Quantitative local estimates for nonlinear elliptic equations involving p-Laplacian type operators

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Abstract

The purpose of this paper is to prove quantitative local upper and lower bounds for weak solutions of elliptic equations of the form $-\Delta_p u = \lambda u^s$, with $p > 1$, $s > 0$ and $\lambda > 0$, defined on bounded domains of $\mathbb{R}^d$, $d \geq 1$, without reference to the boundary behaviour. We give an explicit expression for all the involved constants. As a consequence, we obtain local Harnack inequalities with explicit constants. Finally, we discuss the issue of local absolute bounds, which are new to our knowledge. Such bounds will be true only in a restricted range of s or for a special class of weak solutions, namely for local stable solutions. In the study of local absolute bounds for stable solutions there appears the so-called Joseph-Lundgren exponent as a limit of applicability of such bounds.

Keywords. Nonlinear elliptic equations of p-Laplacian type, local bounds, Harnack inequalities.

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Contents

1 Introduction

In this paper we obtain local upper and lower estimates for the weak solutions of nonlinear elliptic equations of the form

$$-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(u), \tag{1.1}$$

with $p > 1$, posed in a bounded domain $\Omega \subset \mathbb{R}^d$, with $d \geq 1$. The choice of right-hand side that we have in mind is $f(u) = \lambda\, u^s$ with $\lambda, s > 0$. Our main purpose is to obtain local estimates for solutions that are defined inside the domain without reference to their boundary behaviour. The notion of solution that we will use in the whole paper is the following.

Definition 1.1 (Local weak solutions) *Let $\Omega \subset \mathbb{R}^d$ a bounded domain. A function u is a local weak solution to $-\Delta_p u = f(u)$ in Ω if and only if $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $f(u) \in L^1_{\mathrm{loc}}(\Omega)$ and it satisfies*

$$\int_K \left[|\nabla u|^{p-2}\nabla u \cdot \nabla\varphi - f(u)\,\varphi\right]\,\mathrm{d}x = 0$$

for any compact $K \subset \Omega$ and for all bounded $\varphi \in C^1_0(K)$.

The estimate that we prove in this paper are local upper bounds for solutions of any sign, lower bounds for non-negative solutions, and also local Harnack inequalities. The estimates that we obtain are not essentially new from a qualitative point of view, and enjoy a large literature [2, 4, 5, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 25, 30, 28, 29, 31, 32, 37, 39, 41, 47, 48, 51, 52, 53, 54, 55] and the books [33, 34, 36, 49]; however, it is hopeless to give a complete bibliography for this nowadays classical problem. We try to contribute to the general theory on these elliptic equations, with quantitative local bounds; to our knowledge, there does not exists in literature a systematic set of local quantitative estimates in the explicit form given here. By quantitative estimates we mean keeping track of all the constants during the proofs. This paper follows the ideas of [4], in which the authors treated the case $p = 2$, i. e. the case of semilinear equations. Here we extend the techniques and the results of [4] to the more general case represented by the p-Laplacian elliptic equation (1.1).

The interest in obtaining quantitative control of the constants of such inequalities relies in the applications. On one hand, our results are useful in understanding regularity properties of the stationary solutions of the associated parabolic equation (the so-called doubly nonlinear evolution equation); it is needed for instance in the results of [3] on the asymptotic properties of solutions of the fast diffusion equation in bounded domains. On the other hand, it is interesting to see the stability of the estimates (therefore of the regularity of the solutions), when the parameters s or p reach their limiting values; for example, we can consider the (formal) limit $p \to 1^+$ in the local upper estimates of Theorems 4.1 and 4.5 and easily check that the constant is stable under such limiting process; the upper estimates therefore should hold also for the solutions of the 1-Laplacian, often called the Total Variation Flow (TVF). Weak solutions to the TVF have a different definition from the one we provide here for the p-Laplacian, but are sometimes obtained as the limit for $p \to 1^+$ of suitable families $\{u_p\}$ of smooth solution to the p-Laplacian, see [1] for more details; we refrain from doing this limiting process, since it falls out from the scope of this paper.

The range of exponents of interest will be $p > 1$ and $0 < s < r - 1$, where r is the exponent of the Sobolev imbedding of $W^{1,p}$, namely $r = p^* = pd/(d-p)$ if $p < d$ and any $r \in [p^*, \infty)$ if $p \geq d$; it is clear at this point that there is a restriction on the parameter s only when $p < d$. It is worth noticing that the restriction $s < r - 1 = p^* - 1$ appears only when we consider $p < d$, and it is related to several deep aspects of the theory of the equation at hand: for example, when dealing with the homogeneous Dirichlet problem, the existence of bounded weak solutions may fail above that exponent, as well as the absolute upper bounds, see [11, 23, 31, 32, 46, 52]; it is known that when $s \geq p^* - 1$ there exist solutions[1] which are not bounded, therefore not regular, cf. [24, 40, 42, 43, 44, 45]. On the other

[1] for $p = 2$, very weak solutions

hand, when $s < p^* - 1$, bounded solutions are known to be $C^{1,\alpha}$, cf. [25], and the $C^{1,\alpha}$ modulus of continuity directly depends on the local maximum of the solution. Therefore having absolute bounds for the solution allow to have absolute bounds for the $C^{1,\alpha}$ modulus of continuity. We will see that the $C^{1,\alpha}$ modulus of continuity is independent on the solution when $s < s_c^* < p^* - 1$, while it depends on (some L^q-norms of) the solution when $s_c^* < s < p^* - 1$.

Finally, when dealing with quantitative local absolute bounds for the smaller class of stable solutions, cf. Section 7, there will naturally appear the so-called Joseph-Lundgren exponent s_{JL} - which is finite only for "big" dimensions - as a further limit on the range of s to which our absolute bounds apply, as we shall explain in Section 7. As a reference for this topic see for example [13, 14, 16, 28, 29].

1.1 Plan of the paper and main results

We begin with a section devoted to the basic energy estimates. As a consequence, we obtain quantitative Caccioppoli type estimates that allow us to obtain absolute bounds for the $s-1$ -"norm", which to our knowledge have never been observed before, see Corollary 2.5; such absolute bounds will be the key tool needed in Section 7 to derive our local absolute bounds. In Section 3 we recall the Sobolev inequalities that we will use in the paper and derive some preliminary inequalities in the form of reverse Poincaré inequalities, as a direct consequence of the energy estimates and Sobolev inequalities.

We then focus on local upper estimates in Section 4. Our first main result is Theorem 4.1, which can be considered as a smoothing effect with very precise constants. In the case $p-1 < s < r-1$, the estimates of Theorem 4.1 seem to be new to our knowledge. Next, we obtain local upper estimates for $-\Delta_p u = b\, u^{p-1}$ with unbounded coefficient b in Theorems 4.2 and 4.4 and we apply them to the case $b = u^{s-(p-1)}$ in Theorem 4.5. The last upper bounds have the advantage that they do not require the restriction $s < r-1$, they hold for any nonnegative weak solution which moreover belongs to $\mathrm{L}^q_{\rm loc}$, with $q > r[s-(p-1)]/(r-p)$. This last requirement seems to be essential, since in the case $s > r-1$ there are solutions u_∞ which are not bounded, and $u_\infty \in \mathrm{L}^q_{\rm loc}$ with $q < r[s-(p-1)]/(r-p)$, at least when $p = 2$, see [24, 40, 42, 43, 44, 45].

Section 5 is devoted to the local lower estimates. The main result is Theorem 5.1, which holds for all $p > 1$ and $0 < s < r-1$. The proof is based on a quantitative lower Moser iteration, joined with the reverse Hölder inequalities of Appendix 8.1, which are obtained via a simplified John-Nirenberg type Lemma proved in [4] in a quantitative form. Next we prove a more precise quantitative reverse Hölder inequality, Proposition 5.2, but only in the smaller range of exponents $p-1 < s < r(p-1)/p = s_c^*$. The fourth main result of the paper, is Theorem 5.4, in which we use such reverse Hölder inequality to improve the lower bounds of Theorem 5.1 in this smaller range of exponents.

In Section 6 we combine the upper bounds of Section 4 with the lower bounds of Section 5 to obtain various form of Harnack inequalities. The general form, valid in the whole range of exponents, is given in Theorem 6.1, but, unfortunately, the constant of such inequality, depends on a quotient of L^q norms. Next we specialize to the subcritical range $0 < s \le p-1$, Theorem 6.2, and supercritical range $p-1 < s < s_c^*$, Theorem 6.3, and we prove clean versions of the Harnack inequality, i.e. the constant is independent on the solution. In the range $s_c^* < s < r-1$, we are not able to prove such clean forms of Harnack inequalities, and we conjecture that the dependence on some L^q norm of the solution can not be avoided. As far as we know, the Harnack inequality that we derive for $s > p-1$ is not stated explicitly in the literature. The fact that the "constant" involved has to depend on u when $s_c^* \le s < r-1$ is confirmed by the results of [9, 6, 7, 8, 26, 27] applied to separation of variable solutions of parabolic problems. This is also related to the fact that, in the range $s_c^* \le s < r-1$, there exist (very weak) singular solutions, at least when $p = 2$, see [24, 40, 42, 43, 44, 45].

Finally, in Section 7 we derive the quantitative local absolute bounds, which represent the novelty of the paper. In Theorem 7.1 we obtain quantitative local lower bounds when $0 \le s < p-1$ and local absolute upper bounds when $p-1 < s < s_c^*$. We have already discussed why the above absolute bounds cannot be extended $s > s_c^*$ without further assumption on the solution. The last part of the section is devoted to the derivation of absolute upper bounds for all $s>0$, but for the class of local stable solutions. In Theorem 7.6 we obtain quantitative absolute upper bounds for all $s>0$ when the dimension is small, namely $d\le \frac{p(p+3)}{p-1}$, while we reach a bigger exponent $s_{JL}\in(r-1,\infty)$ for bigger dimension. The exponent s_{JL} is the celebrated exponent discovered by Joseph and Lundgren in [35] , see also in [13, 14, 16, 28, 29] . The Appendix contains some technical results used in the paper, complemented with a proof when needed. We will use the notation $\|g\|_{L^q(B_R)}=\|g\|_{q,R}$, $|B_R|=\omega_d\,R^d$ and $\omega_d=|B_1|$.

1.2 More general nonlinearities

We can apply the method used in the proofs to obtain quantitative estimates to a larger class of operators and nonlinearities. We can consider a more general equation, namely

$$A(u)=-\mathrm{div}\,a(x,u,\nabla u)=f(u)\,,$$

where $a(x,\sigma,\xi)$ is a Carathèodory vector valued on $\Omega\times\mathbb{R}\times\mathbb{R}^d$ such that, for some constants $\nu_1\ge\nu_2>0$

1. $|a(x,\sigma,\xi)|\le\nu_1[1+|\xi|^{p-1}]$,
2. $a(x,\sigma,\xi)\xi\ge\nu_2|\xi|^p$,
3. $[a(x,\sigma,\xi)-a(x,\sigma,\eta)][\xi-\eta]>0$,

for a. e. $x\in\Omega$ and $\forall\ \sigma\in\mathbb{R}$, ξ, $\eta\in\mathbb{R}^d$, $\xi\neq\eta$.

The proofs of all the results apply also to this case with minor modifications, but the constants in the estimates will also depend on ν_1 and ν_2. As far as the the right-hand side is concerned, we deal with the model case $f(u)=\lambda u^s$. Indeed, we could have considered a more general nonlinearity $f(u)$ satisfying the following conditions: there exist $0\le b_0\le b_1$, $b_2\ge 0$:

$$b_0\,u^s\le f(u)\le b_1(u+b_2)^s.$$

Also In this case the proofs of all the results apply with minor modifications, and it is not so difficult to keep track of the new constants b_i throughout the proof.

We have decided here to consider the model case, to simplify the exposition and to focus on the main ideas.

2 Local energy estimates and Caccioppoli inequalities

We shall pursue in the sequel the well-known idea that local weak solutions satisfy reverse Sobolev or Poincaré inequalities. Such local reverse inequalities are the key to prove local upper and lower estimates of next sections, and indeed imply that such functions satisfy Harnack inequalities.

Lemma 2.1 (Energy Estimates) *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain and let u be a local nonnegative weak solution to $-\Delta_p u=\lambda\,u^s$ in Ω, $p>1$ and $\lambda,s\ge0$. Then the following energy estimate holds true for any $\alpha\neq-(p-1)$, $\delta>0$ and any test function $\phi\in C^2(\Omega)\cap C^1_0(\overline{\Omega})$, $\phi>0$*

$$\begin{aligned}\frac{|\alpha|}{p}\left|\frac{p}{\alpha+(p-1)}\right|^p\int_\Omega|\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p\phi\,\mathrm{d}x\quad&\le\quad\lambda\int_\Omega(u+\delta)^{\alpha+s}\phi\,\mathrm{d}x\\&\quad+\frac{1}{p\,|\alpha|^{p-1}}\int_\Omega(u+\delta)^{\alpha+(p-1)}\,\frac{|\nabla\phi|^p}{\phi^{p-1}}\,\mathrm{d}x.\end{aligned}\tag{2.1}$$

If $\alpha = -(p-1)$, for any $\delta > 0$, we have the Caccioppoli estimate

$$\frac{(p-1)^2}{p}\int_\Omega |\nabla \log(u+\delta)|^p \phi \,\mathrm{d}x + \lambda \int_\Omega \frac{u^s}{(u+\delta)^{p-1}}\phi \,\mathrm{d}x \le \frac{1}{p}\int_\Omega \frac{|\nabla\phi|^p}{\phi^{p-1}}\,\mathrm{d}x. \tag{2.2}$$

In addition for any $\alpha < 0$ and $\delta > 0$

$$\frac{|\alpha|}{p}\left|\frac{p}{\alpha+(p-1)}\right|^p \int_\Omega |\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p \phi\,\mathrm{d}x \le \frac{1}{p\,|\alpha|^{p-1}}\int_\Omega (u+\delta)^{\alpha+(p-1)}\,\frac{|\nabla\phi|^p}{\phi^{p-1}}\,\mathrm{d}x. \tag{2.3}$$

