frac{\|u_0\|_{L^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{\|u_0\|_{L^{p_c}(B_{R_0})}^{\frac{d}{2m}}}$$

Remarks. (i) This result can be written alternatively as saying that there exists a universal constant $K_9 > 0$ such for all solutions in the above class we have:

$$\frac{\|u_0\|_{L^1(B_R)}}{R^d \left[1 \vee \|u_0\|_{L^{p_c}(B_R)}\right]} \le K_9 \left[\frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{1}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x \in B_{R/2}} u^m(t,x) \right] , \qquad \forall t, R > 0 .$$

(ii) When $s \to 1$, we recover the lower bounds of B.-Vázquez [Adv. Math. 2010], with different proof.

Positivity estimates in the Very Fast Diffusion range

Theorem 8. Local lower bounds II, when $0 < m < m_c$, [BV]

Under the assumptions of the above proposition, let $T(u_0)$ be the finite extinction time for u. Then, for every ball $B_{2R_0} \subset \Omega$, there exists explicit positive universal constants C_* , $K_8 > 0$ (that depend only on m, s, d) and a time t_* such that

$$t_* := C_* \, R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0) \le K_7^{-1} \|u_0\|_{\mathrm{L}^{p_c}(\mathbb{R}^d)}^{1-m} ,$$

and for all $0 \le t \le t_*$

$$\inf_{x \in B_{R_0/2}} u(t,x) \ge K_8 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{\|u_0\|_{\mathrm{L}^{p_c}(B_{R_0})}^{\frac{d}{2m}}}$$

Remarks. (i) This result can be written alternatively as saying that there exists a universal constant $K_9 > 0$ such for all solutions in the above class we have:

$$\frac{\|u_0\|_{\mathrm{L}^1(B_R)}}{R^d \left[1 \vee \|u_0\|_{\mathrm{L}^{p_c}(B_R)}\right]} \le K_9 \left[\frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{1}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x \in B_{R/2}} u^m(t,x) \right] , \qquad \forall t, R > 0 .$$

(ii) When $s \to 1$, we recover the lower bounds of B.-Vázquez [Adv. Math. 2010], with different proof.

Positivity estimates in the Very Fast Diffusion range

Theorem 8. Local lower bounds II, when $0 < m < m_c$, [BV]

Under the assumptions of the above proposition, let $T(u_0)$ be the finite extinction time for u. Then, for every ball $B_{2R_0} \subset \Omega$, there exists explicit positive universal constants C_* , $K_8 > 0$ (that depend only on m, s, d) and a time t_* such that

$$t_* := C_*\, R_0^{2s-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_{R_0})}^{1-m} \le T(u_0) \le K_7^{-1} \|u_0\|_{\mathrm{L}^{p_c}(\mathbb{R}^d)}^{1-m} ,$$

and for all $0 \le t \le t_*$

$$\inf_{x\in B_{R_0/2}} u(t,x) \ge K_8 \frac{\|u_0\|_{\mathrm{L}^1(B_{R_0})}^{\frac{1}{m}}}{R_0^{\frac{d-2s}{m}}} \frac{t^{\frac{1}{1-m}}}{\|u_0\|_{\mathrm{L}^{p_c}(B_{R_0})}^{\frac{d}{2m}}}$$

Remarks. (i) This result can be written alternatively as saying that there exists a universal constant $K_9 > 0$ such for all solutions in the above class we have:

$$\frac{\|u_0\|_{\mathrm{L}^1(B_R)}}{R^d\left[1 \vee \|u_0\|_{\mathrm{L}^{p_c}(B_R)}\right]} \le K_9 \left[\frac{t^{\frac{1}{1-m}}}{R^{\frac{2s}{1-m}}} + \frac{1}{t^{\frac{m}{1-m}} R^{2s}} \inf_{x\in B_{R/2}} u^m(t,x)\right] , \qquad \forall t, R > 0 .$$

(ii) When $s \to 1$, we recover the lower bounds of B.-Vázquez [Adv. Math. 2010], with different proof.

Positivity estimates in the Porous Medium range

Theorem 9. Local lower bounds when $m > 1$, [BV]

Let u be a weak solution to Equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and $m > 1$. We put $\vartheta := 1/[2s + d(m-1)] > 0$. Then there exists a time

$$t_* := C_* \, R^{2s+d(m-1)} \, \|u_0\|_{\mathrm{L}^1(B_R)}^{-(m-1)}$$

such that for every $t \ge t_*$ we have the lower bound

$$\inf_{x \in B_{R/2}} u(t,x) \ge K \, \frac{\|u_0\|_{\mathrm{L}^1(B_R)}^{2s\vartheta}}{t^{d\vartheta}}$$

valid for all $R > 0$. The positive constants C_* and K depend only on m, s and d, and not on R.

Remarks. (i) This result can be written alternatively as saying that there exists a universal constant $C_1 = C_1(d,s,m)$ such that for all solutions in the above class we have

$$\int_{B_R(0)} u_0(x)\,dx \le C_1 \left(R^{1/\vartheta(m-1)} t^{-1/(m-1)} + u(0,t)^{1/2s\vartheta} t^{d/2s} \right).$$

This equivalent version is in complete formal agreement with Aronson-Caffarelli's estimate for $s = 1$, [TAMS 1983]. However, our proof below differs very strongly from the ideas used in Aronson-Caffarelli's case since we cannot use the property of finite propagation of solution with compact support, which is false for $s < 1$.

(ii) In the limit $m \to 1$ of the estimate of Theorem 9 we obtain the (already mentioned) lower bounds for the fractional heat equation.

Positivity estimates in the Porous Medium range

Theorem 9. Local lower bounds when $m > 1$, [BV]

Let u be a weak solution to Equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and $m > 1$. We put $\vartheta := 1/[2s + d(m-1)] > 0$. Then there exists a time

$$t_* := C_* \, R^{2s+d(m-1)} \, \|u_0\|_{\mathrm{L}^1(B_R)}^{-(m-1)}$$

such that for every $t \geq t_*$ we have the lower bound

$$\inf_{x \in B_{R/2}} u(t,x) \geq K \, \frac{\|u_0\|_{\mathrm{L}^1(B_R)}^{2s\vartheta}}{t^{d\vartheta}}$$

valid for all $R > 0$. The positive constants C_* and K depend only on m, s and d, and not on R.

