EXPLICIT CONSTANTS IN HARNACK INEQUALITIES AND REGULARITY ESTIMATES, WITH AN APPLICATION TO THE FAST DIFFUSION EQUATION

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Abstract. *This paper is devoted to the computation of various explicit constants in functional inequalities and regularity estimates for solutions of parabolic equations, which are not available from the literature. We provide new expressions and simplified proofs of the Harnack inequality and the corresponding Hölder continuity of the solution of a linear parabolic equation. We apply these results to the computation of a constructive estimate of a threshold time for the uniform convergence in relative error of the solution of the fast diffusion equation.*

This document is divided into two Parts. Part I is devoted to the explicit computation of the constant in Moser's Harnack inequality based on the method of [25, 26]. As far as we know, no such expression of the constant has yet been published. Part II is devoted to fully explicit and constructive estimates which are needed for proving the stability in some Gagliardo-Nirenberg inequalities in [9]. For a comprehensive introduction to stability issues and a review of the literature, the reader is invited to refer to [9, Section 1]. Boxed inequalities are used to quote results of [9].

Keywords: Harnack Principle; Moser's Harnack inequality; Gagliardo-Nirenberg inequality; stability; entropy methods; fast diffusion equation; asymptotic behavior; intermediate asymptotics; self-similar Barenblatt solutions; rates of convergence
2020 Mathematics Subject Classification. 26D10; 46E35; 35K55; 49J40; 35B40; 49K20; 49K30; 35J20.

Part I

The constant in Moser's Harnack inequality

Let Ω be an open domain and let us consider a positive *weak solution* to

$$\frac{\partial v}{\partial t} = \nabla \cdot \Big(A(t,x)\, \nabla v \Big) \tag{1}$$

on $\Omega_T := (0,T) \times \Omega$, where $A(t,x)$ is a real symmetric matrix with bounded measurable coefficients satisfying the uniform ellipticity condition

$$\lambda_0\, |\xi|^2 \le \sum_{i,j=1}^{d} A_{i,j}(t,x)\, \xi_i\, \xi_j \le \lambda_1\, |\xi|^2 \quad \forall\, (t,x,\xi) \in \mathbb{R}^+ \times \Omega_T \times \mathbb{R}^d\,, \tag{2}$$

for some positive constants λ_0 and λ_1. Let us consider the neighborhoods

$$\begin{aligned} D_R^+(t_0,x_0) &:= (t_0 + \tfrac{3}{4}\, R^2, t_0 + R^2) \times B_{R/2}(x_0)\,, \\ D_R^-(t_0,x_0) &:= \Big(t_0 - \tfrac{3}{4}\, R^2, t_0 - \tfrac{1}{4}\, R^2\Big) \times B_{R/2}(x_0)\,, \end{aligned} \tag{3}$$

and the constant

$$\mathsf{h} := \exp\left[2^{d+4}\, 3^d\, d + c_0^3\, 2^{2\,(d+2)+3} \left(1 + \frac{2^{d+2}}{(\sqrt{2}-1)^{2\,(d+2)}} \right) \sigma \right] \tag{4}$$

where

$$c_0 = 3^{\frac{2}{d}}\, 2^{\frac{(d+2)\,(3\,d^2+18\,d+24)+13}{2\,d}} \left(\frac{(2+d)^{1+\frac{4}{d^2}}}{d^{1+\frac{2}{d^2}}} \right)^{(d+1)(d+2)} \mathcal{K}^{\frac{2\,d+4}{d}}\,, \tag{5}$$

$$\sigma = \sum_{j=0}^{\infty} \left(\tfrac{3}{4}\right)^j \Big((2+j)\,(1+j) \Big)^{2\,d+4}\,. \tag{6}$$

Let $\mathsf{p} := 2\,d/(d-2) = 2^\star$ if $d \ge 3$, $\mathsf{p} := 4$ if $d=2$ and $\mathsf{p} \in (4,+\infty)$ if $d=1$. The constant $\mathcal{K}$ in (5) is the constant in the inequality

$$\|f\|^2_{\mathrm{L}^{\mathsf{p}}(B_R)} \le \mathcal{K} \left(\|\nabla f\|^2_{\mathrm{L}^2(B_R)} + \tfrac{1}{R^2}\, \|f\|^2_{\mathrm{L}^2(B_R)} \right) \quad \forall\, f \in \mathrm{H}^1(B_R)\,. \tag{7}$$

If $d \ge 3$, then $\mathcal{K}$ is independent of R. For $d = 1$, 2, we further assume that $R \le 1$. We learn from [9, Appendices B and C] that

$$\mathcal{K} \le \begin{cases} 2\,\mathcal{S}_1^2 = \frac{2}{\pi}\, \Gamma\left(\frac{d}{2}+1\right)^{\frac{2}{d}} & \text{if} \quad d \ge 3\,, \\ \frac{2}{\sqrt{\pi}} & \text{if} \quad d = 2\,, \\ 2^{1+\frac{2}{\mathsf{p}}} \max\left\{ \frac{\mathsf{p}-2}{\pi^2}, \frac{1}{4} \right\} & \text{if} \quad d = 1\,. \end{cases} \tag{8}$$

Also see Table 1 below. We shall also need some numerical constants associated with balls and spheres. The volume of the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ is

$$\omega_d = |\mathbb{S}^{d-1}| = \frac{2\,\pi^{d/2}}{\Gamma(d/2)} \le \frac{16}{15}\, \pi^3\,. \tag{9}$$

As a consequence, the volume of a d-dimensional unit ball is ω_d/d and

$$\frac{\omega_d}{d} \le \pi^2 \tag{10}$$

whenever sharpness is not needed. Let us define

$$\overline{\mathsf{h}} := \mathsf{h}^{\lambda_1 + 1/\lambda_0} . \tag{11}$$

We claim that the *Harnack inequality* holds with the constant $\overline{\mathsf{h}}$ as follows.

Theorem 1. *Let $T > 0$, $R \in (0, \sqrt{T})$, and take $(t_0, x_0) \in (0, T) \times \Omega$ such that $(t_0 - R^2, t_0 + R^2) \times B_{2R}(x_0) \subset \Omega_T$. Under Assumption (2), if u is a weak solution of (1), then*

$$\sup_{D_R^-(t_0,x_0)} v \le \overline{\mathsf{h}} \inf_{D_R^+(t_0,x_0)} v . \tag{12}$$

Here a *weak solution* is defined as in [26, p. 728], [24, Chapter 3] or [3]. The Harnack inequality of Theorem 1 goes back to J. Moser [25, 26]. The dependence of the constant on the ellipticity constants λ_0 and λ_1 was not clear before the paper of J. Moser [26], where he shows that such a dependence is optimal by providing an explicit example, [26, p. 729]. The fact that h only depends on the dimension d is also pointed out by C.E. Gutierrez and R.L. Wheeden in [22] after the statement of their Harnack inequalities, [22, Theorem A]. However, to our knowledge, a complete constructive proof and an expression like (11) was still missing. We do not claim any originality concerning the strategy but provide for the first time an explicit expression for the constant $\overline{\mathsf{h}}$.

The proof of the above theorem is quite long and technical, and relies on three main ingredients, contained in the first three sections:

- *Moser iteration procedure.* In Section 1, the main local upper and lower smoothing effects are obtained, through the celebrated Moser iteration, in the form of precise $\mathrm{L}^p - \mathrm{L}^\infty$ and $\mathrm{L}^{-p} - \mathrm{L}^{-\infty}$ bounds for arbitrarily small $p > 0$. The next task would be to relate such upper and lower bounds, to produce the desired Harnack inequalities. This can be done by means of parabolic BMO estimates, but in this case one may lose control of the estimates, since not all the proofs of such BMO bounds are constructive. We hence follow the ideas of J. Moser in [26], which avoids the use of BMO spaces, as follows.

- *Logarithmic Estimates.* The idea is to obtain detailed informations on the level sets of solutions. This is done in Section 2 by estimating the logarithm of the solution to (1). This is a fundamental estimate needed both in the approach with BMO spaces (it indeed implies that u has bounded mean oscillation) and in the alternative approach used here.

- *A lemma by E. Bombieri and E. Giusti.* In Section 3, we prove a parabolic version of the Bombieri-Giusti Lemma, following again Moser's proof in [26] (also see [6]). This refinement of the upper bounds may seem trivial at first sight, but it is not and turns out to be crucial for our constructive method.

- *Proof of Moser's Harnack inequality.* We finally prove Theorem 1 in Section 4 using a suitably rescaled solution.

As an important consequence of Theorem 1, we obtain constructive and explicit *Hölder continuity estimates* in Section 5, by following Moser's approach in [25]. We find an explicit expression of the Hölder exponent, which is new and only depends on the dimension and on the ellipticity constants, and a detailed estimate with new and explicit constants.

1 Upper and lower Moser iteration

Let us start by recalling the definition of the parabolic cylinders

$$\begin{aligned} Q_\varrho &= Q_\varrho(0,0) = \left\{|t| < \varrho^2\,,\ |x| < \varrho\right\} = (-\varrho^2, \varrho^2) \times B_\varrho(0)\,,\\ Q_\varrho^+ &= Q_\varrho(0,0) = \left\{0 < t < \varrho^2\,,\ |x| < \varrho\right\} = (0, \varrho^2) \times B_\varrho(0)\,,\\ Q_\varrho^- &= Q_\varrho(0,0) = \left\{0 < -t < \varrho^2\,,\ |x| < \varrho\right\} = (-\varrho^2, 0) \times B_\varrho(0)\,. \end{aligned}$$

In order to perform the celebrated Moser iteration, we establish an important lemma, which relies on (7). We follow the method of [25, 26] and provide a quantitative and constructive proof, with explicit constants. From here on, we assume that u is a positive solution, as was done by J. Moser in [26, p. 729, l. 8-9].

Lemma 2 (Moser iteration, [25, 26])**.** *Assume that r and ρ are such that $1/2 \le \varrho \le r \le 1$ and $\mu = \lambda_1 + 1/\lambda_0$ and let v be a nonnegative solution to* (1)*. Then there exists a positive constant $c_1 = c_1(d)$ such that*

$$\sup_{Q_\varrho} v^p \le \frac{c_1}{(r-\varrho)^{d+2}} \iint_{Q_r} v^p \,\mathrm{d}x\,\mathrm{d}t \quad \forall\, p \in \left(0, \tfrac{1}{\mu}\right) \tag{13}$$

and

$$\sup_{Q_\varrho^-} v^p \le \frac{c_1}{(r-\varrho)^{d+2}} \iint_{Q_r^-} v^p \,\mathrm{d}x\,\mathrm{d}t \quad \forall\, p \in \left(-\tfrac{1}{\mu}, 0\right). \tag{14}$$

Let us observe that the second estimate is a lower bound on v because p is negative. Our contribution is to establish that the constant $c_1 = c_1(d)$ is given by

$$c_1 = 3^{\gamma-1} \left(2^{2\gamma^2+7(\gamma-1)}\, \gamma^{(\gamma+1)(2\gamma-1)}\, d^{(\gamma+1)(\gamma-1)}\, \mathcal{K}^{\gamma-1}\right)^{\frac{\gamma}{(\gamma-1)^2}}\,, \tag{15}$$

where $\gamma = (d+2)/d$ if $d \ge 3$, $\gamma = 5/3$ if $d = 1$ or 2, and $\mathcal{K}$ is the constant of (8).

Proof of Lemma 2. We first notice that it is sufficient to prove the lemma for $\varrho = 1/2$ and $r = 1$. By *admissible transformations*, as they are called in Moser's papers [25, 26], we can change variables according to

$$t \rightsquigarrow \alpha^2\, t + t_0 \quad \text{and} \quad x \rightsquigarrow \alpha\, x + x_0 \tag{16}$$

without changing the class of equations: λ_0 and λ_1 are invariant under (16). Therefore it is sufficient to prove

$$\sup_{Q_{\theta/2}} v^p \le \frac{c_1}{\theta^{d+2}} \iint_{Q_\theta} v^p \,\mathrm{d}x\,\mathrm{d}t \quad \forall\,\theta > 0\,.$$

We recover (13) by setting $\theta = r - \varrho$ and applying the above inequality to all cylinders in Q_r obtained by translation from Q_θ with admissible transformations. The centers of the corresponding cylinders certainly cover Q_ϱ and (13) follows. Analogously, one reduces (14) to the case $\varrho = 1/2$ and $r = 1$.

Step 1. Energy estimates. By definition of weak solutions, we have

$$\iint_{Q_1} \left(-\varphi_t\, v + (\nabla\varphi)^T A\,\nabla v\right) \mathrm{d}x\,\mathrm{d}t = 0 \tag{17}$$

for any test function φ which is compactly supported in $B_1 = \{x \in \mathbb{R}^d : |x| < 1\}$, for any fixed t. For any $p \in \mathbb{R} \setminus \{0, 1\}$, we define

$$w = v^{p/2} \quad \text{and} \quad \varphi = v^{p-1}\,\psi^2\,,$$

where ψ is a C^∞ function which, like φ, has compact support in B_1 for fixed t. We rewrite (17) in terms of w and ψ as

$$\tfrac{1}{4} \iint \psi^2\, \partial_t w^2 \,\mathrm{d}x\,\mathrm{d}t + \tfrac{p-1}{p} \iint \psi^2\, (\nabla w)^T A\,\nabla w \,\mathrm{d}x\,\mathrm{d}t = -\iint \psi\, w\, (\nabla\psi)^T A\,\nabla w \,\mathrm{d}x\,\mathrm{d}t \tag{18}$$

where we may integrate over a slice $t_1 < t < t_2$ of Q_1. From here on we adopt the convention that the integration domain is not specified whenever we integrate compactly supported functions on $\mathbb{R}^d$ or on $\mathbb{R} \times \mathbb{R}^d$. Setting $p \neq 1$,

$$\varepsilon = \tfrac{1}{2} \left|1 - \tfrac{1}{p}\right|$$

and recalling that

$$\psi\, w\, (\nabla\psi)^T A\,\nabla w \le \frac{1}{4\,\varepsilon}\, w^2 (\nabla\psi)^T A\,\nabla\psi + \varepsilon\,\psi^2\, (\nabla w)^T A\,\nabla w\,,$$

we deduce from (18) that

$$\pm\,\frac{1}{4} \iint \partial_t \left(\psi^2\, w^2\right) \mathrm{d}x\,\mathrm{d}t + \varepsilon \iint \psi^2\, (\nabla w)^T A\,\nabla w \,\mathrm{d}x\,\mathrm{d}t \\ \le \frac{1}{4} \iint \left(\frac{1}{\varepsilon}\, (\nabla\psi)^T A\,\nabla\psi + 2\,|\psi\,\psi_t|\right) w^2 \,\mathrm{d}x\,\mathrm{d}t\,, \tag{19}$$

where the plus sign in front of the first integral corresponds to the case $1/p < 1$, while the minus sign corresponds to $1/p > 1$. Recall that p can take negative values. Using the ellipticity condition (2) and (18), we deduce

$$\pm\,\frac{1}{4}\iint \partial_t\left(\psi^2\,w^2\right)\,\mathrm{d}x\,\mathrm{d}t+\lambda_0\,\varepsilon\iint \psi^2\,|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{4}\iint\left(\frac{\lambda_1}{\varepsilon}\,|\nabla\psi|^2+2\,|\psi\,\psi_t|\right)w^2\,\mathrm{d}x\,\mathrm{d}t\,. \tag{20}$$

By choosing a suitable test function ψ, compactly supported in Q_1, and such that

$$\|\nabla\psi\|_{\mathrm{L}^\infty(Q_1)}\le\frac{2}{r-\varrho}\quad\text{and}\quad\|\psi_t\|_{\mathrm{L}^\infty(Q_1)}\le\frac{4}{r-\varrho}$$

(see Lemma 15 in Appendix A.3), we have

$$\begin{aligned}\frac{1}{4}\iint\left(\frac{\lambda_1}{\varepsilon}\,|\nabla\psi|^2+2\,|\psi\,\psi_t|\right)w^2\,\mathrm{d}x\,\mathrm{d}t &\le\left(\frac{\lambda_1}{\varepsilon}\,\frac{1}{(r-\varrho)^2}+\frac{1}{r-\varrho}\right)\iint_{\mathrm{supp}(\psi)}w^2\,\mathrm{d}x\,\mathrm{d}t\\ &\le\frac{1}{(r-\varrho)^2}\left(\frac{\lambda_1}{\varepsilon}+1\right)\iint_{\mathrm{supp}(\psi)}w^2\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{21}$$

for any r and ϱ such that $0<\varrho<r\le 1$. If $1/p>1$, let us take $\tilde t\in(-\varrho^2,\varrho^2)$ to be such that

$$\int_{B_\varrho}w^2(\tilde t,x)\,\mathrm{d}x\ge\frac{1}{4}\sup_{0<|t|<\varrho^2}\int_{B_\varrho}w^2(t,x)\,\mathrm{d}x$$

and choose ψ such that $\psi(0,x)=1$ on Q_ϱ and $\psi(0,x)=0$ outside Q_r, so that

$$\begin{aligned}\sup_{0<|t|<\varrho^2}\int_{B_\varrho}w^2(t,x)\,\mathrm{d}x &\le 4\int_{B_\varrho}w^2(\tilde t,x)\,\mathrm{d}x\\ &\le 4\int_{B_r}w^2(\tilde t,x)\,\psi^2(\tilde t,x)\,\mathrm{d}x\le 4\iint_{Q_r}\partial_t\left(\psi^2\,w^2\right)\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned} \tag{22}$$

The same holds true if we replace Q_r by Q_r^+ and $0<|t|<\varrho^2$ by $0<t<\varrho^2$.

If $1/p<1$ (which includes the case $p<0$), similar arguments yield

$$\sup_{-\varrho^2<t<0}\int_{B_\varrho}w^2(t,x)\,\mathrm{d}x\le 4\iint_{Q_r^-}\partial_t\left(\psi^2\,w^2\right)\,\mathrm{d}x\,\mathrm{d}t\,. \tag{23}$$

Step 2. Sobolev's inequality. For any $f\in\mathrm{H}^1(Q_R)$, we have

$$\iint_{Q_R}f^{2\,\gamma}\,\mathrm{d}x\,\mathrm{d}t\le\,2\,\pi^2\,\mathcal{K}\left[\frac{1}{R^2}\iint_{Q_R}f^2\,\mathrm{d}x\,\mathrm{d}t+\iint_{Q_R}\left|\nabla f\right|^2\mathrm{d}x\,\mathrm{d}t\right] \times\sup_{|s|\in(0,\varrho^2)}\left[\int_{B_R}f^2(s,x)\,\mathrm{d}x\right]^{\frac{2}{d}} \tag{24}$$

with $\gamma = 1 + 2/d$ if $d \geq 3$. If $d = 1$ or 2, we rely on (7), take $\gamma = 5/3$, use Hölder's inequality with $2\,\gamma = 10/3 < 4$ and $\mathsf{p} \geq 4$ if $d = 2$, $\mathsf{p} > 4$ if $d = 1$. In order to fix ideas, we take $\mathsf{p} = 4$ if $d = 2$ and $\mathsf{p} = 8$ if $d = 1$. Hence

$$\iint_{Q_R} f^{2\,\gamma}\,\mathrm{d}x\,\mathrm{d}t \leq |Q_1|^{1-\frac{2\,\gamma}{\mathsf{p}}} \left(\iint_{Q_R} f^{\mathsf{p}}\,\mathrm{d}x\,\mathrm{d}t\right)^{1-\frac{2\,\gamma}{\mathsf{p}}} .$$

According to (10), we know that $|Q_1| = |(-1,1)|\,|B_1| \leq 2\,\pi^2$ in any dimension.

Step 3. The case $p > 0$ and $p \neq 1$. Assume that $1/2 \leq \varrho < r \leq 1$. We work in the cylinder $Q_r = \mathrm{supp}(\psi)$. Here, we choose $\psi(t,x) = \varphi_{\rho,r}(|x|)\,\varphi_{\rho^2,r^2}(|t|)$ where $\varphi_{\rho,r}$ and φ_{ρ^2,r^2} are defined in Appendix A.3, so that $\psi = 1$ on Q_ϱ and $\psi = 0$ outside Q_r.