Remark 2.2 *We underline that when $\alpha > -(p-1)$ we can let $\delta \to 0^+$ in the energy estimates* (2.1) *and* (2.3) *to get*

$$\frac{p^{p-1}\,|\alpha|}{[\alpha+(p-1)]^p}\int_\Omega |\nabla(u^{\frac{\alpha+(p-1)}{p}})|^p\phi\,\mathrm{d}x \le \lambda\int_\Omega u^{\alpha+s}\phi\,\mathrm{d}x + \frac{1}{p\,|\alpha|^{p-1}}\int_\Omega u^{\alpha+(p-1)}\,\frac{|\nabla\phi|^p}{\phi^{p-1}}\,\mathrm{d}x \tag{2.4}$$

and for $-(p-1) < \alpha < 0$,

$$\frac{p^{p-1}\,|\alpha|}{[\alpha+(p-1)]^p}\int_\Omega |\nabla(u^{\frac{\alpha+(p-1)}{p}})|^p\phi\,\mathrm{d}x \le \frac{1}{p\,|\alpha|^{p-1}}\int_\Omega u^{\alpha+(p-1)}\,\frac{|\nabla\phi|^p}{\phi^{p-1}}\,\mathrm{d}x. \tag{2.5}$$

Proof of Lemma 2.1. Let $0 < \phi \in C^2(\Omega)\cap C_0^1(\overline{\Omega})$ and $\delta > 0$. Multiply the equation by $(u+\delta)^\alpha\phi$, $\alpha \neq -(p-1)$ and integrate by parts on Ω to get

$$\begin{aligned}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\phi\,(u+\delta)^\alpha\,\mathrm{d}x + \alpha\int_\Omega |\nabla u|^p(u+\delta)^{\alpha-1}\phi\,\mathrm{d}x &= \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla[(u+\delta)^\alpha\phi]\,\mathrm{d}x\\ &= -\int_\Omega \Delta_p u(u+\delta)^\alpha\phi\,\mathrm{d}x\\ &= \lambda\int_\Omega u^s(u+\delta)^\alpha\phi\,\mathrm{d}x.\end{aligned} \tag{2.6}$$

So, for any $\alpha \neq -(p-1)$, we have

$$\begin{aligned}\frac{|\alpha|\,p^p}{|\alpha+(p-1)|^p}\int_\Omega |\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p\phi\,\mathrm{d}x &= |\alpha|\int_\Omega |\nabla u|^p(u+\delta)^{\alpha-1}\phi\,\mathrm{d}x\\ &\le \lambda\int_\Omega u^s(u+\delta)^\alpha\phi\,\mathrm{d}x + \int_\Omega |\nabla u|^{p-1}(u+\delta)^\alpha|\nabla\phi|\,\mathrm{d}x.\end{aligned} \tag{2.7}$$

Now applying the inequality (8.5) with $\sigma = \frac{p}{p-1} > 1$ to the second term in the right hand side of (2.7), we obtain

$$\begin{aligned}\int_\Omega |\nabla u|^{p-1}\,|\nabla\phi|(u+\delta)^\alpha\,\mathrm{d}x \le{}& \frac{\varepsilon(p-1)}{p}\left|\frac{p}{\alpha+(p-1)}\right|^p\int_\Omega |\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p\phi\,\mathrm{d}x\\ &+\frac{1}{p\,\varepsilon^{p-1}}\int_\Omega (u+\delta)^{\alpha+(p-1)}\frac{|\nabla\phi|^p}{\phi^{p-1}}\,\mathrm{d}x.\end{aligned}$$

Simplifying and choosing $\varepsilon > 0$ such that

$$|\alpha| - \frac{\varepsilon(p-1)}{p} > 0,$$

for example $\varepsilon = |\alpha|$, we arrive at the following energy estimate

$$\begin{aligned}\frac{|\alpha|}{p}\left|\frac{p}{\alpha+(p-1)}\right|^p\int_\Omega |\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p\phi\,\mathrm{d}x \le{}& \lambda\int_\Omega (u+\delta)^{\alpha+s}\phi\,\mathrm{d}x\\ &+\frac{1}{p\,|\alpha|^{p-1}}\int_\Omega (u+\delta)^{\alpha+(p-1)}\frac{|\nabla\phi|^p}{\phi^{p-1}}\,\mathrm{d}x.\end{aligned} \tag{2.8}$$

In the particular case $\alpha < 0$, since u is assumed to be nonnegative, we get from (2.6)

$$|\alpha| \int_\Omega |\nabla u|^p (u+\delta)^{\alpha-1} \phi \, \mathrm{d}x \quad \leq \quad \int_\Omega |\nabla u|^{p-1} |\nabla \phi| (u+\delta)^\alpha \, \mathrm{d}x.$$

So, proceeding as above, we arrive at

$$\frac{|\alpha|}{p} \left(\frac{p}{|\alpha + (p-1)|} \right)^p \int_\Omega |\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p \phi \, \mathrm{d}x \leq \frac{1}{p\,|\alpha|^{p-1}} \int_\Omega (u+\delta)^{\alpha+(p-1)} \frac{|\nabla \phi|^p}{\phi^{p-1}} \, \mathrm{d}x.$$

Now let us consider the case $\alpha = -(p-1)$, as before, multiplying the equation by $(u+\delta)^{-(p-1)}\phi$, $\delta > 0$, and integrating by parts on Ω, we get

$$\begin{aligned}
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \frac{\nabla \phi}{(u+\delta)^{p-1}} \, \mathrm{d}x - (p-1) \int_\Omega \frac{|\nabla u|^p}{(u+\delta)^p} \phi \, \mathrm{d}x &= \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla[(u+\delta)^{-(p-1)} \phi] \, \mathrm{d}x \\
&= -\int_\Omega \Delta_p u (u+\delta)^{-(p-1)} \phi \, \mathrm{d}x \\
&= \lambda \int_\Omega u^s (u+\delta)^{-(p-1)} \phi \, \mathrm{d}x
\end{aligned}$$

So

$$(p-1) \int_\Omega \frac{|\nabla u|^p}{(u+\delta)^p} \phi \, \mathrm{d}x + \lambda \int_\Omega \frac{u^s}{(u+\delta)^{p-1}} \phi \, \mathrm{d}x \quad \leq \quad \int_\Omega |\nabla u|^{p-1} \frac{|\nabla \phi|}{(u+\delta)^{p-1}} \, \mathrm{d}x.$$

Applying Young inequality to the last term of the previous inequality and rewriting

$$\int_\Omega \frac{|\nabla u|^p}{(u+\delta)^p} \phi \, \mathrm{d}x = \int_\Omega |\nabla \log(u+\delta)|^p \phi \, \mathrm{d}x$$

we get (2.2). □

Now we can compute some useful calculations in order to get an explicit expression for all the constants.

Lemma 2.3 (A test function) *Fix two balls $B_{R_1} \subset B_{R_0} \subset\subset \Omega$. Then there exists a test function $\phi \in C^2(\Omega) \cap C^1_0(\overline{\Omega})$ which is radially symmetric,* $\operatorname{supp}(\phi) = B_{R_0}$*, $\phi = 1$ on B_{R_1}, and satisfies*

$$\frac{|\nabla \phi|^p}{\phi^{p-1}} \leq \frac{2^{p-1} p^p}{(R_0 - R_1)^p} \qquad \text{and} \qquad \|\nabla \phi\|_\infty \leq \frac{p}{R_0 - R_1} \qquad \text{for any } p > 1. \tag{2.9}$$

Proof. Consider the radial test function defined on $B_{R_0} \subset\subset \Omega$

$$\phi(|x|) = \begin{cases} 1\,, & \text{if } 0 \leq |x| \leq R_1 \\ 1 - \frac{2^{p-1}(|x|-R_1)^p}{(R_0-R_1)^p}\,, & \text{if } R_1 < |x| \leq \frac{R_0+R_1}{2} \\ \frac{2^{p-1}(R_0-|x|)^p}{(R_0-R_1)^p}\,, & \text{if } \frac{R_0+R_1}{2} < |x| \leq R_0 \\ 0\,, & \text{if } |x| > R_0 \end{cases}$$

for any $0 < R_1 < R_0$. We have

$$\nabla \phi(|x|) = \begin{cases} 0\,, & \text{if } 0 \leq |x| \leq R_1 \text{ or if } |x| > R_0 \\ -\frac{2^{p-1}p(|x|-R_1)^{p-1}}{(R_0-R_1)^p} \frac{x}{|x|}\,, & \text{if } R_1 < |x| \leq \frac{R_0+R_1}{2} \\ -\frac{2^{p-1}p(R_0-|x|)^{p-1}}{(R_0-R_1)^p} \frac{x}{|x|}\,, & \text{if } \frac{R_0+R_1}{2} < |x| \leq R_0 \end{cases}$$

So we easily obtain the bounds (2.9). □

Remark 2.4 *As a consequence of the first inequality in* (2.9)*, we have that*

$$\int_\Omega \frac{|\nabla\phi|^p}{\phi^{p-1}}\,\mathrm{d}x \le \frac{2^{p-1}\,p^p\,|B_{R_0}|}{(R_0-R_1)^p} = \frac{2^{p-1}\,p^p\,R_0^d\,\omega_d}{(R_0-R_1)^p}.$$

Corollary 2.5 (Quantitative Caccioppoli Estimates) *Let $\delta>0$. Under the assumption of Lemma 2.1 and using the test function ϕ of Lemma 2.3, we have the* quantitative Caccioppoli estimates, for any $\delta>0$:

$$\frac{(p-1)^2}{p}\int_{B_{R_1}} \big|\nabla\log(u+\delta)\big|^p\,\mathrm{d}x + \lambda\int_{B_{R_1}} \frac{u^s}{(u+\delta)^{p-1}}\,\mathrm{d}x \le \frac{p^{p-1}\,2^{p-1}\,R_0^d\,\omega_d}{(R_0-R_1)^p}. \tag{2.10}$$

Proof. Using (2.2) with ϕ as in Lemma 2.3 and recalling Remark 2.4 we easily obtain the desired result. □

Remark 2.6 *Letting $\delta\to 0^+$ in* (2.10)*, we get*

$$\lambda\int_{B_{R_1}} u^{s-(p-1)}\,\mathrm{d}x \le \frac{p^{p-1}\,2^{p-1}\omega_d R_0^d}{(R_0-R_1)^p}\,. \tag{2.11}$$

As a consequence of this fact in Section 7 we obtain a local absolute upper bound in the range $p-1<s<s_c^*=r(p-1)/p$, r defined in (3.2) *and a local absolute lower bound if $u\not\equiv 0$ on B_{R_0} and $0<s<p-1$.*

3 Sobolev and reverse Poincaré inequalities

In this section we will recall the Sobolev inequalities that will be used throughout the paper and we also show how they combine with the energy inequalities of the previous section to give a kind of reverse Poincaré inequalities, that will be necessary for the upper bounds when dealing with unbounded coefficients.

Sobolev inequalities. Our local bounds will be a consequence of the Sobolev imbedding theorems on balls $B_\rho\subset\mathbb{R}^d$. Indeed the following Sobolev type inequalities hold true:

$$\|g\|^p_{\mathrm{L}^r(B_\rho)} \le \mathcal{S}_p^p\left(\|\nabla g\|^p_{\mathrm{L}^p(B_\rho)} + \frac{1}{\rho^p}\|g\|^p_{\mathrm{L}^p(B_\rho)}\right) \tag{3.1}$$

for any $g\in W^{1,p}(\Omega)$, where Ω is a bounded open domain of $\mathbb{R}^d$ with smooth boundary, $B_\rho\subset\Omega$, and

$$\begin{cases} \text{if } p<d, & r=p^*=\dfrac{pd}{d-p}\,, & \mathcal{S}_p\,,\\ \text{if } p=d, & r\in(p,+\infty)\,, & \mathcal{S}_p=\mathcal{S}_p'\mathrm{diam}(\Omega)^{\frac{d}{r}}\\ \text{if } p>d, & r=+\infty\,, & \mathcal{S}_p=\mathcal{S}_p'\mathrm{diam}(\Omega)^{1-\frac{d}{p}} \end{cases} \tag{3.2}$$

and $\mathcal{S}_p,\mathcal{S}_p'>0$ only depends on p,d , see e.g. Theorem 3.11 and 3.12 of [34]. On the other hand, whenever $g\in W_0^{1,p}(B_\rho)$ we have

$$\|g\|^p_{\mathrm{L}^r(B_\rho)} \le \mathcal{S}_p^p\,\|\nabla g\|^p_{\mathrm{L}^p(B_\rho)}, \tag{3.3}$$

where r is defined in (3.2) and $\mathcal{S}_p$ is the Sobolev constant, which only depends on p,d , see e.g. Theorem 3.9 of [34]. We will denote by r the Sobolev exponent corresponding to $W_0^{1,p}(B_\rho)$ through all the paper.

Now we state and prove a lemma originally due to Trudinger [53] (see Lemma 5.1 p. 745 there). For a proof in the case $p=2$ see Lemma 3.2 in [4].