Remarks. (i) This result can be written alternatively as saying that there exists a universal constant $C_1 = C_1(d,s,m)$ such for all solutions in the above class we have

$$\int_{B_R(0)} u_0(x)\,dx \leq C_1 \left(R^{1/\vartheta(m-1)} t^{-1/(m-1)} + u(0,t)^{1/2s\vartheta} t^{d/2s} \right).$$

This equivalent version is in complete formal agreement with Aronson-Caffarelli's estimate for $s = 1$, [TAMS 1983]. However, our proof below differs very strongly from the ideas used in Aronson-Caffarelli's case since we cannot use the property of finite propagation of solution with compact support, which is false for $s < 1$.

(ii) In the limit $m \to 1$ of the estimate of Theorem 9 we obtain the (already mentioned) lower bounds for the fractional heat equation.

Positivity estimates in the Porous Medium range

Theorem 9. Local lower bounds when $m > 1$, [BV]

Let u be a weak solution to Equation (EQ), corresponding to $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ and $m > 1$. We put $\vartheta := 1/[2s + d(m-1)] > 0$. Then there exists a time

$$t_* := C_* \, R^{2s+d(m-1)} \, \|u_0\|_{\mathrm{L}^1(B_R)}^{-(m-1)}$$

such that for every $t \geq t_*$ we have the lower bound

$$\inf_{x \in B_{R/2}} u(t,x) \geq K \, \frac{\|u_0\|_{\mathrm{L}^1(B_R)}^{2s\vartheta}}{t^{d\vartheta}}$$

valid for all $R > 0$. The positive constants C_* and K depend only on m, s and d, and not on R.

Remarks. (i) This result can be written alternatively as saying that there exists a universal constant $C_1 = C_1(d,s,m)$ such for all solutions in the above class we have

$$\int_{B_R(0)} u_0(x)\,dx \leq C_1 \left(R^{1/\vartheta(m-1)} t^{-1/(m-1)} + u(0,t)^{1/2s\vartheta} t^{d/2s} \right).$$

This equivalent version is in complete formal agreement with Aronson-Caffarelli's estimate for $s = 1$, [TAMS 1983]. However, our proof below differs very strongly from the ideas used in Aronson-Caffarelli's case since we cannot use the property of finite propagation of solution with compact support, which is false for $s < 1$.

(ii) In the limit $m \to 1$ of the estimate of Theorem 9 we obtain the (already mentioned) lower bounds for the fractional heat equation.

Understanding the problem

The existence of solutions of the Cauchy Problem (CP) can be extended to the case where the initial datum is a finite and nonnegative Radon measure. We denote by $\mathcal{M}^+(\mathbb{R}^d)$ the space of such measures on $\mathbb{R}^d$.

Theorem 10. Existence with measure data , [V]

For every $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ there exists a nonnegative and continuous weak solution of Equation (EQ) in $Q = (0,\infty) \times \mathbb{R}^d$ taking initial data (trace) μ in the sense that for every $\varphi \in C_c^2(\mathbb{R}^d)$ we have

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} \varphi(x)d\mu(x)\,.$$

In this part of the talk we address the reverse problem:

Given a solution, can we find the initial trace?

In the case $s = 1$ such question was solved thanks to the works of Aronson-Caffarelli [TAMS 1983], Dahlberg-Kenig [RMI 1988], Pierre [NLTMA 1982] and others, see a presentation in Vázquez's book [Vbk]. When $m = 1$, this kind of problem is part of the Widder theory [TAMS 1944, book 1975].

Understanding the problem

The existence of solutions of the Cauchy Problem (CP) can be extended to the case where the initial datum is a finite and nonnegative Radon measure. We denote by $\mathcal{M}^+(\mathbb{R}^d)$ the space of such measures on $\mathbb{R}^d$.

Theorem 10. Existence with measure data , [V]

For every $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ there exists a nonnegative and continuous weak solution of Equation (EQ) in $Q = (0,\infty) \times \mathbb{R}^d$ taking initial data (trace) μ in the sense that for every $\varphi \in C_c^2(\mathbb{R}^d)$ we have

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} \varphi(x)d\mu(x)\,.$$

In this part of the talk we address the reverse problem:

Given a solution, can we find the initial trace?

In the case $s = 1$ such question was solved thanks to the works of Aronson-Caffarelli [TAMS 1983], Dahlberg-Kenig [RMI 1988], Pierre [NLTMA 1982] and others, see a presentation in Vázquez's book [Vbk]. When $m = 1$, this kind of problem is part of the Widder theory [TAMS 1944, book 1975].

Understanding the problem

The existence of solutions of the Cauchy Problem (CP) can be extended to the case where the initial datum is a finite and nonnegative Radon measure. We denote by $\mathcal{M}^+(\mathbb{R}^d)$ the space of such measures on $\mathbb{R}^d$.

Theorem 10. Existence with measure data , [V]

For every $\mu \in \mathcal{M}^+(\mathbb{R}^d)$ there exists a nonnegative and continuous weak solution of Equation (EQ) in $Q = (0,\infty) \times \mathbb{R}^d$ taking initial data (trace) μ in the sense that for every $\varphi \in C_c^2(\mathbb{R}^d)$ we have

$$\lim_{t\to 0^+} \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,dx = \int_{\mathbb{R}^d} \varphi(x)d\mu(x)\,.$$

In this part of the talk we address the reverse problem:

Given a solution, can we find the initial trace?

In the case $s = 1$ such question was solved thanks to the works of Aronson-Caffarelli [TAMS 1983], Dahlberg-Kenig [RMI 1988], Pierre [NLTMA 1982] and others, see a presentation in Vázquez's book [Vbk]. When $m = 1$, this kind of problem is part of the Widder theory [TAMS 1944, book 1975].