Collecting inequalities (20), (21) and (22), we obtain

$$\sup_{0<|t|<\varrho^2} \int_{B_\varrho} w^2(t,x)\,\mathrm{d}x + \lambda_0\,\varepsilon \iint_{Q_\varrho} |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t \leq \frac{1}{(r-\varrho)^2}\left(\frac{\lambda_1}{\varepsilon}+1\right)\iint_{Q_r} w^2\,\mathrm{d}x\,\mathrm{d}t\,.$$

Now apply (24) to $f = w$ and use the above estimates to get

$$\begin{aligned}
&\iint_{Q_\varrho} w^{2\,\gamma}\,\mathrm{d}x\,\mathrm{d}t \\
&\leq 2\,\pi^2\,\mathcal{K}\left[\frac{1}{\varrho^2}\iint_{Q_\varrho} w^2\,\mathrm{d}x\,\mathrm{d}t + \iint_{Q_\varrho}\left|\nabla w\right|^2\mathrm{d}x\,\mathrm{d}t\right] \sup_{|s|\in(0,\varrho^2)}\left(\int_{B_\varrho} w^2(s,x)\,\mathrm{d}x\right)^{\frac{2}{d}} \\
&\leq 2\,\pi^2\,\mathcal{K}\left[\frac{1}{\varrho^2}\iint_{Q_\varrho} w^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{(r-\varrho)^2\,\lambda_0\,\varepsilon}\left(\tfrac{\lambda_1}{\varepsilon}+1\right)\iint_{Q_r} w^2\,\mathrm{d}x\,\mathrm{d}t\right] \\
&\qquad\qquad \times\left(\frac{1}{(r-\varrho)^2}\left(\tfrac{\lambda_1}{\varepsilon}+1\right)\iint_{Q_r} w^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{d}} \\
&\leq 2\,\pi^2\,\mathcal{K}\left[\frac{1}{\varrho^2} + \frac{1}{(r-\varrho)^2\,\lambda_0\,\varepsilon}\left(\tfrac{\lambda_1}{\varepsilon}+1\right)\right]\left[\frac{1}{(r-\varrho)^2}\left(\tfrac{\lambda_1}{\varepsilon}+1\right)\right]^{\frac{2}{d}}\left(\iint_{Q_r} w^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2}{d}+1} \\
&\qquad\qquad := A(d,\varrho,r,\lambda_0,\lambda_1,\varepsilon,2\,\pi^2\,\mathcal{K})\left(\iint_{Q_r} w^2\,\mathrm{d}x\,\mathrm{d}t\right)^{\gamma} .
\end{aligned}$$

Using the fact that $\mu = \lambda_1 + 1/\lambda_0 > 1$ and $1/2 \leq \varrho < r \leq 1$, we can estimate the constant A as follows:

$$\begin{aligned}
A &\leq 2\,\pi^2\,\mathcal{K}\left[\frac{1}{\varrho^2} + \frac{1}{(r-\varrho)^2\,\lambda_0\,\varepsilon}\left(\tfrac{\lambda_1}{\varepsilon}+1\right)\right]\left(\frac{1}{(r-\varrho)^2}\left(\tfrac{\lambda_1}{\varepsilon}+1\right)\right)^{\frac{2}{d}} \\
&\leq \frac{2\,\pi^2\,\mathcal{K}}{(r-\varrho)^{2\,\gamma}}\left(\tfrac{1}{2}+\tfrac{\lambda_1}{\varepsilon^2\,\lambda_0}\right)\left(\tfrac{\lambda_1}{\varepsilon}\right)^{\frac{2}{d}} \\
&\leq \frac{2\,\pi^2\,\mathcal{K}}{(r-\varrho)^{2\,\gamma}}\left(1+\tfrac{\mu^2}{\varepsilon^2}\right)\left(\tfrac{\mu}{\varepsilon}\right)^{\frac{2}{d}} \leq \frac{2^5\,\mathcal{K}}{(r-\varrho)^{2\,\gamma}}\left(1+\tfrac{\mu}{\varepsilon}\right)^{\gamma+1}
\end{aligned}$$

where we have used that $\lambda_1/\lambda_0 \leq \frac{1}{2}(\lambda_1^2 + 1/\lambda_0^2) \leq \frac{1}{2}(\lambda_1 + 1/\lambda_0)^2 = \mu^2$ and $\pi \leq 4$.

First iteration step. Recall that $w=v^{p/2}$, $\varepsilon=\frac{1}{2}\left|1-\frac{1}{p}\right|$, and $\gamma=1+\frac{2}{d}$ if $d\geq 3$, $\gamma=5/3$ if $d=1$ or 2, $\mu=\lambda_1+1/\lambda_0>1$ and $1/2\leq\varrho<r\leq 1$. We can summarize these results by

$$\left(\iint_{Q_\varrho} v^{\gamma\,p}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{\gamma\,p}}\leq\left(\frac{(2^5\,\mathcal{K})^{\frac{1}{\gamma}}}{(r-\varrho)^2}\right)^{\frac{1}{p}}\left(1+\tfrac{\mu}{\varepsilon}\right)^{\frac{\gamma+1}{\gamma\,p}}\left(\iint_{Q_r} v^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}$$

for any $p>0$ such that $p\neq 1$. For any $n\in\mathbb{N}$, let

$$\varrho_n=\frac{1}{2}\left(1-2^{-n}\right)\,,\quad p_n=\frac{\gamma+1}{2}\,\gamma^{n-n_0}=p_0\,\gamma^n\,,\quad \varepsilon_n=\frac{1}{2}\left|1-\frac{1}{p_n}\right|$$

for some fixed $n_0\in\mathbb{N}$. Note that $\varrho_0=1$, $p_0=\frac{1+\gamma}{2\,\gamma^{n_0}}$, ϱ_n monotonically decrease to $1/2$, and p_n monotonically increase to ∞. We observe that for all n, $n_0\in\mathbb{N}$, we have $p_n\neq 1$ and, as a consequence, $\varepsilon_n>0$. Indeed, if $d\geq 3$, $p_n=1$ would mean that

$$n_0-n=\frac{\log\left(\frac{1+\gamma}{2}\right)}{\log\gamma}=\frac{\log\left(1+\frac{1}{d}\right)}{\log\left(1+\frac{2}{d}\right)}$$

and, as a consequence, $0<n_0-n\leq\log(4/3)/\log(5/3)<1$, a contradiction with the fact that n and n_0 are integers. The same argument holds if $d=1$ or $d=2$ with $n_0-n=\log(4/3)/\log(5/3)$, as $\gamma=5/3$ corresponds to the value of γ for $d=1$, 2 or 3. It is easy to check that for any $n\geq 0$,

$$|p_n-1|\geq\min\{p_{n_0}-1,1-p_{n_0-1}\}=\min\left\{\tfrac{1}{d},\tfrac{1}{d+2}\right\}=\tfrac{1}{d+2}\,.$$

For an arbitrary $p\in(0,1/\mu)$, we choose

$$n_0=\text{i.p.}\left(\frac{\log\left(\frac{1+\gamma}{2\,p}\right)}{\log\gamma}\right)+1$$

where i.p. denotes the integer part, so that $0<p_0\leq p<\gamma\,p_0$. By monotonicity of the L^q norms, that is,

$$\left(\iint_{Q_r} v^{p_0}\,\frac{\mathrm{d}x\,\mathrm{d}t}{|Q_r|}\right)^{\frac{1}{p_0}}\leq\left(\iint_{Q_r} v^{p}\,\frac{\mathrm{d}x\,\mathrm{d}t}{|Q_r|}\right)^{\frac{1}{p}}\leq\left(\iint_{Q_r} v^{\gamma\,p_0}\,\frac{\mathrm{d}x\,\mathrm{d}t}{|Q_r|}\right)^{\frac{1}{\gamma\,p_0}}\,,$$

it is sufficient to prove inequality (13) for $p=p_0$.

Let us define $p_\mu\in(p_0\,\mu,1]$ such that

$$1+\frac{\mu}{\varepsilon_n}=1+\frac{2\,\mu\,p_n}{|p_n-1|}=1+\frac{2\,\mu\,p_0\,\gamma^n}{|p_n-1|}\leq 1+2\,(d+2)\,\gamma^n\leq 4\,(d+2)\,\gamma^n=4\,d\,\gamma^{n+1}\quad(25)$$

because $d+2=d\,\gamma$ if $d\geq 3$ and $\gamma=5/3$ if $d\leq 3$. Finally, let us define

$$Y_n:=\left(\iint_{Q_{\varrho_n}} v^{p_n}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_n}}\,,\ I_0=(2^5\,\mathcal{K})^{\frac{1}{\gamma}}(4\,d\,\gamma^2)^{\frac{\gamma+1}{\gamma}}$$

and $C = 4\,\gamma^{\frac{\gamma+1}{\gamma}}$, $\theta = \frac{1}{\gamma} \in (0,1)$, and $\xi = \frac{1}{p_0}$.

Iteration. Summing up, we have the following iterative inequality

$$Y_n \le \left(\frac{(2^5\,\mathcal{K})^{\frac{1}{\gamma}}}{(\varrho_{n-1}-\varrho_n)^2} \left(1+\tfrac{\mu}{\varepsilon_n}\right)^{\frac{\gamma+1}{\gamma}} \right)^{\frac{1}{p_{n-1}}} Y_{n-1}\,.$$

Using $\varrho_{n-1}-\varrho_n = 2^{-n}$ and inequality (25), we obtain

$$Y_n \le I_{n-1}^{\xi\,\theta^{n-1}}\,Y_{n-1} \quad \text{with} \quad I_{n-1} \le I_0\,C^{n-1}\,. \tag{26}$$

Lemma 3 (See [10]). *The sequence* $(Y_n)_{n\in\mathbb{N}}$ *is a bounded sequence such that*

$$Y_\infty := \limsup_{n\to+\infty} Y_n \le I_0^{\frac{\xi}{1-\theta}}\,C^{\frac{\xi\,\theta}{(1-\theta)^2}}\,Y_0\,.$$

The proof follows from the observation that

$$\begin{aligned} Y_n \le I_{n-1}^{\xi\,\theta^{n-1}} Y_{n-1} \le \left(I_0\,C^{n-1}\right)^{\xi\,\theta^{n-1}} Y_{n-1} = I_0^{\xi\,\theta^{n-1}}\,C^{\xi\,(n-1)\,\theta^{n-1}}\,Y_{n-1} \\ \le \prod_{j=0}^{n-1} I_0^{\xi\,\theta^j} C^{\xi\,j\,\theta^j}\,Y_0 = I_0^{\xi\sum_{j=0}^{n-1}\theta^j}\,C^{\xi\sum_{j=0}^{n-1} j\,\theta^j}\,Y_0\,. \end{aligned}$$

With the estimates

$$\left(\iint_{Q_1} v^{p_0}\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p_0}} \le |Q_1|^{\frac{1}{p_0}-\frac{1}{p}} \left(\iint_{Q_1} v^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}\,,$$

$\frac{1}{p_0} - \frac{1}{p} \le \frac{\gamma-1}{p}$ and $|Q_1| = 2\,|B_1| \le 2\,\pi^2$, we obtain

$$\sup_{Q_{1/2}} v \le \left(2^5\,\mathcal{K}\,(4\,d\,\gamma^2)^{\gamma+1}\right)^{\frac{1}{p}\,\frac{\gamma}{\gamma-1}} \left(4^\gamma\,\gamma^{\gamma+1}\right)^{\frac{1}{p}\,\frac{\gamma}{(\gamma-1)^2}} \left(2\,\pi^2\right)^{\frac{\gamma-1}{p}} \left(\iint_{Q_1} v^p\,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{1}{p}}$$

which, using $2\,\pi^2 \le 24$ and after raising to the power p, is (13) with c_1 given by (15).

Step 4. The case $p<0$. Assume that $1/2 \le \varrho < r \le 1$. We work in the cylinder $Q_r^- = \mathrm{supp}(\psi)$. Here, we choose $\phi(t,x) = \varphi_{\rho,r}(|x|)\,\varphi_{\rho^2,r^2}(-t)$, where φ is defined as in Appendix A.3, so that $\psi = 1$ on Q_ϱ^- and $\psi = 0$ outside Q_r^-.

After collecting (20), (21) and (23), we obtain

$$\sup_{-\varrho^2<t<0} \int_{B_\varrho} w^2(t,x) + \lambda_0\,\varepsilon \iint_{Q_\varrho^-} |\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t \le \frac{1}{(r-\varrho)^2}\left(\tfrac{\lambda_1}{\varepsilon}+1\right) \iint_{Q_r^-} w^2\,\mathrm{d}x\,\mathrm{d}t\,.$$

Then the proof follows exactly the same scheme as for $p>0$, with the simplification that we do not have to take extra precautions in the choice of p. The constant c_1 is the same. □

2 Logarithmic Estimates

We prove now fine level set estimates on the solutions by Caccioppoli-type energy estimates. These estimates are based on a weighted Poincaré inequality (see Step 2 of the proof of Lemma 4) originally due to F. John, as explained by J. Moser in [25]). This is a fundamental step for this approach and for the more standard approach based on BMO and John-Nirenberg estimates. The level set estimates are better understood in terms of

$$w = -\log v\,,$$

the *logarithm of* v, solution to (1), which satisfies the nonlinear equation

$$w_t = -\frac{v_t}{v} = \sum_{i,j=1}^{d} \partial_i\Big(A_{i,j}(t,x)\,\partial_j(-\log v)\Big) - \sum_{i,j=1}^{d} (\partial_i \log v)\,A_{i,j}(t,x)\,(\partial_j \log v)\,,$$

i.e.,

$$w_t = \nabla\cdot(A\,\nabla w) - (\nabla w)^T\,A\,\nabla w\,. \tag{27}$$

All computations can be justified by computing with $-\log(\delta+v)$ for an arbitrarily small $\delta>0$ and passing to the limit as $\delta\to 0_+$. We recall that $\mu=\lambda_1+1/\lambda_0$. Let us choose a test function ψ as follows:

$$\psi(x) := \prod_{\nu=1}^{d}\chi_\nu(x_\nu),\quad \text{where}\quad \chi_\nu(z) := \begin{cases} 1 & \text{if } |z|\le 1\\ 2-|z| & \text{if } 1\le|z|\le 2\\ 0 & \text{if } |z|\ge 2\end{cases}\,. \tag{28}$$

Note that this test function has convex super-level sets, or equivalently said, on any straight line segment, $\psi(x)$ assumes its minimum at an end point.

Even if (27) is a nonlinear equation, the nonlinear term actually helps. The reason for that lies in the following result.

Lemma 4. *Assume that ψ is a smooth compactly supported test function as in (28). If w is a (sub)solution to (27) in*

$$\Big\{(t,x)\in\mathbb{R}\times\mathbb{R}^d\,:\,|t|<1\,,\;|x|<2\Big\} = (-1,1)\times B_2(0)\,,$$

then there exist positive constants a and $c_2(d)$ such that, for all $s>0$,

$$\Big|\Big\{(t,x)\in Q_1^+\,:\,w(t,x)>s-a\Big\}\Big| + \Big|\Big\{(t,x)\in Q_1^-\,:\,w(t,x)<-s-a\Big\}\Big| \le c_2\,|B_1|\,\frac{\mu}{s}\,, \tag{29}$$

where

$$c_2 = 2^{d+2}\,3^d\,d \quad \textit{and} \quad a = -\frac{\int w(0,x)\,\psi^2(x)\,\mathrm{d}x}{\int \psi^2(x)\,\mathrm{d}x}\,. \tag{30}$$

Equivalently, the above inequality stated in terms of v reads

$$\left|\left\{(t,x)\in Q_1^+\,:\,\log v(t,x)<-s+a\right\}\right| \\ +\left|\left\{(t,x)\in Q_1^-\,:\,\log v(t,x)>s+a\right\}\right|\le c_2\,|B_1|\,\frac{\mu}{s}\,, \quad (31)$$

where $a=\int\log v(0,x)\,\psi^2(x)\,\mathrm{d}x/\int\psi^2(x)\,\mathrm{d}x$.

Proof. We follow the proof of Lemma 2 of [26], which in turn refers to [25, p. 121-123]. We provide some minor improvements and quantify all constants. For better readability, we split the proof into several steps.

Step 1. Energy estimates. Testing equation (or inequality) (27) with $\psi^2(x)$, we obtain

$$\int\psi^2\,w(t_2)\,\mathrm{d}x-\int\psi^2\,w(t_1)\,\mathrm{d}x+\frac{1}{2}\iint\psi^2\,(\nabla w)^T\,A\,\nabla w\,\mathrm{d}x\,\mathrm{d}t \\ \le 2\iint(\nabla\psi)^T\,A\,\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\,. \quad (32)$$

Using the conditions (2), we have that

$$\lambda_0\iint\psi^2\,|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t\le\iint\psi^2\,(\nabla w)^T\,A\,\nabla w\,\mathrm{d}x\,\mathrm{d}t\,,$$
$$\iint(\nabla\psi)^T\,A\,\nabla\psi\,\mathrm{d}x\,\mathrm{d}t\le\lambda_1\iint|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t\,.$$

Combining the above two inequalities, we obtain

$$\int\psi^2\,w(t_2)\,\mathrm{d}x-\int\psi^2\,w(t_1)+\frac{\lambda_0}{2}\iint\psi^2\,|\nabla w|^2\,\mathrm{d}x\,\mathrm{d}t \\ \le 2\,\lambda_1\iint|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t\le 2^d\,\lambda_1\,(t_2-t_1)\,|B_1|\,\|\nabla\psi\|^2_{\mathrm{L}^\infty}\,. \quad (33)$$

Step 2. Weighted Poincaré inequalities. Let $b\ge 0$ be a continuous function with support of diameter $D=\mathrm{diam}(\mathrm{supp}(\psi))$ such that the domains $\{x\in\mathbb{R}^d\,:\,b(x)\ge \mathrm{const}\}$ are convex. Then for any function $f\in\mathrm{L}^2_b$ with $|\nabla f|\in\mathrm{L}^2_b$, we have that

$$\int\left|f(x)-\overline{f}_b\right|^2 b(x)\,\mathrm{d}x\le\lambda_b\,D^2\int|\nabla f(x)|^2 b(x)\,\mathrm{d}x$$

where

$$\lambda_b=\frac{|\mathrm{supp}(b)|\,\|b\|_{\mathrm{L}^\infty}}{2\int b(x)\,\mathrm{d}x}\quad\text{and}\quad\overline{f}_b=\frac{\int f(x)\,b(x)\,\mathrm{d}x}{\int b(x)\,\mathrm{d}x}\,.$$

The proof follows from the unweighted Poincaré inequality: see for instance [25, Lemma 3].

Poincaré inequality with weight ψ^2. We have that $D = 2\,d$ in the particular case of $b = \psi^2$, where ψ is given in (28) and such that $0 \le \psi \le 1$, as for the constant λ_b, we have

$$\lambda_b \le \frac{|B_2|}{2\int_{B_1} \psi^2\,\mathrm{d}x} = \frac{|B_2|}{2\,|B_1|} = 2^{d-1}\,.$$

Since $\|b\|_{\mathrm{L}^\infty} = \|\psi^2\|_{\mathrm{L}^\infty} = 1$, $|B_1| \le \int \psi^2\,\mathrm{d}x \le 3^d\,|B_1|$, we obtain

$$\iint \left|w(t,x) - \overline{w(t)}_\psi\right|^2 \psi^2(x)\,\mathrm{d}x\,\mathrm{d}t \le 2^d\,d \iint |\nabla w(t,x)|^2\,\psi^2(x)\,\mathrm{d}x\,\mathrm{d}t\,, \tag{34}$$

with

$$\overline{w(t)}_\psi := \frac{\int w(t,x)\,\psi^2(x)\,\mathrm{d}x}{\int \psi^2(x)\,\mathrm{d}x}\,.$$

Step 3. Differential inequality. Let us recall that $\|\nabla\psi\|^2_{\mathrm{L}^\infty} \le 1$. We combine inequalities (33) and (34) into

$$\int \psi^2\,w(t_2)\,\mathrm{d}x - \int \psi^2\,w(t_1) + \frac{\lambda_0}{2^{d+1}\,d}\int_{t_1}^{t_2}\int \left|w(t,x) - \overline{w(t)}_\psi\right|^2 \psi^2(x)\,\mathrm{d}x\,\mathrm{d}t \\ \le 2^d\,\lambda_1\,(t_2 - t_1)\,|B_1|\,.$$

Recalling that $\psi = 1$ on B_1 and the expression of $\overline{w(t)}_\psi$ given in (34), we obtain

$$\frac{\overline{w(t_2)}_\psi - \overline{w(t_1)}_\psi}{t_2 - t_1} + \frac{\lambda_0}{2^{d+1}\,3^d\,d}\frac{1}{(t_2 - t_1)\,|B_1|}\int_{t_1}^{t_2}\int_{B_1}\left|w(t,x) - \overline{w(t)}_\psi\right|^2 \mathrm{d}x\,\mathrm{d}t \\ \le \frac{2^d\,\lambda_1\,|B_1|}{\int \psi^2\,\mathrm{d}x} \le 2^d\,\lambda_1\,.$$

Here we have used that $|B_1| \le \int \psi^2\,\mathrm{d}x \le 3^d\,|B_1|$. Recalling that $\mu = \lambda_1 + 1/\lambda_0$, so that $\lambda_0\,\mu > 1$, we obtain

$$\frac{\overline{w(t_2)}_\psi - \overline{w(t_1)}_\psi}{t_2 - t_1} + \frac{1}{2^{d+1}\,3^d\,d\,\mu}\frac{1}{(t_2 - t_1)\,|B_1|}\int_{t_1}^{t_2}\int_{B_1}\left|w(t,x) - \overline{w(t)}_\psi\right|^2 \mathrm{d}x\,\mathrm{d}t \\ \le \frac{2^d\,\lambda_1\,|B_1|}{\int \psi^2\,\mathrm{d}x} \le 2^d\,\mu\,.$$

Letting $t_2 \to t_1$ we obtain the following differential inequality for $\overline{w(t)}_\psi$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{w(t)}_\psi + \frac{1}{2^{d+1}\,3^d\,d\,\mu}\frac{1}{|B_1|}\int_{B_1}\left|w(t,x) - \overline{w(t)}_\psi\right|^2 \mathrm{d}x \le 2^d\,\mu\,. \tag{35}$$

The above inequality can be applied to

$$\underline{w}(t,x) = w(t,x) - \overline{w(0)}_\psi - 2^d\,\mu\,t\,.$$

Notice that $\underline{w}$ is a subsolution to (27) since w is. With $a = -\overline{w(0)}_\psi$, we can write (35) in terms of

$$W(t) = \overline{w(t)}_\psi + a - 2^d\,\mu\,t \quad \text{such that} \quad W(0) = 0$$

as

$$\frac{\mathrm{d}}{\mathrm{d}t}W(t) + \frac{1}{2^{d+1}\,3^d\,d\,\mu}\frac{1}{|B_1|}\int_{B_1}|\underline{w}(t,x) - W(t)|^2\,\mathrm{d}x \le 0\,.$$

An immediate consequence of the above inequality is that $W(t) \le W(0) = 0$ for all $t \in (0,1)$.

Let $Q_s(t) = \{x \in B_1\,:\,w(t,x) > s\}$, for a given $t \in (0,1)$. For any $s > 0$, we have

$$\underline{w}(t,x) - W(t) \ge s - W(t) \ge 0 \quad \forall\,x \in Q_s(t)\,,$$

because $W(t) \le 0$ for $t \in (0,1)$. Using $\frac{\mathrm{d}}{\mathrm{d}t}W = -\frac{\mathrm{d}}{\mathrm{d}t}(s - W)$, the integration restricted to Q_s in (35) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(s - W(t)\Big) \ge \frac{1}{2^{d+1}\,3^d\,d\,\mu}\frac{|B_s(t)|}{|B_1|}\Big(s - W(t)\Big)^2\,.$$

By integrating over $(0,1)$, it follows that

$$\begin{aligned}\Big|\Big\{(t,x) \in Q_1^+\,:\,\underline{w}(t,x) > s\Big\}\Big| &= \iint_{\{\underline{w}>s\}\cap Q_1^+}\mathrm{d}x\,\mathrm{d}t = \int_0^1|Q_s(t)|\,\mathrm{d}t\\ &\le 2^{d+1}\,3^d\,d\,\mu\,|B_1|\left(\frac{1}{s - W(0)} - \frac{1}{s - W(1)}\right) \le 2^{d+1}\,3^d\,d\,|B_1|\,\frac{\mu}{s}\,,\end{aligned}$$

which proves the first part of inequality (29).