Lemma 3.1 *Let $v \in L^r(B_R)$ and $b \in L^m(B_R)$ for $m > r/(r-p)$ with $r > p$ defined in (3.2). Then for any $\gamma > 0$ the following inequality holds*

$$\int_{B_R} b\, v^p \,\mathrm{d}x \leq \gamma \left(\int_{B_R} v^r \,\mathrm{d}x\right)^{\frac{p}{r}} + \frac{K_{m,p,r}}{\gamma^{\frac{mp+r}{m(r-p)-r}}} |B_R|^{\frac{p}{r}} \left(\int_{B_R} b^m \,\mathrm{d}x\right)^{\frac{r}{m(r-p)-r}} \int_{B_R} v^p \,\mathrm{d}x, \tag{3.4}$$

where

$$K_{m,p,r} := \frac{m(r-p)-r}{rm} \left(\frac{pm+r}{rm}\right)^{\frac{mp+r}{m(r-p)-r}} . \tag{3.5}$$

Proof. Let us estimate for any $0 < \gamma_1 < p$,

$$\begin{aligned}
\int_{B_R} b v^{(p-\gamma_1)+\gamma_1} \,\mathrm{d}x &\leq_{(a)} \left(\int_{B_R} v^{(p-\gamma_1)\frac{r}{p}} \,\mathrm{d}x\right)^{\frac{p}{r}} \left(\int_{B_R} b^{\frac{r}{r-p}} v^{\gamma_1 \frac{r}{r-p}} \,\mathrm{d}x\right)^{\frac{r-p}{r}} \\
&\leq_{(b)} \left(\int_{B_R} v^r \,\mathrm{d}x\right)^{\frac{p-\gamma_1}{r}} |B_R|^{\frac{\gamma_1}{r}} \left(\int_{B_R} b^{\frac{r}{r-p}} v^{\gamma_1 \frac{r}{r-p}} \,\mathrm{d}x\right)^{\frac{r-p}{r}} \\
&\leq_{(c)} \frac{\gamma_0(p-\gamma_1)}{p} \left(\int_{B_R} v^r \,\mathrm{d}x\right)^{\frac{p}{r}} + \frac{\gamma_1}{p} \frac{1}{\gamma_0^{\frac{p-\gamma_1}{\gamma_1}}} |B_R|^{\frac{p}{r}} \left(\int_{B_R} b^{\frac{r}{r-p}} v^{\gamma_1 \frac{r}{r-p}} \,\mathrm{d}x\right)^{\frac{p(r-p)}{\gamma_1 r}} \\
&\leq_{(d)} \frac{\gamma_0(p-\gamma_1)}{p} \left(\int_{B_R} v^r \,\mathrm{d}x\right)^{\frac{p}{r}} + \frac{\gamma_1}{p} \frac{1}{\gamma_0^{\frac{p-\gamma_1}{\gamma_1}}} |B_R|^{\frac{p}{r}} \left(\int_{B_R} b^m \,\mathrm{d}x\right)^{\frac{p}{\gamma_1 m}} \\
&\quad \times \left(\int_{B_R} v^{\frac{\gamma_1 rm}{m(r-p)-r}}\right)^{\frac{p[m(r-p)-r]}{\gamma_1 rm}} \\
&=_{(e)} \frac{(pm+r)\gamma_0}{rm} \left(\int_{B_R} v^r \,\mathrm{d}x\right)^{\frac{p}{r}} + \frac{m(r-p)-r}{mr} \frac{1}{\gamma_0^{\frac{mp+r}{m(r-p)-r}}} |B_R|^{\frac{p}{r}} \\
&\quad \times \left(\int_{B_R} b^m \,\mathrm{d}x\right)^{\frac{r}{m(r-p)-r}} \int_{B_R} v^p \,\mathrm{d}x,
\end{aligned}$$

where in (a) we have used Hölder inequality with exponents r/p and $r/(r-p)$, in (b) with $p/(p-\gamma_1)$ and p/γ_1; in (c) we have applied the Young inequality (8.5), with $\varepsilon = \gamma_0$, $\sigma = p/(p-\gamma_1)$. In (d) we have used again Hölder inequality with exponents $m(r-p)/r > 1$, since we are assuming $m > r/(r-p)$, and $m(r-p)/[m(r-p)-r]$ and in (e) we have put

$$0 < \gamma_1 = \frac{p[m(r-p)-r]}{rm} < p.$$

To obtain the desired result it is sufficient take

$$\gamma = \gamma_0 \frac{pm+r}{rm}.$$

□

Theorem 3.2 (Reverse Poincaré inequality) *Let u be a weak solution to $-\Delta_p u = b\, u^{p-1}$ on B_R, with $p > 1$. Let $b \in L^m(B_R)$, for $m > r/(r-p)$ and $r > p$ defined in (3.2). Suppose that $u \in L^{\alpha+(p-1)}(B_R)$. Then for any positive test function $1 \geq \phi \in C^2(B_R) \cap C_0^1(\overline{B_R})$ and any $\alpha > 0$ the following estimate holds true*

$$\int_{B_R} \left|\nabla \left(u^{\frac{\alpha+(p-1)}{p}}\right)\right|^p \phi^p \,\mathrm{d}x \leq \Lambda(b) \int_{B_R} u^{\alpha+(p-1)} \,\mathrm{d}x,$$

with

$$\Lambda(b) := \left\{ \left[1 + 2\left(\frac{\alpha + (p-1)}{\alpha}\right)^p\right] \|\nabla\phi\|_\infty^p + K_{m,p,r}\,\frac{2p}{\alpha}\left(\frac{2^p\,p\,S_p^p}{\alpha}\right)^{\frac{mp+r}{m(r-p)-r}} \left(\frac{\alpha+(p-1)}{p}\right)^{\frac{mpr}{m(r-p)-r}} |B_R|^{\frac{p}{r}} \left(\int_{B_R} b^m \,\mathrm{d}x\right)^{\frac{r}{m(r-p)-r}} \right\},$$

$K_{m,p,r}$ *given in* (3.5).

Remark 3.3 *We underline that the requirement $u \in L^{\alpha+(p-1)}(B_R)$ will be dispensed later, without further comment by using a Moser iteration technique.*

Proof. We divide the proof in few steps.

• Step 1. *Energy estimates.* Multiplying $-\Delta_p u = b\, u^{p-1}$ by $(u+\delta)^\alpha \phi^p$, $1 \geq \phi \in C^2(B_R) \cap C_0^1(\overline{B_R})$, $\alpha > 0$, $\delta > 0$ and integrating by parts on B_R, we get

$$\alpha \int_{B_R} |\nabla u|^p\,(u+\delta)^{\alpha-1}\phi^p \,\mathrm{d}x \leq p\int_{B_R} |\nabla u|^{p-1}\,|\nabla\phi|\,\phi^{p-1}(u+\delta)^\alpha \,\mathrm{d}x + \int_{B_R} b\,u^{p-1}\,(u+\delta)^\alpha\phi^p \,\mathrm{d}x. \tag{3.6}$$

Using (8.5) with $\sigma = p/(p-1)$ to estimate the first term in the right hand side of (3.6), we obtain

$$[\alpha - \varepsilon(p-1)]\left(\frac{p}{\alpha+(p-1)}\right)^p \int_{B_R} |\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}})]^p\,\phi^p\,\mathrm{d}x \leq \int_{B_R} b\,(u+\delta)^{\alpha+(p-1)}\,\phi^p\,\mathrm{d}x + \frac{1}{\varepsilon^{p-1}}\int_{B_R} (u+\delta)^{\alpha+(p-1)}\,|\nabla\phi|^p\,\mathrm{d}x.$$

Choosing $\varepsilon = \alpha/p$, we arrive at

$$\frac{\alpha}{p}\left(\frac{p}{\alpha+(p-1)}\right)^p \int_{B_R} |\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p\,\phi^p\,\mathrm{d}x \leq \int_{B_R} b\,(u+\delta)^{\alpha+(p-1)}\,\phi^p\,\mathrm{d}x + \frac{p^{p-1}}{\alpha^{p-1}}\int_{B_R} (u+\delta)^{\alpha+(p-1)}\,|\nabla\phi|^p\,\mathrm{d}x. \tag{3.7}$$

• Step 2. *Sobolev inequality in $W_0^{1,p}(B_R)$.* We apply inequality (3.4) of Lemma 3.1 to $v = (u+\delta)^{\frac{\alpha+(p-1)}{p}}\phi \in W_0^{1,p}(B_R)$, so that for any $\gamma > 0$:

$$\int_{B_R} b\,(u+\delta)^{\alpha+(p-1)}\,\phi^p\,\mathrm{d}x \leq \gamma\left(\int_{B_R} (u+\delta)^{[\alpha+(p-1)]\frac{r}{p}}\,\phi^r\,\mathrm{d}x\right)^{\frac{p}{r}} + \frac{K_{m,p,r}}{\gamma^{\frac{mp+r}{m(r-p)-r}}}\,|B_R|^{\frac{p}{r}}\left(\int_{B_R} b^m\,\mathrm{d}x\right)^{\frac{r}{m(r-p)-r}} \int_{B_R} (u+\delta)^{\alpha+(p-1)}\,\phi^p\,\mathrm{d}x, \tag{3.8}$$

where $K_{m,p,r}$ is given in (3.5). Since $v = (u+\delta)^{\frac{\alpha+(p-1)}{p}}\phi \in W_0^{1,p}(B_R)$, the Sobolev inequality (3.3) reads

$$\left(\int_{B_R} [(u+\delta)^{\frac{\alpha+(p-1)}{p}}\phi]^r\,\mathrm{d}x\right)^{\frac{p}{r}} \leq S_p^p\,2^{p-1}\int_{B_R} |\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p\,\phi^p\,\mathrm{d}x + S_p^p\,2^{p-1}\int_{B_R} (u+\delta)^{\alpha+(p-1)}\,|\nabla\phi|^p\,\mathrm{d}x.$$

We combine the above Sobolev inequality with (3.8) to get

$$\begin{aligned}\int_{B_R} b\,(u+\delta)^{\alpha+(p-1)}\,\phi^p\,\mathrm{d}x \quad &\le \quad \gamma\, S_p^p 2^{p-1}\int_{B_R}|\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p\,\phi^p\,\mathrm{d}x \\ &\quad + \gamma\, S_p^p 2^{p-1}\int_{B_R}(u+\delta)^{\alpha+(p-1)}\,|\nabla\phi|^p\,\mathrm{d}x \\ &\quad + \frac{K_{m,p,r}}{\gamma^{\frac{mp+r}{m(r-p)-r}}}\,|B_R|^{\frac{p}{r}}\left(\int_{B_R} b^m\,\mathrm{d}x\right)^{\frac{r}{m(r-p)-r}}\int_{B_R}(u+\delta)^{\alpha+(p-1)}\,\phi^p\,\mathrm{d}x. \end{aligned}\tag{3.9}$$

• Step 3. Putting together (3.7) and (3.9) and recalling that $\phi \le 1$, we obtain

$$\left[\frac{\alpha}{p}\left(\frac{p}{\alpha+(p-1)}\right)^p - \gamma\, S_p^p\, 2^{p-1}\right]\int_{B_R}|\nabla[(u+\delta)^{\frac{\alpha+(p-1)}{p}}]|^p\,\phi^p\,\mathrm{d}x$$

$$\le \left\{\left[\gamma\, S_p^p\, 2^{p-1} + \frac{p^{p-1}}{\alpha^{p-1}}\right]\|\nabla\phi\|_\infty^p + \frac{K_{m,p,r}}{\gamma^{\frac{mp+r}{m(r-p)-r}}}\,|B_R|^{\frac{p}{r}}\left(\int_{B_R} b^m\,\mathrm{d}x\right)^{\frac{r}{m(r-p)-r}}\right\}\int_{B_R}(u+\delta)^{\alpha+(p-1)}\,\mathrm{d}x.$$

Choosing

$$\gamma = \left(\frac{p}{\alpha+(p-1)}\right)^p\,\frac{\alpha}{p\,2^p\,S_p^p}$$

and letting $\delta \to 0$ we obtain the desired result. □

Remark 3.4 *If we take ϕ as in Lemma 2.3 we obtain*

$$\int_{B_{R_1}}|\nabla(u^{\frac{\alpha+(p-1)}{p}})|^p\,\mathrm{d}x \le \Lambda(b)\int_{B_{R_0}} u^{\alpha+(p-1)}\,\mathrm{d}x,$$

with

$$\begin{aligned}\Lambda(b) \quad :=& \quad \left\{\left[1+2\left(\frac{\alpha+(p-1)}{\alpha}\right)^p\right]\frac{p^p}{(R_0-R_1)^p} + K_{m,r,p}\,\frac{2p}{\alpha}\left(\frac{2^p\,p\,S_p^p}{\alpha}\right)^{\frac{mp+r}{m(r-p)-r}}\right. \\ &\quad \left.\times\left(\frac{\alpha+(p-1)}{p}\right)^{\frac{mpr}{m(r-p)-r}}\,\omega_d^{\frac{p}{r}}\,R_0^{\frac{dp}{r}}\left(\int_{B_{R_0}} b^m\,\mathrm{d}x\right)^{\frac{r}{m(r-p)-r}}\right\},\end{aligned}\tag{3.10}$$

for all $\alpha > 0$, $m > r/(r-p)$, and $K_{m,p,r}$ as in (3.5).

4 Local upper bounds

This section is devoted to the proof of quantitative local upper bounds for local nonnegative weak solutions to $-\Delta_p u = \lambda u^s$, for any $\lambda > 0$ and any $s \ge 0$. We also get quantitative local estimates for solutions to the problem $-\Delta_p u = b(x)u^{p-1}$ with $b \in L^m$, eventually unbounded. We prove our results for nonnegative solutions, but the careful reader can realize that almost the same proof holds for nonnegative subsolutions, or for solutions with any sign.

4.1 Local upper bounds I. The upper Moser iteration

The first form of the upper bounds that we present in this section, is a consequence of energy estimates, Caccioppoli inequalities and the "local" Sobolev inequality on balls.

Theorem 4.1 (Local Upper Estimates) *Let $\Omega\subset\mathbb{R}^d$ and $\lambda>0$. Let u be a local nonnegative weak solution to $-\Delta_p u=\lambda u^s$ in Ω with $p>1$, $0\le s<r-1$ and r as in* (3.2)*. Then the following bound holds true for any $B_{R_\infty}\subset B_{R_0}\subseteq\Omega$ and for any $q>\overline{q}:=[s-(p-1)]_+\,r/(r-p)$*

$$\|u\|_{\infty,R_\infty}\le I_{\infty,q}\left(\fint_{B_{R_0}} u^q\,\mathrm{d}x\right)^{\frac{(r-p)\mu}{r}}\left(\fint_{B_{R_\infty}} u^{[s-(p-1)]_+}\,\mathrm{d}x\right)^{-\mu}\tag{4.1}$$

where $\mu=r/\{(r-p)q-r[s-(p-1)]_+\}$ and the constant $I_{\infty,q}$ is depends on d,p,s,q,r,R_0,R_∞, and when $s\neq p-1$ does not depend on λ, see an explicit expression in formula (4.4) *below.*
Moreover, when $0\le s\le p-1$, the above estimate takes the simplified form:

$$\|u\|_{\infty,R_\infty}\le I_{\infty,q}\left(\fint_{B_{R_0}} u^q\,\mathrm{d}x\right)^{\frac{1}{q}},\tag{4.2}$$

and holds for all $q>0$. The constant $I_{\infty,q}$ is the same as above and is given in formula (4.4)*.*

Remark on the result. Inequality (4.1) is a kind of reverse Hölder inequality, indeed we can rewrite it as:

$$\|u\|_{[s-(p-1)]_+,R_\infty}^{\mu[s-(p-1)]_+}\|u\|_{\infty,R_\infty}\le C\,\|u\|_{q,R_0}^{\frac{q(r-p)\mu}{r}}.\tag{4.3}$$

This form makes clearer the fact that if there is a constant that makes true (4.3) for a $q\ge\overline{q}$, then by Hölder inequality, the same inequality holds true for all $q'>q$, with the same constant. The same applies to (4.1).