Understanding the problem

Lemma. Conditions for existence and uniqueness of initial traces , [BV]

Let $m > 0$ and let u be a solution to equation (EQ) in $(0, T] \times \mathbb{R}^d$. Assume that there exist a time $0 < T_1 \le T$, some positive constants $K_1, K_2, \alpha > 0$ and a continuous function $\omega : [0, +\infty) \to [0, +\infty)$, with $\omega(0) = 0$ such that

$$(i) \qquad \sup_{t\in(0,T_1]} \int_{B_R(x_0)} u(t,x)\, \mathrm{d}x \le K_1 , \qquad \forall\, R > 0 ,\; x_0 \in \mathbb{R}^d ,$$

[(i) guarantees *existence* , (ii) guarantees *uniqueness*], and such that

$$(ii) \qquad \left[\int_{\mathbb{R}^d} u(t,x)\varphi(x)\, \mathrm{d}x\right]^\alpha \le \left[\int_{\mathbb{R}^d} u(t',x)\varphi(x)\, \mathrm{d}x\right]^\alpha + K_2\, \omega(|t-t'|)$$

for all $0 < t, t' \le T_1$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \varphi\, \mathrm{d}\mu = \lim_{t\to 0^+} \int_{\mathbb{R}^d} u(t,x)\varphi(x)\, \mathrm{d}x , \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d) .$$

Moreover the initial trace μ satisfies the bound (i) with the same constant, namely $\mu(B_R(x_0)) \le K_1$.

Remark. The constants K_1 and K_2 may depend on u and φ, through some norm.

Understanding the problem

Lemma. Conditions for existence and uniqueness of initial traces , [BV]

Let $m > 0$ and let u be a solution to equation (EQ) in $(0, T] \times \mathbb{R}^d$. Assume that there exist a time $0 < T_1 \le T$, some positive constants $K_1, K_2, \alpha > 0$ and a continuous function $\omega : [0, +\infty) \to [0, +\infty)$, with $\omega(0) = 0$ such that

$$(i) \qquad \sup_{t\in(0,T_1]} \int_{B_R(x_0)} u(t,x)\,\mathrm{d}x \le K_1\,, \qquad \forall\, R > 0\,,\ x_0 \in \mathbb{R}^d\,,$$

[(i) guarantees *existence* , (ii) guarantees *uniqueness*], and such that

$$(ii) \qquad \left[\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,\mathrm{d}x\right]^\alpha \le \left[\int_{\mathbb{R}^d} u(t',x)\varphi(x)\,\mathrm{d}x\right]^\alpha + K_2\,\omega(|t-t'|)$$

for all $0 < t, t' \le T_1$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \varphi\,\mathrm{d}\mu = \lim_{t\to 0^+} \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,\mathrm{d}x\,, \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d)\,.$$

Moreover the initial trace μ satisfies the bound (i) with the same constant, namely $\mu(B_R(x_0)) \le K_1$.

Remark. The constants K_1 and K_2 may depend on u and φ, through some norm.

Understanding the problem

Lemma. Conditions for existence and uniqueness of initial traces , [BV]

Let $m > 0$ and let u be a solution to equation (EQ) in $(0, T] \times \mathbb{R}^d$. Assume that there exist a time $0 < T_1 \le T$, some positive constants $K_1, K_2, \alpha > 0$ and a continuous function $\omega : [0, +\infty) \to [0, +\infty)$, with $\omega(0) = 0$ such that

$$(i) \qquad \sup_{t\in(0,T_1]} \int_{B_R(x_0)} u(t,x)\,\mathrm{d}x \le K_1\,, \qquad \forall\, R > 0\,,\ x_0 \in \mathbb{R}^d\,,$$

[(i) guarantees *existence* , (ii) guarantees *uniqueness*], and such that

$$(ii) \qquad \left[\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,\mathrm{d}x\right]^\alpha \le \left[\int_{\mathbb{R}^d} u(t',x)\varphi(x)\,\mathrm{d}x\right]^\alpha + K_2\,\omega(|t-t'|)$$

for all $0 < t, t' \le T_1$ and for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \varphi\,\mathrm{d}\mu = \lim_{t\to 0^+} \int_{\mathbb{R}^d} u(t,x)\varphi(x)\,\mathrm{d}x\,, \qquad \text{for all } \varphi \in C_c^\infty(\mathbb{R}^d)\,.$$

Moreover the initial trace μ satisfies the bound (i) with the same constant, namely $\mu(B_R(x_0)) \le K_1$.

Remark. The constants K_1 and K_2 may depend on u and φ, through some norm.

Understanding the problem

Lemma. Conditions for existence and uniqueness of initial traces , [BV]

Let $m>0$ and let u be a solution to equation (EQ) in $(0,T]\times\mathbb{R}^d$. Assume that there exist a time $0<T_1\le T$, some positive constants $K_1,K_2,\alpha>0$ and a continuous function $\omega:[0,+\infty)\to[0,+\infty)$, with $\omega(0)=0$ such that

$$(i)\qquad \sup_{t\in(0,T_1]}\int_{B_R(x_0)} u(t,x)\,\mathrm{d}x\le K_1\,,\qquad \forall\,R>0\,,\ x_0\in\mathbb{R}^d\,,$$

[(i) guarantees *existence* , (ii) guarantees *uniqueness*], and such that

$$(ii)\qquad \left[\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,\mathrm{d}x\right]^{\alpha}\le\left[\int_{\mathbb{R}^d} u(t',x)\varphi(x)\,\mathrm{d}x\right]^{\alpha}+K_2\,\omega(|t-t'|)$$

for all $0<t,t'\le T_1$ and for all $\varphi\in C_c^\infty(\mathbb{R}^d)$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d}\varphi\,\mathrm{d}\mu=\lim_{t\to0^+}\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,\mathrm{d}x\,,\qquad \text{for all } \varphi\in C_c^\infty(\mathbb{R}^d)\,.$$

Moreover the initial trace μ satisfies the bound (i) with the same constant, namely $\mu(B_R(x_0))\le K_1$.

Remark. The constants K_1 and K_2 may depend on u and φ, through some norm.