Step 4. Estimating the second term of inequality (29). We just replace t by $-t$ and repeat the same proof. Upon setting $a = -\overline{w(0)}_\psi$, we obtain

$$\Big|\Big\{(t,x) \in Q_1^-\,:\,w < -s - a\Big\}\Big| \le 2^{d+1}\,3^d\,d\,|B_1|\,\frac{\mu}{s}\,.$$

□

3 A lemma by Bombieri and Giusti

To avoid the direct use of BMO spaces (whose embeddings and inequalities, like the celebrated John-Nirenberg inequality, may not have explicit constants), we use the parabolic version, due to J. Moser, of a Lemma attributed to E. Bombieri and E. Giusti, in the elliptic setting: see [6]. We use the version of [26, Lemma 3], which applies to measurable functions f, not necessarily solutions to a PDE, and to any family of domains $(\Omega_r)_{0<r<R}$ such that $\Omega_r \subset \Omega_R$.

Lemma 5 (Bombieri-Giusti [7], Moser [26]). *Let β, c_1, $\mu > 0$, $c_2 \geq 1/\mathrm{e}$, $\theta \in [1/2, 1)$ and $p \in (0, 1/\mu)$ be positive constants, and let $f > 0$ be a positive measurable function defined on a neighborhood of Ω_1 for which*

$$\sup_{\Omega_\varrho} f^p < \frac{c_1}{(r-\varrho)^\beta \, |\Omega_1|} \iint_{\Omega_r} f^p \, \mathrm{d}x \, \mathrm{d}t \tag{36}$$

for any r and ϱ such that $\theta \leq \varrho < r \leq 1$, and

$$\left| \left\{ (t,x) \in \Omega_1 \, : \, \log f > s \right\} \right| < c_2 \, |\Omega_1| \, \frac{\mu}{s} \quad \forall \, s > 0 \, . \tag{37}$$

Let σ be as in (6). Then we have

$$\sup_{\Omega_\theta} f < \kappa_0^\mu \, , \quad \textit{where} \quad \kappa_0 := \exp\left[2 \, c_2 \vee \frac{8 \, c_1^3}{(1-\theta)^{2\,\beta}} \right] . \tag{38}$$

The difference between the upper bounds (36) and (38) is subtle. The first inequality depends on the solution on the whole space-time set Ω_r and is somehow implicit. By assumption (37), if the set where f is super-exponential has small measure, then on a slightly smaller set the solution is quantitatively bounded by an explicit and uniform constant, given by (38).

Proof of Lemma 5. We sketch the relevant steps of the proof of [26, Lemma 3]. Our goal is to provide some minor technical improvements and quantify all constants. Without loss of generality, after replacing s by $s\,\mu$, we reduce the problem to the case $\mu = 1$. Analogously, we also assume that $|\Omega_1| = 1$. We define the nondecreasing function

$$\varphi(\varrho) = \sup_{\Omega_\varrho} (\log f) \quad \forall \, \varrho \in [\theta, 1) \, .$$

We will prove that assumptions (36) and (37) imply the following dichotomy:
– either $\varphi(r) \leq 2\, c_2$ and there is nothing to prove: $\kappa_0 = \mathrm{e}^{2\,c_2}$,
– or $\varphi(r) > 2\, c_2$ and we have

$$\varphi(\varrho) \leq \frac{3}{4} \, \varphi(r) + \frac{8 \, c_1^3}{(r-\varrho)^{2\,\beta}} \tag{39}$$

for any r and ϱ such that $\theta \leq \varrho < r \leq 1$. We postpone the proof of (39) and observe that (39) can be iterated along a monotone increasing sequence $(\varrho_k)_{k \geq 0}$ such that

$$\theta \leq \varrho_0 < \varrho_1 < \cdots < \varrho_k \leq 1$$

for any $k \in \mathbb{N}$ to get

$$\varphi(\varrho_0) < \frac{3}{4} \, \varphi(\varrho_k) + 8 \, c_1^3 \sum_{j=0}^{k-1} \left(\tfrac{3}{4} \right)^j \frac{1}{(\varrho_{j+1} - \varrho_j)^{2\,\beta}} \, .$$

By monotonicity, we have that $\varphi(\varrho_k) \le \varphi(1) < \infty$, so that in the limit as $k \to +\infty$, we obtain

$$\varphi(\theta) \le \varphi(\varrho_0) \le 8\, c_1^3 \sum_{j=0}^{\infty} \left(\tfrac{3}{4}\right)^j \frac{1}{\left(\varrho_{j+1} - \varrho_j\right)^{2\,\beta}}$$

provided the right-hand side converges. This convergence holds true for the choice

$$\varrho_j = 1 - \frac{1-\theta}{1+j}\,,$$

and in that case, the estimate

$$\varphi(\theta) \le \frac{8\, c_1^3\, \sigma}{(1-\theta)^{2\,\beta}} := \tilde{\kappa}_0$$

implies inequality (38) with $\mu = 1$ because

$$\sup_{\Omega_\theta} f \le \exp\left(\sup_{\Omega_\theta}(\log f)\right) = \mathrm{e}^{\varphi(\theta)} \le \mathrm{e}^{\tilde{\kappa}_0} := \kappa_0\,.$$

In order to complete the proof, we have to prove inequality (39).

Proof of Inequality (39). We are now under the assumption $\varphi(r) > 2\, c_2$. We first estimate the integral

$$\begin{aligned}\iint_{\Omega_r} f^p \,\mathrm{d}x\,\mathrm{d}t &= \iint_{\{\log f > \frac{1}{2}\varphi(r)\}} f^p \,\mathrm{d}x\,\mathrm{d}t + \iint_{\{\log f \le \frac{1}{2}\varphi(r)\}} f^p \,\mathrm{d}x\,\mathrm{d}t \\ &\le \mathrm{e}^{p\,\varphi(r)} \left|\left\{(t,x) \in \Omega_1 \,:\, \log f > \tfrac{1}{2}\,\varphi(r)\right\}\right| + |\Omega_1|\,\mathrm{e}^{\frac{p}{2}\,\varphi(r)} \\ &\le \frac{2\,c_2}{\varphi(r)}\,\mathrm{e}^{p\,\varphi(r)} + \mathrm{e}^{\frac{p}{2}\,\varphi(r)}\,,\end{aligned} \tag{40}$$

where we have estimated the first integral using that

$$\sup_{\Omega_r} f^p \le \sup_{\Omega_r} \mathrm{e}^{p \log f} \le \mathrm{e}^{p \sup_{\Omega_r} \log f} = \mathrm{e}^{p\,\varphi(r)}\,.$$

In the present case, assumption (37) reads:

$$\left|\left\{(t,x) \in \Omega_1 \,:\, \log f > \tfrac{1}{2}\,\varphi(r)\right\}\right| < \frac{2\,c_2}{\varphi(r)}.$$

We choose

$$p = \frac{2}{\varphi(r)} \log\left(\frac{\varphi(r)}{2\,c_2}\right)$$

such that the last two terms of (40) are equal, which gives

$$\iint_{\Omega_r} f^p \,\mathrm{d}x\,\mathrm{d}t \le 2\,\mathrm{e}^{\frac{p}{2}\,\varphi(r)}\,. \tag{41}$$

The exponent p is admissible, that is, $0 < p < 1/\mu = 1$, if $\varphi(r) > 2/\mathrm{e}$, which follows from the assumption $c_2 > 1/\mathrm{e}$. Now, using assumption (36) and inequality (41), we obtain

$$\begin{aligned}\varphi(\varrho) = \frac{1}{p}\sup_{\Omega_\varrho}\log(f^p) = \frac{1}{p}\log\left(\sup_{\Omega_\varrho} f^p\right)\\ \le \frac{1}{p}\log\left(\frac{c_1}{(r-\varrho)^\beta}\iint_{\Omega_r} f^p\,\mathrm{d}x\,\mathrm{d}t\right)\\ \le \frac{1}{p}\log\left(\frac{2\,c_1\,\mathrm{e}^{\frac{p}{2}\varphi(r)}}{(r-\varrho)^\beta}\right) = \frac{1}{p}\log\left(\frac{2\,c_1}{(r-\varrho)^\beta}\right) + \frac{1}{2}\varphi(r)\\ = \frac{1}{2}\varphi(r)\left(1 + \frac{\log(2\,c_1) - \log(r-\varrho)^\beta}{\log(\varphi(r)) - \log(2\,c_1)}\right)\\ \le \frac{1}{2}\varphi(r)\left(1+\frac{1}{2}\right) = \frac{3}{4}\varphi(r)\,.\end{aligned}$$

In the last line, we take

$$\varphi(r) \ge \frac{8\,c_1^3}{(r-\varrho)^{2\,\beta}}$$

so that

$$\frac{\log(2\,c_1) - \log(r-\varrho)^\beta}{\log(\varphi(r)) - \log(2\,c_1)} \le \frac{1}{2}\,. \tag{42}$$

We again have that either $\varphi(r) < \frac{8\,c_1^3}{(r-\varrho)^{2\,\beta}}$ and (39) holds, or $\varphi(r) \ge \frac{8\,c_1^3}{(r-\varrho)^{2\,\beta}}$ and (42) holds, hence $\varphi(\varrho) \le \frac{3}{4}\varphi(r)$. We conclude that (39) holds in all cases and this completes the proof. □

4 Proof of Moser's Harnack inequality

Proof of Theorem 1. We prove the Harnack inequality

$$\sup_{D^-} v \le \mathsf{h}^\mu \inf_{D^+} v \tag{43}$$

where h is as in (4) and $D^\pm$ are the parabolic cylinders given by

$$\begin{aligned} D &= \{|t| < 1\,,\ |x| < 2\} = (-1,1)\times B_2\,,\\ D^+ &= \left\{\tfrac{3}{4} < t < 1\,,\ |x| < \tfrac{1}{2}\right\} = \left(\tfrac{3}{4},1\right)\times B_{1/2}(0)\,,\\ D^- &= \left\{-\tfrac{3}{4} < t < -\tfrac{1}{4}\,,\ |x| < \tfrac{1}{2}\right\} = \left(-\tfrac{3}{4},-\tfrac{1}{4}\right)\times B_{1/2}(0)\,.\end{aligned}$$

The general inequality (12) follows by applying the admissible transformations corresponding to (16), which do not alter the values of λ_0, λ_1 and $\mu = \lambda_1 + 1/\lambda_0$.

Let v be a positive solution to (1) and $a \in \mathbb{R}$ to be fixed later. In order to use Lemma 2 and Lemma 4, we apply Lemma 5 to

$$v_+(t,x) = \mathrm{e}^{-a}\, v(t,x) \quad \text{and} \quad v_-(t,x) = \frac{\mathrm{e}^{+a}}{v(t,x)}\,.$$

Step 1. Upper estimates. Let us prove that

$$\sup_{D^-} v_+ \le \overline{\kappa}_0^{\,\mu} \tag{44}$$

where $\overline{\kappa}_0$ has an explicit expression, given below in (46). For all $\varrho \in [1/2, 1)$, let

$$\begin{aligned}\Omega_\varrho &:= \left\{(t,x) \in \Omega_1 \,:\, \left|t + \tfrac12\right| < \tfrac12\,\varrho^2\,,\ |x| < \varrho/\sqrt{2}\right\}\\ &= \left(-\tfrac12\,(\varrho^2+1), \tfrac12\,(\varrho^2-1)\right) \times B_{\varrho/\sqrt{2}}(0) = Q_{\varrho/\sqrt{2}}\left(-\tfrac12, 0\right)\,.\end{aligned}$$

Note that if $\varrho = 1/\sqrt{2}$, then $\Omega_\varrho = (-3/4, -1/4) \times B_{1/2}(0) = D^-$, and also that $\Omega_\varrho \subset \Omega_1 = (-1,0) \times B_1(0) = Q_1^-$ for any $\varrho \in [1/2, 1)$.

The first assumption of Lemma 5, namely inequality (36) with $\beta = d+2$ is nothing but inequality (13) of Lemma 2 applied to $\Omega_\varrho = Q_{\varrho/\sqrt{2}}(-1/2, 0)$, that is,

$$\sup_{\Omega_\varrho} v_+^p \le \frac{c_1\, 2^{\frac{d+2}{2}}}{(r-\varrho)^{d+2}} \iint_{\Omega_r} v_+^p \,\mathrm{d}x\,\mathrm{d}t \quad \forall\, p \in (0, 1/\mu)\,. \tag{45}$$

Note that the results of Lemma 2 hold true for these cylinders as well, with the same constants, since $Q_{\varrho/\sqrt{2}}(-1/2, 0)$ can be obtained from $Q_\varrho(0,0)$ by means of admissible transformations (16) which leave the class of equations unchanged, *i.e.*, such that λ_1, λ_0 and μ are the same.

The second assumption of Lemma 5, namely inequality (37) of Lemma 5, if stated in terms of super-level sets of $\log v_+$, reads

$$|\{x \in \Omega_1 \,:\, \log v_+ > s\}| = \left|\left\{(t,x) \in Q_1^- \,:\, \log v > s + a\right\}\right| \le c_2\, |B_1|\, \frac{\mu}{s}$$

according to Lemma 4. Hence we are in the position to apply Lemma 5 with $\theta = 1/\sqrt{2}$ to conclude that (44) is true with

$$\overline{\kappa}_0 := \exp\left[2\, c_2 \vee \frac{8\, c_1^3 (\sqrt{2})^{3\,(d+2)}\, \sigma}{(1 - 1/\sqrt{2})^{2\,(d+2)}}\right]\,. \tag{46}$$

This concludes the first step.

Step 2. Lower estimates. Let us prove that

$$\sup_{D^+} v_- \le \underline{\kappa}_0^{\,\mu} \tag{47}$$

where $\underline{\kappa}_0$ has an explicit expression, given below in (48). For all $\varrho \in [1/2, 1)$, let

$$\Omega_\varrho = \Big\{(t,x) \in \Omega_1 \,:\, 0 < 1-t < \varrho^2\,,\ |x| < \varrho\Big\} = \Big(1-\varrho^2, 1\Big) \times B_\varrho(0) = Q_\varrho^-(1,0)\,.$$

Note that if $\varrho = 1/2$ then $\Omega_\varrho = (3/4, 1) \times B_{1/2}(0) = D^+$, and $\Omega_\varrho \subset \Omega_1 = (0,1) \times B_1(0) = Q_1^+$ for any $\varrho \in [1/2, 1)$.

The first assumption of Lemma 5, namely inequality (36) with $\beta = d+2$ is nothing but inequality (14) of Lemma 2 applied to $\Omega_\varrho = Q_\varrho^-(1,0)$

$$\sup_{\Omega_\varrho} v_-^p \le \frac{c_1}{(r-\varrho)^{d+2}} \iint_{\Omega_r} v_-^p \,\mathrm{d}x\,\mathrm{d}t \quad \forall\, p \in (-\tfrac{1}{\mu}, 0)\,.$$

Note that the results of Lemma 2 hold true for these cylinders as well, with the same constants, since $Q_\varrho^-(1,0)$ can be obtained from $Q_\varrho(0,0)$ by means of admissible transformations (16).

The second assumption of Lemma 5, namely inequality (37) of Lemma 5, if stated in terms of super-level sets of $\log v_-$, reads

$$|\{x \in \Omega_1 \,:\, \log v_- > s\}| = \Big|\{(t,x) \in Q_1^+ \,:\, \log v < -s + a\}\Big| \le c_2\, |B_1|\, \frac{\mu}{s}\,.$$

and follows from inequality (31) of Lemma 4. With the same a and c_2, we are in the position to apply Lemma 5 with $\theta = 1/2$ to conclude that (47) is true with

$$\underline{\kappa}_0 := \exp\left[2\, c_2 \vee c_1^3 2^{2\,(d+2)+3}\, \sigma\right]\,. \tag{48}$$

This concludes the second step.

Step 3. Harnack inequality and its constant. We deduce from (44) and (47) that

$$\overline{\kappa}_0^{-\mu} \sup_{D^-} v \ \le \mathrm{e}^a \le \underline{\kappa}_0^{\mu} \inf_{D^+} v$$

or, equivalently,

$$\sup_{D^-} v \ \le (\overline{\kappa}_0\, \underline{\kappa}_0)^{\mu} \inf_{D^+} v = \widetilde{\mathsf{h}}^{\mu} \inf_{D^+} v\,.$$

Using (46) and (48), we compute

$$\begin{aligned}
\widetilde{\mathsf{h}} = \overline{\kappa}_0\, \underline{\kappa}_0 &= \exp\left[2\, c_2 \vee c_1^3\, 2^{2\,(d+2)+3}\, \sigma\right] \exp\left[2\, c_2 \vee \tfrac{8\, c_1^3\, (\sqrt{2})^{3\,(d+2)}}{(1-1/\sqrt{2})^{2\,(d+2)}}\, \sigma\right] \\
&\le \exp\left[4\, c_2 + c_1^3 \left(2^{2\,(d+2)+3} + \tfrac{8\, (\sqrt{2})^{3\,(d+2)}}{(1-1/\sqrt{2})^{2\,(d+2)}}\right)\, \sigma\right] \\
&= \exp\left[4\, c_2 + c_1^3\, 2^{2\,(d+2)+3} \left(1 + \tfrac{2^{d+2}}{(\sqrt{2}-1)^{2\,(d+2)}}\right)\, \sigma\right] := \mathsf{h}\,.
\end{aligned}$$

The expressions of c_1 and c_2 are given in (15) and (30) respectively. The above expression of h agrees with the simplified expression of (4), which completes the proof. □

5 Harnack inequality implies Hölder continuity

In this section, we shall show a standard application of the Harnack inequality (12). It is well known that (12) implies Hölder continuity of solutions to (1), as in Moser's celebrated paper [25, pp. 108-109]. The novelty is that, here, we keep track of all constants and obtain a quantitative expression of the Hölder continuity exponent, which only depends on the Harnack constant, *i.e.*, only depends on the dimension d and on the ellipticity constants λ_0 and λ_1 in (2).

Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$ two bounded domains and let us consider $Q_1 := (T_2, T_3) \times \Omega_1 \subset (T_1, T_4) \times \Omega_2 =: Q_2$, where $0 \le T_1 < T_2 < T_3 < T < 4$. We define the *parabolic distance* between Q_1 and Q_2 as

$$d(Q_1, Q_2) := \inf_{\substack{(t,x)\in Q_1 \\ (s,y)\in[T_1,T_4]\times\partial\Omega_2\cup\{T_1,T_4\}\times\Omega_2}} |x-y| + |t-s|^{\frac{1}{2}}\,. \tag{49}$$

In what follows, for simplicity, we shall consider Ω_1 and Ω_2 as convex sets. However, this is not necessary and the main result of this section holds without such restriction.

Theorem 6. *Let v be a nonnegative solution of (1) on Q_2 and assume that u satisfies (2). Then we have*

$$\sup_{(t,x),(s,y)\in Q_1} \frac{|v(t,x)-v(s,y)|}{\left(|x-y|+|t-s|^{1/2}\right)^{\nu}} \le 2\left(\frac{128}{d(Q_1,Q_2)}\right)^{\nu} \|v\|_{\mathrm{L}^\infty(Q_2)} \tag{50}$$

where

$$\nu := \log_4\left(\frac{\overline{\mathsf{h}}}{\overline{\mathsf{h}}-1}\right),$$

and $\overline{\mathsf{h}}$ is as in (11).

From the expression of h in (4) it is clear that $\overline{\mathsf{h}} \ge \frac{4}{3}$, from which we deduce that $\nu \in (0,1)$.

Proof. We proceed in steps: in step 1 we shall show that inequality (12) implies a *reduction of oscillation* on cylinders of the form (3). In step 2 we will iterate such reduction of oscillation and directly show estimate (50).

Step 1. Reduction of oscillation. Let us define $D_R(t_0,x_0) = (t_0 - R^2, t_0 + R^2) \times B_{2R}(x_0)$ and let $D_R^+(t_0,x_0)$, $D_R^-(t_0,x_0)$ be as in (3). Let us define

$$M := \max_{D_R(t_0,x_0)} v\,, \quad M^\pm = \max_{D_R^\pm(t_0,x_0)} v\,, \quad m = \min_{D_R(t_0,x_0)} v\,, \quad m^\pm = \max_{D_R^\pm(t_0,x_0)} v\,,$$

and let us define the oscillations ω and ω^+ namely

$$\omega = M - m \quad \text{and} \quad \omega^+ = M^+ - m^+\,.$$

We observe that the function $M-u$ and $u-m$ are nonnegative solution to (1) which also satisfy (2) with λ_0 and λ_1 as in (2). We are therefore in the position to apply inequality (12) to those functions and get

$$M-m^- = \sup_{D_R^-(t_0,x_0)} M-u \le \bar{\mathsf{h}} \inf_{D_R^+(t_0,x_0)} u-M = \bar{\mathsf{h}}\left(M-M^+\right),$$
$$M^- - m = \sup_{D_R^-(t_0,x_0)} u-m \le \bar{\mathsf{h}} \inf_{D_R^+(t_0,x_0)} u-m = \bar{\mathsf{h}}\left(m^+-m\right).$$

Summing up the two above inequalities we get

$$\omega \le \omega + (M^- - m^-) \le \bar{\mathsf{h}}\,\omega - \bar{\mathsf{h}}\,\omega^+$$

which can be rewritten as

$$\omega^+ \le \frac{\bar{\mathsf{h}}-1}{\bar{\mathsf{h}}}\,\omega =: \zeta\,\omega\,, \tag{51}$$

which means that the oscillation on $D_R^+(t_0,x_0)$ is smaller then the oscillation on $D_R(t_0,x_0)$, recall that $\zeta<1$. In the next step we will iterate such inequality in a sequence of nested cylinders to get a *geometric* reduction of oscillations.