Remark on the constant. The proof below allow us to find an explicit expression of the constant:

$$\begin{aligned}I_{\infty,q}\;&=\;\left[\frac{|B_{R_0}|^{\frac{r-p}{r}+1}}{|B_{R_\infty}|}\,\frac{S_p^p\,q^p\,p^p\,c_1}{r^p\,(R_0-R_\infty)^p}\left\{\frac{\Lambda_{s,0}\,r}{p\,q}\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\right.\right.\\&\qquad\left.\left.+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p c_2+\frac{c_1^{p-1}\,2^{p-1}}{c_0^p}\right\}\left(\frac{r}{p}\right)^{\frac{pr}{r-p}}\right]^{\frac{r}{q(r-p)-r[s-(p-1)]_+}}\end{aligned}\tag{4.4}$$

with

$$\frac{1}{c_0^p}\le\left(\frac{r}{(r-p)\,q-[s-(p-1)]_+\,r}\right)^p,$$

$$c_1:=\begin{cases}\dfrac{pq}{pq-r(p-1)}, & \text{if } q>s_c^*=\frac{r(p-1)}{p},\\[2ex] \displaystyle\max_{i=0,1}\frac{\left(\frac{r}{p}\right)^{j_0+i-1}q}{\left|\left(\frac{r}{p}\right)^{j_0+i-1}q-(p-1)\right|}, & \text{if } 0<q<s_c^*=\frac{r(p-1)}{p},\end{cases}$$

$$c_2:=\max\left\{\frac{|p\,q-r(p-1)|\,r^p}{(p\,q)^{p+1}},\left(\frac{p}{p-1}\right)^p\frac{1}{(p+1)^{p+1}}\right\}$$

and

$$\Lambda_{s,0}=\frac{\lambda}{p^{p-1}}\,R_0^p\ \text{ if } s=p-1\ \text{ and }\ \Lambda_{s,0}=2^{p-1+d}\ \text{ if } s\neq p-1.\tag{4.5}$$

Moreover when $0<q<s_c^*$ we require the additional condition

$$j_0:=i.p.\left[\frac{\log\frac{r(p-1)}{qp}}{\log\frac{r}{p}}\right]\neq\frac{\log\frac{r(p-1)}{qp}}{\log\frac{r}{p}}\tag{4.6}$$

where $i.p.[t]$ denotes the integer part of $t \in \mathbb{R}$.

Finally, we would like to remark that this latter condition (4.6) is not really essential: indeed, we can obtain an explicit constant $I_{\infty,q}$ for a $q > \overline{q}$ such that $i.p.\left[(\log \frac{r(p-1)}{qp})/(\log \frac{r}{p})\right] = (\log \frac{r(p-1)}{qp})/(\log \frac{r}{p})$, simply by considering a $q' \in (\overline{q}, q)$ such that condition (4.6) holds so that the explicit constant is given by $I_{\infty,q'}$; then by the remark after formula (4.3), we obtain the desired bound also for $q > q'$ with the same constant ($I_{\infty,q} = I_{\infty,q'}$) as a consequence of Hölder inequality.

Proof of Theorem 4.1. We are going to use the energy estimate (2.1) for any $\alpha > -(p-1)$, $\alpha \neq 0$, to prove $L^q - L^\infty$ local estimates via Moser iteration, keeping track all the constants. We divide the proof in several steps.

• STEP 1. Let ϕ the function defined in Lemma 2.3. The local Sobolev inequality (3.1) on the ball B_{R_1} applied to $g = u^{[\alpha+(p-1)]/p}$ together with the energy estimate (2.4), gives, using the properties of the function ϕ, established in Lemma 2.3,

$$
\begin{aligned}
\left[\int_{B_{R_1}} u^{[\alpha+(p-1)]\frac{r}{p}}\,\mathrm{d}x\right]^{\frac{p}{r}} &\le S_p^p\left[\int_{B_{R_1}} |\nabla(u^{\frac{\alpha+(p-1)}{p}})|^p\,\mathrm{d}x + \frac{1}{R_1^p}\int_{B_{R_0}} u^{\alpha+(p-1)}\,\mathrm{d}x\right]\\
&\le S_p^p\left\{\frac{p}{|\alpha|}\left(\frac{\alpha+(p-1)}{p}\right)^p \lambda\int_{B_{R_0}} u^{\alpha+s}\,\mathrm{d}x\right.\\
&\quad\left.+\left[\frac{1}{R_1^p}+\left(\frac{\alpha+(p-1)}{|\alpha|}\right)^p\frac{2^{p-1}}{(R_0-R_1)^p}\right]\int_{B_{R_0}} u^{\alpha+(p-1)}\,\mathrm{d}x\right\}.
\end{aligned}
\tag{4.7}
$$

• STEP 2. *Caccioppoli estimates and first iteration step.* Now we need to split into two cases, namely $0 \le s \le p-1$ and $p-1 < s < r-1$.

Superlinear case: $p-1 < s < r-1$. We continue estimate (4.7) as follows:

$$
\begin{aligned}
\left[\int_{B_{R_1}} u^{[\alpha+(p-1)]\frac{r}{p}}\,\mathrm{d}x\right]^{\frac{p}{r}} &\le S_p^p\left\{\frac{p}{|\alpha|}\left(\frac{\alpha+(p-1)}{p}\right)^p\lambda + \left[\frac{1}{R_1^p}+\left(\frac{\alpha+(p-1)}{|\alpha|}\right)^p\right.\right.\\
&\quad\left.\left.\times\frac{2^{p-1}}{(R_0-R_1)^p}\right]\frac{\int_{B_{R_0}} u^{\alpha+(p-1)}\,\mathrm{d}x}{\int_{B_{R_0}} u^{\alpha+s}\,\mathrm{d}x}\right\}\int_{B_{R_0}} u^{\alpha+s}\,\mathrm{d}x\\
&\le_{(a)} S_p^p\frac{|B_{R_0}|}{\|u\|_{s-(p-1),R_0}^{s-(p-1)}}\left\{\frac{p}{|\alpha|}\left(\frac{\alpha+(p-1)}{p}\right)^p\lambda\frac{\|u\|_{s-(p-1),R_0}^{s-(p-1)}}{|B_{R_0}|}\right.\\
&\quad\left.+\left[\frac{1}{R_1^p}+\left(\frac{\alpha+(p-1)}{|\alpha|}\right)^p\frac{2^{p-1}}{(R_0-R_1)^p}\right]\right\}\int_{B_{R_0}} u^{\alpha+s}\,\mathrm{d}x\\
&\le_{(b)} \frac{S_p^p|B_{R_0}|}{(R_0-R_1)^p\|u\|_{s-(p-1),R_0}^{s-(p-1)}}\left\{\frac{[\alpha+(p-1)]^p\,2^{p-1+d}}{|\alpha|}\left(\frac{R_0-R_1}{R_0}\right)^p\right.\\
&\quad\left.+\left(\frac{R_0-R_1}{R_1}\right)^p+\frac{[\alpha+(p-1)]^p\,2^{p-1}}{|\alpha|^p}\right\}\int_{B_{R_0}} u^{\alpha+s}\,\mathrm{d}x.
\end{aligned}
\tag{4.8}
$$

In (a) we have used the convexity in the variable $t > 0$ of the function $N(t) = \log\|u\|_t^t$. Hence, since the incremental quotient is increasing (see for example [50] for more details), choosing $\alpha + (p-1) \ge \overline{\alpha} > 0$, we obtain

$$
\frac{N(\overline{\alpha}+s-(p-1))-N(\overline{\alpha})}{s-(p-1)} \le \frac{N(\alpha+s)-N(\alpha+(p-1))}{s-(p-1)} \quad \text{that is} \quad \frac{\|u\|_{\overline{\alpha}+s-(p-1)}^{\overline{\alpha}+s-(p-1)}}{\|u\|_{\overline{\alpha}}^{\overline{\alpha}}} \le \frac{\|u\|_{\alpha+s}^{\alpha+s}}{\|u\|_{\alpha+(p-1)}^{\alpha+(p-1)}}.
$$

Then, using Hölder inequality with exponents $[\overline{\alpha}+s-(p-1)]/[s-(p-1)] > 1$ and $[\overline{\alpha}+s-(p-1)]/\overline{\alpha} > 1$, since we are assuming $s > p-1$, we have

$$\begin{aligned}\frac{\|u\|_{\alpha+s,R_0}^{\alpha+s}}{\|u\|_{\alpha+(p-1),R_0}^{\alpha+(p-1)}} &\geq \frac{\|u\|_{\overline{\alpha}+s-(p-1),R_0}^{\overline{\alpha}+s-(p-1)}}{\|u\|_{\overline{\alpha},R_0}^{\overline{\alpha}}} = \frac{\|u\|_{\overline{\alpha}+s-(p-1),R_0}^{\overline{\alpha}}}{\|u\|_{\overline{\alpha},R_0}^{\overline{\alpha}}} \|u\|_{\overline{\alpha}+s-(p-1)}^{s-(p-1)} \\ &\geq |B_{R_0}|^{-\frac{s-(p-1)}{\overline{\alpha}+s-(p-1)}} \|u\|_{s-(p-1),R_0}^{s-(p-1)} |B_{R_0}|^{-\frac{\overline{\alpha}}{\overline{\alpha}+s-(p-1)}} = \frac{\|u\|_{s-(p-1),R_0}^{s-(p-1)}}{|B_{R_0}|},\end{aligned}$$

In (b) we have used the Caccioppoli estimate (2.11) with R_0 and $2R_0$, that is

$$\frac{\lambda\|u\|_{s-(p-1),R_0}^{s-(p-1)}}{|B_{R_0}|} \leq \frac{p^{p-1}\,2^{p-1}}{(2R_0-R_0)^p}\,\frac{|B_{2R_0}|}{|B_{R_0}|} = \frac{p^{p-1}\,2^{p-1+d}}{R_0^p}. \tag{4.9}$$

Sublinear case: $0 \leq s \leq p-1$. We first assume $0 \leq s < p-1$, we discuss the case $s = p-1$ separately. We continue estimate (4.7) as follows:

$$\begin{aligned}\left[\int_{B_{R_1}} u^{[\alpha+(p-1)]\frac{r}{p}}\ \mathrm{d}x\right]^{\frac{p}{r}} &\leq S_p^p \left\{\frac{p}{|\alpha|}\left(\frac{\alpha+(p-1)}{p}\right)^p \lambda\,\frac{\int_{B_{R_0}} u^{\alpha+s}\ \mathrm{d}x}{\int_{B_{R_0}} u^{\alpha+(p-1)}\ \mathrm{d}x}\right.\\ &\quad \left. + \frac{1}{R_1^p} + \frac{[\alpha+(p-1)]^p}{|\alpha|^p}\frac{2^{p-1}}{(R_0-R_1)^p}\right\}\int_{B_{R_0}} u^{\alpha+(p-1)}\ \mathrm{d}x \\ &\leq \frac{S_p^p}{(R_0-R_1)^p}\left\{\frac{[\alpha+(p-1)]^p\,2^{p-1+d}}{|\alpha|}\left(\frac{R_0-R_1}{R_0}\right)^p\right.\\ &\quad \left. + \left(\frac{R_0-R_1}{R_1}\right)^p + \frac{[\alpha+(p-1)]^p\,2^{p-1}}{|\alpha|^p}\right\}\int_{B_{R_0}} u^{\alpha+(p-1)}\ \mathrm{d}x.\end{aligned} \tag{4.10}$$

Indeed the properties of the function $N(t) = \log\|u\|_t^t$ give

$$\frac{N(\overline{\alpha})-N(\overline{\alpha}+s-(p-1))}{(p-1)-s} \leq \frac{N(\alpha+(p-1))-N(\alpha+s)}{(p-1)-s} \quad \text{that is} \quad \frac{\|u\|_{\alpha+s}^{\alpha+s}}{\|u\|_{\alpha+(p-1)}^{\alpha+(p-1)}} \leq \frac{\|u\|_{\overline{\alpha}+s-(p-1)}^{\overline{\alpha}+s-(p-1)}}{\|u\|_{\overline{\alpha}}^{\overline{\alpha}}}.$$

Then, using Hölder inequality, the following reverse Hölder inequality

$$\left(\int_{B_{R_0}} u^{\overline{\alpha}}\ \mathrm{d}x\right)^{-\frac{(p-1)-s}{\overline{\alpha}}} \leq \frac{\int_{B_{R_0}} u^{s-(p-1)}\ \mathrm{d}x}{|B_{R_0}|^{\frac{\overline{\alpha}+(p-1)-s}{\overline{\alpha}}}}$$

and (4.9) give

$$\frac{\|u\|_{\alpha+s,R_0}^{\alpha+s}}{\|u\|_{\alpha+(p-1),R_0}^{\alpha+(p-1)}} \leq \frac{\|u\|_{\overline{\alpha}+s-(p-1),R_0}^{\overline{\alpha}+s-(p-1)}}{\|u\|_{\overline{\alpha},R_0}^{\overline{\alpha}}} \leq \frac{|B_{R_0}|^{\frac{(p-1)-s}{\overline{\alpha}}}}{\|u\|_{\overline{\alpha},R_0}^{(p-1)-s}} \leq \frac{\int_{B_{R_0}} u^{s-(p-1)}}{|B_{R_0}|} \leq \frac{1}{\lambda}\,\frac{p^{p-1}\,2^{p-1+d}}{R_0^p}.$$

Notice that when $s = p-1$, we obtain from (4.7) directly

$$\begin{aligned}\left[\int_{B_{R_1}} u^{[\alpha+(p-1)]\frac{r}{p}}\ \mathrm{d}x\right]^{\frac{p}{r}} &\leq \frac{S_p^p}{(R_0-R_1)^p}\left\{\frac{p}{|\alpha|}\frac{[\alpha+(p-1)]^p\,\lambda}{p^p}(R_0-R_1)^p\right.\\ &\quad \left. + \left(\frac{R_0-R_1}{R_1}\right)^p + \frac{[\alpha+(p-1)]^p\,2^{p-1}}{|\alpha|^p}\right\}\int_{B_{R_0}} u^{\alpha+(p-1)}\ \mathrm{d}x.\end{aligned} \tag{4.11}$$

• Step 3. *The first iteration step.* Now we are ready to write the first iteration step for all $s \ge 0$. Let $\beta = \alpha + (p-1) \ge \beta_0 > 0$ and recall that we are requiring $\beta \ne (p-1)$ as well, then inequalities (4.8), (4.10) and (4.11) can be written as

$$\left[\int_{B_{R_1}} u^{\beta\,\frac{r}{p}}\,\mathrm{d}x\right]^{\frac{p}{r}} \le I(\beta, s, p, R_1, R_0)\int_{B_{R_0}} u^{\beta+[s-(p-1)]_+}\,\mathrm{d}x$$

with

$$\begin{aligned} I(\beta,s,p,R_1,R_0) \quad &:= \quad \frac{S_p^p}{(R_0-R_1)^p}\,\frac{|B_{R_0}|}{\int_{B_{R_0}} u^{[s-(p-1)]_+}\,\mathrm{d}x}\left\{\frac{\beta^p\,\Lambda_s}{|\beta-(p-1)|}\left(\frac{R_0-R_1}{R_0}\right)^p\right.\\ &\qquad\left. +\left(\frac{R_0-R_1}{R_1}\right)^p+\frac{\beta^p\,2^{p-1}}{|\beta-(p-1)|^p}\right\}, \end{aligned} \tag{4.12}$$

and

$$\Lambda_s = \frac{\lambda}{p^{p-1}}\,R_0^p \ \text{ if } s = p-1 \ \text{ and } \ \Lambda_s = 2^{p-1+d} \ \text{ if } s \ne p-1. \tag{4.13}$$

• Step 4. *The Moser iteration.* Let us define the sequence of exponents $\beta_n > 0$ so that

$$\beta_n + [s-(p-1)]_+ = \beta_{n-1}\,\frac{r}{p} \quad \Rightarrow \quad \beta_n = \beta_{n-1}\,\frac{r}{p} - [s-(p-1)]_+$$

it turns out that, for any given $\beta_0 > 0$ and all $n \ge 1$, by (8.4),

$$\begin{aligned} \beta_n \quad &= \quad \beta_0\left(\frac{r}{p}\right)^n - [s-(p-1)]_+\sum_{k=0}^{n-1}\left(\frac{r}{p}\right)^k = \left(\frac{r}{p}\right)^n\left[\beta_0 - [s-(p-1)]_+\sum_{j=1}^{n}\left(\frac{p}{r}\right)^j\right]\\ &= \quad \left(\frac{r}{p}\right)^n\left\{\beta_0 - [s-(p-1)]_+\frac{p}{r-p}\right\} + [s-(p-1)]_+\frac{p}{r-p}. \end{aligned}$$

Moreover we have that for all $s \ge 0$,

$$\beta_n\left(\frac{p}{r}\right)^n \to \beta_0 - [s-(p-1)]_+\frac{p}{r-p}, \quad \text{as } n \to +\infty.$$

Requiring that $\beta_0 > p[s-(p-1)]_+/(r-p)$, which will be assumed from now on, then implies that $\beta_n \to +\infty$ as $n \to +\infty$. We shall also require $\beta_n \ne (p-1)$ for any n.