Understanding the problem

Lemma. Conditions for existence and uniqueness of initial traces , [BV]

Let $m>0$ and let u be a solution to equation (EQ) in $(0,T]\times\mathbb{R}^d$. Assume that there exist a time $0<T_1\le T$, some positive constants $K_1, K_2, \alpha>0$ and a continuous function $\omega:[0,+\infty)\to[0,+\infty)$, with $\omega(0)=0$ such that

$$(i)\qquad \sup_{t\in(0,T_1]}\int_{B_R(x_0)} u(t,x)\,\mathrm{d}x \le K_1\,,\qquad \forall\, R>0\,,\; x_0\in\mathbb{R}^d\,,$$

[(i) guarantees *existence* , (ii) guarantees *uniqueness*], and such that

$$(ii)\qquad \left[\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,\mathrm{d}x\right]^\alpha \le \left[\int_{\mathbb{R}^d} u(t',x)\varphi(x)\,\mathrm{d}x\right]^\alpha + K_2\,\omega(|t-t'|)$$

for all $0<t,t'\le T_1$ and for all $\varphi\in C_c^\infty(\mathbb{R}^d)$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d}\varphi\,\mathrm{d}\mu = \lim_{t\to 0^+}\int_{\mathbb{R}^d} u(t,x)\varphi(x)\,\mathrm{d}x\,,\qquad \text{for all } \varphi\in C_c^\infty(\mathbb{R}^d)\,.$$

Moreover the initial trace μ satisfies the bound (i) with the same constant, namely $\mu(B_R(x_0))\le K_1$.

Remark. The constants K_1 and K_2 may depend on u and φ, through some norm.

Existence and uniqueness of initial trace for $0 < m < 1$.

Theorem 11. Existence and uniqueness of initial trace for FDE, [BV]

Let $0 < m < 1$ and let u be a nonnegative weak solution of equation (EQ) in $(0, T] \times \mathbb{R}^d$. Assume that $\|u(T)\|_{L^1(\mathbb{R}^d)} < \infty$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \psi \, d\mu = \lim_{t \to 0^+} \int_{\mathbb{R}^d} u(t,x)\psi(x) \, dx \,, \qquad \text{for all } \psi \in C_0(\mathbb{R}^d) \,.$$

Moreover, the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le \|u(T)\|_{L^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s}\, T \,.$$

where $C_1 = C_1(m, d, s) > 0$ as in Theorem 1.

Remarks. (i) The proof applies with minor modification to the class of very weak solutions with data $u_0 \in L^1(\mathbb{R}^d, \varphi \, dx)$ constructed in Theorem 2 .

(ii) The existence of solutions and traces for the standard FDE with (not necessarily locally finite) Borel measures as data is studied in Chasseigne-Vázquez [ARMA 2002],. We do not address the corresponding question here.

(iii) The above results holds also when $m = 1$, if $\|u(T)\|_{L^1(\mathbb{R}^d, \varphi)} < \infty$ with $\varphi \in \mathcal{W}_\alpha$, $\alpha = d + 2s$, but the last bound becomes

$$\mu(B_R(x_0)) \le K_1 = e^{K_0\, T} \|u(T)\|_{L^1(\mathbb{R}^d, \varphi)} \,.$$

Existence and uniqueness of initial trace for $0 < m < 1$.

Theorem 11. Existence and uniqueness of initial trace for FDE , [BV]

Let $0 < m < 1$ and let u be a nonnegative weak solution of equation (EQ) in $(0,T] \times \mathbb{R}^d$. Assume that $\|u(T)\|_{L^1(\mathbb{R}^d)} < \infty$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \psi \,d\mu = \lim_{t\to 0^+} \int_{\mathbb{R}^d} u(t,x)\psi(x)\,dx\,, \qquad \text{for all } \psi \in C_0(\mathbb{R}^d)\,.$$

Moreover, the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le \|u(T)\|_{L^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s}\, T\,.$$

where $C_1 = C_1(m,d,s) > 0$ as in Theorem 1.

Remarks. (i) The proof applies with minor modification to the class of very weak solutions with data $u_0 \in L^1(\mathbb{R}^d, \varphi\,dx)$ constructed in Theorem 2 .

(ii) The existence of solutions and traces for the standard FDE with (not necessarily locally finite) Borel measures as data is studied in Chasseigne-Vázquez [ARMA 2002],. We do not address the corresponding question here.

(iii) The above results holds also when $m = 1$, if $\|u(T)\|_{L^1(\mathbb{R}^d,\varphi)} < \infty$ with $\varphi \in \mathcal{W}_\alpha$, $\alpha = d + 2s$, but the last bound becomes

$$\mu(B_R(x_0)) \le K_1 = e^{K_0 T}\, \|u(T)\|_{L^1(\mathbb{R}^d,\varphi)}\,.$$

Existence and uniqueness of initial trace for $0 < m < 1$.

Theorem 11. Existence and uniqueness of initial trace for FDE , [BV]

Let $0 < m < 1$ and let u be a nonnegative weak solution of equation (EQ) in $(0, T] \times \mathbb{R}^d$. Assume that $\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d)} < \infty$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \psi \, \mathrm{d}\mu = \lim_{t \to 0^+} \int_{\mathbb{R}^d} u(t,x)\psi(x) \, \mathrm{d}x \,, \qquad \text{for all } \psi \in C_0(\mathbb{R}^d) \,.$$

Moreover, the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le \|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s} \, T \,.$$

where $C_1 = C_1(m, d, s) > 0$ as in Theorem 1.

Remarks. (i) The proof applies with minor modification to the class of very weak solutions with data $u_0 \in \mathrm{L}^1(\mathbb{R}^d, \varphi \, \mathrm{d}x)$ constructed in Theorem 2 .
(ii) The existence of solutions and traces for the standard FDE with (not necessarily locally finite) Borel measures as data is studied in Chasseigne-Vázquez [ARMA 2002],.
We do not address the corresponding question here.
(iii) The above results holds also when $m = 1$, if $\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d, \varphi)} < \infty$ with $\varphi \in \mathcal{W}_\alpha$, $\alpha = d + 2s$, but the last bound becomes

$$\mu(B_R(x_0)) \le K_1 = \mathrm{e}^{K_0 T} \, \|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d, \varphi)} \,.$$

Existence and uniqueness of initial trace for $0 < m < 1$.