Step 2. Iteration. Let us define $\delta = d(Q_1,Q_2)/64$. The number δ has the following property:

$$\begin{aligned}&\text{Let } (t,x)\in Q_1 \text{ and } (s,y)\in(0,\infty)\times\mathbb{R}^d.\\ &\text{If } |x-y|+|t-s|^{\frac12}\le\delta \text{ then, } (s,y)\in Q_2\,.\end{aligned} \tag{P}$$

Let us consider $(t,x),(s,y)\in Q_1$, then either

$$|x-y|+|t-s|^{\frac12}<\delta\,, \tag{A}$$

or

$$|x-y|+|t-s|^{\frac12}\ge\delta\,. \tag{B}$$

If (A) happens, then there exists an integer $k\ge0$ such that

$$\frac{\delta}{4^{k+1}} \le |x-y|+|t-s|^{\frac12} \le \frac{\delta}{4^k}\,.$$

Let us define $z=\frac{x+y}{2}$ and $\tau_0=\frac{t+s}{2}$. Since Q_1 is a convex set we have that $z,\tau_0\in Q_1$. Let us define,

$$R_{i+1} := 4\,R_i \quad \tau_{i+1} := \tau_i - 14\,R_i^2 \quad \forall\, i\in\{0,\cdots,k-1\} \text{ where } R_0=\frac{\delta}{4^{k-1}}\,.$$

With such choices we have that

$$D_{R_i}(z,\tau_i)\subset D^+_{R_{i+1}}(z,\tau_{i+1}) \quad \forall\, i\in\{0,\cdots,k-1\}\,, \tag{52}$$

and

$$(t,x)\,,(s,y)\in D_{R_0}(z,\tau_0)\subset D^+_{R_1}(z,\tau_1)\,.$$

We also observe that, as a consequence of property (P) we have that $D_{R_k}(z,\tau_k)\subset Q_2$. Let us define, for any $i\in\{0,\cdots,k-1\}$

$$\omega_i:=\max_{D_{R_i}(z,\tau_i)}u-\min_{D_{R_i}(z,\tau_i)}u\quad\text{and}\quad\omega_i^+:=\max_{D^+_{R_i}(z,\tau_i)}u-\min_{D^+_{R_i}(z,\tau_i)}u\,.$$

As a consequence of (52)

$$\omega_i\le\omega^+_{i+1}\,. \tag{53}$$

By iterating inequalities (53) - (51), we obtain that

$$\begin{aligned}|v(t,x)-v(s,y)|\le\omega_0\le\omega_1^+\le\xi\,\omega_1&\\ &\le\ \xi^k\,\omega_k=\left(\frac{1}{4}\right)^{k\,\nu}\omega_k\\ &\le\left(\frac{4}{\delta}\right)^{\nu}\left(\frac{\delta}{4^{k+1}}\right)^{\nu}\omega_k\\ &\le 2\left(\frac{4}{\delta}\right)^{\nu}\left(|x-y|+|t-s|^{\frac12}\right)^{\nu}\,\|v\|_{\mathrm{L}^\infty(Q_2)}\,.\end{aligned}$$

This concludes the proof of (50) under assumption (A).

Let us now assume that (B) happens. In this case we have that

$$\begin{aligned}|v(t,x)-v(s,y)|\le 2\,\|u\|_{\mathrm{L}^\infty(Q_2)}\,\frac{\delta^\nu}{\delta^\nu}&\le 2\,\|v\|_{\mathrm{L}^\infty(Q_2)}\left(\frac{|x-y|+|t-s|^{\frac12}}{\delta}\right)^{\nu}\\ &\le\ 2\left(\frac{4}{\delta}\right)^{\nu}\left(|x-y|+|t-s|^{\frac12}\right)^{\nu}\,\|v\|_{\mathrm{L}^\infty(Q_2)}\,.\end{aligned}$$

The proof is then completed. □

Part II

The fast diffusion equation: a handbook, with proofs

1 Scope of the handbook

One of the difficulties in the study of parabolic and especially nonlinear parabolic equations is the computation of the constants in the functional inequalities. Here we focus on various estimates for the fast diffusion equation. We have in mind the explicit computation of a threshold time for the uniform convergence in relative error, which is essential for the results of improved decay rates of the free energy and stability estimates in Gagliardo-Nirenberg inequalities (see Appendix A.1), but we also state various constructive estimates which are of independent interest.

1.1 Definitions and notations

Let us consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t} = \Delta u^m\,, \quad u(t=0,\cdot) = u_0 \tag{54}$$

on $\mathbb{R}^d$ with $d \geq 1$ and $m \in (m_1, 1)$ with $m_1 := (d-1)/d$. We also introduce the parameter α that will be of constant use in this document

$$\alpha = 2 - d\,(1-m) = 1 + d\,(m - m_1) = d\,(m - m_c)\,, \quad m_c = \frac{d}{d-2}. \tag{55}$$

1.2 Outline

In Section 2, we provide details on the comparison of the entropy - entropy production inequality with its linearized counterpart, *i.e.*, the Hardy-Poincaré inequality: see Proposition 7. Section 3 is devoted to various results on the solutions of the fast diffusion equation (54) which are needed to establish the uniform convergence in relative error.

1. The local L^1 bound of Lemma 8, known as Herrero-Pierre estimate, is established with explicit constants in Section 3.1.

2. An explicit local upper bound is proved in Lemma 9 in Section 3.2.

3. The Aleksandrov reflection principle is applied in Proposition 10 to prove a first local lower bound in Section 3.3, which is extended in Section 3.4: see Lemma 11.

4. Details on the inner estimate in terms of the free energy are collected in Section 3.5: see Proposition 12.

5. In the Appendix A, some useful observations are summarized or detailed: a *user guide for the computation of the threshold time* collects in Appendix A.1 all necessary informations for the computation of the threshold time $t_\star$ of [9, Theorem 4] and [9, Proposition 12]; the numerical value of the optimal constant in the Gagliardo-Nirenberg inequality on the disk is the established in Appendix A.2; details on the truncation functions are provided in Appendix A.3.

2 Relative entropy and fast diffusion flow

Here we deal with *the asymptotic time layer improvement* of [9, Section 2.3]. Let us consider the Barenblatt profile

$$\mathcal{B}(x)=\Big(1+|x|^2\Big)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d$$

of mass $\mathcal{M}:=\int_{\mathbb{R}^d}\mathcal{B}(x)\,dx$ and a nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}v(x)\,dx=\mathcal{M}$. The *free energy* (or *relative entropy*) and the *Fisher information* (or *relative entropy production*) are defined respectively by

$$\mathcal{F}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\Big(v^m-\mathcal{B}^m-m\,\mathcal{B}^{m-1}\,(v-\mathcal{B})\Big)\,dx$$

and

$$\mathcal{I}[v]:=\frac{m}{1-m}\int_{\mathbb{R}^d}v\,\Big|\nabla v^{m-1}-\nabla\mathcal{B}^{m-1}\Big|^2\,dx\,.$$

We also define the *linearized free energy* and the *linearized Fisher information* by

$$\mathsf{F}[g]:=\frac{m}{2}\int_{\mathbb{R}^d}|g|^2\,\mathcal{B}^{2-m}\,dx\quad\text{and}\quad\mathsf{I}[g]:=m\,(1-m)\int_{\mathbb{R}^d}|\nabla g|^2\,\mathcal{B}\,dx\,,$$

in such a way that

$$\mathsf{F}[g]=\lim_{\varepsilon\to0}\varepsilon^{-2}\,\mathcal{F}[\mathcal{B}+\varepsilon\,\mathcal{B}^{2-m}\,g]\quad\text{and}\quad\mathsf{I}[g]=\lim_{\varepsilon\to0}\varepsilon^{-2}\,\mathcal{I}[\mathcal{B}+\varepsilon\,\mathcal{B}^{2-m}\,g]\,.\tag{56}$$

By the *Hardy-Poincaré inequality* of [5], for any function $g\in\mathrm{L}^2(\mathbb{R}^d,\mathcal{B}^{2-m}\,dx)$ such that $\nabla g\in\mathrm{L}^2(\mathbb{R}^d,\mathcal{B}\,dx)$ and $\int_{\mathbb{R}^d}g\,\mathcal{B}^{2-m}\,dx=0$, if $d\ge1$ and $m\in(m_1,1)$, then we have

$$\mathsf{I}[g]\ge4\,\mathsf{F}[g]\,.$$

This inequality can be proved by spectral methods as in [16, 17] or obtained as a consequence of the *entropy - entropy production inequality*

$$\mathcal{I}[v]\ge4\,\mathcal{F}[v]\tag{57}$$

of [15], using (56). If additionally we assume that $\int_{\mathbb{R}^d} x\, g\, \mathcal{B}^{2-m}\, dx = 0$, then we have the *improved Hardy-Poincaré inequality*

$$\mathsf{I}[g] \geq 4\,\alpha\, \mathsf{F}[g]\,. \tag{58}$$

where $\alpha = 2 - d\,(1-m) = d\,(m-m_c)$. Details can be found in [8, Lemma 1] (also see [20, Proposition 1] and [27, 16, 17] for related spectral results).

Now let us consider

$$g := v\, \mathcal{B}^{m-2} - \mathcal{B}^{m-1} \tag{59}$$

and notice that $\int_{\mathbb{R}^d} x\, v(x)\, dx = 0$ if and only if $\int_{\mathbb{R}^d} x\, g\, \mathcal{B}^{2-m}\, dx = 0$. Our goal is to deduce an improved version of (57) from (58), in a neighborhood of the Barenblatt functions determined by a relative error measured in the uniform convergence norm. We choose the following numerical constant

$$\chi := \frac{1}{322} \quad \text{if} \quad d \geq 2\,, \quad \chi := \frac{m}{266 + 56\, m} \quad \text{if} \quad d = 1\,.$$

In view of [9], notice that $\chi \geq m/(266 + 56\, m)$ in any dimension.

Proposition 7. *Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$ and $\eta := 2\,d\,(m - m_1)$. If $v \in \mathrm{L}^1(\mathbb{R}^d)$ is nonnegative and such that $\int_{\mathbb{R}^d} v(x)\, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\, v(x)\, dx = 0$, and*

$$(1-\varepsilon)\, \mathcal{B} \leq v \leq (1+\varepsilon)\, \mathcal{B} \quad a.e. \tag{$H_{\varepsilon,T}$}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathcal{I}[v] \geq (4 + \eta)\, \mathcal{F}[v]\,. \tag{60}$$

Proof. We estimate the free energy $\mathcal{F}$ and the Fisher information $\mathcal{I}$ in terms of their linearized counterparts F and I as in [5]. Let g be as in (59). Under Assumption ($H_{\varepsilon,T}$), we deduce by a simple Taylor expansion that

$$(1+\varepsilon)^{-a}\, \mathsf{F}[g] \leq \mathcal{F}[v] \leq (1-\varepsilon)^{-a}\, \mathsf{F}[g] \tag{61}$$

as in [5, Lemma 3], where $a = 2 - m$. Slightly more complicated but still elementary computations based on [5, Lemma 7] show that

$$\mathsf{I}[g] \leq s_1(\varepsilon)\, \mathcal{I}[v] + s_2(\varepsilon)\, \mathsf{F}[g]\,, \tag{62}$$

where

$$s_1(\varepsilon) := \frac{(1+\varepsilon)^{2\,a}}{1-\varepsilon} \quad \text{and} \quad s_2(\varepsilon) := \frac{2\,d}{m}\,(1-m)^2 \left(\frac{(1+\varepsilon)^{2\,a}}{(1-\varepsilon)^{2\,a}} - 1\right).$$

Collecting (58), (61) and (62), elementary computations show that (60) holds with $\eta = f(\varepsilon)$, where

$$f(\varepsilon) = \frac{4\,\alpha\,(1-\varepsilon)^a\; - 4\, s_1(\varepsilon) - (1+\varepsilon)^a\, s_2(\varepsilon)}{s_1(\varepsilon)}\,.$$

We claim that

$$\max_{\varepsilon\in(0,\chi\,\eta)} f(\varepsilon)\ge 2\,d\,(m-m_1)\,.$$

Let us consider

$$g(\varepsilon):=1-\frac{(1-\varepsilon)^{1+a}}{(1+\varepsilon)^{2\,a}}\quad\text{and}\quad h(\varepsilon):=\frac{1-\varepsilon}{(1+\varepsilon)^a}\left(\frac{(1+\varepsilon)^{2\,a}}{(1-\varepsilon)^{2\,a}}-1\right)$$

and observe that g is concave and $g(\varepsilon)\le g'(0)\,\varepsilon=(1+3\,a)\,\varepsilon\le 7\,\varepsilon$ for any $\varepsilon\in[0,1]$ and $a\in[1,2]$, while h is convex and such that $h(\varepsilon)\le h'(1/2)\,\varepsilon$ for any $\varepsilon\in[0,1/2]$ with $h'(1/2)\le 133$ for any $a\in[1,2]$. By writing

$$f(\varepsilon)=2\,\eta-4\,\alpha\,g(\varepsilon)-2\,\frac{d}{m}\,(1-m)^2\,h(\varepsilon)\,,$$

and after observing that $4\,\alpha\le 8$ and $\frac{d}{m}\,(1-m)^2\le 1$ if $d\ge 2$ and $m\in(m_1,1)$, $\frac{d}{m}\,(1-m)^2\le\frac{1}{m}$ if $d=1$ and $m\in(0,1)$, we conclude that

$$f(\varepsilon)\ge 2\,\eta-\tfrac{\varepsilon}{\chi}\ge\eta\quad\forall\,\varepsilon\in(0,\chi\,\eta)\,.$$

□

Let us conclude this section by a list of observations:

▷ Proposition 7 is an *improved entropy - entropy production inequality*. It can be understood as a *stability result* for the standard entropy - entropy production inequality, which is equivalent to the Gagliardo-Nirenberg inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathcal{C}_{\rm GN}\,\|f\|_{2\,p}\quad\forall\,f\in\mathcal{D}(\mathbb{R}^d)\,,\tag{63}$$

where the exponent is $\theta=\frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, p is in the range $(1,p^*)$ with $p^*=+\infty$ if $d=1$ or 2, and $p^*=d/(d-2)$ if $d\ge 3$, and $\mathcal{D}(\mathbb{R}^d)$ denotes the set of smooth functions on $\mathbb{R}^d$ with compact support. Similar results with less explicit estimates can be found in [5, 8]. Compared to [9, Theorems 1 and 15], this is a much weaker result in the sense that the admissible neighborhood in which we can state the stability result is somewhat artificial, or at least very restrictive. However, this makes sense in the asymptotic time layer as $t\to+\infty$, from the point of view of the nonlinear flow.

▷ According to [15], it is known that (63) is equivalent to (57) if m and p are such that

$$p=\frac{1}{2\,m-1}\,.$$

The fact that p is in the interval $(1,p^*)$ is equivalent to $m\in(m_1,1)$ if $d\ge 2$ and $m\in(1/2,1)$ if $d=1$. In order to define $\int_{\mathbb{R}^d}|x|^2\,\mathcal{B}\,dx$, there is the condition that $m>d/(d+2)$, which is an additional restriction only in dimension $d=1$. This is why in Section 3 we shall only consider the case $m>1/3$ if $d=1$.

$\triangleright$ In [9, Proposition 3], the result is stated for a solution to the *fast diffusion equation* in self-similar variables

$$\frac{\partial v}{\partial t} + \nabla \left(v \, \nabla v^{m-1} \right) = 2 \, \nabla \cdot (x \, v) \,, \quad v(t=0,\cdot) = v_0 \,. \tag{64}$$

With the same assumptions and definitions as in Proposition 7, *if v is a non-negative solution to (64) of mass $\mathcal{M}$, with*

$$(1-\varepsilon) \, \mathcal{B} \le v(t,\cdot) \le (1+\varepsilon) \, \mathcal{B} \quad \forall \, t \ge T \tag{65}$$

for some $\varepsilon \in (0, \chi \, \eta)$ and $T > 0$, and such that $\int_{\mathbb{R}^d} x \, v(t,x) \, dx = 0$, then we have

$$\mathcal{I}[v(t,\cdot)] \ge (4+\eta) \, \mathcal{F}[v(t,\cdot)] \quad \forall \, t \ge T \,. \tag{66}$$

This result is equivalent to Proposition 7.

$\triangleright$ The admissible neighborhood of $\mathcal{B}$ is in fact stable under the action of the flow defined by (64). The improved inequality (66) holds if (65) holds at $t = T$ and if v is a non-negative solution to (64) of mass $\mathcal{M}$, with $\int_{\mathbb{R}^d} x \, v_0(x) \, dx = 0$. The condition (65) is very restrictive if we impose it with $T = 0$ as in [5, 8]. A key observation in [9] is that it is satisfied in the asymptotic time layer as $t \to +\infty$ and that we can provide an explicit estimate of T.

$\triangleright$ Condition (65) is slightly different from the one appearing in [5, 8]. In those papers the initial data is assumed to be such that

$$\left(c_1 + |x|^2 \right)^{\frac{1}{m-1}} \le v(0,x) \le \left(c_2 + |x|^2 \right)^{\frac{1}{m-1}} \quad \forall \, x \in \mathbb{R}^d$$

for some positive c_1 and c_2 such that $0 < c_2 \le 1 \le c_1$. The above condition is much stronger than (65) as it guarantees that $(v/\mathcal{B} - 1) \in \mathrm{L}^q(\mathbb{R}^d)$ for some $q < \infty$. In [9], we only need that $(v/\mathcal{B} - 1) \in \mathrm{L}^\infty(\mathbb{R}^d)$.

3 Uniform convergence in relative error

We state and prove here the local upper and lower bounds used in [9, Section 3.2], with explicit constants. The method follows the proofs of [12, 13]. Comparing to the existing literature we give simpler proofs and provide explicit constants. Some of the results presented here were already contained in [28, Chapters 1, 2 and 6].

In this section, we consider solutions to the Cauchy problem for the Fast Diffusion Equation posed in the whole Euclidean space $\mathbb{R}^d$, in the range $m_1 < m < 1$, $d \ge 1$. Global existence of non-negative solutions of (54) is established in [23]. Much more is known on (54) and we refer to [29] for a general overview. Recall that we always assume $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$.

3.1 Mass displacement estimates: local L^1 bounds

We prove the Lemma as needed in the proof of [12, Theorem 1.1], a slightly modified version of the result of M.A. Herrero and M. Pierre in [23, Lemma 3.1]). Our main task is to derive have an explicit expression of the constants.

Lemma 8. *Let $m \in (0,1)$ and $u(t,x)$ be a nonnegative solution to the Cauchy problem (54). Then, for any $t, \tau \geq 0$ and $r, R > 0$ such that $\varrho_0 r \geq 2\,R$ for some $\varrho_0 > 0$, we have*

$$\int_{B_{2\,R}(x_0)} u(t,x)\,\mathrm{d}x \leq 2^{\frac{m}{1-m}} \int_{B_{2\,R+r}(x_0)} u(\tau,x)\,\mathrm{d}x + \mathsf{c}_3 \frac{|t-\tau|^{\frac{1}{1-m}}}{r^{\frac{2-d\,(1-m)}{1-m}}}\,, \tag{67}$$

where

$$\mathsf{c}_3 := 2^{\frac{m}{1-m}}\,\omega_d \left(\frac{16\,(d+1)\,(3+m)}{1-m}\right)^{\frac{1}{1-m}} (\varrho_0+1)\,. \tag{68}$$

Proof. Let $\phi = \varphi^\beta$, for some $\beta > 0$ (sufficiently large, to be chosen later) be a radial cut-off function supported in $B_{2\,R+r}(x_0)$ and let $\varphi = 1$ in $B_{2\,R}(x_0)$. We can take, for instance, $\varphi = \varphi_{2\,R,2\,R+r}$, where $\varphi_{2\,R,2\,R+r}$ is defined in (114). We know that, see for instance (113) of Lemma 15 in Appendix A.3 that,

$$\|\nabla\varphi\|_\infty \leq \frac{2}{r} \quad \text{and} \quad \|\Delta\varphi\|_\infty \leq \frac{4\,d}{r^2}. \tag{69}$$

In what follow we will write B_R instead of $B_R(x_0)$ when no confusion arises. Let us compute

$$\begin{aligned}
\left|\frac{d}{dt}\int_{B_{2\,R+r}} u(t,x)\phi\,(x)\,\mathrm{d}x\right| &= \left|\int_{B_{2R+r}} \Delta\,(u^m)\,\phi\,\mathrm{d}x\right| = \left|\int_{B_{2R+r}} u^m \Delta\phi\,\mathrm{d}x\right| \\
&\leq \int_{B_{2R+r}} u^m \left|\Delta\phi\right| \mathrm{d}x \\
&\leq \left(\int_{B_{2R+r}} u\,\phi\,\mathrm{d}x\right)^m \left(\int_{B_{2\,R+r}} \frac{|\Delta\phi|^{\frac{1}{1-m}}}{\phi^{\frac{m}{1-m}}}\,\mathrm{d}x\right)^{1-m} \\
&:= C\,(\phi)\left(\int_{B_{2\,R+r}} u\,\phi\,(x)\,\mathrm{d}x\right)^m,
\end{aligned} \tag{70}$$

where we have used Hölder's inequality with conjugate exponents $\frac{1}{m}$ and $\frac{1}{1-m}$. We have obtained the following closed differential inequality

$$\left|\frac{d}{dt}\int_{B_{2R+r}} u\,(t,x)\,\phi\,(x)\,\mathrm{d}x\right| \leq C(\phi)\left(\int_{B_{2R+r}} u\,(t,x)\,\phi\,(x)\,\mathrm{d}x\right)^m.$$

An integration in time shows that, for all $t,\,\tau \geq 0$, we have

$$\left(\int_{B_{2R}} u\,(t,x)\,\phi\,(x)\,\mathrm{d}x\right)^{1-m} \leq \left(\int_{B_{2R}} u\,(\tau,x)\,\phi\,(x)\,\mathrm{d}x\right)^{1-m} + (1-m)\,C\,(\phi)\,|t-\tau|\,.$$

Since ϕ is supported in ${}_{2\,R+r}$ and equal to 1 in $B_{2\,R}$, this implies (67), indeed, using

$$(a+b)^{\frac{1}{1-m}} \le 2^{\frac{1}{1-m}-1}\left(a^{\frac{1}{1-m}}+b^{\frac{1}{1-m}}\right),$$

we get

$$\begin{aligned}\int_{B_{2R}} u(t,x)\,\mathrm{d}x &\le 2^{\frac{m}{1-m}}\left(\int_{B_{2R+r}} u(\tau,x)\,\mathrm{d}x + \Big((1-m)\,C(\phi)\Big)^{\frac{1}{1-m}}\,|t-\tau|^{\frac{1}{1-m}}\right)\\ &\le 2^{\frac{m}{1-m}}\int_{B_{2R+r}} u(\tau,x)\,\mathrm{d}x + \mathsf{c}_3\,\frac{|t-\tau|^{\frac{1}{1-m}}}{r^{\frac{2-d\,(1-m)}{1-m}}}\,,\end{aligned}$$

where

$$\mathsf{c}_3(r) := 2^{\frac{m}{1-m}}((1-m)C(\phi))^{\frac{1}{1-m}}r^{\frac{2-d\,(1-m)}{1-m}}.$$

The above proof is formal when considering weak or very weak solutions, in which case, it is quite lengthy (although standard) to make it rigorous, cf. [23, Proof of Lemma 3.1]; indeed, it is enough to consider the time-integrated version of estimates (70), and conclude by a Grownwall-type argument.