We will explicitly choose a decreasing sequence of radii $0 < R_\infty < \ldots < R_n < R_{n-1} < \ldots < R_0$ in the next step, in order to estimate the constants. The first iteration step reads:

$$\|u\|_{\beta_n\frac{r}{p},R_n} \le I(\beta_n,s,p,R_n,R_{n-1})^{\frac{1}{\beta_n}}\left[\int_{B_{R_{n-1}}} u^{\beta_n+[s-(p-1)]_+}\,\mathrm{d}x\right]^{\frac{1}{\beta_n}} = I_n^{\frac{1}{\beta_n}}\,\|u\|_{\beta_{n-1}\frac{r}{p},R_{n-1}}^{\frac{r}{p}\frac{\beta_{n-1}}{\beta_n}} \tag{4.14}$$

where the constants $I(\beta_n,s,p,R_n,R_{n-1})$ are as (4.12), that is

$$\begin{aligned} I_n \quad &= \quad \frac{S_p^p}{(R_{n-1}-R_n)^p}\,\frac{|B_{R_{n-1}}|}{\int_{B_{R_{n-1}}} u^{[s-(p-1)]_+}\,\mathrm{d}x}\left\{\frac{\beta_n^p\,\Lambda_{s,n}}{|\beta_n-(p-1)|}\left(\frac{R_{n-1}-R_n}{R_{n-1}}\right)^p\right.\\ &\qquad\left. +\left(\frac{R_{n-1}-R_n}{R_n}\right)^p+\frac{\beta_n^p\,2^{p-1}}{|\beta_n-(p-1)|^p}\right\}, \end{aligned}$$

with

$$\Lambda_{s,n} = \frac{\lambda}{p^{p-1}}\,R_{n-1}^p \ \text{ if } s = p-1 \ \text{ and } \ \Lambda_{s,n} = \Lambda_s = 2^{p-1+d} \ \text{ if } s \ne p-1.$$

Iterating (4.14), we get

$$\|u\|_{\beta_n \frac{r}{p},R_n} \le ... \le I_n^{\frac{1}{\beta_n}}\, I_{n-1}^{\frac{r}{p}\frac{1}{\beta_n}} ... I_1^{\left(\frac{r}{p}\right)^{n-1}\frac{1}{\beta_n}}\, \|u\|_{\frac{r}{p}\beta_0,R_0}^{\left(\frac{r}{p}\right)^{n}\frac{\beta_0}{\beta_n}} = \prod_{j=1}^{n} I_j^{\left(\frac{r}{p}\right)^{n-j}\frac{1}{\beta_n}}\, \|u\|_{\frac{r}{p}\beta_0,R_0}^{\left(\frac{r}{p}\right)^{n}\frac{\beta_0}{\beta_n}},$$

with

$$\beta_0 > [s-(p-1)]_+ \frac{p}{r-p} \quad \text{or} \quad q = \frac{r}{p}\beta_0 > \overline{q} := [s-(p-1)]_+ \frac{r}{r-p}.$$

Taking the limit as $n \to +\infty$ we obtain

$$\|u\|_{\infty,R_\infty} = \lim_{n\to\infty} \|u\|_{\beta_n \frac{r}{p},R_n} \le \lim_{n\to\infty} \prod_{j=1}^{n} I_j^{\left(\frac{r}{p}\right)^{n-j}\frac{1}{\beta_n}}\, \|u\|_{\frac{r}{p}\beta_0,R_0}^{\left(\frac{r}{p}\right)^{n}\frac{\beta_0}{\beta_n}} = I_\infty\, \|u\|_{q,R_0}^{\frac{(r-p)q}{(r-p)q-r[s-(p-1)]_+}}.$$

Notice that the last step follows because we shall see below that

$$\prod_{j=1}^{n} I_j^{\left(\frac{r}{p}\right)^{n-j}\frac{1}{\beta_n}}$$

has a limit I_∞ as $n \to \infty$. As a consequence of the above estimate we obtain the boundedness of the solution u so that the previous bound holds for any $q > [s-(p-1)]_+\, r/(r-p)$, as stated.

• Step 5. *Estimating all the constants.* Now it remains to estimate I_∞. We will prove later that

$$I_j \le I_0 \left(\frac{r}{p}\right)^{pj}. \tag{4.15}$$

Using such bound we show that

$$\begin{aligned} I_\infty &= \lim_{n\to\infty} \prod_{j=1}^{n} I_j^{\left(\frac{r}{p}\right)^{n-j}\frac{1}{\beta_n}} = \lim_{n\to\infty} \exp\left[\left(\frac{r}{p}\right)^n \frac{1}{\beta_n}\sum_{j=1}^{n}\left(\frac{p}{r}\right)^j \log(I_j)\right] \\ &\le \lim_{n\to\infty} \exp\left[\left(\frac{r}{p}\right)^n \frac{1}{\beta_n}\sum_{j=1}^{n}\left(\frac{p}{r}\right)^j \log\left[I_0\left(\frac{r}{p}\right)^{pj}\right]\right] \\ &= \exp\left\{\frac{p}{\beta_0(r-p)-[s-(p-1)]_+\,p}\left[\log(I_0) + \frac{rp}{r-p}\log\left(\frac{r}{p}\right)\right]\right\} \\ &= I_0^{\frac{p}{\beta_0(r-p)-[s-(p-1)]_+\,p}} \cdot \left(\frac{r}{p}\right)^{\frac{pr}{r-p}\frac{p}{\beta_0(r-p)-[s-(p-1)]_+\,p}} = \left[I_0 \cdot \left(\frac{r}{p}\right)^{\frac{pr}{r-p}}\right]^{\frac{p}{\beta_0(r-p)-[s-(p-1)]_+\,p}} \end{aligned}$$

where we have used the identities (8.2) and (8.3). Now we have to prove (4.15) and so an explicit estimate for I_0 in order to finally obtain (4.4).

Estimating I_j. To obtain (4.15) for any j we choose a decreasing sequence of radii $0 < R_\infty < ... < R_j < R_{j-1} < ... < R_0$ such that

$$(R_{j-1}-R_j)^p = (R_0 - R_\infty)^p \frac{c_0^p}{\beta_j^p}, \quad \text{with } c_0 = \left[\sum_{j=1}^{\infty}\frac{1}{\beta_j}\right]^{-1} < \infty,$$

so that

$$\sum_{j=1}^{\infty}(R_{j-1}-R_j) = R_0 - R_\infty.$$

So

$$
\begin{aligned}
I_j \quad \le \quad & \frac{S_p^p\,\beta_j^p}{c_0^p\,(R_0-R_\infty)^p}\,\frac{|B_{R_0}|}{\int_{B_{R_\infty}} u^{[s-(p-1)]_+}}\left\{\frac{\Lambda_{s,0}\,c_0^p}{|\beta_j-(p-1)|}\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\right.\\
&\left.+\frac{c_0^p}{\beta_j^p}\left(\frac{R_0-R_\infty}{R_\infty}\right)^p+\frac{\beta_j^p\,2^{p-1}}{|\beta_j-(p-1)|^p}\right\}
\end{aligned}
$$

recalling that

$$
\Lambda_{s,0}=\frac{\lambda}{p^{p-1}}\,R_0^p \;\text{ if } s=p-1 \;\text{ and }\; \Lambda_{s,0}=2^{p-1+d}\;\text{ if } s\neq p-1.
$$

Notice that

$$
\frac{|\beta_j-(p-1)|}{\beta_j^{p+1}}\le \max\left\{\frac{|\beta_0-(p-1)|}{\beta_0^{p+1}},\ \left(\frac{p}{p-1}\right)^p\ \frac{1}{(p+1)^{p+1}}\right\}=:c_2,
$$

and

$$
\beta_j=\left(\frac{r}{p}\right)^j\left\{\beta_0-[s-(p-1)]_+\frac{p}{r-p}\right\}+[s-(p-1)]_+\frac{p}{r-p}\le \beta_0\left(\frac{r}{p}\right)^j,\quad \forall\ j.
$$

Moreover

$$
\frac{\beta_j}{|\beta_j-(p-1)|}\le c_1:=\begin{cases}\dfrac{\beta_0}{\beta_0-(p-1)}, & \text{if } \beta_0>p-1,\\[2ex] \displaystyle\max_{i=0,1}\frac{\beta_{j_0+i}}{|\beta_{j_0+i}-(p-1)|}, & \text{if } 0<\beta_0<p-1.\end{cases}
$$

As a matter of fact, when $\beta_0>p-1$, we have

$$
\frac{\beta_j}{|\beta_j-(p-1)|}\le\frac{\beta_0}{\beta_0-(p-1)}
$$

since the one-variable real function

$$
\frac{t}{|t-(p-1)|}=\frac{t}{t-(p-1)}\quad\text{ is decreasing for } t\ge\beta_0>p-1.
$$

The case $0<\beta_0<p-1$ deserves a further explanation. We define j_0 to be the greatest integer for which $\beta_{j_0}<p-1$, so that $\beta_{j_0+1}>p-1$, that is

$$
\beta_{j_0}<p-1<\beta_{j_0+1}\quad\text{if and only if}\quad j_0=i.p.\left[\frac{\log\frac{p-1}{\beta_0}}{\log\frac{r}{p}}\right]
$$

and we shall take $\beta_0\in(0,p-1)$ such that

$$
\frac{\log\frac{p-1}{\beta_0}}{\log\frac{r}{p}}\neq i.p.\left[\frac{\log\frac{p-1}{\beta_0}}{\log\frac{r}{p}}\right] \tag{4.16}
$$

Summing up, when we consider $0<\beta_0<p-1$, we have to be careful to choose it so that $\beta_j\neq p-1$ for all j which amounts (4.16), then we can assure that $\beta_{j_0}<p-1<\beta_{j_0+1}$ and we can estimate

$$
\frac{\beta_j}{|\beta_j-(p-1)|}\le\max_{i=0,1}\frac{\beta_{j_0+i}}{|\beta_{j_0+i}-(p-1)|}=\max_{i=0,1}\frac{\left(\frac{r}{p}\right)^{j_0+i}\beta_0}{\left|\left(\frac{r}{p}\right)^{j_0+i}\beta_0-(p-1)\right|}.
$$

Hence, we can go on estimating I_j, we get

$$\begin{aligned}
I_j \quad &\leq \quad \frac{S_p^p\,\beta_j^p}{(R_0-R_\infty)^p}\,\frac{|B_{R_0}|}{\int_{B_{R_\infty}} u^{[s-(p-1)]_+}}\,\frac{\beta_j}{|\beta_j-(p-1)|}\\
&\quad\times\left\{\frac{\Lambda_{s,0}}{\beta_j}\left(\frac{R_0-R_\infty}{R_\infty}\right)^p+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\frac{|\beta_j-(p-1)|}{\beta_j^{p+1}}+\left(\frac{\beta_j}{|\beta_j-(p-1)|}\right)^{p-1}\frac{2^{p-1}}{c_0^p}\right\}\\
&\leq \quad \frac{S_p^p\,\beta_0^p\,c_1}{(R_0-R_\infty)^p}\,\frac{|B_{R_0}|}{\int_{B_{R_\infty}} u^{[s-(p-1)]_+}}\left\{\frac{\Lambda_{s,0}}{\beta_0}\left(\frac{R_0-R_\infty}{R_\infty}\right)^p+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p c_2+\frac{c_1^{p-1}\,2^{p-1}}{c_0^p}\right\}\left(\frac{r}{p}\right)^{jp}\\
&= \quad I_0\left(\frac{r}{p}\right)^{jp}.
\end{aligned}$$

Estimate (4.15) is now proved.

After some simple calculations, the proof is concluded by letting $\beta_0 = pq/r$. $\square$

4.2 Local upper bounds II. The case of unbounded coefficients.

In this section we establish upper bounds for nonnegative solution to $-\Delta_p u = b(x)u^{p-1}$ on B_R with $b \in L^m(B_R)$ eventually unbounded. These estimates follow from the energy estimates together with the Reverse Poincaré inequalities, which are consequence of Sobolev inequality on balls, see Section 3 .