Theorem 11. Existence and uniqueness of initial trace for FDE, [BV]

Let $0 < m < 1$ and let u be a nonnegative weak solution of equation (EQ) in $(0, T] \times \mathbb{R}^d$. Assume that $\|u(T)\|_{L^1(\mathbb{R}^d)} < \infty$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \psi \, d\mu = \lim_{t \to 0^+} \int_{\mathbb{R}^d} u(t,x)\psi(x) \, dx \,, \qquad \text{for all } \psi \in C_0(\mathbb{R}^d) \,.$$

Moreover, the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le \|u(T)\|_{L^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s} T \,.$$

where $C_1 = C_1(m,d,s) > 0$ as in Theorem 1.

Remarks. (i) The proof applies with minor modification to the class of very weak solutions with data $u_0 \in L^1(\mathbb{R}^d, \varphi\, dx)$ constructed in Theorem 2 .
(ii) The existence of solutions and traces for the standard FDE with (not necessarily locally finite) Borel measures as data is studied in Chasseigne-Vázquez [ARMA 2002],. We do not address the corresponding question here.
(iii) The above results holds also when $m = 1$, if $\|u(T)\|_{L^1(\mathbb{R}^d, \varphi)} < \infty$ with $\varphi \in \mathcal{W}_\alpha$, $\alpha = d + 2s$, but the last bound becomes

$$\mu(B_R(x_0)) \le K_1 = e^{K_0 T} \|u(T)\|_{L^1(\mathbb{R}^d, \varphi)} \,.$$

Existence and uniqueness of initial trace for $0 < m < 1$.

Proof. The proof is divided into three steps.

• STEP 1. *Weighted estimates I. Existence when* $0 < m < 1$. First we recall the weighted estimates of Theorem 1 , which imply for all $0 \le t \le T_1 \le T$

$$\left(\int_{\mathbb{R}^d} u(t,x)\phi_R(x)\,\mathrm{d}x\right)^{1-m} \le \left(\int_{\mathbb{R}^d} u(T,x)\phi_R(x)\,\mathrm{d}x\right)^{1-m} + C_1 R^{d(1-m)-2s}\,|T-T_1|$$
$$\le \|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s}\,T := K_1$$

where $C_1 > 0$ depends only on α, m, d as in Theorem 1 . This implies hypothesis (i) of previous Lemma, therefore it guarantees the existence of an initial trace that satisfies the bound $\mu(B_R(x_0)) \le K_1 = \|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s}\,T$.

• STEP 1. *Weighted estimates I. Existence when* $m = 1$. First we recall the weighted estimates when $m = 1$, which imply for all $0 \le t \le T_1 \le T$

$$\left(\int_{\mathbb{R}^d} u(t,x)\phi_R(x)\,\mathrm{d}x\right)^{1-m} \le \mathrm{e}^{K_0(T-t)}\left(\int_{\mathbb{R}^d} u(T,x)\phi_R(x)\,\mathrm{d}x\right)^{1-m} := K_1$$

The proof is easy:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u\phi_R\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^d} u\,(-\Delta)^s\phi_R\,\mathrm{d}x\right| \le \left\|\frac{(-\Delta)^s\phi_R}{\phi_R}\right\|_\infty \int_{\mathbb{R}^d} u\,\varphi_R\,\mathrm{d}x\,.$$

This implies hypothesis (i) of the previous Lemma, therefore it guarantees the existence of an initial trace that satisfies the bound $\mu(B_R(x_0)) \le K_1 = \mathrm{e}^{K_0\,T}\,\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)}$.

Existence and uniqueness of initial trace for $0 < m < 1$.

Proof. The proof is divided into three steps.

• STEP 1. *Weighted estimates I. Existence when* $0 < m < 1$. First we recall the weighted estimates of Theorem 1 , which imply for all $0 \le t \le T_1 \le T$

$$\left(\int_{\mathbb{R}^d} u(t,x)\phi_R(x)\,\mathrm{d}x\right)^{1-m} \le \left(\int_{\mathbb{R}^d} u(T,x)\phi_R(x)\,\mathrm{d}x\right)^{1-m} + C_1 R^{d(1-m)-2s}\,|T-T_1|$$
$$\le \|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s}\,T := K_1$$

where $C_1 > 0$ depends only on α, m, d as in Theorem 1 . This implies hypothesis (i) of previous Lemma, therefore it guarantees the existence of an initial trace that satisfies the bound $\mu(B_R(x_0)) \le K_1 = \|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d)} + C_1 R^{d(1-m)-2s}\,T$.

• STEP 1. *Weighted estimates I. Existence when* $m = 1$. First we recall the weighted estimates when $m = 1$, which imply for all $0 \le t \le T_1 \le T$

$$\left(\int_{\mathbb{R}^d} u(t,x)\phi_R(x)\,\mathrm{d}x\right)^{1-m} \le \mathrm{e}^{K_0(T-t)}\left(\int_{\mathbb{R}^d} u(T,x)\phi_R(x)\,\mathrm{d}x\right)^{1-m} := K_1$$

The proof is easy:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u\phi_R\,\mathrm{d}x\right| = \left|\int_{\mathbb{R}^d} u\,(-\Delta)^s\phi_R\,\mathrm{d}x\right| \le \left\|\frac{(-\Delta)^s\phi_R}{\phi_R}\right\|_\infty \int_{\mathbb{R}^d} u\,\varphi_R\,\mathrm{d}x\,.$$

This implies hypothesis (i) of the previous Lemma, therefore it guarantees the existence of an initial trace that satisfies the bound $\mu(B_R(x_0)) \le K_1 = \mathrm{e}^{K_0\,T}\,\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)}$.

Existence and uniqueness of initial trace for $0 < m < 1$.