The proof is completed once we show that the quantity $\mathsf{c}_3(r)$ is bounded and provide the expression (68). Recall that $\phi = \varphi^\beta$, so that

$$\begin{aligned}|\Delta\,(\phi(x))|^{\frac{1}{1-m}}\,\phi(x)^{-\frac{m}{1-m}} &= \varphi(x)^{-\frac{\beta\,m}{1-m}}\left|\beta\,(\beta-1)\,\varphi^{\beta-2}\,|\nabla\,\varphi\,|^2+\beta\,\varphi^{\beta-1}\,\Delta\varphi\right|^{\frac{1}{1-m}}\\ &\le \Big(\beta\,(\beta-1)\Big)^{\frac{1}{1-m}}\,\varphi^{\frac{\beta-2-\beta\,m}{1-m}}\left|\,|\nabla\,\varphi\,|^2+|\Delta\varphi|\right|^{\frac{1}{1-m}}\\ &\le \left(\tfrac{4\,(3+m)}{(1-m)^2}\right)^{\frac{1}{1-m}}\left(\tfrac{4\,(d+1)}{r^2}\right)^{\frac{1}{1-m}}\,.\end{aligned} \tag{71}$$

The first inequality follow from the fact that we are considering a radial function $0 \le \varphi(x) \le 1$, and we take $\beta = \frac{4}{1-m} > \frac{2}{1-m}$. The last one follows by (69). Finally:

$$\begin{aligned}&\Big((1-m)\,C(\phi)\Big)^{\frac{1}{1-m}}\,r^{\frac{2-d\,(1-m)}{1-m}}\\ &= (1-m)^{\frac{1}{1-m}}\left(\int_{B_{2\,R+r}\setminus B_{2\,R}}\frac{|\Delta\phi|^{\frac{1}{1-m}}}{\phi^{\frac{m}{1-m}}}\,\mathrm{d}x\right)r^{\frac{2-d\,(1-m)}{1-m}}\\ &\le (1-m)^{\frac{1}{1-m}}\left(\tfrac{4\,(3+m)}{(1-m)^2}\right)^{\frac{1}{1-m}}\left(\tfrac{4\,(d+1)}{r^2}\right)^{\frac{1}{1-m}}\left|B_{2\,R+r}\setminus B_{2\,R}\right|r^{\frac{2-d\,(1-m)}{1-m}}\\ &\quad= \omega_d\left(\tfrac{16\,(d+1)\,(3+m)}{1-m}\right)^{\frac{1}{1-m}}\frac{(2\,R+r)^d-(2\,R)^d}{d\,r^d}\\ &\quad\le \omega_d\left(\tfrac{16\,(d+1)\,(3+m)}{1-m}\right)^{\frac{1}{1-m}}(\varrho_0+1)\end{aligned}$$

where we have used that the support of $\Delta\phi$ is contained in the annulus $B_{2\,R+r}\setminus B_{2\,R}$, inequality (71) and in the last step we have used that $\varrho_0\,r \ge 2\,R$ and

$$(2\,R+r)^d-(2\,R)^d \le d\,(2\,R+r)^{d-1}\,r \le d\,(\varrho_0+1)\,r^d\,.$$

The proof is now completed. □

3.2 Local upper bounds

Lemma 9. *Assume that* $d \geq 1$, $m \in (m_1, 1)$. *If* u *is a solution of* (54) *with non-negative initial datum* $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, *then there exists a positive constant* $\overline{\kappa}$ *such that any solution* u *of* (54) *satisfies for all* $(t, R) \in (0, +\infty)^2$ *the estimate*

$$\sup_{y \in B_{R/2}(x)} u(t, y) \leq \overline{\kappa} \left(\frac{1}{t^{d/\alpha}} \left(\int_{B_R(x)} u_0(y)\, dy \right)^{2/\alpha} + \left(\frac{t}{R^2} \right)^{\frac{1}{1-m}} \right) . \tag{72}$$

The above estimate is well known, *cf.* [18, 19, 14, 13], but the point is that we provide an explicit expression of the constant

$$\overline{\kappa} = \mathsf{k}\, \mathcal{K}^{\frac{2q}{\beta}} \tag{73}$$

where $\mathsf{k} = \mathsf{k}(m, d, \beta, q)$ is such that

$$\mathsf{k}^\beta = \left(\tfrac{4\beta}{\beta+2}\right)^\beta \left(\tfrac{4}{\beta+2}\right)^2 \pi^{8(q+1)}\, e^{8 \sum_{j=0}^\infty \log(j+1) \left(\frac{q}{q+1}\right)^j}\, 2^{\frac{2m}{1-m}} \left(1 + \mathsf{a}\, \omega_d\right)^2 \mathsf{b}$$

with $\mathsf{a} = \frac{3\,(16\,(d+1)\,(3+m))^{\frac{1}{1-m}}}{(2-m)\,(1-m)^{\frac{m}{1-m}}} + \frac{2^{\frac{d-m\,(d+1)}{1-m}}}{3^d\, d}$ and $\mathsf{b} = \frac{3 8^{2\,(q+1)}}{\left(1-(2/3)^{\frac{\beta}{4\,(q+1)}}\right)^{4\,(q+1)}}$.
The constant $\mathcal{K}$ is the same constant as in (7) and corresponds to the inequality

$$\|f\|^2_{\mathrm{L}^{p_m}(B)} \leq \mathcal{K} \left(\|\nabla f\|^2_{\mathrm{L}^2(B)} + \|f\|^2_{\mathrm{L}^2(B)} \right) . \tag{74}$$

In other words, (74) is (7) written for $R = 1$. The other parameters are given in Table 1 (see [9] for details on optimality and proofs).

	p_m	$\mathcal{K}$	q	β
$d \geq 3$	$\frac{2d}{d-2}$	$\frac{2}{\pi}\, \Gamma(\frac{d}{2}+1)^{2/d}$	$\frac{d}{2}$	α
$d = 2$	4	$\frac{2}{\sqrt{\pi}}$	2	$2\,(\alpha-1)$
$d = 1$	$\frac{4}{m}$	$2^{1+\frac{m}{2}}\ \max\left(\frac{2\,(2-m)}{m\,\pi^2}, \frac{1}{4}\right)$	$\frac{2}{2-m}$	$\frac{2m}{2-m}$

Table 1: Table of the parameters and the constant $\mathcal{K}$ in dimensions $d = 1$, $d = 2$ and $d \geq 3$. The latter case corresponds to the critical Sobolev exponent while the inequality for $d \leq 2$ is subcritical. In dimension $d = 1$, $p_m = 4/m$, which makes the link with (8).

Proof of Lemma 9. Our proof follows the scheme of [13] so we shall only sketch its main steps, keeping track of the explicit expression of the constants. The point

$x \in \mathbb{R}^d$ is arbitrary and by translation invariance it is not restrictive to assume that $x = 0$ and write $B_R = B_R(0)$. We also recall that u always possesses the regularity needed to perform all computations throughout the following steps.

Let us introduce the rescaled function

$$\hat{u}(t,x) = \left(\frac{R^2}{\tau}\right)^{\frac{1}{1-m}} u(\tau\, t, R\, x) \tag{75}$$

which solves (54) on the cylinder $(0,1] \times B_1$. In Steps 1-3 we establish on $\hat{v} = \max(\hat{u}, 1)$ a $\mathrm{L}^2 - \mathrm{L}^\infty$ smoothing inequality which we improve to a $\mathrm{L}^1 - \mathrm{L}^\infty$ smoothing in Step 4, using a *de Giorgi*-type iteration. In Step 5, we scale back the estimate to get the result on u.

Step 1. We observe that $\hat{v} = \max\{\hat{u}, 1\}$ solves $\frac{\partial \hat{v}}{\partial t} \leq \Delta \hat{v}^m$. According to [13, Lemma 2.5], we know that

$$\sup_{s\in[T_1,T]} \int_{B_{R_1}} \hat{v}^{p_0}(s,x)\, dx + \iint_{Q_1} \left|\nabla \hat{v}^{\frac{p_0+m-1}{2}}\right|^2 dx\, dt \leq \frac{8}{c_{m,p_0}} \iint_{Q_0} \left(\hat{v}^{m+p_0-1} + \hat{v}^{p_0}\right) dx\, dt$$

where $Q_k = (T_k, T] \times B_{R_k}$ with $0 < T_0 < T_1 < T \leq 1$, $0 < R_1 < R_0 \leq 1$ and $c_{m,p_0} = \min\left\{1 - \frac{1}{p_0}, \frac{2\,(p_0-1)}{p_0+m-1}\right\} \geq \frac{1}{2}$. We have $\hat{v}^{m+p_0-1} \leq \hat{v}^{p_0}$ because $\hat{v} \geq 1$, so that

$$\sup_{s\in[T_1,T]} \int_{B_{R_1}} \hat{v}^{p_0}(s,x)\, dx + \iint_{Q_1} \left|\nabla \hat{v}^{\frac{p_0+m-1}{2}}\right|^2 dx\, dt \leq \mathcal{C}_0 \iint_{Q_0} \hat{v}^{p_0}\, dx\, dt \tag{76}$$

where

$$\mathcal{C}_0 = 32\left(\frac{1}{(R_0 - R_1)^2} + \frac{1}{T_1 - T_0}\right).$$

Step 2. Let p_m be as defined in Section 1.1 and $\mathcal{K}$ be the constant in the inequality (7). Let $q = p_m/(p_m - 2)$ and $Q_i = (T_i, T] \times B_{R_i}$ as in Step 1. We claim that

$$\iint_{Q_1} \hat{v}^{p_1}\, dx\, dt \leq \mathcal{K}_0 \left(\iint_{Q_0} \hat{v}^{p_0}\, dx\, dt\right)^{1+\frac{1}{q}} \quad \text{with} \quad \mathcal{K}_0 = \mathcal{K}\left(R_1^{-2} + \mathcal{C}_0\right)^{1+\frac{1}{q}}. \tag{77}$$

Let us proven (77). Using Hölder's inequality, for any $a \in (2, p_m)$ we may notice that

$$\int_{B_{R_1}} |f(s,x)|^a\, dx = \int_{B_{R_1}} |f(s,x)|^2\, |f(s,x)|^{a-2}\, dx \leq \|f\|^2_{\mathrm{L}^{p_m}(B_{R_1})}\, \|f\|^{a-2}_{\mathrm{L}^b(B_{R_1})}$$

with $b = q\,(a-2)$. Using (7), this leads to

$$\iint_{Q_1} |f(t,x)|^a\, dx\, dt \leq \mathcal{K}\left(\|\nabla f\|^2_{\mathrm{L}^2(Q_1)} + \tfrac{1}{R_1^2}\, \|f\|^2_{\mathrm{L}^2(Q_1)}\right) \sup_{s\in(T_1,T)} \left(\int_{B_{R_1}} |f(s,x)|^b\, dx\right)^{\frac{1}{q}}.$$

Choosing $f^2 = \hat{v}^{p_0+m-1}$ with $a = 2\,p_1/(p_0+m-1)$ and $b = 2\,p_0/(p_0+m-1)$ we get

$$\iint_{Q_1} \hat{v}^{p_1}\,dx\,dt \le \mathcal{K} \iint_{Q_1} \left(\left| \nabla \hat{v}^{\frac{p_0+m-1}{2}} \right|^2 + \frac{\hat{v}^{p_0}}{R_1^2} \right) dx\,dt \sup_{s\in(T_1,T)} \left(\int_{B_{R_1}} \hat{v}^{p_0}\,dx \right)^{\frac{1}{q}}$$

where

$$p_1 = \left(1 + \frac{1}{q}\right) p_0 - 1 + m > p_0\,.$$

Letting $X = \left\| \nabla \hat{v}^{(p_0+m-1)/2} \right\|_2^2$, $Y_i = \iint_{Q_1} \hat{v}^{p_i}\,dx\,dt$ and $Z = \sup_{s\in(T_1,T)} \int_{B_{R_1}} \hat{v}^{p_0}\,dx$, we get $Y_1 \le \mathcal{K}\,(X + R_1^{-2}\,Y_0)\,Z^{1/q}$, while (76) reads $X + Z \le \mathcal{C}_0\,Y_0$. Hence $Y_1 \le \mathcal{K}\left((R_1^{-2} + \mathcal{C}_0)\,Y_0 - Z\right) Z^{1/q} \le \mathcal{K}\left((R_1^{-2} + \mathcal{C}_0)\,Y_0\right)^{(q+1)/q}$, that is inequality (77).

Step 3. We perform a Moser-type iteration. In order to iterate (77), fix $R_\infty < R_0 < 1$, $T_0 < T_\infty < 1$ and also assume that $2\,R_\infty \ge R_0$. We shall consider the sequences $(p_k)_{k\in\mathbb{N}}$, $(R_k)_{k\in\mathbb{N}}$, $(T_k)_{k\in\mathbb{N}}$ and $(\mathcal{K}_k)_{k\in\mathbb{N}}$ defined as follows:

$$\begin{aligned}
&p_k = \left(1 + \frac{1}{q}\right)^k (2 - q\,(1-m)) + q\,(1-m)\,,\\
&R_k - R_{k+1} = \frac{6}{\pi^2}\,\frac{R_0 - R_\infty}{(k+1)^2}\,, \quad T_{k+1} - T_k = \frac{90}{\pi^4}\,\frac{T_\infty - T_0}{(k+1)^4}\,,\\
&\mathcal{K}_k = \mathcal{K}\left(R_{k+1}^{-2} + \mathcal{C}_k\right)^{1+\frac{1}{q}}\,, \quad \mathcal{C}_k = 32\left(\frac{1}{(R_k - R_{k+1})^2} + \frac{1}{T_{k+1} - T_k}\right),
\end{aligned}$$

using the Riemann sums $\sum_{k\in\mathbb{N}} (k+1)^{-2} = \frac{\pi^2}{6}$ and $\sum_{k\in\mathbb{N}} (k+1)^{-4} = \frac{\pi^4}{90}$. It is clear that $\lim_{k\to+\infty} R_k = R_\infty$, $\lim_{k\to+\infty} T_k = T_\infty$ and $\mathcal{C}_k$ diverge as $k \to +\infty$. In addition, the assumption $2\,R_\infty \ge R_0$ leads to $R_{k+1}^{-2} \le (R_0 - R_\infty)^{-2}$ hence $\mathcal{K}_k$ is explicitly bounded by

$$\mathcal{K}_k \le \mathcal{K}\left(\pi^4\,(k+1)^4 L_\infty\right)^{1+\frac{1}{q}}\,, \quad \text{where} \quad L_\infty := \frac{1}{(R_0 - R_\infty)^2} + \frac{1}{(T_\infty - T_0)}\,.$$

Set $Q_\infty = (T_\infty, T) \times B_{R_\infty}$ and notice that $Q_\infty \subset Q_k$ for any $k \ge 0$. By iterating (77), we find that

$$\|\hat{v}\|_{\mathrm{L}^{p_{k+1}}(Q_\infty)} \le \|\hat{v}\|_{\mathrm{L}^{p_{k+1}}(Q_{k+1})} \le \mathcal{K}_k^{\frac{1}{p_{k+1}}} \, \|\hat{v}\|_{\mathrm{L}^{p_k}(Q_k)}^{\frac{(q+1)\,p_k}{q\,p_{k+1}}} \le \prod_{j=0}^{k} \mathcal{K}_j^{\frac{1}{p_{k+1}}\left(\frac{q+1}{q}\right)^{k-j}} \, \|\hat{v}\|_{\mathrm{L}^2(Q_0)}^{\frac{2\,(q+1)^{k+1}}{q^{k+1}\,p_{k+1}}}$$

and

$$\prod_{j=0}^{k} \mathcal{K}_j^{\frac{1}{p_{k+1}}\left(\frac{q+1}{q}\right)^{k-j}} \le \left[\mathcal{K}\left(\pi^4\,L_\infty\right)^{1+\frac{1}{q}}\right]^{\frac{1}{p_{k+1}}\sum_{j=0}^{k}\left(\frac{q+1}{q}\right)^j} \prod_{j=1}^{k+1} j^{\frac{4\left(\frac{q+1}{q}\right)^{k+2-j}}{p_{k+1}}}\,.$$

By lower semicontinuity of the L^∞ norm, letting $k \to +\infty$, we obtain

$$\|\hat{v}\|_{\mathrm{L}^\infty((T_\infty,T]\times B_{R_\infty})} \le \mathcal{C}\,\|\hat{v}\|_{\mathrm{L}^2((T_0,T]\times B_{R_0})}^{\frac{2}{2-q\,(1-m)}} \tag{78}$$

where $0 < T_0 < T_\infty < T \le 1$, $1/2 < R_\infty < R_0 \le 1$, $R_0 \le 2\,R_\infty$, and

$$\mathcal{C} = \mathcal{K}^{\frac{q}{2-q\,(1-m)}}\left(\pi^4\,L_\infty\right)^{\frac{(q+1)}{2-q\,(1-m)}}\;e^{\frac{4\,(q+1)}{q(2-q\,(1-m))}\sum_{j=1}^\infty\left(\frac{q}{q+1}\right)^j\log j}\,.$$

Step 4. We show how to improve the $\mathrm{L}^2-\mathrm{L}^\infty$ smoothing estimate (78) to a $\mathrm{L}^1-\mathrm{L}^\infty$ estimate, using a de Giorgi-type iteration. Let us set

$$\beta = 2-2\,q\,(1-m)\;=\begin{cases}\alpha & \text{if}\quad d\ge 3\,,\\ 2\,(\alpha-1) & \text{if}\quad d=2\,,\\ \frac{2\,m}{2-m} & \text{if}\quad d=1\,,\end{cases} \tag{79}$$

we recall that $\beta > 0$ for any $m \in (m_1, 1)$ and $d \ge 1$. Then, from (78), we obtain, using Hölder's and Young's inequalities,

$$\begin{aligned}\|\hat{v}\|_{\mathrm{L}^\infty((1/9,1]\times B_{1/2})} &\le \mathcal{C}\,\|\hat{v}\|_{\mathrm{L}^\infty((\tau_1,1]\times B_{r_1})}^{\frac{1}{2-q\,(1-m)}}\,\|\hat{v}\|_{\mathrm{L}^1((\tau_1,1]\times B_{r_1})}^{\frac{1}{2-q\,(1-m)}}\\ &\le \frac{1}{2}\,\|\hat{v}\|_{\mathrm{L}^\infty((\tau_1,1]\times B_{r_1})} + \mathfrak{C}_1\,\|\hat{v}\|_{\mathrm{L}^1((\tau_1,1]\times B_{r_1}))}^{\frac{2}{\beta}}\end{aligned} \tag{80}$$

where $1/9 < \tau_1 < 1$, $1/2 < r_1 < 1$ and

$$\mathfrak{C}_1 = X\left(\frac{1}{\left(r_1-\frac{1}{2}\right)^2}+\frac{1}{\frac{1}{9}-\tau_1}\right)^{\frac{2\,(q+1)}{\beta}}$$

with

$$X = \tfrac{\beta}{\beta+2}\left(\tfrac{4}{\beta+2}\right)^{\frac{2}{\beta}}\mathcal{K}^{\frac{2\,q}{\beta}}\left(\pi^q\,e^{\sum_{j=1}^\infty\left(\frac{q}{q+1}\right)^j\log j}\right)^{\frac{8\,(q+1)}{q\,\beta}}\,.$$

To iterate (80) we shall consider sequences $(r_i)_{i\in\mathbb{N}}, (\tau_i)_{i\in\mathbb{N}}$ such that

$$r_{i+1}-r_i = \tfrac{1}{6}\,(1-\xi)\,\xi^i\,,\qquad \tau_i-\tau_{i+1} = \tfrac{1}{9}\,(1-\xi^2)\,\xi^{2i}\,.$$

with $\xi = (2/3)^{\frac{\beta}{4\,(q+1)}}$. Since $2/3 \le \xi \le 1$, we have

$$\frac{1}{1-\xi^2} \le \frac{1}{5\,(1-\xi)^2}\,,$$

and this iteration gives us

$$\|\hat{v}\|_{\mathrm{L}^\infty((1/9,1]\times B_{1/2})} \le \frac{1}{2^k}\,\|\hat{v}\|_{\mathrm{L}^\infty((\tau_k,1]\times B_{r_k})} + \|\hat{v}\|_{\mathrm{L}^1((\tau_k,1]\times B_{r_k})}^{\frac{2}{\beta}}\sum_{i=0}^{k-1}\frac{\mathfrak{C}_{i+1}}{2^i}$$

where for all $i \geq 0$

$$\frac{\mathfrak{C}_{i+1}}{2^i} \leq \left(\tfrac{38}{(1-\xi)^2}\right)^{\frac{2\,(q+1)}{\beta}} X \left(\tfrac{3}{4}\right)^i .$$

In the limit $k \to \infty$ we find

$$\|\hat{v}\|_{\mathrm{L}^\infty((1/9,1]\times B_{1/2})} \leq \mathfrak{C}\, \|\hat{v}\|^{\frac{2}{\beta}}_{\mathrm{L}^1((0,1]\times B_{2/3})} \tag{81}$$

where

$$\mathfrak{C} = 4\left(\tfrac{38}{(1-\xi)^2}\right)^{\frac{2\,(q+1)}{\beta}} X\,. \tag{82}$$