Theorem 4.2 (The Moser iteration) *Let u be a nonnegative weak (sub)solution to $-\Delta_p u = b\,u^{p-1}$ on B_R, with $b \in L^m(B_R)$, $m > r/(r-p)$ and $r > p$ as in (3.2). Then the following bound holds true for any $R_\infty < R_0 < R$ and $q > p-1$*

$$\|u\|_{\infty,R_\infty} \leq \frac{I_{\infty,q}\,(b)}{(R_0-R_\infty)^{\frac{pr}{q(r-p)}}}\,\|u\|_{q,R_0}, \tag{4.17}$$

with constant

$$\begin{aligned}
I_{\infty,q}(b) \quad &= \quad \left(S_p^p\,q^{\frac{mr(p-1)}{m(r-p)-r}}\right)^{\frac{r}{q(r-p)}}\left(\frac{p}{r-p}\right)^{\frac{mr^2(p-1)}{q(r-p)[m(r-p)-r]}}\left(\frac{r}{p}\right)^{\frac{mr^2(p-1)(r+p)}{q(r-p)^2[m(r-p)-r]}}\\
&\quad\times\left[3\left(\frac{q}{q-(p-1)}\right)^p\frac{p^p}{q^{\frac{rm(p-1)}{m(r-p)-r}}}+K_{m,p,r}\,\frac{2^{\frac{mr+(mp+r)(p-1)}{m(r-p)-r}}}{p^{\frac{rm(p-1)}{m(r-p)-r}}}\left(\frac{q}{q-(p-1)}\right)^{\frac{mr}{m(r-p)-r}}\right.\\
&\quad\left.\times\;(R_0-R_\infty)^p\,|B_{R_0}|^{\frac{p}{r}}\,\|b\|_{m,R_0}^{\frac{mr}{m(r-p)-r}}+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\frac{1}{(p-1)^{\frac{mr(p-1)}{m(r-p)-r}}}\right]^{\frac{r}{q(r-p)}}
\end{aligned}$$

and $K_{m,p,r}$ as in (3.5).

Remark 4.3 *Notice that in the case of bounded coefficients, i. e. $b \in L^\infty(B_R)$, we can pass to the limit as $m \to \infty$ in the above expression $I_{\infty,q}\,(b)$ to get*

$$\begin{aligned}
I_{\infty,q}(b) \quad &= \quad \left(S_p^p\,q^{\frac{r(p-1)}{r-p}}\right)^{\frac{r}{q(r-p)}}\left(\frac{p}{r-p}\right)^{\frac{r^2(p-1)}{q(r-p)^2}}\left(\frac{r}{p}\right)^{\frac{r^2(p-1)(r+p)}{q(r-p)^3}}\\
&\quad\times\left[3\left(\frac{q}{q-(p-1)}\right)^p\frac{p^p}{q^{\frac{r(p-1)}{r-p}}}+\frac{r-p}{r}\left(\frac{p}{r}\right)^{\frac{p}{r-p}}\frac{2^{\frac{r+p(p-1)}{r-p}}}{p^{\frac{r(p-1)}{r-p}}}\left(\frac{q}{q-(p-1)}\right)^{\frac{r}{r-p}}\right.\\
&\quad\left.\times\;(R_0-R_\infty)^p\,|B_{R_0}|^{\frac{p}{r}}\,\|b\|_{\infty,R_0}^{\frac{r}{r-p}}+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\frac{1}{(p-1)^{\frac{r(p-1)}{r-p}}}\right]^{\frac{r}{q(r-p)}}.
\end{aligned}$$

Proof. We divide the proof in two steps.

• Step 1. *Sobolev and Reverse Poincaré inequalities.* We start considering the radii $R_\infty < \rho_1 < \rho_0 < R_0$ and we use (3.1) on the ball B_{ρ_1} with $g = u^{[\alpha+(p-1)]/p}$, for some $\alpha > 0$, to get

$$\left[\int_{B_{\rho_1}} u^{\frac{\alpha+(p-1)}{p} r}\,\mathrm{d}x\right]^{\frac{p}{r}} \le S_p^p \left[\int_{B_{\rho_1}} |\nabla(u^{\frac{\alpha+(p-1)}{p}})|^p\,\mathrm{d}x + \frac{1}{\rho_1^p}\int_{B_{\rho_1}} u^{\alpha+(p-1)}\,\mathrm{d}x\right].$$

To estimate the first term in the right hand side of the previous inequality we use Theorem 3.2 (see Remark 3.4), we get

$$\left[\int_{B_{\rho_1}} u^{\frac{\alpha+(p-1)}{p} r}\,\mathrm{d}x\right]^{\frac{p}{r}} \le S_p^p \left[\Lambda(b) + \frac{1}{\rho_1^p}\right]\int_{B_{\rho_0}} u^{\alpha+(p-1)}\,\mathrm{d}x \tag{4.18}$$

with $\Lambda(b)$ as in (3.10) and $K_{m,p,r}$ in (3.5).

Notice that

$$\begin{aligned}
\Lambda(b) \quad \le \quad & \frac{[\alpha+(p-1)]^{\frac{mr(p-1)}{m(r-p)-r}}}{(\rho_0-\rho_1)^p}\left\{3\left(\frac{\alpha+(p-1)}{\alpha}\right)^p \frac{p^p}{[\alpha+(p-1)]^{\frac{mr(p-1)}{m(r-p)-r}}} + K_{m,p,r}\frac{2^{\frac{mr+(mp+r)(p-1)}{m(r-p)-r}}}{p^{\frac{mr(p-1)}{m(r-p)-r}}}\right.\\
&\left.\times\left(\frac{\alpha+(p-1)}{\alpha}\right)^{\frac{mr}{m(r-p)-r}}(R_0-R_\infty)^p\,|B_{R_0}|^{\frac{p}{r}}\,\|b\|_{m,R_0}^{\frac{rm}{m(r-p)-r}}\right\}.
\end{aligned}$$

Hence, we get, since $\alpha > 0$,

$$\begin{aligned}
S_p^p\left[\Lambda(b)+\frac{1}{\rho_1^p}\right] \quad \le \quad & \frac{S_p^p\,[\alpha+(p-1)]^{\frac{mr(p-1)}{m(r-p)-r}}}{(\rho_0-\rho_1)^p}\left\{3\left(\frac{\alpha+(p-1)}{\alpha}\right)^p\frac{p^p}{[\alpha+(p-1)]^{\frac{mr(p-1)}{m(r-p)-r}}}\right.\\
&+K_{m,p,r}\frac{2^{\frac{mr+(mp+r)(p-1)}{m(r-p)-r}}}{p^{\frac{mr(p-1)}{m(r-p)-r}}}\left(\frac{\alpha+(p-1)}{\alpha}\right)^{\frac{mr}{m(r-p)-r}}(R_0-R_\infty)^p|B_{R_0}|^{\frac{p}{r}}\,\|b\|_{m,R_0}^{\frac{rm}{m(r-p)-r}}\\
&\left.+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\frac{1}{(p-1)^{\frac{mr(p-1)}{m(r-p)-r}}}\right\}.
\end{aligned}$$

• Step 2. *The Moser iteration.* We now fix $\beta_0 = \alpha + (p-1) > p-1$ and we define the sequence

$$\beta_n = \frac{r}{p}\,\beta_{n-1} = \left(\frac{r}{p}\right)^n \beta_0$$

and that of radii $R_\infty = \rho_\infty < ... < \rho_n < \rho_{n-1} < ... < \rho_0 = R_0$ such that

$$(\rho_{n-1}-\rho_n)^p = c_3^p\,(R_0-R_\infty)^p\,\left(\frac{p}{r}\right)^{\frac{mr(p-1)n}{m(r-p)-r}} \tag{4.19}$$

with

$$c_3 := \left[\sum_{k=1}^{\infty}\left(\frac{p}{r}\right)^{\frac{mr(p-1)k}{p[m(r-p)-r]}}\right]^{-1}$$

so that

$$\sum_{k=1}^{\infty}(\rho_{k-1}-\rho_k) = (R_0-R_\infty).$$

Moreover, recalling (8.2), we get the following estimate for c_3

$$c_3 = \left(\frac{r}{p}\right)^{\frac{mr(p-1)}{p[m(r-p)-r]}} - 1 \ge \left(\frac{r}{p}-1\right)^{\frac{mr(p-1)}{p[m(r-p)-r]}} = \left(\frac{r-p}{p}\right)^{\frac{mr(p-1)}{p[m(r-p)-r]}}. \tag{4.20}$$

With these choices inequality (4.18), in which $\alpha+(p-1)$ is replaced by $\beta_{n-1}>p-1$ and ρ_1, ρ_0 by ρ_n, ρ_{n-1} respectively, reads

$$\begin{aligned}\left[\int_{B_{\rho_n}} u^{\beta_n}\,\mathrm{d}x\right]^{\frac{p}{r}} \leq\;& \frac{S_p^p\,\beta_{n-1}^{\frac{mr(p-1)}{m(r-p)-r}}}{(\rho_{n-1}-\rho_n)^p}\left\{3\left(\frac{\beta_{n-1}}{\beta_{n-1}-(p-1)}\right)^p \frac{p^p}{\beta_{n-1}^{\frac{mr(p-1)}{m(r-p)-r}}}+K_{m,p,r}\,\frac{2^{\frac{mr+(mp+r)(p-1)}{m(r-p)-r}}}{p^{\frac{mr(p-1)}{m(r-p)-r}}}\right.\\ &\times\left(\frac{\beta_{n-1}}{\beta_{n-1}-(p-1)}\right)^{\frac{mr}{m(r-p)-r}}(R_0-R_\infty)^p\,|B_{R_0}|^{\frac{p}{r}}\,\|b\|_{m,R_0}^{\frac{mr}{m(r-p)-r}}\\ &\left.+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\frac{1}{(p-1)^{\frac{mr(p-1)}{m(r-p)-r}}}\right\}\int_{B_{\rho_{n-1}}} u^{\beta_{n-1}}\,\mathrm{d}x = I_{n-1}\int_{B_{\rho_{n-1}}} u^{\beta_{n-1}}\,\mathrm{d}x.\end{aligned}$$

Letting $Y_n:=\|u\|_{\beta_n,\rho_n}$, we have obtained

$$Y_n\leq I_{n-1}^{\frac{1}{\beta_{n-1}}}\|u\|_{\beta_{n-1},\rho_{n-1}} = I_{n-1}^{\left(\frac{p}{r}\right)^{n-1}\frac{1}{\beta_0}}\,Y_{n-1} = I_{n-1}^{\sigma\,\theta^{n-1}}\,Y_{n-1},$$

where we have set $\sigma=1/\beta_0$ and $\theta=p/r\in(0,1)$. We shall prove that $I_{n-1}\leq C^{n-1}\,I_0$, for some $C>0$, in order to apply Lemma 8.5. Indeed

$$\begin{aligned}I_{n-1}\;\leq\;& \frac{S_p^p\,\beta_0^{\frac{mr(p-1)}{m(r-p)-r}}}{(R_0-R_\infty)^p}\left(\frac{p}{r-p}\right)^{\frac{mr(p-1)}{m(r-p)-r}}\left\{3\left(\frac{\beta_0}{\beta_0-(p-1)}\right)^p\frac{p^p}{\beta_0^{\frac{mr(p-1)}{m(r-p)-r}}}+\frac{2^{\frac{mr+(mp+r)(p-1)}{m(r-p)-r}}}{p^{\frac{mr(p-1)}{m(r-p)-r}}}\right.\\ &\times K_{m,p,r}\left(\frac{\beta_0}{\beta_0-(p-1)}\right)^{\frac{mr}{m(r-p)-r}}(R_0-R_\infty)^p\,|B_{R_0}|^{\frac{p}{r}}\,\|b\|_{m,R_0}^{\frac{mr}{m(r-p)-r}}\\ &\left.+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\frac{1}{(p-1)^{\frac{mr(p-1)}{m(r-p)-r}}}\right\}\left(\frac{r}{p}\right)^{\frac{mr(p-1)}{m(r-p)-r}}\left(\frac{r}{p}\right)^{\frac{2mr(p-1)}{m(r-p)-r}(n-1)} =: I_0\,C^{n-1},\end{aligned}$$

where we have used (4.19), (4.20), the definition of β_{n-1} and the following facts

$$\frac{\beta_{n-1}}{\beta_{n-1}-(p-1)}\leq\frac{\beta_0}{\beta_0-(p-1)},\quad\text{for }\beta_0>p-1\ \text{ and }\ \frac{1}{\beta_{n-1}}\leq\frac{1}{\beta_0}.$$

So by Lemma 8.5 with the above choices of σ, θ, I_0 and C, we get

$$Y_\infty\leq I_0^{\frac{\sigma}{1-\theta}}\,C^{\frac{\sigma\,\theta}{(1-\theta)^2}}\,Y_0\quad\text{which is}\quad \|u\|_{\infty,R_\infty}\leq I_\infty(b)\,\|u\|_{\beta_0,R_0}$$

with

$$\begin{aligned}I_\infty(b)\;=\;&\left\{\frac{S_p^p\,\beta_0^{\frac{mr(p-1)}{m(r-p)-r}}}{(R_0-R_\infty)^p}\left(\frac{p}{r-p}\right)^{\frac{mr(p-1)}{m(r-p)-r}}\left[3\left(\frac{\beta_0}{\beta_0-(p-1)}\right)^p\frac{p^p}{\beta_0^{\frac{mr(p-1)}{m(r-p)-r}}}\right.\right.\\ &+\frac{2^{\frac{mr+(mp+r)(p-1)}{m(r-p)-r}}}{p^{\frac{mr(p-1)}{m(r-p)-r}}}K_{m,p,r}\left(\frac{\beta_0}{\beta_0-(p-1)}\right)^{\frac{mr}{m(r-p)-r}}(R_0-R_\infty)^p\,|B_{R_0}|^{\frac{p}{r}}\,\|b\|_{m,R_0}^{\frac{mr}{m(r-p)-r}}\\ &\left.\left.+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\frac{1}{(p-1)^{\frac{mr(p-1)}{m(r-p)-r}}}\right]\right\}^{\frac{r}{\beta_0(r-p)}}\left(\frac{r}{p}\right)^{\frac{mr^2(p-1)(r+p)}{\beta_0[m(r-p)-r](r-p)^2}}.\end{aligned}$$

The proof is concluded once we let $\beta_0=q>p-1$. □

Theorem 4.4 (Extending Local Upper Bounds) *Let u be a nonnegative weak solution to $-\Delta_p u = b\,u^{p-1}$ on B_R, with $p>1$, $b\in L^m(B_R)$, $m>r/(r-p)$ and $r>p$ as in (3.2). Then the following bound holds for any $R_\infty<R_0<R$ and for any $q_0>0$*

$$\|u\|_{\infty,R_\infty}\leq\frac{A_{q_0}^{(1)}}{(R_0-R_\infty)^{\frac{pr}{q_0(r-p)}}}\left[A_{q_0}^{(2)}+A_{q_0}^{(3)}\|b\|_{m,R_0}^{\frac{mr}{m(r-p)-r}}\right]^{\frac{r}{q_0(r-p)}}\|u\|_{q_0,R_0}$$

the constants $A_{q_0}^{(i)}$, for $i = 1,2,3$ depend on d,p,s,q,r,R_0,R_∞, see an explicit expression in formulas (4.21) *below.*