• STEP 2. *Pseudo-local estimates. Uniqueness.* In order to prove uniqueness of the initial trace is is sufficient to prove hypothesis (ii) of previous Lemma, namely we need to prove that

$$\left[\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right]^\alpha \le \left[\int_{\mathbb{R}^d} u(t',x)\psi(x)\,\mathrm{d}x\right]^\alpha + K_2\,\omega(|t-t'|)$$

for all $0 < t, t' \le T_1 \le T$ and for all $\psi \in C_c^\infty(\mathbb{R}^d)$. We will see that this is true for $\alpha = 1$ and $\omega(|t-t'|) = |t-t'|$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$, then we have

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^d} (-\Delta)^s u^m \psi\,\mathrm{d}x\right| =_{(a)} \left|\int_{\mathbb{R}^d} u^m (-\Delta)^s \psi\,\mathrm{d}x\right| \\
&\le \int_{\mathbb{R}^d} u^m\,\phi_R(x)\,\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\,\mathrm{d}x \\
&\le_{(b)} \left\|\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d}\phi_R\,\mathrm{d}x\right)^{1-m}\left(\int_{\mathbb{R}^d} u\phi_R\,\mathrm{d}x\right)^m \\
&\le_{(c)} k_7\,\|\phi_R\|_{\mathrm{L}^1(\mathbb{R}^d)}\,K_1 := K_2\,.
\end{aligned}$$

Notice that in (a) we have used the fact that $(-\Delta)^s$ is a symmetric operator. In (b) we have chosen $\phi_R(x) := \phi(x/R)$, with $\phi \in \mathcal{W}_\alpha$ with $\alpha = d + 2s$. It then follows that $\| |(-\Delta)^s\psi(x)| / \phi_R(x)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le k_7$. In (c) we have used that the L^m-norm $(m < 1)$ is smaller than the L^1 norm, since the measure $\phi_R\,\mathrm{d}x$ is finite. Finally we have used the bound of Step 1: $\left(\int_{\mathbb{R}^d} u(t,x)\varphi_R(x)\,\mathrm{d}x\right)^{1-m} \le K_1$, for all $0 \le t \le T_1$.

Existence and uniqueness of initial trace for $0 < m < 1$.

• STEP 2. *Pseudo-local estimates. Uniqueness.* In order to prove uniqueness of the initial trace is is sufficient to prove hypothesis (ii) of previous Lemma, namely we need to prove that

$$\left[\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right]^\alpha \le \left[\int_{\mathbb{R}^d} u(t',x)\psi(x)\,\mathrm{d}x\right]^\alpha + K_2\,\omega(|t-t'|)$$

for all $0 < t, t' \le T_1 \le T$ and for all $\psi \in C_c^\infty(\mathbb{R}^d)$. We will see that this is true for $\alpha = 1$ and $\omega(|t-t'|) = |t-t'|$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$, then we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^d} (-\Delta)^s u^m \psi\,\mathrm{d}x\right| =_{(a)} \left|\int_{\mathbb{R}^d} u^m (-\Delta)^s\psi\,\mathrm{d}x\right| \\ &\le \int_{\mathbb{R}^d} u^m\,\phi_R(x)\,\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\,\mathrm{d}x \\ &\le_{(b)} \left\|\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d}\phi_R\,\mathrm{d}x\right)^{1-m}\left(\int_{\mathbb{R}^d} u\phi_R\,\mathrm{d}x\right)^m \\ &\le_{(c)} k_7\,\|\phi_R\|_{\mathrm{L}^1(\mathbb{R}^d)}\,K_1 := K_2\,.\end{aligned}$$

Notice that in (a) we have used the fact that $(-\Delta)^s$ is a symmetric operator. In (b) we have chosen $\phi_R(x) := \phi(x/R)$, with $\phi \in \mathcal{W}_\alpha$ with $\alpha = d + 2s$. It then follows that $\||(-\Delta)^s\psi(x)|\,/\phi_R(x)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le k_7$. In (c) we have used that the L^m-norm $(m < 1)$ is smaller than the L^1 norm, since the measure $\phi_R\,\mathrm{d}x$ is finite. Finally we have used the bound of Step 1: $\left(\int_{\mathbb{R}^d} u(t,x)\varphi_R(x)\,\mathrm{d}x\right)^{1-m} \le K_1$, for all $0 \le t \le T_1$.

Existence and uniqueness of initial trace for $0 < m < 1$.

• STEP 2. *Pseudo-local estimates. Uniqueness.* In order to prove uniqueness of the initial trace is is sufficient to prove hypothesis (ii) of previous Lemma, namely we need to prove that

$$\left[\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right]^\alpha \le \left[\int_{\mathbb{R}^d} u(t',x)\psi(x)\,\mathrm{d}x\right]^\alpha + K_2\,\omega(|t-t'|)$$

for all $0 < t, t' \le T_1 \le T$ and for all $\psi \in C_c^\infty(\mathbb{R}^d)$. We will see that this is true for $\alpha = 1$ and $\omega(|t-t'|) = |t-t'|$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$, then we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^d}(-\Delta)^s u^m\psi\,\mathrm{d}x\right| =_{(a)} \left|\int_{\mathbb{R}^d} u^m(-\Delta)^s\psi\,\mathrm{d}x\right| \\ &\le \int_{\mathbb{R}^d} u^m\,\phi_R(x)\,\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\,\mathrm{d}x \\ &\le_{(b)} \left\|\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}\phi_R\,\mathrm{d}x\right)^{1-m}\left(\int_{\mathbb{R}^d}u\phi_R\,\mathrm{d}x\right)^m \\ &\le_{(c)} k_7\,\|\phi_R\|_{\mathrm{L}^1(\mathbb{R}^d)}\,K_1 := K_2\,.\end{aligned}$$

Notice that in (a) we have used the fact that $(-\Delta)^s$ is a symmetric operator. In (b) we have chosen $\phi_R(x) := \phi(x/R)$, with $\phi \in \mathcal{W}_\alpha$ with $\alpha = d + 2s$. It then follows that $\||(-\Delta)^s\psi(x)|\,/\phi_R(x)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le k_7$. In (c) we have used that the L^m-norm $(m < 1)$ is smaller than the L^1 norm, since the measure $\phi_R\,\mathrm{d}x$ is finite. Finally we have used the bound of Step 1: $\left(\int_{\mathbb{R}^d} u(t,x)\varphi_R(x)\,\mathrm{d}x\right)^{1-m} \le K_1$, for all $0 \le t \le T_1$.