Step 5. In this step we complete the proof of (72). We recall that $\hat{v} = \max\{\hat{u}, 1\}$ and then, using inequality (81) and the fact that $\hat{u} \leq \hat{v} \leq \hat{u}d\,(1-m)+1$, we find

$$\sup_{y\in B_{1/2}} \hat{u}(1,y) \leq \|\hat{u}\|_{\mathrm{L}^\infty\left((1/9,1]\times B_{1/2}\right)} \leq \mathfrak{C}\,\|\hat{u}+1\|^{\frac{2}{\beta}}_{\mathrm{L}^1\left((0,1]\times B_{2/3}\right)}\,. \tag{83}$$

The function $\hat{u}$ satisfies the following inequality for any $s \in [0,1]$

$$\int_{B_{2/3}} \hat{u}(s,x)\,dx \leq 2^{\frac{m}{1-m}} \int_{B_1} \hat{u}_0\,dx + \mathscr{C}\, s^{\frac{1}{1-m}}\,, \tag{84}$$

where

$$\mathscr{C} = 2^{\frac{m}{1-m}} \left(3\omega_d \left[\frac{16\,(d+1)\,(3+m)}{1-m}\right]^{\frac{1}{1-m}}\right). \tag{85}$$

We recall that $\omega_d = |\mathbb{S}^{d-1}| = \frac{2\,\pi^{d/2}}{\Gamma(d/2)}$. Inequality (84) is obtained by aplying Lemma 8 with $R = 1/3$, $r = 1/3$ and $\rho = 2$. Integrating inequality (84) over $[0,1]$ we find

$$\|\hat{u}\|_{\mathrm{L}^1((0,1]\times B_{2/3})} \leq 2^{\frac{m}{1-m}} \int_{B_1} \hat{u}_0\,dx + \tfrac{1-m}{2-m}\,\mathscr{C}\,. \tag{86}$$

We deduce from inequalities (83)-(86) that

$$\sup_{y\in B_{1/2}(x)} \hat{u}(1,y) \leq \mathfrak{C}\left[2^{\frac{m}{1-m}}\left(\int_{B_1} \hat{u}_0\,dx\right) + \tfrac{1-m}{2-m}\,\mathscr{C} + \left(\tfrac{2}{3}\right)^d \frac{\omega_d}{d}\right]^{\frac{2}{\beta}}\,. \tag{87}$$

where β is as in (79). Let us define

$$\overline{\kappa} := \mathfrak{C}\left[2^{\frac{m}{1-m}} + \tfrac{1-m}{2-m}\,\mathscr{C} + \left(\tfrac{2}{3}\right)^d \frac{\omega_d}{d}\right]^{\frac{2}{\beta}}\,,$$

with $\mathfrak{C}$ given in (82) and $\mathscr{C}$ in (85). We first prove inequality (72) assuming

$$\tau \geq \tau_\star := R^\alpha \|u_0\|^{1-m}_{\mathrm{L}^1(B_R)}\,,$$

which, by (75), is equivalent to the assumption $\|\hat{u}_0\|_{\mathrm{L}^1(B_1)} \le 1$. Indeed, together with (87), we get

$$\sup_{y\in B_{R/2}} u(\tau,y) \le \overline{\kappa}\left(\frac{\tau}{R^2}\right)^{\frac{1}{1-m}} \le \overline{\kappa}\left(\frac{1}{\tau^{\frac{d}{\alpha}}}\|u_0\|_{\mathrm{L}^1(B_R)}^{\frac{2}{\alpha}} + \left(\frac{\tau}{R^2}\right)^{\frac{1}{1-m}}\right), \tag{88}$$

which is exactly (72). Now, for any $0 < t \le \tau_\star$, we use the time monotonicity estimate

$$u(\tau) \le u(\tau_\star)\left(\frac{\tau_\star}{\tau}\right)^{\frac{d}{\alpha}}$$

obtained by integrating in time the estimate $u_t \ge -(d/\alpha)(u/t)$ of Aronson and Benilan (see [1]). Combined with the estimate (88) at time $\tau_\star$, this leads to

$$\begin{aligned}\sup_{y\in B_{R/2}} u(\tau,y) &\le \sup_{y\in B_{R/2}} u(\tau_\star,y)\left(\frac{\tau_\star}{\tau}\right)^{\frac{d}{\alpha}} \le \overline{\kappa}\left(\frac{\tau_\star}{R^2}\right)^{\frac{1}{1-m}}\left(\frac{\tau_\star}{\tau}\right)^{\frac{d}{\alpha}} \\ &= \overline{\kappa}\frac{\|u_0\|_{\mathrm{L}^1(B_R)}^{\frac{2}{\alpha}}}{\tau^{\frac{d}{\alpha}}} \le \overline{\kappa}\left(\frac{1}{\tau^{\frac{d}{\alpha}}}\|u_0\|_{\mathrm{L}^1(B_R)}^{\frac{2}{\alpha}} + \left(\frac{\tau}{R^2}\right)^{\frac{1}{1-m}}\right)\end{aligned}$$

and concludes the proof. □

3.3 A comparison result based on the Aleksandrov reflection principle

In this section, we use the Aleksandrov reflection principle, a key tool for proving lower bounds (see Lemma 11).

Proposition 10. *Let $B_{\lambda R}(x_0) \subset \mathbb{R}^d$ be an open ball with center in $x_0 \in \mathbb{R}^d$ of radius λR with $R > 0$ and $\lambda > 2$. Let u be a solution to problem*

$$\begin{cases} u_t = \Delta(u^m) & \text{in } (0,+\infty)\times\mathbb{R}^d\,, \\ u(0,x) = u_0(x) & x \in \mathbb{R}^d\,. \end{cases} \tag{89}$$

with $\mathrm{supp}(u_0) \subset B_R(x_0)$. *Then, one has:*

$$u(t,x_0) \ge u(t,x) \tag{90}$$

for any $t > 0$ and for any $x \in D_{\lambda,R}(x_0) = B_{\lambda R}(x_0) \setminus B_{2R}(x_0)$. Hence,

$$u(t,x_0) \ge |D_{\lambda,R}(x_0)|^{-1}\int_{D_{\lambda,R}(x_0)} u(t,x)\,\mathrm{d}x\,. \tag{91}$$

We use the mean value inequality (91) in following form:

$$\int_{B_{2R+r}(x_0)\setminus B_{2^bR}(x_0)} u(t,x)\,\mathrm{d}x \le A_d\, r^d\, u(t,x_0)\,, \tag{92}$$

with $b = 2 - (1/d)$, $r > 2\,R(2^{1-\frac{1}{d}} - 1) =: r_0$ and a suitable positive constant A_d. This inequality can easily be obtained from (91). Let us first assume $d \geq 2$, note that in this case $b - 1 \geq 1/2$ and therefore $r \geq 2\,R\left(\sqrt{2} - 1\right)$. By Taylor expansion we obtain that for some $\xi \in (r_0, r)$ that

$$|B_{2\,R+r}(x_0) \setminus B_{2^b R}(x_0)| = \frac{\omega_d}{d}\left[(2\,R + r)^d - 2^{bd}\,R^d\right] = \omega_d\,(2\,R + \xi)^{d-1}\,(r - r_0)$$
$$\leq \omega_d\,(2\,R + \xi)^{d-1}\,r \leq \omega_d\,r^d \left(\frac{\sqrt{2}}{\sqrt{2} - 1}\right)^{d-1},$$

a simple computation shows that $\sqrt{2}/\left(\sqrt{2} - 1\right) \approx 3.4142135 \leq 4$. In the case $d = 1$, we have that $b = 1$ and thefore

$$|B_{2\,R+r}(x_0) \setminus B_{2\,R}(x_0)| = \omega_1\,r\,.$$

In conclusion we obtain that, for $r \geq 2\,R\,(2^{1-\frac{1}{d}} - 1)$, we have

$$|B_{2\,R+r}(x_0) \setminus B_{2^b R}(x_0)| \leq A_d\,r^d \quad \text{where} \quad A_d := \omega_d\,4^{d-1}\,. \tag{93}$$

Proof. This proof borrows some ideas of [21]. Without loss of generality we may assume that $x_0 = 0$ and write B_R instead of $B_R(0)$. Let us recall that the support of u_0 is contained in B_R. Let us consider an hyperplane Π of equation $\Pi = \{x \in \mathbb{R}^d \mid x_1 = a\}$ with $a \geq R > 0$, in this way Π is tangent to the the sphere of radius a centered in the origin. Let us as well define $\Pi_+ = \{x \in \mathbb{R}^d \mid x_1 > a\}$ and $\Pi_- = \{x \in \mathbb{R}^d \mid x_1 < a\}$, and the reflection $\sigma(z) = \sigma(z_1, z_2, \ldots, z_n) = (2a - z_1, z_2, \ldots, z_n)$. By these definitions we have that $\sigma(\Pi_+) = \Pi_-$ and $\sigma(\Pi_-) = \Pi_+$. Let us denote $Q = (0, \infty) \times \Pi_-$ and the parabolic boundary $\partial_p Q := \partial Q$. We now consider the *Boundary Value Problem* (BVP) defined as

$$\begin{cases} u_t = \Delta(u^m) & \text{in } Q, \\ u(t,x) = g(t,x) & \text{in } \partial_p Q, \end{cases} \tag{BVP}$$

for some (eventually continuous) function $g(t,x)$. Let us define $u_1(t,x)$ to be the restriction of $u(t,x)$ to Q and $u_2(t,x) = u_1(t, \sigma(x))$. We recall that $u_2(t,x)$ is still a solution to problem (89). Also, both $u_1(t,x)$ and $u(t,x)$ are solutions to (BVP) with boundary values $g_1(t,x)$ and $g_2(t,x)$. Furthermore, for any $t > 0$ and for any $x \in \Pi$, we have that $g_1(t,x) = g_2(t,x)$, as well $g_1(t,x) = u_0 \geq g_2(t,x) = 0$ for any $x \in \Pi_-$. By comparison principle we obtain for any $(t,x) \in Q$

$$u_1(t,x) \geq u_2(t,x)\,. \tag{94}$$

The comparison principle for generic boundary value problems is classical in the literature, however we were not able to find the exact reference for a version on a hyperplane. We refer to the books [14, 30, 29, 21], see also [23, Lemma 3.4] for a very similar comparison principle, and also [2, Remark 1.5] for a general remark about such principles.

Inequality (94) implies for any $t > 0$ that

$$u(t,0) \geq u(t,(2a,\ldots,0)).$$

By moving a in the range $(R, \lambda R/2)$ we find that $u(t,0) \geq u(t,x)$ for any $x \in D_{\lambda,R}$ such that $x = (x_1, 0, \ldots, 0)$. It is clear that by rotating the hyperplane Π we can generalize the above argument and obtain inequality (90). Lastly, we observe that inequality (91) can be easily deduced by averaging inequality (90). The proof is complete. □

3.4 Local lower bounds

We recall Lemma [9, Lemma 6] which follows from [12, Theorem 1.1].

Lemma 11 (test)**.** *Let $u(t,x)$ be a solution to (54) and let $R > 0$ such that $M_R(x_0) := \|u_0\|_{\mathrm{L}^1(B_R(x_0))} > 0$. Then the inequality*

$$\inf_{|x-x_0|\leq R} u(t,x) \geq \kappa \left(R^{-2}\, t\right)^{\frac{1}{1-m}} \quad \forall\, t \in [0, 2\,\underline{t}] \tag{95}$$

holds with

$$\underline{t} = \tfrac{1}{2}\,\kappa_\star\, M_R^{1-m}(x_0)\, R^\alpha\,.$$

This estimate is based on the results of Sections 3.1 and 3.3. Our contribution here is to establish that the constants are

$$\kappa_\star = 2^{\,3\,\alpha+2}\, d^{\alpha} \quad \text{and} \quad \kappa = \alpha\,\omega_d \left(\frac{(1-m)^4}{2^{38}\, d^{\,4}\, \pi^{16\,(1-m)\,\alpha}\, \overline{\kappa}^{\,\alpha^2\,(1-m)}}\right)^{\frac{2}{(1-m)^2\,\alpha\, d}}\,. \tag{96}$$

Proof. Without loss of generality we assume that $x_0 = 0$. The proof is a combination of several steps. Different positive constants that depend on m and d are denoted by C_i.

Step 1. Reduction. By comparison we may assume $\mathrm{supp}(u_0) \subset B_R(0)$. Indeed, a general $u_0 \geq 0$ is greater than $u_0\chi_{B_R}$, χ_{B_R} being the characteristic function of B_R. If v is the solution of the fast diffusion equation with initial data $u_0\chi_{B_R}$ (existence and uniqueness are well known in this case), then we obtain by comparison:

$$\inf_{x\in B_R} u(t,x) \geq \inf_{x\in B_R} v(t,x)\,.$$

Step 2. A priori estimates. The so called smoothing effect (see e.g. [23, Theorem 2.2] , or [29]) asserts that, for any $t > 0$ and $x \in \mathbb{R}^d$, we have:

$$u(t,x) \leq \overline{\kappa}\, \frac{\|u_0\|_1^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}}\,. \tag{97}$$

where $\alpha = 2-d(1-m)$. We remark that (97) can be deduced from inequality (72) of Lemma 9 by simply takin the limit $R \to \infty$. The explicit expression of the constant $\overline{\kappa}$ is given in (73). We remark that $\|u_0\|_1 = M_R$ since u_0 is nonnegative and supported in B_R, so that we get $u(t,x) \le \overline{\kappa} M_R^{\frac{2}{\alpha}} t^{-\frac{d}{\alpha}}$. Let $b = 2 - 1/d$, an integration over $B_{2^b R}$ gives then:

$$\int_{B_{2^b R}} u(t,x)\,\mathrm{d}x \le \overline{\kappa}\,\frac{\omega_d}{d}\,\frac{M_R^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}}\left(2^b\,R\right)^d \le C_2\,\frac{M_R^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}}\,R^d\,, \tag{98}$$

where C_2 can be chosen as

$$C_2 := 2^d\,\max\left\{1, \overline{\kappa}\,\frac{\omega_d}{d}\right\}. \tag{99}$$

Step 3. In this step we use the so-called *Aleksandrov reflection principle*, see Proposition 10 in section 3.3 for its proof. This principle reads:

$$\int_{B_{2R+r}\setminus B_{2^b R}} u(t,x)\,\mathrm{d}x \le A_d\,r^d u(t,0) \tag{100}$$

where A_d is as in (93) and $b = 2 - 1/d$. One has to remember of the condition

$$r \ \ge\ (2^{(d-1)/d} - 1)\,2\,R. \tag{101}$$

We refer to Proposition 10 and formula (92) in section 3.3 for more details.

Step 4. Integral estimate. Thanks to Lemma 8, for any $R, r > 0$ and $s, t \ge 0$ one has

$$\int_{B_{2R}} u(s,x)\,\mathrm{d}x \le C_3\left[\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x + \frac{|s-t|^{1/(1-m)}}{r^{(2-d(1-m))/(1-m)}}\right],$$

where the constant C_3 has to satisfy $C_3 \ge \max(1, \mathsf{c}_3)$ and c_3 is defined in (68). In what follows we prefer to take a larger constant (for reasons that will be clarified later) and put

$$C_3 = \left(\frac{16}{1-m}\right)^{\frac{1}{1-m}} \max\left(1, 2\,\omega_d\,\left[\frac{16\,(d+1)\,(3+m)}{1-m}\right]^{\frac{1}{1-m}}\right).$$

We let $s = 0$ and rewrite it in a form more useful for our purposes:

$$\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x \ge \frac{M_R}{C_3} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{\alpha}{1-m}}}\,. \tag{102}$$

We recall that $M_{2R} = M_R$ since u_0 is nonnegative and supported in B_R.

Step 5. We now put together all previous calculations:

$$\begin{aligned}\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x &= \int_{B_{2R}} u(t,x)\,\mathrm{d}x + \int_{B_{2R+r}\setminus B_{2^b R}} u(t,x)\,\mathrm{d}x \\ &\le C_2\,\frac{M_R^{\frac{2}{\alpha}}\,R^d}{t^{\frac{d}{\alpha}}} + A_d\,r^d\,u(t,0)\,.\end{aligned}$$

This follows from (98) and (100). Next, we use (102) to obtain:

$$\frac{M_R}{C_3} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{\alpha}{1-m}}} \le \int_{B_{2\,R+r}} u(t,x)\,\mathrm{d}x \le C_2\,\frac{M_R^{\frac{2}{\alpha}}\,R^d}{t^{\frac{d}{\alpha}}} + A_d\,r^d u(t,0)\,.$$

Finally we obtain

$$u(t,0) \ge \frac{1}{A_d}\left[\left(\frac{M_R}{C_3} - C_2\,\frac{M_R^{\frac{2}{\alpha}}\,R^d}{t^{\frac{d}{\alpha}}}\right)\frac{1}{r^d} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2}{1-m}}}\right] = \frac{1}{A_d}\left[\frac{B(t)}{r^d} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2}{1-m}}}\right].$$

Step 6. The function $B(t)$ is positive when

$$B(t) = \frac{M_R}{C_3} - C_2\,\frac{M_R^{\frac{2}{\alpha}}\,R^d}{t^{\frac{d}{\alpha}}} > 0 \Longleftrightarrow t > (C_3\,C_2)^{\frac{\alpha}{d}}\;.M_R^{1-m}R^\alpha$$

Let us define

$$\tilde{\kappa}_\star := 4\;(C_3\,C_2)^{\frac{\alpha}{d}} \quad \text{and} \quad \underline{\tilde{t}} = \tfrac{1}{2}\,\tilde{\kappa}_\star\,M_R^{1-m}\,R^\alpha\,. \tag{103}$$

We assume that $t \ge 2\,\underline{\tilde{t}}$ and optimize the function

$$f(r) = \frac{1}{A_d}\left[\frac{B(t)}{r^d} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2}{1-m}}}\right]$$

with respect to $r(t) = r > 0$. The function f reaches its maximum at $r = r_{max}(t)$ given by

$$r_{max}(t) = \left(\frac{2}{d\,(1-m)}\right)^{\frac{1-m}{\alpha}}\;\frac{t^{\frac{1}{\alpha}}}{B(t)^{\frac{1-m}{\alpha}}}\,.$$

We recall that we have to verify that r_{max} satisfies condition (101), namely that $r_{max}(t) > \left(2^{(d-1)/d} - 1\right)2\,R$. To check this we optimize in t the function $r_{max}(t)$ with respect to $t \in (2\underline{\tilde{t}}, +\infty)$. The minimum of $r_{max}(t)$ is attained at a time $t = t_{min}$ given by

$$t_{min} = \left(\frac{2}{\alpha}\,C_2\,C_3\right)^{\frac{\alpha}{d}}\;M_R^{1-m}\,R^\alpha\,.$$

We compute $r_{max}(t_{min})$ and find that

$$r_{max}(t_{min}) = \left(\frac{2}{d\,(1-m)}\right)^{\frac{2\,(1-m)}{\alpha}}\;\left(\frac{2}{\alpha}\,C_2\right)^{\frac{1}{d}}\;C_3^{\frac{2}{d\alpha}}\,R\,.$$

Therefore the condition $r_{max}(t_{min}) > \left(2^{(d-1)/d} - 1\right)2\,R$ is nothing more than a lower bound on the constants C_2 and C_3, namely that

$$\left(\frac{2}{d\,(1-m)}\right)^{\frac{2\,(1-m)}{\alpha}}\;\left(\frac{2}{\alpha}\,C_2\right)^{\frac{1}{d}}\;C_3^{\frac{2}{d\alpha}} \ge 2^{(d-1)/d} - 1\,.$$

Such a lower bound is easily verified, by using the fact $m \in (m_1, 1)$, we have $(1-m)^{-1} > d$ and therefore we have the following inequalities

$$\frac{2}{d\,(1-m)} \geq 2\,, \quad \frac{2}{\alpha} = \frac{2}{2-d\,(1-m)} \geq 1\,, \quad C_2 \geq 2^d \quad \text{and} \quad C_3 \geq 16^d\, d^d\,, \quad (104)$$

therefore, from the above inequalities we find that

$$\left(\frac{2}{d\,(1-m)}\right)^{\frac{2\,(1-m)}{\alpha}} \left(\frac{2}{\alpha}\,C_2\right)^{\frac{1}{d}} C_3^{\frac{2}{d\alpha}} \geq 32\,d \geq 2^{(d-1)/d} - 1\,,$$

and so the such a lower bound is verified. Let us now continue with the proof.