Remark on the constant. The proof below allow us to find an explicit expression of the constant:

(4.21)

$$
\begin{aligned}
A_{q_0}^{(1)} &:= \left(S_p^p\, q_0^{\frac{mr(p-1)}{m(r-p)-r}}\right)^{\frac{r}{q_0(r-p)}} \left(\frac{p}{r-p}\right)^{\frac{mr^2(p-1)}{q_0(r-p)[m(r-p)-r]}} \left(\frac{r}{p}\right)^{\frac{mr^2(p-1)(r+p)}{q_0(r-p)^2[m(r-p)-r]}} && \text{if } q_0 > p-1\\
A_{q_0}^{(1)} &:= 3\left(S_p^p\, [q_0+(p-1)]^{\frac{mr(p-1)}{m(r-p)-r}}\right)^{\frac{r}{q_0(r-p)}} \left(\frac{p}{r-p}\right)^{\frac{mr^2(p-1)}{q_0(r-p)[m(r-p)-r]}} \left(\frac{r}{p}\right)^{\frac{mr^2(p-1)(r+p)}{q_0(r-p)^2[m(r-p)-r]}}\\
&\quad\times \left(\frac{pr}{q_0(r-p)}\right)^{\frac{pr}{q_0(r-p)}} 2^{\frac{p-1}{q_0}+\frac{2pr}{q_0(r-p)}} && \text{if } 0 < q_0 \le p-1\\
A_{q_0}^{(2)} &:= 3\left(\frac{q_0}{q_0-(p-1)}\right)^p \frac{p^p}{q_0^{\frac{mr(p-1)}{m(r-p)-r}}} + \left(\frac{R_0-R_\infty}{R_\infty}\right)^p \frac{1}{(p-1)^{\frac{mr(p-1)}{m(r-p)-r}}} && \text{if } q_0 > p-1\\
A_{q_0}^{(2)} &:= 3\left(\frac{q_0+(p-1)}{q_0}\right)^p \frac{p^p}{[q_0+(p-1)]^{\frac{mr(p-1)}{m(r-p)-r}}}\\
&\quad+ \left(\frac{R_0-R_\infty}{R_\infty}\right)^p \frac{1}{(p-1)^{\frac{mr(p-1)}{m(r-p)-r}}} && \text{if } 0 < q_0 \le p-1
\end{aligned}
$$

and

$$
\begin{aligned}
A_{q_0}^{(3)} &:= K_{m,p,r}\frac{2^{\frac{mr+(mp+r)(p-1)}{m(r-p)-r}}}{p^{\frac{mr(p-1)}{m(r-p)-r}}} \left(\frac{q_0}{q_0-(p-1)}\right)^{\frac{mr}{m(r-p)-r}} (R_0-R_\infty)^p\,|B_{R_0}|^{\frac{p}{r}} && \text{if } q_0 > p-1\\
A_{q_0}^{(3)} &:= K_{m,p,r}\frac{2^{\frac{mr+(mp+r)(p-1)}{m(r-p)-r}}}{p^{\frac{mr(p-1)}{m(r-p)-r}}} \left(\frac{q_0+(p-1)}{q_0}\right)^{\frac{mr}{m(r-p)-r}} (R_0-R_\infty)^p|B_{R_0}|^{\frac{p}{r}} && \text{if } 0 < q_0 \le p-1,
\end{aligned}
$$

with $K_{m,p,r}$ as in (3.5).

Proof. The statement of the theorem, in the case $q_0 > p-1$, easy follows from Theorem 4.2. When $0 < q_0 \le p-1$ we have to apply Lemma 8.7. By Theorem 4.2 (with $q = q_0 + (p-1) > p-1$), we have

$$\|u\|_{\infty,R_\infty} \le \frac{I_{\infty,q_0+(p-1)}(b)}{(R_0-R_\infty)^{\frac{pr}{[q_0+(p-1)](r-p)}}}\|u\|_{q_0+(p-1),R_0}$$

and hence we arrive at the desired result using Lemma 8.7 with $\overline{q} = \infty$, $\underline{q} = q_0 + (p-1)$, $K = I_{\infty,q_0+(p-1)}(b)$ and $\gamma = pr/\{[q_0+(p-1)](r-p)\}$. $\square$

The above theorem has the following important consequence, when applied to the equation $-\Delta_p u = \lambda\, u^s$.

Theorem 4.5 (Local Upper Bounds, second form) *Let u be a nonnegative weak solution to $-\Delta_p u = \lambda\, u^s$ on B_R, with $p > 1$ $\lambda > 0$, $s > p-1$ and $r > p$ as in* (3.2). *If $u \in L^{\overline{m}}(B_{R_0})$, with $\overline{m} > r[s-(p-1)]/(r-p)$, then the following bound holds for any $R_\infty < R_0 < R$ and for any $q_0 > 0$*

$$\|u\|_{\infty,R_\infty} \le \frac{A_{q_0}^{(1)}}{(R_0-R_\infty)^{\frac{pr}{q_0(r-p)}}}\left[A_{q_0}^{(2)} + A_{q_0}^{(3)}\,\lambda^{\frac{\overline{m}r}{\overline{m}(r-p)-r[s-(p-1)]}}\,\|u\|_{\overline{m},R_0}^{\frac{\overline{m}r[s-(p-1)]}{\overline{m}(r-p)-r[s-(p-1)]}}\right]^{\frac{r}{q_0(r-p)}}\|u\|_{q_0,R_0},$$

where $A_{q_0}^{(1)}$, $A_{q_0}^{(2)}$ and $A_{q_0}^{(3)}$ are as in Theorem 4.4.

Proof. Since u is a solution to $-\Delta_p u = \lambda\, u^s$ on B_R, then u is also a solution to $-\Delta_p u = b\, u^{p-1}$ on B_R with $b = \lambda\, u^{s-(p-1)}$. Therefore we need to assume $u^{s-(p-1)} \in L^m(B_R)$, with $m > r/(r-p)$ that it is equivalent to require $u \in L^{\overline{m}}(B_R)$, with $\overline{m} = m[s-(p-1)] > r[s-(p-1)]/(r-p)$. So that

$$\|b\|_{m,R_0}^{\frac{mr}{m(r-p)-r}} = \left[\lambda^m \int_{B_{R_0}} u^{m[s-(p-1)]}\,\mathrm{d}x\right]^{\frac{r}{m(r-p)-r}} = \lambda^{\frac{\overline{m}r}{\overline{m}(r-p)-r[s-(p-1)]}}\, \|u\|_{\overline{m},R_0}^{\frac{\overline{m}r[s-(p-1)]}{\overline{m}(r-p)-r[s-(p-1)]}}.$$

Hence the result follows from Theorem 4.4. □

5 Local lower bounds via Moser iteration

In this section we prove quantitative local lower bounds for nonnegative weak solutions to $-\Delta_p u = \lambda u^s$. The strategy to prove the lower bounds is classical, and combines a lower Moser iteration with some reverse Hölder inequalities obtained via a John-Nirenberg type Lemma. Since we are interested keeping track of all (the relevant) constants, we need a quantitative version of a John-Nirenberg type Lemma to obtain quantitative reverse Hölder inequalities; this has been done in [4] and the proofs of [4] also adapt to the current setting with minor modifications that we give in Appendix 8.1.

We first show how the lower Moser iteration proves quantitative local lower bounds in a general form, that hold in the whole range $0 \le s < r-1$. In the next subsection, we will improve the results in a smaller range, namely $p-1 < s < s_c^* = r(p-1)/p$.

Theorem 5.1 (Local Lower Estimates) *Let $\Omega \subset \mathbb{R}^d$. Let u be a nonnegative local weak solution to $-\Delta_p u = \lambda u^s$ in Ω, with $p > 1$, $\lambda \ge 0$ and $0 \le s < s_c = r-1$, r as in (3.2). Then for any $\varepsilon > 0$, for any*

$$0 < \underline{q} \le \frac{(p-1)^{\frac{2}{p}}\, 2^{\frac{(d-1)(p-1)}{p}}}{p\,\omega_d^2\, d\,[e(d-1)+\varepsilon]} = q_0$$

and for any $B_{R_\infty} \subset B_{R_0} \subseteq \Omega$ the following bound holds

$$\inf_{x\in B_{R_\infty}} u(x) = \|u\|_{-\infty,R_\infty} \ge I_{-\infty,\underline{q}}\, \frac{\|u\|_{\underline{q},R_0}}{|B_{R_0}|^{\frac{1}{\underline{q}}}}$$

where

$$I_{-\infty,\underline{q}} = \left[\frac{(R_0-R_\infty)R_\infty}{R_0^{\frac{d(r-p)}{rp}}}\right]^{\frac{pr}{\underline{q}(r-p)}} \left[\frac{1}{S_p^p[2^{p-1}R_\infty^p + (R_0-R_\infty)^p]\,2^{\frac{pr}{r-p}}}\right]^{\frac{r}{\underline{q}\,(r-p)}} \left[\frac{\varepsilon}{2^d(e\,d+\varepsilon)\,\sqrt{\omega_d}}\right]^{\frac{2}{\underline{q}}}. \tag{5.1}$$

Proof. The proof is divided in two steps.

• Step 1. In this step we want to prove $L^{-q} - L^{-\infty}$ local estimates via Moser iteration. Consider $\alpha < -(p-1)$, choosing ϕ as in Lemma 2.3 in the estimate (2.3), we obtain

$$\int_{B_{R_1}} |\nabla[(u+\delta)^{\frac{\alpha+p-1}{p}}]|^p\,\mathrm{d}x \le \frac{2^{p-1}\,|\alpha+(p-1)|^p}{|\alpha|^p\,(R_0-R_1)^p} \int_{B_{R_0}} (u+\delta)^{\alpha+p-1}\,\mathrm{d}x.$$

Applying now the Sobolev inequality (3.1) on the ball B_{R_1}, one gets

$$\begin{aligned}
\left[\int_{B_{R_1}} (u+\delta)^{\frac{\alpha+p-1}{p}r}\,\mathrm{d}x\right]^{\frac{p}{r}} &\le S_p^p \left[\int_{B_{R_1}} |\nabla[(u+\delta)^{\frac{\alpha+p-1}{p}}]|^p\,\mathrm{d}x + \frac{1}{R_1^p}\int_{B_{R_1}} (u+\delta)^{\alpha+p-1}\,\mathrm{d}x\right]\\
&\le S_p^p \left[\frac{2^{p-1}\,|\alpha+(p-1)|^p}{|\alpha|^p\,(R_0-R_1)^p} + \frac{1}{R_1^p}\right] \int_{B_{R_0}} (u+\delta)^{\alpha+p-1}\,\mathrm{d}x\\
&\le S_p^p \left[\frac{2^{p-1}}{(R_0-R_1)^p} + \frac{1}{R_1^p}\right] \int_{B_{R_0}} (u+\delta)^{\alpha+p-1}\,\mathrm{d}x,
\end{aligned}$$

since $|\alpha+(p-1)|/|\alpha|<1$ for any $\alpha<-(p-1)$. Let for a given $\gamma_0<0$,

$$\gamma_n=\frac{r}{p}\,\gamma_{n-1}=\left(\frac{r}{p}\right)^n\gamma_0.$$

Notice that $\gamma_n\to-\infty$ monotonically. The above inequality, with $\alpha=\alpha_n$ and $\gamma_{n-1}=\alpha_n+(p-1)<0$ reads

$$\begin{aligned}\|u+\delta\|_{\gamma_n,R_n} &\geq \left[S_p^p\left(\frac{2^{p-1}}{(R_{n-1}-R_n)^p}+\frac{1}{R_n^p}\right)\right]^{\frac{1}{\gamma_{n-1}}}\left[\int_{B_{R_{n-1}}}(u+\delta)^{\gamma_{n-1}}\,\mathrm{d}x\right]^{\frac{1}{\gamma_{n-1}}}\\ &=: I_n^{\frac{1}{\gamma_{n-1}}}\,\|u+\delta\|_{\gamma_{n-1},R_{n-1}}.\end{aligned}$$

The iteration is simple now, and gives

$$\|u+\delta\|_{\gamma_n,R_n}\geq I_n^{\frac{1}{\gamma_{n-1}}}\,I_{n-1}^{\frac{1}{\gamma_{n-2}}}\,...I_1^{\frac{1}{\gamma_0}}\,\|u+\delta\|_{\gamma_0,R_0}=\prod_{j=1}^n I_j^{\frac{1}{\gamma_{j-1}}}\,\|u+\delta\|_{\gamma_0,R_0},\tag{5.2}$$

where we have chosen $0<R_\infty<...<R_n<R_{n-1}<...<R_0$ such that

$$\sum_{j=1}^\infty(R_{j-1}-R_j)=R_0-R_\infty\quad\text{and}\quad R_{j-1}-R_j=\frac{R_0-R_\infty}{2^j}$$

so that

$$I_j=S_p^p\left(\frac{2^{p-1}\,2^{pj}}{(R_0-R_\infty)^p}+\frac{1}{R_j^p}\right)\leq\frac{S_p^p}{(R_0-R_\infty)^p}\left[2^{p-1}+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\right]2^{pj}=:I_0\,2^{pj}$$

and

$$\begin{aligned}\prod_{j=1}^n(I_0\,2^{pj})^{\frac{1}{\gamma_{j-1}}} &= \exp\left\{\sum_{j=1}^n\frac{1}{\gamma_{j-1}}\log(I_0\,2^{pj})\right\}=\exp\left\{\frac{1}{\gamma_0}\,\frac{r}{p}\sum_{j=1}^n\left(\frac{p}{r}\right)^j\log(I_0\,2^{pj})\right\}\\ &= \exp\left\{\frac{1}{\gamma_0}\,\frac{r}{p}\log I_0\sum_{j=1}^n\left(\frac{p}{r}\right)^j+\frac{1}{\gamma_0}\,r\log 2\sum_{j=1}^n j\left(\frac{p}{r}\right)^j\right\}\\ &= I_0^{\frac{1}{\gamma_0}\,\frac{r}{p}\,\sum_{j=1}^n\left(\frac{p}{r}\right)^j}\,2^{\frac{r}{\gamma_0}\,\sum_{j=1}^n j\left(\frac{p}{r}\right)^j}\xrightarrow[n\to\infty]{}I_0^{\frac{1}{\gamma_0}\,\frac{r}{p}\,\sum_{j=1}^\infty\left(\frac{p}{r}\right)^j}\,2^{\frac{r}{\gamma_0}\,\sum_{j=1}^\infty j\left(\frac{p}{r}\right)^j}\end{aligned}$$