Existence and uniqueness of initial trace for $0 < m < 1$.

• STEP 2. *Pseudo-local estimates. Uniqueness.* In order to prove uniqueness of the initial trace is is sufficient to prove hypothesis (ii) of previous Lemma, namely we need to prove that

$$\left[\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right]^\alpha \le \left[\int_{\mathbb{R}^d} u(t',x)\psi(x)\,\mathrm{d}x\right]^\alpha + K_2\,\omega(|t-t'|)$$

for all $0 < t, t' \le T_1 \le T$ and for all $\psi \in C_c^\infty(\mathbb{R}^d)$. We will see that this is true for $\alpha = 1$ and $\omega(|t-t'|) = |t-t'|$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$, then we have

$$\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^d} (-\Delta)^s u^m \psi\,\mathrm{d}x\right| =_{(a)} \left|\int_{\mathbb{R}^d} u^m (-\Delta)^s \psi\,\mathrm{d}x\right| \\
&\le \int_{\mathbb{R}^d} u^m\,\phi_R(x)\,\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\,\mathrm{d}x \\
&\le_{(b)} \left\|\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d}\phi_R\,\mathrm{d}x\right)^{1-m}\left(\int_{\mathbb{R}^d} u\phi_R\,\mathrm{d}x\right)^m \\
&\le_{(c)} k_7\,\|\phi_R\|_{\mathrm{L}^1(\mathbb{R}^d)}\,K_1 := K_2\,.
\end{aligned}$$

Notice that in (a) we have used the fact that $(-\Delta)^s$ is a symmetric operator. In (b) we have chosen $\phi_R(x) := \phi(x/R)$, with $\phi \in \mathcal{W}_\alpha$ with $\alpha = d+2s$. It then follows that $\||(-\Delta)^s\psi(x)|\,/\phi_R(x)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le k_7$. In (c) we have used that the L^m-norm $(m<1)$ is smaller than the L^1 norm, since the measure $\phi_R\,\mathrm{d}x$ is finite. Finally we have used the bound of Step 1: $\left(\int_{\mathbb{R}^d} u(t,x)\varphi_R(x)\,\mathrm{d}x\right)^{1-m} \le K_1$, for all $0 \le t \le T_1$.

Existence and uniqueness of initial trace for $0 < m < 1$.

• STEP 2. *Pseudo-local estimates. Uniqueness.* In order to prove uniqueness of the initial trace is is sufficient to prove hypothesis (ii) of previous Lemma, namely we need to prove that

$$\left[\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right]^{\alpha} \le \left[\int_{\mathbb{R}^d} u(t',x)\psi(x)\,\mathrm{d}x\right]^{\alpha} + K_2\,\omega(|t-t'|)$$

for all $0 < t, t' \le T_1 \le T$ and for all $\psi \in C_c^\infty(\mathbb{R}^d)$. We will see that this is true for $\alpha = 1$ and $\omega(|t-t'|) = |t-t'|$. Let $\psi \in C_c^\infty(\mathbb{R}^d)$, then we have

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}^d}(-\Delta)^s u^m\psi\,\mathrm{d}x\right| =_{(a)} \left|\int_{\mathbb{R}^d} u^m(-\Delta)^s\psi\,\mathrm{d}x\right| \\ &\le \int_{\mathbb{R}^d} u^m\,\phi_R(x)\,\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\,\mathrm{d}x \\ &\le_{(b)} \left\|\frac{|(-\Delta)^s\psi(x)|}{\phi_R(x)}\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d}\phi_R\,\mathrm{d}x\right)^{1-m}\left(\int_{\mathbb{R}^d} u\phi_R\,\mathrm{d}x\right)^{m} \\ &\le_{(c)} k_7\,\|\phi_R\|_{\mathrm{L}^1(\mathbb{R}^d)}\,K_1 := K_2\,.\end{aligned}$$

Notice that in (a) we have used the fact that $(-\Delta)^s$ is a symmetric operator. In (b) we have chosen $\phi_R(x) := \phi(x/R)$, with $\phi \in \mathcal{W}_\alpha$ with $\alpha = d + 2s$. It then follows that $\left\||(-\Delta)^s\psi(x)|\,/\phi_R(x)\right\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le k_7$. In (c) we have used that the L^m-norm $(m < 1)$ is smaller than the L^1 norm, since the measure $\phi_R\,\mathrm{d}x$ is finite. Finally we have used the bound of Step 1: $\left(\int_{\mathbb{R}^d} u(t,x)\varphi_R(x)\,\mathrm{d}x\right)^{1-m} \le K_1$, for all $0 \le t \le T_1$.

Existence and uniqueness of initial trace for $0 < m < 1$.

- STEP 2. *Continued...* Summing up, we have obtained:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| \le K_2\,.$$

Integrating the above differential inequality we obtain for any $\tau, t \ge 0$ and all $\psi \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d} u(\tau,x)\psi(x)\,\mathrm{d}x + K_2\,|t-\tau|\,.$$

- STEP 3. We still have to pass from test functions $\psi \in C_c^\infty(\mathbb{R}^d)$ to $\psi \in C_c^0(\mathbb{R}^d)$, but this is easy by approximation (mollification). □

Theorem 12. Existence and uniqueness of initial trace for Fractional HE

Let $m = 1$ and let u be a nonnegative weak solution of equation (EQ) in $(0,T] \times \mathbb{R}^d$. Assume that $\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)} < \infty$ with $\varphi \in \mathcal{W}_\alpha$, $\alpha = d + 2s$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \psi\,\mathrm{d}\mu = \lim_{t\to 0^+}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\,, \qquad \text{for all } \psi \in C_0(\mathbb{R}^d)\,.$$

Moreover, the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le \mathrm{e}^{C_1 T}\,\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)}\,, \qquad \text{for some explicit } C_1 = C_1(m,d,s) > 0.$$

Existence and uniqueness of initial trace for $0 < m < 1$.