Step 7. After a few straightforward computations, we show that the maximum value is attained for all $t > 2\,\underline{\tilde{t}}$ as follows:

$$f(r_{max}) = \alpha\, A_d \frac{[d\,(1-m)]^{\frac{d\,(1-m)}{\alpha}}}{2^{\frac{2}{\alpha}}} \left[\frac{1}{C_3} - C_2\,\frac{M_R^{\frac{d\,(1-m)}{\alpha}} R^d}{t^{\frac{d}{\alpha}}}\right]^{\frac{2}{\alpha}} \frac{M_R^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}} > 0\,.$$

We get in this way the estimate:

$$u(t,0) \geq K_1\, H_1(t)\, \frac{M_R^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}}\,,$$

where

$$H_1(t) = \left[\frac{1}{C_3} - C_2\,\frac{M_R^{\frac{d\,(1-m)}{\alpha}} R^d}{t^{\frac{d}{\alpha}}}\right]^{\frac{2}{\alpha}} \quad \text{and} \quad K_1 = \alpha\, A_d \frac{[d\,(1-m)]^{\frac{d\,(1-m)}{\alpha}}}{2^{\frac{2}{\alpha}}}.$$

A straightforward calculation shows that the function is non-decreasing in time, thus if $t \geq 2\,\underline{\tilde{t}}$:

$$H_1(t) \geq H_1(2\,\underline{\tilde{t}}) = C_3^{-\frac{2}{\alpha}} \left(1 - 4^{-\frac{d}{\alpha}}\right)^{\frac{2}{\alpha}}\,,$$

and finally we obtain for $t \geq 2\,\underline{\tilde{t}}$ that

$$u(t,0) \geq \; K_1\, C_3^{-\frac{2}{\alpha}} \left(1 - 4^{-\frac{d}{\alpha}}\right)^{\frac{2}{\alpha}} \frac{M_R^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}} = \underline{\tilde{\kappa}}\, \frac{M_R^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}}\,. \quad (105)$$

Step 8. From the center to the infimum. Now we want to obtain a positivity estimate for the infimum of the solution u in the ball $B_R = B_R(0)$. Suppose that the infimum is attained in some point $x_m \in \overline{B_R}$, so that $\inf_{x\in B_R} u(t,x) = u(t,x_m)$, then one can apply (105) to this point and obtain:

$$u(t,x_m) \geq \underline{\tilde{\kappa}}\, \frac{M_{2\,R}(x_m)^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}}$$

for $t > \tilde{\kappa}_\star M_R^{1-m}(x_m) R^\alpha$. Since the point $x_m \in \overline{B_R(0)}$ then it is clear that $B_R(0) \subset B_{2\,R}(x_m) \subset B_{4R}(x_0)$, and this leads to the inequality:

$$M_{2\,R}(x_m) \ge M_R(0) \quad \text{and} \quad M_{2\,R}(x_m) \le M_{4R}(0)$$

since $M_\varrho(y) = \int_{B_\varrho(y)} u_0(x)\,\mathrm{d}x$ and $u_0 \ge 0$. Thus, we have found that:

$$\inf_{x\in B_R(0)} u(t,x) = u(t,x_m) \ge \underline{\tilde{\kappa}}\, \frac{M_{2\,R}^{\frac{2}{\alpha}}(x_m)}{t^{\frac{d}{\alpha}}} \ge \underline{\tilde{\kappa}}\, \frac{M_{2\,R}^{\frac{2}{\alpha}}(0)}{t^{\frac{d}{\alpha}}} = \underline{\tilde{\kappa}}\, \frac{M_R^{\frac{2}{\alpha}}(0)}{t^{\frac{d}{\alpha}}} .$$

for $t > 2\,\underline{\tilde{t}}(0) = \tilde{\kappa}_\star M_{4R}^{1-m}(0) R^\alpha = \tilde{\kappa}_\star\, M_R^{1-m}(0) R^\alpha$, after noticing that $M_{4R}(0) = M_{2\,R}(0) = M_R(0)$, since $\mathrm{supp}(u_0) \subset B_R(0)$. Finally we obtain the claimed estimate

$$\inf_{x\in B_R(0)} u(t,x) \ge \underline{\tilde{\kappa}}\, \frac{M_R^{\frac{2}{\alpha}}}{t^{\frac{d}{\alpha}}} \quad \forall\, t \ge 2\,\underline{\tilde{t}} .$$

Step 9. The last step consists in obtaining a lower estimate when $0 \le t \le 2\,\underline{\tilde{t}}$. To this end we consider the fundamental estimate of Bénilan-Crandall [4]:

$$u_t(t,x) \le \frac{u(t,x)}{(1-m)t} .$$

This easily implies that the function:

$$u(t,x) t^{-1/(1-m)}$$

is non-increasing in time. Thus, for any $t \in (0, 2\,\underline{\tilde{t}})$, we have that

$$u(t,x) \ge u(2\,\underline{t}, x)\, \frac{t^{1/(1-m)}}{(2\,\underline{\tilde{t}})^{1/(1-m)}} \ge \underline{\tilde{\kappa}}\, \tilde{\kappa}_\star^{-\frac{2}{1-m}} \left(t\, R^{-2}\right)^{\frac{1}{1-m}} .$$

which is exactly inequality (95). It is straightforward to verify that the constant $\tilde{\kappa}$ has the value

$$\tilde{\kappa} = \underline{\tilde{\kappa}}\, \tilde{\kappa}_\star^{-\frac{2}{1-m}} = \alpha\, A_d \frac{\left[d\,(1-m)\right]^{\frac{d\,(1-m)}{\alpha}}}{2^{\frac{2}{\alpha}}}\, C_3^{-\frac{2}{\alpha}} \left(1 - 4^{-\frac{d}{\alpha}}\right)^{\frac{2}{\alpha}} \tilde{\kappa}_\star^{-\frac{2}{1-m}} . \tag{106}$$

Step 10. Simplification of the constants. In this step we are going to simplify the expression of some constants in order to obtain the expression in (96). This translates into estimates from below of the actual values of constants $\underline{\tilde{\kappa}}$ and $\tilde{\kappa}_\star$, and in order to do so, we need to estimate C_2 and C_3. Let us begin with C_2, since we only need an estimate from below. We learn from (10) that $\omega_d/d \le \pi^2$ for any $d \ge 1$. It is then clear from (99) that

$$2^d \le C_2 \le 2^d\, \overline{\kappa}\, \pi^2 .$$

In the case of C_3 we already have a lower bound given in (104), in what follows we compute the upper bound. Let us recall that from (9) we have that for any $d \geq 1$, $\omega_d \leq 16\pi^3/15$. Since $m < 1$, we have that

$$16\,(d+1)\,(3+m) \leq 64\,(d+1) \leq 128\,d\,.$$

Combining the above inequality, with the estimates on ω_d and the defintion of C_3 we get

$$(4d)^d \leq C_3 \leq \left(\frac{128\,d}{1-m}\right)^{\frac{2}{1-m}} 4\,\pi^3\,.$$

Therefore, we can estimate $\tilde{\kappa}_\star$ and obtain the expression of $\kappa_\star$

$$\tilde{\kappa}_\star = 4\,(C_2\,C_3)^{\frac{\alpha}{d}} \geq 2^2\left(2^{5\,d}\,d^d\right)^{\frac{\alpha}{d}} = 2^{3\,\alpha+2}\,d^\alpha =: \kappa_\star\,.$$

Let us simplify $\tilde{\kappa}$. By combining (106), (103) and (93), we get that

$$\tilde{\kappa} \geq \alpha\,\omega_d\,2^{2d-2-\frac{2\,(1-m)+4\,\alpha}{\alpha(1-m)}}\,[d\,(1-m)]^{\frac{d\,(1-m)}{\alpha}}\,C_3^{-\frac{4}{\alpha\,d\,(1-m)}}\,C_2^{-\frac{2\,\alpha}{d\,(1-m)}}\left(1-4^{-\frac{d}{\alpha}}\right)^{\frac{2}{\alpha}}\,.$$

Let us begin simplifying the expression $\left(1-4^{-\frac{d}{\alpha}}\right)$. We first notice that, since $\alpha \in (1,2)$, we have that $1-4^{-\frac{d}{\alpha}} \geq 1-4^{-\frac{d}{2}}$, which is an expression monotone increasing in d. We have therefore that

$$\left(1-4^{-\frac{d}{\alpha}}\right)^{\frac{2}{\alpha}} \geq \left(1-4^{-\frac{1}{2}}\right)^{\frac{2}{\alpha}} = 2^{-\frac{2}{\alpha}}\,.$$

Combining all together we find

$$\tilde{\kappa} \geq \alpha\,\omega_d\,2^{-\mathfrak{a}}\,\pi^{-\mathfrak{b}}\,\overline{\kappa}^{-\frac{2\,\alpha}{d\,(1-m)}}\,d^{\frac{d\,(1-m)}{\alpha}-\frac{8}{\alpha(1-m)^2 d}}\,(1-m)^{\frac{d^2\,(1-m)^3+8}{\alpha\,(1-m)^2\,d}}\,,$$

where

$$\mathfrak{a} = \frac{56+8\,(1-m)+2\,\alpha^2\,d\,(1-m)+2\,\alpha\,(1-m)^2\,d}{\alpha\,(1-m)^2\,d} - 2\,d \quad \text{and} \quad \mathfrak{b} = \frac{12+4\,\alpha^2}{d\,(1-m)}\,.$$

Since $m_1 < m < 1$, and $d\,(1-m) < 1$, we can simplify the expression of $\mathfrak{a}$ and $\mathfrak{b}$ into

$$\mathfrak{a} \leq \frac{76}{\alpha\,(1-m)^2\,d} \quad \text{and} \quad \mathfrak{b} \leq \frac{32}{d\,(1-m)}\,.$$

By summing up all estimates above and estimating the exponents of $(1-m)$ and d, we get

$$\tilde{\kappa} \geq \frac{\alpha\,\omega_d\left(\frac{1-m}{d}\right)^{\frac{8}{\alpha\,(1-m)^2\,d}}}{2^{\frac{76}{\alpha\,(1-m)^2\,d}}\,\pi^{\frac{32}{d\,(1-m)}}\,\overline{\kappa}^{\frac{2\,\alpha}{d\,(1-m)}}} = \kappa\,.$$

□

3.5 Details on the inner estimate in terms of the free energy

Let us recall the result of [9, Propostion 11].

Proposition 12. *Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough and $G > 0$ be given. There exist a numerical constant $\mathsf{K} > 0$ and an exponent $\vartheta \in (0,1)$ such that, for any $t \geq 4\,T(\varepsilon)$, any solution u of (54) with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} \mathcal{B}\,dx$ and $\mathcal{F}[u_0] \leq G$, then satisfies*

$$\left|\frac{u(t,x)}{B(t,x)} - 1\right| \leq \frac{\mathsf{K}}{\varepsilon^{\frac{1}{1-m}}} \left(\frac{1}{t} + \frac{\sqrt{G}}{R(t)}\right)^{\vartheta} \quad \text{if} \quad |x| \leq 2\,\rho(\varepsilon)\,R(t)\,. \tag{107}$$

The values of the parameters $\rho(\varepsilon)$, $T(\varepsilon)$can be found in Appendix A.1, where the expression of

$$\begin{aligned}\mathsf{K} := 2^{\frac{3\,d}{\alpha} + \frac{3+6\,\alpha}{\alpha\,(1-m)} + \vartheta + 10}\,\frac{(\alpha + \mathcal{M})^{\vartheta}}{m^{\vartheta}(1-m)^{2\,(1+\vartheta) + \frac{2}{1-m}}} \\ \times \left[1 + \mathsf{b}^d\,C_{d,\nu,1}\left(\left(\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,\frac{2^{\nu}}{2^{\nu}-1} + c_2\right)^{\frac{d}{d+\nu}} + \frac{\mu^{2d}}{\alpha^{\frac{d}{\alpha}}}\,\mathcal{M}^{\frac{d}{d+\nu}}\right)\right]\end{aligned}$$

is also detailed.

Proof. We refer to [9] for the main steps of the proof and focus on the details of the computation of K. We have to explain how to compute the constant c_1, to prove that λ_0 and λ_1 are bounded and bounded away from zero and to obtain the final form of the constant $\overline{C}$ which finally allows us to compute K.

Computation of c_1. Let us recall the cylinders

$$Q_1 := (1/2, 3/2) \times B_1(0)\,, \quad Q_2 := (1/4, 2) \times B_8(0)\,, \tag{108}$$

$$Q_3 := (1/2, 3/2) \times B_1(0) \setminus B_{1/2}(0) \quad \text{and} \quad Q_4 := (1/4, 2) \times B_8(0) \setminus B_{\frac{1}{4}}(0)\,. \tag{109}$$

and let us assume that v is a solution to (1) which satisfies (2) for some $0 < \lambda_0 \leq \lambda_1 < \infty$. In what follows we shall explain how to compute the constant c_1 in:

[9, Inequality (66)],

$$\sup_{(t,x),(s,y) \in Q_i} \frac{|v(t,x) - v(s,y)|}{\left(|x-y| + |t-s|^{1/2}\right)^{\nu}} \leq c_1\,\|v\|_{\mathrm{L}^\infty(Q_{i+1})} \quad \forall\, i \in \{1,2\}\,.$$

By applying Theorem 6 it is clear that the only ingredient needed is to estimate from below $d(Q_1, Q_2)$ and $d(Q_3, Q_4)$, where $d(\cdot,\cdot)$ is defined in (49). Let us consider the case of $d(Q_1, Q_2)$. By symmetry, it is clear that the infimum in (49) is achieved by a couple of points $(t,x)\in\overline{Q_1}$, $(s,y)\in\partial Q_2$ such that either $|x|=1, t\in(1/2,3/2)$ and $|y|=8, s\in(1/2,3/2)$ or $t=1/2, y=1/4$ and $x,y\in B_1$. In both cases we have that $d(Q_1,Q_2)=|x-y|+|t-s|^{\frac12}\ge 1/4$. By a very similar argument we can also conclude that $d(Q_3,Q_4)\ge 1/4$. Therefore, we conclude that, in both cases, c_1 can be taken (accordingly to inequality (50)).

$$2\,(128)^{\nu}\max\left\{\frac{1}{d(Q_1,Q_2)}\,,\frac{1}{d(Q_3,Q_4)}\right\}^{\nu}\le 2\,(512)^{\nu}\le 2^{10}=:c_1\,,$$

where we have used the fact that $\nu\in(0,1)$.

Estimates of λ_0 and λ_1 of [9, formula (73)]. In *Step 2* of [9, Proposition 11], we consider a solution $u(t,x)$ to (54) as a solution to the linear equation (1) with coefficients

$$a(t,x)=m\,u^{m-1}(t,x)\,,\quad A(t,x)=a(t,x)\,\mathrm{Id}\,,$$

where Id is the identity matrix on $\mathbb{R}^d$. We also observe that $u(t,x)$ (and its rescaled version $\hat{u}_{\tau,k}$) satisfies the condition (2) (with the coefficient $a(t,x)$ given by the above expression above) and with

[9, Definition (73)]

$$\lambda_0^{\frac{1}{m-1}}:=m^{\frac{1}{m-1}}\,\overline{C}\,\max\left\{\sup_{Q_2}B(t-\tfrac{1}{\alpha},x),\,\sup_{k\ge1}\sup_{Q_4}B(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\mathcal{M})\right\},$$
$$\lambda_1^{\frac{1}{m-1}}:=m^{\frac{1}{m-1}}\,\underline{C}\,\min\left\{\inf_{Q_2}B(t-\tfrac{1}{\alpha},x)\,,\inf_{k\ge1}\inf_{Q_4}B(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\mathcal{M})\right\}.$$

where Q_i are as in (108). Our task here is to give an estimate on λ_0, λ_1 and to show that they are bounded and bounded away from zero. Let us consider first the case of $B(t-\frac{1}{\alpha},x)$: for any $(t,x)\in(0,\infty)\times\mathbb{R}^d$, we have that

$$B(t-\tfrac{1}{\alpha},x)=\frac{t^{\frac{1}{1-m}}}{\mathsf{b}^{\frac{\alpha}{1-m}}}\left(\frac{t^{\frac{2}{\alpha}}}{\mathsf{b}^2}+|x|^2\right)^{\frac{1}{m-1}}\quad\text{where}\quad \mathsf{b}=\left(\tfrac{1-m}{2\,m\,\alpha}\right)^{\frac{1}{\alpha}}\,.$$

We deduce therefore that, for any $(t,x)\in Q_2$, we have that

$$\frac{1}{4^{\frac{1}{1-m}}\,\mathsf{b}^{\frac{\alpha}{1-m}}}\left(\frac{2^{\frac{2}{\alpha}}}{\mathsf{b}^2}+2^6\right)^{\frac{1}{m-1}}\le B(t-\tfrac{1}{\alpha},x)\le \mathsf{b}^d\,4^{\frac{d}{\alpha}}\,.$$

This is enough to prove that $\lambda_0>0$ and $\lambda_1<\infty$. Let us consider $B(t-\frac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\mathcal{M})$, we recall that

$$B(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\mathcal{M})=\frac{t^{\frac{1}{1-m}}}{\mathsf{b}^{\frac{\alpha}{1-m}}}\left(\frac{t^{\frac{2}{\alpha}}}{k^2\,\mathsf{b}^2}+|x|^2\right)^{\frac{1}{m-1}}\,.$$

Let us consider $(t,x)\in Q_4$, we have therefore

$$\frac{1}{4^{\frac{1}{1-m}}\,\mathsf{b}^{\frac{\alpha}{1-m}}}\left(\frac{2^{\frac{2}{\alpha}}}{k^2\,\mathsf{b}^2}+64\right)^{\frac{1}{m-1}}\le B(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\,\mathcal{M})\le\frac{2^{\frac{1}{1-m}}}{\mathsf{b}^{\frac{\alpha}{1-m}}}\left(\frac{1}{\mathsf{b}^2\,k^2\,4^{\frac{2}{\alpha}}}+\frac{1}{16}\right)^{\frac{1}{m-1}}.$$

From the above computation we deduce that

$$\sup_{k\ge1}\sup_{Q_4}B(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\,\mathcal{M})\big\}\le\frac{2^{\frac{1}{1-m}}}{\mathsf{b}^{\frac{\alpha}{1-m}}}\left(\frac{1}{\mathsf{b}^2\,4^{\frac{2}{\alpha}}}+\frac{1}{16}\right)^{\frac{1}{m-1}},$$

while

$$\frac{1}{2^{\frac{7}{1-m}}\,\mathsf{b}^{\frac{\alpha}{1-m}}}\le\inf_{k\ge1}\inf_{Q_4}B(t-\tfrac{1}{\alpha},x;k^{\frac{\alpha}{1-m}}\,\mathcal{M})\,.$$

Combining all estimates together we obtain that

$$0<\lambda_0\le\lambda_1<\infty\,,$$

this completes the proof of this part.

Simplification of the constant $\overline{C}$. Recall that $\alpha\in(1,2)$ so that

$$\bar{c}_3+\frac{2\,\bar{c}_2}{\alpha}=\frac{1}{1-m}+\frac{2m}{2\,(1-m)^2\,\alpha^3}\le\frac{1}{1-m}+\frac{m}{(1-m)^2}=\frac{1}{(1-m)^2}$$

hence,

$$\begin{aligned}\overline{C}&=2^{\frac{d}{\alpha}}\,\left(C+\left(\bar{c}_3+\tfrac{2}{\alpha}\,\bar{c}_2\right)\right)\left(\sqrt{\tfrac{4\,\alpha\,\mathcal{M}}{m}}+\left(\bar{c}_3+\tfrac{2}{\alpha}\,\bar{c}_2\right)\mathcal{M}\right)^{\vartheta}\\&\le2^{\frac{d}{\alpha}}\,\left(C+\tfrac{1}{(1-m)^2}\right)\left(\frac{2\,\alpha}{m}+\mathcal{M}+\frac{\mathcal{M}}{(1-m)^2}\right)^{\vartheta}\\&\le2^{\frac{d}{\alpha}+\vartheta}\,\frac{(1+C)}{m^{\vartheta}(1-m)^{2(1+\vartheta)}}(\alpha+\mathcal{M})^{\vartheta}\end{aligned}$$

where the constant C is given by

$$\begin{aligned}C:=\mathsf{b}^d\left(1+4\,\mathsf{b}^2\,Z^2\,\rho(\varepsilon)^2\right)^{\frac{1}{1-m}}\,C_{d,\nu,1}&\\\times\left(\left(c_1\,4^{\frac{d}{\alpha}}\,\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,\frac{2^{\nu}}{2^{\nu}-1}+c_2\right)^{\frac{d}{d+\nu}}+\frac{1}{(2\,Z\,\rho(\varepsilon))^d}\,(2\,\mathcal{M})^{\frac{d}{d+\nu}}\right)&.\end{aligned}$$

where

$$c_1:=2^{10}\,,\quad c_2:=2\,\max\left\{\mathsf{b}\,,|||\nabla B(1-\tfrac{1}{\alpha},x)|||_{\mathrm{L}^{\infty}(\mathbb{R}^d)}\right\}\quad\text{and}\quad Z=(2\,\alpha)^{\frac{1}{\alpha}}\,.$$

For any $\varepsilon\in(0,\varepsilon_{m,d})\subset(0,1/2)$, we have that

$$\sqrt{\tfrac{(1-\varepsilon)^m(\varepsilon-\varepsilon)}{2^{1+m}}}\,\frac{1}{\mu\,\sqrt{\varepsilon}}\le\underline{\rho}(\varepsilon)=\frac{1}{\mu}\left(\left(1+(1+\varepsilon)^{1-m}\right)\frac{\left(\frac{1-\varepsilon}{1-\varepsilon}\right)^{1-m}-1}{1-(1-\varepsilon)^{1-m}}\right)^{1/2}\le\frac{2\,\sqrt{\varepsilon}}{\mu\,\sqrt{\varepsilon}}$$

and

$$\frac{1}{\sqrt{1-m}}\,\frac{1}{\mu\,\sqrt{\varepsilon}}\le\overline{\rho}(\varepsilon)=\frac{1}{\mu}\left(\frac{(1+\varepsilon)^{1-m}+1}{(1+\varepsilon)^{1-m}-1}\right)^{\frac{1}{2}}\le\frac{4}{\sqrt{1-m}}\frac{1}{\mu\,\sqrt{\varepsilon}}\,.$$

We recall that $\underline{\varepsilon}<1$, we obtain therefore that

$$\rho(\varepsilon)^2:=\max\{\overline{\rho}(\varepsilon),\underline{\rho}(\varepsilon)\}^2\le\max\left\{\frac{4}{\mu^2\,\varepsilon}\,,\,\frac{16}{(1-m)}\frac{1}{\mu^2\,\varepsilon}\right\}\le\frac{16}{(1-m)^2\,\mu^2\,\varepsilon}$$

and also, since $\varepsilon<1/2$, we have that

$$\rho(\varepsilon)^2\ge\frac{1}{(1-m)\mu^2\,\varepsilon}\ge\frac{2}{(1-m)\,\mu^2}\,.$$

Combining all above estimates together we find that

$$\begin{aligned}\left(1+4\,\mathsf{b}^2\,(2\,\alpha)^{\frac{2}{\alpha}}\,\rho(\varepsilon)^2\right)^{\frac{1}{1-m}}&\le\left(\frac{\mu^2+2^{\frac{6\,\alpha+2}{\alpha}}\mathsf{b}^2}{(1-m)^2,\mu^2\,\varepsilon}\right)^{\frac{1}{1-m}}\\&\le\frac{2^{\frac{2+6\alpha}{\alpha(1-m)}}}{(1-m)^{\frac{2}{1-m}}\varepsilon^{\frac{1}{1-m}}}\left(\frac{\mu^2+\alpha^{\frac{2}{\alpha}}\mathsf{b}^2}{\mu^2}\right)^{\frac{1}{1-m}}\\&\le\frac{2^{\frac{3+6\alpha}{\alpha(1-m)}}}{(1-m)^{\frac{2}{1-m}}\varepsilon^{\frac{1}{1-m}}}\,,\end{aligned}$$

where in the last step we have used the identity $\mu=\mathsf{b}\,\alpha^{\frac{1}{\alpha}}$.

Altogether, we finally obtain

$$\begin{aligned}\overline{C}\le&\;\frac{2^{\frac{d}{\alpha}+\frac{3+6\alpha}{\alpha(1-m)}+\vartheta}}{\varepsilon^{\frac{1}{1-m}}}\,\frac{(\alpha+\mathcal{M})^{\vartheta}}{m^{\vartheta}(1-m)^{2(1+\vartheta)+\frac{2}{1-m}}}\\&\times\left[1+\mathsf{b}^d\,C_{d,\nu,1}\left(\left(2^{10+\frac{2d}{\alpha}}\,\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,\frac{2^{\nu}}{2^{\nu}-1}+c_2\right)^{\frac{d}{d+\nu}}+\left(\frac{\mu^2}{\alpha^{\frac{1}{\alpha}}}\right)^{d}\,(2\,\mathcal{M})^{\frac{d}{d+\nu}}\right)\right]\\\le&\;\frac{2^{\frac{3d}{\alpha}+\frac{3+6\alpha}{\alpha(1-m)}+\vartheta+10}}{\varepsilon^{\frac{1}{1-m}}}\,\frac{(\alpha+\mathcal{M})^{\vartheta}}{m^{\vartheta}(1-m)^{2(1+\vartheta)+\frac{2}{1-m}}}\\&\times\left[1+\mathsf{b}^d\,C_{d,\nu,1}\left(\left(\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,\frac{2^{\nu}}{2^{\nu}-1}+c_2\right)^{\frac{d}{d+\nu}}+\frac{\mu^{2d}}{\alpha^{\frac{d}{\alpha}}}\,\mathcal{M}^{\frac{d}{d+\nu}}\right)\right]=:\frac{\mathsf{K}}{\varepsilon^{\frac{1}{1-m}}}\,.\end{aligned}$$

The proof is completed. □

A Further estimates and additional results

Here we collect additional material concerning various estimates: the "user guide" of Appendix A.1 collects the formulas needed for the computation of $t_\star$ in Theorem 13; Appendix A.2 details how the numerical value of the constant on the disk in [9, Appendix C.2] is computed; Appendix A.3 is devoted to the precise definition of the truncation functions used in Section 3.1.

A.1 A user guide for the computation of the threshold time

Let us recall what the threshold time is. The results of [9, Theorem 4] and [9, Proposition 12] can be summarized as follows.

Theorem 13 ([9]). *Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$. There is a numerical constant $\varepsilon_{m,d} \in (0, 1/2)$, a real number $\nu > 0$, and a positive numerical constant $\mathsf{c}_\star = \mathsf{c}_\star(m, d)$ with $\lim_{m\to 1_-} \mathsf{c}_\star(m, d) = +\infty$ such that the following property holds: for any $A > 0$ and $G > 0$, if u is a solution of (54) with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} \mathcal{B}\,dx$, $\mathcal{F}[u_0] \leq G$ and*

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \tag{H_A}$$

and if $\varepsilon \in (0, \varepsilon_{m,d})$, then

$$\sup_{x\in\mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq t_\star\,,$$

where

$$t_\star = \frac{\mathsf{c}_\star}{\varepsilon^{\mathsf{a}}} \left(1 + A^{1-m} + G^{\frac{\alpha}{2}}\right) \quad \text{with} \quad \mathsf{a} = \frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}\,. \tag{110}$$

We do not reproduce the proof here but establish the expression of $\mathsf{c}_\star$ by collecting all intermediate constants and formulas. We assume from now on that u is a solution of (54) which satisfies all assumptions of Theorem 13.

Let us start by recalling the definition of the main numerical parameters:

$$m_c = \frac{d-2}{d}\,,\ m_1 = \frac{d-1}{d}\,,\ \alpha = d\,(m-m_c)\,,\ \mu = \left(\frac{1-m}{2\,m}\right)^{\frac{1}{\alpha}} \text{ and } \mathsf{b} = \left(\frac{1-m}{2\,m\,\alpha}\right)^{\frac{1}{\alpha}}.$$

[9, Corollary 9]

$$u(t,x) \geq (1-\varepsilon)\,B(t\,,\,x\,) \quad \text{if} \quad |x| \geq R(t)\,\underline{\rho}(\varepsilon) \quad \text{and} \quad \varepsilon \in (0, \underline{\varepsilon})\,. \tag{49}$$

Here

$$R(t) = (1+\alpha\,t)^{1/\alpha} \qquad \text{[9, Eq. (21)]}$$

and

$$\underline{\varepsilon} := 1 - (\underline{M}\,/\mathcal{M})^{\frac{2}{\alpha}}$$

is a numerical constant which is computed from

$$\underline{M} := \min\left\{2^{-d/2}\left(\frac{\kappa}{\mathsf{b}^d}\right)^{\alpha/2}, \frac{\kappa}{\big(d\,(1-m)\big)^{d/2}\,\alpha^{\frac{\alpha}{2\,(1-m)}}}\right\} \kappa_\star^{\frac{1}{1-m}}\,\mathcal{M}^2\,. \qquad \text{[9, Eq. (47)]}$$

We recall that κ and $\kappa_\star$ have been defined in (96) and are given by

$$\kappa_\star = 2^{3\,\alpha+2}\,d^\alpha \quad \text{and} \quad \kappa = \alpha\,\omega_d \left(\frac{(1-m)^4}{2^{38}\,d^4\,\pi^{16\,(1-m)\,\alpha}\,\overline{\kappa}^{\alpha^2\,(1-m)}}\right)^{\frac{2}{(1-m)^2\,\alpha\,d}} .$$

As a byproduct of [9, Proposition 8], by integrating over $\mathbb{R}^d$, we deduce from

$$u(t,x) \ge B\Big(t-\underline{t}-\tfrac{1}{\alpha}\,,\,x\,;\,\underline{M}\Big) \qquad [9, \text{Eq. } (38)]$$

that $\underline{M}\,/\mathcal{M} < 1$, which proves that $\underline{\varepsilon} > 0$. The two other constants of Corollary 9 are given by

$$\underline{\rho}(\varepsilon) := \frac{1}{\mu}\left(\Big(1+(1+\varepsilon)^{1-m}\Big)\,\frac{\Big(\frac{1-\varepsilon}{1-\underline{\varepsilon}}\Big)^{1-m}-1}{1-(1-\varepsilon)^{1-m}}\right)^{1/2} \qquad [9, \text{Eq. } (52)]$$

and

$$\underline{T}(\varepsilon) := \frac{\kappa_\star\,(2\,A)^{1-m}+\frac{2}{\alpha}}{1-(1-\varepsilon)^{1-m}}\,. \qquad [9, \text{Eq. } (51)]$$

[9, Corollary 10]

$$u(t,x) \le (1+\varepsilon)\,B(t\,,\,x\,) \quad \text{if} \quad |x| \ge R(t)\,\overline{\rho}(\varepsilon) \quad \text{and} \quad \varepsilon \in (0,\overline{\varepsilon})\,. \qquad [9, \text{Eq. } (54)]$$

As a byproduct of Proposition 7, by integrating over $\mathbb{R}^d$, we deduce from

$$u(t,x) \le B\Big(t+\overline{t}-\tfrac{1}{\alpha}\,,\,x\,;\,\overline{M}\Big) \qquad [9, \text{Eq. } (32)]$$

that $\overline{M}\,/\mathcal{M} > 1$, which proves that

$$\overline{\varepsilon} := \Big(\overline{M}/\mathcal{M}\Big)^{\frac{2}{\alpha}} - 1 > 0\,.$$

Notice that $\overline{\varepsilon}$ is a numerical constant. The two other constants of Corollary 10 are given by

$$\overline{\rho}(\varepsilon) := \frac{1}{\mu}\left(\frac{(1+\varepsilon)^{1-m}+1}{(1+\varepsilon)^{1-m}-1}\right)^{\frac{1}{2}} \qquad [9, \text{Eq. } (57)]$$

and

$$\overline{T}(\varepsilon) := \frac{2\,\overline{t}}{(1+\varepsilon)^{1-m}-1} \qquad [9, \text{Eq. } (56)]$$

where

$$c := \max\Big\{1, 2^{5-m}\,\overline{\kappa}^{1-m}\,\mathsf{b}^\alpha\Big\}\,, \quad \overline{t} := c\,t_0\,, \qquad [9, \text{Eq. } (36)]$$

$\overline{\kappa}$ is given by (73), and

$$t_0 := A^{1-m}\,. \qquad [9, \text{Eq. } (34)]$$

[9, Proposition 11]

$$\left|\frac{u(t,x)}{B(t,x)}-1\right| \le \mathsf{K}\left(\frac{1}{t}+\frac{\sqrt{G}}{R(t)}\right)^{\vartheta} \quad \text{if} \quad |x|\le 2\,\rho(\varepsilon)\,R(t) \quad \text{and} \quad \varepsilon\in(0,\varepsilon_{m,d})\ . \qquad [9, \text{Eq. (61)}]$$

The range of admissible ε is determined by

$$\varepsilon_{m,d} := \min\left\{\overline{\varepsilon},\, \underline{\varepsilon},\, \tfrac{1}{2}\right\} \qquad [9, \text{Eq. (59)}]$$

and $\varepsilon \le \chi\,\eta$ where $\eta = 2\,d\,(m-m_1)$ and $\chi := \frac{m}{266+56\,m}$. The exponent

$$\vartheta = \frac{\nu}{d+\nu}\,. \qquad [9, \text{Eq. (78)}]$$

is defined as follows. Let

$$\nu := \log_4\left(\frac{\overline{\mathsf{h}}}{\overline{\mathsf{h}}-1}\right) \quad \text{with} \quad \overline{\mathsf{h}} := \mathsf{h}^{\lambda_1+1/\lambda_0}\,. \qquad [9, \text{Eq. (67)}]$$

The value of the constant h has been computed in [11] (also see (4)) and is given by

$$\mathsf{h} := \exp\left[2^{d+4}\,3^d\,d + c_0^3\,2^{2\,d+7}\left(1+\frac{2^{d+2}}{(\sqrt{2}-1)^{2\,d+4}}\right)\sigma\right] \qquad [9, \text{Eq. (65)}]$$

where

$$c_0 = 3^{\frac{2}{d}}\,2^{\frac{(d+2)\,(3\,d^2+18\,d+24)+13}{2\,d}}\left(\frac{(2+d)^{1+\frac{4}{d^2}}}{d^{1+\frac{2}{d^2}}}\right)^{(d+1)(d+2)}\mathcal{K}^{\frac{2\,d+4}{d}}\,,$$

$$\sigma = \sum_{j=0}^{\infty}\left(\tfrac{3}{4}\right)^j\,\Big((2+j)\,(1+j)\Big)^{2\,d+4}$$

and $\mathcal{K}$ is the optimal constant in the interpolation inequality (74), that is,

$$\|f\|^2_{\mathrm{L}^{p_m}(B)} \le \mathcal{K}\left(\|\nabla f\|^2_{\mathrm{L}^2(B)} + \|f\|^2_{\mathrm{L}^2(B)}\right).$$

The values of $\mathcal{K}$ are given in Table 1. We refer to Section 3.5 for the values of λ_0 and λ_1.

As for the other constants, we have

$$\rho(\varepsilon) := \max\left\{\overline{\rho}(\varepsilon),\, \underline{\rho}(\varepsilon)\right\}, \quad T(\varepsilon) := \max\left\{\overline{T}(\varepsilon),\, \underline{T}(\varepsilon)\right\} \qquad [9, \text{Eq. (60)}]$$

and

$$\mathsf{K} := 2^{\frac{3\,d}{\alpha}+\frac{3+6\,\alpha}{\alpha\,(1-m)}+\vartheta+10}\,\frac{(\alpha+\mathcal{M})^{\vartheta}}{m^{\vartheta}(1-m)^{2\,(1+\vartheta)+\frac{2}{1-m}}}$$
$$\times\left[1+\mathsf{b}^d\,C_{d,\nu,1}\left(\left(\overline{\kappa}\,\mathcal{M}^{\frac{2}{\alpha}}\,\frac{2^{\nu}}{2^{\nu}-1}+c_2\right)^{\frac{d}{d+\nu}}+\frac{\mu^{2d}}{\alpha^{\frac{d}{\alpha}}}\,\mathcal{M}^{\frac{d}{d+\nu}}\right)\right]. \qquad [9, \text{Eq. (84)}]$$

The exponent ϑ is the same as above. The other constants in the expression of K are

$$c_2 = 2\,\max\left\{\mathsf{b}\,,\,\|\nabla B(1-\tfrac{1}{\alpha},\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\right\} \qquad \text{[9, Eq. (71)]}$$

where

$$\|\nabla B(1-\tfrac{1}{\alpha},\cdot)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = \left(\tfrac{\mu}{\alpha^{1/\alpha}}\right)^{d+1}\sup_{z>0}\tfrac{2\,z}{1-m}\left(1+z^2\right)^{-2\,\frac{2-m}{1-m}} = \frac{\mu^{d+1}}{\alpha^{\frac{d+1}{\alpha}}}\,\frac{2^{\frac{1}{m-1}}}{\sqrt{(1-m)(3-m)}}\left(\tfrac{3-m}{2-m}\right)^{\frac{2-m}{1-m}}\,,$$

and $C_{d,\nu,1}$ corresponds to the optimal constant for $p=1$ in

$$\|u\|_{\mathrm{L}^\infty(B_R(x))} \le C_{d,\nu,p}\left(\lfloor u\rfloor_{C^\nu(B_{2\,R}(x))}^{\frac{d}{d+p\,\nu}}\,\|u\|_{\mathrm{L}^p(B_{2\,R}(x))}^{\frac{p\,\nu}{d+p\,\nu}} + R^{-\frac{d}{p}}\,\|u\|_{\mathrm{L}^p(B_{2\,R}(x))}\right). \qquad \text{[9, Eq. (102)]}$$

We know from [9, Appendix A] that $C_{d,\nu,p}$ is independent of $R>0$.

The last step is to collect the above estimates and compute

$$\mathsf{c}_\star(m,d) = \sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\,\varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\,\varepsilon\,\kappa_3(\varepsilon,m)\right\} \qquad \text{[9, Eq. (89)]}$$

where

$$\kappa_1(\varepsilon,m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1}\,,\,\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\},$$

$$\kappa_2(\varepsilon,m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}} \quad\text{and}\quad \kappa_3(\varepsilon,m) := \frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}\,.$$

We recall that $c := \max\left\{1, 2^{5-m}\,\overline{\kappa}^{1-m}\,\mathsf{b}^\alpha\right\}$ as above (also see [9, Eq. (36)]).

A.2 A numerical estimate of the constant in the Gagliardo-Nirenberg inequality on the disk

The following two-dimensional Gagliardo-Nirenberg inequality has been established in [9, Lemma 18].

Lemma 14. *Let $d=2$. For any $R>0$, we have*

$$\|u\|^2_{\mathrm{L}^4(B_R)} \le \frac{2\,R}{\sqrt{\pi}}\left(\|\nabla u\|^2_{\mathrm{L}^2(B_R)} + \frac{1}{R^2}\,\|u\|^2_{\mathrm{L}^2(B_R)}\right) \quad \forall\,u\in\mathrm{H}^1(B_R)\,. \tag{111}$$

The optimal constant is approximatively

$$0.0564922... < 2/\sqrt{\pi} \approx 1.12838\,.$$

We know from the proof that $\mathcal{C} \leq 2/\sqrt{\pi} \approx 1.12838$. Let us explain how we can compute the numerical value of the optimal constant $\mathcal{C}$ in the inequality (111). To compute $\mathcal{C}$ numerically, we observe that it is achieved among radial functions by symmetrization. The equality case is achieved by some radial function u, by standard compactness considerations. It is therefore enough to solve the Euler-Lagrange equation

$$- u'' - \frac{u'}{r} + u = u^3 \,, \quad u(0) = a > 0 \,, \quad u'(0) = 0 \,. \tag{112}$$

To emphasize the dependence of the solution in the shooting parameter a, we denote by u_a the solution of (112) with $u(0) = a$. We look for the value of a for which u_a changes sign only once (as it is orthogonal to the constants) and such that $u'(1) = 0$, which is our shooting criterion. Let $s(a) = u_a'(1)$ for the solution of (112). With $a = 1$, we find that $u_a \equiv 1$.

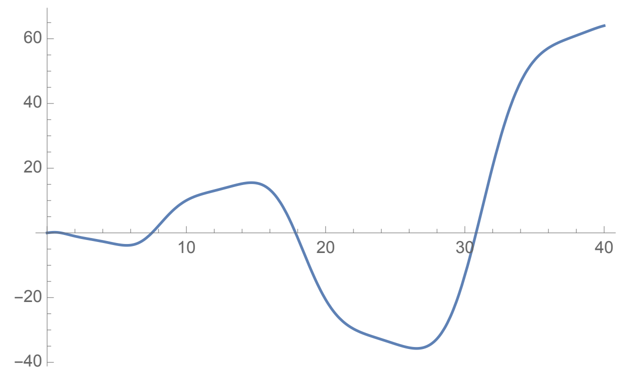

Figure 1: Plot of $a \mapsto s(a)$. We find that $s(1) = 0$ and also $s(a_\star) = 0$ for some $a_\star \approx 7.52449$ which provides us with a solution $u_{a_\star}$ with only one sign change.

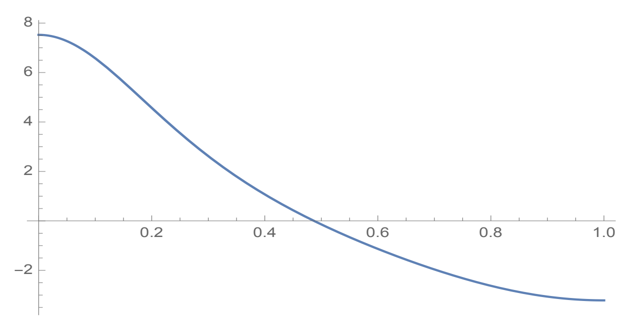

Figure 2: Plot of the solution $u_{a_\star}$ of (112).

Numerically, we obtain that

$$2\,\pi \int_0^1 \left(|u_{a_\star}'|^2 + |u_{a_\star}|^2\right) r\,dr = 2\,\pi \int_0^1 |u_{a_\star}|^4\, r\,dr = \frac{1}{\mathcal{C}} \left(2\,\pi \int_0^1 |u_{a_\star}|^4\, r\,dr\right)^{1/2}$$

which means

$$\mathcal{C} = \left(2\,\pi\int_0^1 |u_{a_\star}|^4\,r\,dr =\right)^{-1/2} \approx 0.0564922\,.$$

A.3 Truncation functions

Here are some details on the truncation functions used in this document.

Lemma 15 (Lemma 2.2 of [10]). *Fix two balls $B_{R_1} \subset B_{R_0} \subset\subset \Omega$. Then there exists a test function $\varphi_{R_1,R_0} \in C_0^1(\Omega)$, with $\nabla\varphi_{R_1,R_0} \equiv 0$ on $\partial\Omega$, which is radially symmetric and piecewise C^2 as a function of r, satisfies* $\mathrm{supp}(\varphi_{R_1,R_0}) = B_{R_0}$ *and $\varphi_{R_1,R_0} = 1$ on B_{R_1}, and moreover satisfies the bounds*

$$\|\nabla\varphi_{R_1,R_0}\|_\infty \le \frac{2}{R_0 - R_1} \quad \textit{and} \quad \|\Delta\varphi_{R_1,R_0}\|_\infty \le \frac{4\,d}{(R_0 - R_1)^2}. \tag{113}$$

Proof. With a standard abue of notation, we write indifferently that a radial function is a function of x or of $|x|$. Let us consider the radial test function defined on B_{R_0}

$$\varphi_{R_1,R_0}(|x|) = \begin{cases} 1 & \text{if } 0 \le |x| \le R_1 \\ 1 - \frac{2(|x|-R_1)^2}{(R_0-R_1)^2} & \text{if } R_1 < |x| \le \frac{R_0+R_1}{2} \\ \frac{2(R_0-|x|)^2}{(R_0-R_1)^2} & \text{if } \frac{R_0+R_1}{2} < |x| \le R_0 \\ 0 & \text{if } |x| > R_0 \end{cases} \tag{114}$$

for any $0 < R_1 < R_0$. We have

$$\nabla\varphi_{R_1,R_0}(|x|) = \begin{cases} 0 & \text{if } 0 \le |x| \le R_1 \text{ or if } |x| > R_0 \\ -\frac{4(|x|-R_1)}{(R_0-R_1)^2}\frac{x}{|x|} & \text{if } R_1 < |x| \le \frac{R_0+R_1}{2} \\ -\frac{4(R_0-|x|)}{(R_0-R_1)^2}\frac{x}{|x|} & \text{if } \frac{R_0+R_1}{2} < |x| \le R_0 \end{cases}$$

and, recalling that $\Delta\varphi(|x|) = \varphi''(|x|) + (d-1)\varphi'(|x|)/|x|$, we have

$$\Delta\varphi_{R_1,R_0}(|x|) = \begin{cases} 0 & \text{if } 0 \le |x| \le R_1 \text{ or if } |x| > R_0 \\ -\frac{4}{(R_0-R_1)^2} - \frac{d-1}{|x|}\frac{4(|x|-R_1)}{(R_0-R_1)^2} & \text{if } R_1 < |x| \le \frac{R_0+R_1}{2} \\ -\frac{4}{(R_0-R_1)^2} - \frac{d-1}{|x|}\frac{4(R_0-|x|)}{(R_0-R_1)^2} & \text{if } \frac{R_0+R_1}{2} < |x| \le R_0 \end{cases}$$

and easily obtain the bounds (113). □

Acknowledgments: This work has been partially supported by the Project EFI (ANR-17-CE40-0030) of the French National Research Agency (ANR). M.B. and N.S. were partially supported by Projects MTM2017-85757-P (Spain) and by the E.U. H2020 MSCA programme, grant agreement 777822. N.S. was partially funded by the FPI-grant BES-2015-072962, associated to the project MTM2014-52240-P (Spain).

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