- STEP 2. *Continued...* Summing up, we have obtained:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| \le K_2\,.$$

Integrating the above differential inequality we obtain for any $\tau, t \ge 0$ and all $\psi \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d} u(\tau,x)\psi(x)\,\mathrm{d}x + K_2\,|t-\tau|\,.$$

- STEP 3. We still have to pass from test functions $\psi \in C_c^\infty(\mathbb{R}^d)$ to $\psi \in C_c^0(\mathbb{R}^d)$, but this is easy by approximation (mollification). □

Theorem 12. Existence and uniqueness of initial trace for Fractional HE

Let $m = 1$ and let u be a nonnegative weak solution of equation (EQ) in $(0,T]\times\mathbb{R}^d$. Assume that $\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)} < \infty$ with $\varphi \in \mathcal{W}_\alpha$, $\alpha = d+2s$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \psi\,\mathrm{d}\mu = \lim_{t\to 0^+}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\,, \qquad \text{for all } \psi \in C_0(\mathbb{R}^d)\,.$$

Moreover, the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le \mathrm{e}^{C_1 T}\,\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)}\,, \qquad \text{for some explicit } C_1 = C_1(m,d,s) > 0.$$

Existence and uniqueness of initial trace for $0 < m < 1$.

• STEP 2. *Continued...* Summing up, we have obtained:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| \le K_2\,.$$

Integrating the above differential inequality we obtain for any $\tau, t \ge 0$ and all $\psi \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d} u(\tau,x)\psi(x)\,\mathrm{d}x + K_2\,|t-\tau|\,.$$

• STEP 3. We still have to pass from test functions $\psi \in C_c^\infty(\mathbb{R}^d)$ to $\psi \in C_c^0(\mathbb{R}^d)$, but this is easy by approximation (mollification). $\square$

Theorem 12. Existence and uniqueness of initial trace for Fractional HE

Let $m = 1$ and let u be a nonnegative weak solution of equation (EQ) in $(0,T]\times\mathbb{R}^d$. Assume that $\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)} < \infty$ with $\varphi \in \mathcal{W}_\alpha\,,\ \alpha = d+2s$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d}\psi\,\mathrm{d}\mu = \lim_{t\to 0+}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\,,\qquad \text{for all } \psi \in C_0(\mathbb{R}^d)\,.$$

Moreover, the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le \mathrm{e}^{C_1 T}\,\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)}\,,\qquad \text{for some explicit } C_1 = C_1(m,d,s) > 0.$$

Existence and uniqueness of initial trace for $0 < m < 1$.

• STEP 2. *Continued...* Summing up, we have obtained:

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\right| \le K_2\,.$$

Integrating the above differential inequality we obtain for any $\tau, t \ge 0$ and all $\psi \in C_c^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x \le \int_{\mathbb{R}^d} u(\tau,x)\psi(x)\,\mathrm{d}x + K_2\,|t-\tau|\,.$$

• STEP 3. We still have to pass from test functions $\psi \in C_c^\infty(\mathbb{R}^d)$ to $\psi \in C_c^0(\mathbb{R}^d)$, but this is easy by approximation (mollification). □

Theorem 12. Existence and uniqueness of initial trace for Fractional HE

Let $m = 1$ and let u be a nonnegative weak solution of equation (EQ) in $(0,T]\times\mathbb{R}^d$. Assume that $\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)} < \infty$ with $\varphi \in \mathcal{W}_\alpha$, $\alpha = d + 2s$. Then there exists a unique nonnegative Radon measure μ as initial trace, that is

$$\int_{\mathbb{R}^d}\psi\,\mathrm{d}\mu = \lim_{t\to 0^+}\int_{\mathbb{R}^d} u(t,x)\psi(x)\,\mathrm{d}x\,, \qquad \text{for all } \psi \in C_0(\mathbb{R}^d)\,.$$

Moreover, the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le \mathrm{e}^{C_1\,T}\,\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d,\varphi)}\,, \qquad \text{for some explicit } C_1 = C_1(m,d,s) > 0.$$

Existence and uniqueness of initial trace for $m > 1$.

Theorem 13. Existence and uniqueness of initial trace, PME case, [BV]

Let $m > 1$ and let u be a solution to the Cauchy problem (CP) on $(0, T] \times \mathbb{R}^d$. Assume that $\|u(T)\|_{\mathrm{L}^1(\mathbb{R}^d)} + \|u(T)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} < +\infty$. Then there exists a unique nonnegative Borel measure μ as initial trace, that is

$$\int_{\mathbb{R}^d} \psi \, \mathrm{d}\mu = \lim_{t \to 0^+} \int_{\mathbb{R}^d} u(t,x)\psi(x) \, \mathrm{d}x \,, \qquad \text{for all } \psi \in C_0(\mathbb{R}^d) \,.$$

Moreover the initial trace μ satisfies the bound

$$\mu(B_R(x_0)) \le C_1 \left[\left(\frac{R^{2s+d(m-1)}}{T} \right)^{\frac{1}{m-1}} + T^{\frac{d}{2s}} \, u(x_0, T)^{\frac{1}{2s\vartheta}} \right] \,,$$

where $C_1 = C_1(m, d, s) > 0$ as in Theorem 9.

The proof is different from the case $m \le 1$: uses the smoothing effect, mass conservation and the lower estimates in the Aronson-Caffarelli form:

$$\int_{B_R(x_0)} u_0(x) \, dx \le C_1 \left[\left(\frac{R^{2s+d(m-1)}}{T} \right)^{\frac{1}{m-1}} + T^{\frac{d}{2s}} \, u(x_0, T)^{\frac{1}{2s\vartheta}} \right] .$$

Existence and uniqueness of initial trace for $m > 1$.

The End

Thank you!!!

Muchas Gracias!!!

Grazie Mille!!!