Using (8.2) and (8.3), we get

$$\prod_{j=1}^n(I_0\,2^{pj})^{\frac{1}{\gamma_{j-1}}}\xrightarrow[n\to\infty]{}I_0^{\frac{1}{\gamma_0}\,\frac{r}{r-p}}\,2^{\frac{r}{\gamma_0}\,\frac{pr}{(r-p)^2}}=(I_0\,2^{\frac{rp}{r-p}})^{\frac{r}{\gamma_0\,(r-p)}}.$$

We can now take the limit in (5.2) to get for any $\gamma_0<0$

$$\begin{aligned}\|u+\delta\|_{-\infty,R_\infty} &\geq \lim_{n\to\infty}\|u+\delta\|_{\gamma_n,R_n}\geq\lim_{n\to\infty}\prod_{j=1}^n(I_0\,2^{pj})^{\frac{1}{\gamma_{j-1}}}\,\|u+\delta\|_{\gamma_0,R_0}\\ &= \left\{\frac{S_p^p}{(R_0-R_\infty)^p}\left[2^{p-1}+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\right]2^{\frac{pr}{r-p}}\right\}^{\frac{r}{\gamma_0\,(r-p)}}\|u+\delta\|_{\gamma_0,R_0}\end{aligned}\tag{5.3}$$

• Step 2. *Reverse Hölder inequalities.* Joining inequality (5.3) and (8.1) and letting $\gamma_0=-\underline{q}$, for any

$$0<\underline{q}\leq\frac{(p-1)^{\frac{2}{p}}\,2^{\frac{(d-1)(p-1)}{p}}}{p\,\omega_d^2\,d\,[e(d-1)+\varepsilon]}.$$

we obtain

$$\|u+\delta\|_{-\infty,R_\infty} \geq \left[\frac{(R_0-R_\infty)^p\,R_\infty^p}{S_p^p[2^{p-1}R_\infty^p+(R_0-R_\infty)^p]2^{\frac{rp}{r-p}}\,R_0^{\frac{d(r-p)}{r}}}\right]^{\frac{r}{\underline{q}\,(r-p)}} \left[\frac{\varepsilon}{2^d(e\,d+\varepsilon)\,\sqrt{\omega_d}}\right]^{\frac{2}{\underline{q}}} \frac{\|u+\delta\|_{\underline{q},R_0}}{|B_{R_0}|^{\frac{1}{\underline{q}}}}.$$

To conclude the proof it is sufficient let $\delta\to 0^+$. $\square$

5.1 Reverse Hölder inequalities and additional local lower bounds

In this section we will prove first a more precise quantitative reverse Hölder inequality, that holds in the smaller range of exponents $s>p-1$. We have in mind to join local upper and lower estimates to get a clean form of Harnack inequality (see next section). The difficulty here is that the lower bound of the previos section has the form of reverse smoothing effect from $L^{\underline{q}}$ to $L^{-\infty}$ for a suitable explicit $\underline{q}$, which can be very small, sometimes too small: we need to reach higher values of q, namely above $r[s-(p-1)]/(r-p)$ and this will be possible through a reverse Hölder inequality, that holds only when $p-1<s<r(p-1)/p=s_c^*$. Under no further assumptions on the solution at hand, it is impossible -to our knowledge- to extend this reverse Hölder inequality to higher values of s in a quantitative way.

Proposition 5.2 (Reverse Hölder inequalities) *Let $\Omega\subset\mathbb{R}^d$ and let $\lambda>0$. Let u be a nonnegative local weak solution to $-\Delta_p u=\lambda u^s$ in Ω, with $p-1<s<r(p-1)/p=s_c^*$. Let $B_{\bar R}\subset B_{R_0}\subseteq\Omega$. Then*

$$\frac{\|u\|_{\overline{q},\bar R}}{|B_{\bar R}|^{\frac{1}{\overline{q}}}} \leq I_{\overline{q},q_0}\,\frac{\|u\|_{q_0,R_0}}{|B_{R_0}|^{\frac{1}{q_0}}}, \tag{5.4}$$

for any $q_0\in(0,\overline{q}]$, with

$$\frac{r[s-(p-1)]}{r-p}<\overline{q}<\frac{r(p-1)}{p}=s_c^*,$$

$$I_{\overline{q},q_0}:=\left\{S_p^p\left[\frac{2^{p-1}\,p^p\,\overline{q}^p}{[r(p-1)-p\overline{q}]^p}+\left(\frac{R_0-\bar R}{\bar R}\right)^p\right]\right\}^{\frac{r}{p\overline{q}}}\left[\frac{R_0^{d\left(\frac1p-\frac1r\right)}\,\omega_d^{\frac1p-\frac1r}}{R_0-\bar R}\right]^{\frac{r}{\overline{q}}}\left(\frac{R_0}{\bar R}\right)^{\frac{d}{\overline{q}}} \tag{5.5}$$

if $p\overline{q}/r\leq q_0\leq\overline{q}$ and

$$\begin{aligned} I_{\overline{q},q_0} &:= 3\cdot 2^{\frac{p\overline{q}-rq_0}{(r-p)q_0}}\left\{S_p^p\left[\frac{2^{p-1}\,p^p\,\overline{q}^p}{[r(p-1)-p\overline{q}]^p}\left(\frac{\bar R}{R_0-\bar R}\right)^p+1\right]\right\}^{\frac{r(\overline{q}-q_0)}{(r-p)q_0\overline{q}}}\\ &\quad\times\left(\frac{4\,r\,p\,(\overline{q}-q_0)\,\omega_d^{\frac{r-p}{rp}}}{(r-p)\,\overline{q}\,q_0}\right)^{\frac{rp(\overline{q}-q_0)}{(r-p)\overline{q}q_0}}\frac{R_0^{\frac{d}{q_0}}}{\bar R^{\frac{pr(\overline{q}-q_0)}{(r-p)\overline{q}q_0}+\frac{d}{\overline{q}}}}, \end{aligned} \tag{5.6}$$

if $0<q_0<p\overline{q}/r$.

Remark 5.3 *We note that the interval in which $\overline{q}$ can vary is not empty, since we are assuming $s<s_c^*$.*

Proof. Let $-(p-1)<\alpha<0$. Consider the energy estimate (2.5). It implies, using ϕ as in Lemma 2.3, with $R_\infty<R_0$

$$\int_{B_{R_\infty}}|\nabla(u^{\frac{\alpha+(p-1)}{p}})|^p\,\mathrm{d}x\leq\frac{2^{p-1}\,[\alpha+(p-1)]^p}{|\alpha|^p\,(R_0-R_\infty)^p}\int_{B_{R_0}}u^{\alpha+(p-1)}\,\mathrm{d}x.$$

Applying now Sobolev inequality (3.1) with $g = u^{[\alpha+(p-1)]/p}$ on the ball B_{R_∞} we arrive at

$$\begin{aligned}\left[\int_{B_{R_\infty}} u^{\frac{\alpha+(p-1)}{p}r}\,\mathrm{d}x\right]^{\frac{p}{r}} &\le S_p^p\left[\int_{B_{R_\infty}} |\nabla(u^{\frac{\alpha+(p-1)}{p}})|^p\,\mathrm{d}x + \frac{1}{R_\infty^p}\int_{B_{R_\infty}} u^{\alpha+(p-1)}\,\mathrm{d}x\right]\\ &\le S_p^p\left[\frac{2^{p-1}\,[\alpha+(p-1)]^p}{|\alpha|^p\,(R_0-R_\infty)^p}+\frac{1}{R_\infty^p}\right]\int_{B_{R_\infty}} u^{\alpha+(p-1)}\,\mathrm{d}x.\end{aligned}$$

Letting $0 < \alpha + (p-1) =: \beta < p-1$, we get

$$\left[\int_{B_{R_\infty}} u^{\beta\frac{r}{p}}\,\mathrm{d}x\right]^{\frac{p}{r\beta}} \le \frac{S_p^{\frac{p}{\beta}}}{(R_0-R_\infty)^{\frac{p}{\beta}}}\left[\frac{2^{p-1}\,\beta^p}{|\beta-(p-1)|^p}+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\right]^{\frac{1}{\beta}}\left[\int_{B_{R_\infty}} u^{\beta}\,\mathrm{d}x\right]^{\frac{1}{\beta}}. \tag{5.7}$$

Let $\overline{q} = r\beta/p$, then

$$\frac{r[s-(p-1)]}{r-p} < \overline{q} < \frac{r}{p}(p-1) \quad \text{imply} \quad \frac{p[s-(p-1)]}{r-p} < \beta < p-1.$$

We note that the interval in which β can vary is compatible with the request $0 < \beta < p-1$ and it is not empty since we are assuming $p-1 < s < s_c^*$. With this choice, from (5.7) we get, for any $R_\infty \le \rho < R \le R_0$,

$$\|u\|_{\overline{q},\rho} \le \left\{S_p^p\left[\frac{2^{p-1}\,p^p\,\overline{q}^p}{[r(p-1)-p\overline{q}]^p}+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\right]\right\}^{\frac{r}{p\overline{q}}}\frac{\|u\|_{\frac{p}{r}\overline{q},R}}{(R-\rho)^{\frac{r}{\overline{q}}}}. \tag{5.8}$$

Let $\underline{q} = p\overline{q}/r < \overline{q}$. We consider separately the case $\underline{q} \le q_0 \le \overline{q}$ and the case $0 < q_0 < \underline{q} < \overline{q}$. In the first case we can use Hölder inequality in (5.8), to obtain

$$\|u\|_{\overline{q},\rho} \le \left\{S_p^p\left[\frac{2^{p-1}\,p^p\,\overline{q}^p}{[r(p-1)-p\overline{q}]^p}+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\right]\right\}^{\frac{r}{p\overline{q}}}\left[\frac{R^{\frac{d}{p}}\,\omega_d^{\frac{1}{p}-\frac{1}{r}}}{(R-\rho)\,\rho^{\frac{d}{r}}}\right]^{\frac{r}{\overline{q}}}|B_\rho|^{\frac{1}{\overline{q}}}\frac{\|u\|_{q_0,R}}{|B_R|^{\frac{1}{q_0}}},$$

which is (5.4) when $\underline{q} \le q_0 \le \overline{q}$, once we let $R = R_0$ and $\rho = R_\infty = \bar{R}$. On the other hand, when $0 < q_0 < \underline{q} < \overline{q}$, we can use inequality (5.8) rewritten as

$$\|u\|_{\overline{q},\rho} \le \left\{S_p^p\left[\frac{2^{p-1}\,p^p\,\overline{q}^p}{[r(p-1)-p\overline{q}]^p}+\left(\frac{R_0-R_\infty}{R_\infty}\right)^p\right]\right\}^{\frac{r}{p\overline{q}}}\frac{\|u\|_{\frac{p}{r}\overline{q},R}}{(R-\rho)^{\frac{r}{\overline{q}}}} =: \frac{K}{(R-\rho)^{\frac{r}{\overline{q}}}}\|u\|_{\frac{p}{r}\overline{q},R},$$

so that Lemma 8.7 applied with $\gamma = r/\overline{q}$ and $q = \underline{q} = p\,\overline{q}/r$ gives that for all $0 < q_0 < \underline{q} < \overline{q}$

$$\begin{aligned}\|u\|_{\overline{q},R_\infty} &\le 3\cdot 2^{\frac{p\overline{q}-rq_0}{q_0(r-p)}}\left\{S_p^p\left[\frac{2^{p-1}\,p^p\,\overline{q}^p}{[r(p-1)-p\overline{q}]^p}\left(\frac{R_\infty}{R_0-R_\infty}\right)^p+1\right]\right\}^{\frac{r(\overline{q}-q_0)}{q_0\overline{q}(r-p)}}\\ &\quad\times\left(\frac{4\,r\,p(\overline{q}-q_0)}{\overline{q}\,q_0(r-p)}\right)^{\frac{pr(\overline{q}-q_0)}{q_0\overline{q}(r-p)}}\frac{R_0^{\frac{d}{q_0}}\,\omega_d^{\frac{1}{q_0}}}{R_\infty^{\frac{d}{\overline{q}}+\frac{rp(\overline{q}-q_0)}{\overline{q}q_0(r-p)}}\,\omega_d^{\frac{1}{\overline{q}}}}|B_{R_\infty}|^{\frac{1}{\overline{q}}}\frac{\|u\|_{q_0,R_0}}{|B_{R_0}|^{\frac{1}{q_0}}},\end{aligned}$$

therefore the statement follows putting $R_\infty = \bar{R}$. □

As a consequence of the above proposition we can improve the local lower bounds of Theorem 5.1 in this good supercritical range.

Theorem 5.4 (Local Lower Estimates) *Let $\Omega \subset \mathbb{R}^d$ and let $\lambda > 0$. Let u be a nonnegative local weak solution to $-\Delta_p u = \lambda u^s$ in Ω, with $p-1 < s < s_c^* = r(p-1)/p$, r as in (3.2). Then for any $B_{R_\infty} \subset B_{\bar{R}} \subset B_{R_0} \subseteq \Omega$, the following bound holds*

$$\inf_{x\in B_{R_\infty}} u(x) = \|u\|_{-\infty,R_\infty} \ge \frac{I_{-\infty,\underline{q}}}{I_{\overline{q},\underline{q}}}\frac{\|u\|_{\overline{q},\bar{R}}}{|B_{\bar{R}}|^{\frac{1}{\overline{q}}}}, \quad \textit{with} \quad \frac{r[s-(p-1)]}{r-p} < \overline{q} < \frac{r}{p}(p-1)$$

where $\underline{q} \in (0, q_0 \wedge \overline{q}]$ *with*

$$q_0 = \frac{(p-1)^{\frac{2}{p}}\, 2^{\frac{(d-1)(p-1)}{p}}}{p\,\omega_d^2\, d^2\, e}, \tag{5.9}$$

$I_{-\infty,\underline{q}}$ *as in* (5.1) *and* $I_{\overline{q},\underline{q}}$ *as in* (5.5)–(5.6) *with* $q_0 = \underline{q}$ *there.*

Proof. It is suffices combine the local lower bound proved in Theorem 5.1, with $\varepsilon = e$ and the reverse Hölder inequality of Proposition 5.2 with $q_0 = \underline{q}$ and $0 < R_\infty < \bar{R} < R_0$. □

6 Harnack inequalities

In this section we combine the upper bounds of Section 4 with the lower bounds of Section 5 to obtain various form of Harnack inequalities. The general form, valid in the whole range of exponents, is given in Theorem 6.1. As far as we know, the Harnack inequality that we derive for $s > p-1$ is not stated explicitly in the literature. Unfortunately, the constant of the general Harnack inequality of Theorem 6.1 depends on u through a quotient of L^q norms. Such quotient simplifies to a constant in some cases and gives clean versions of the Harnack inequality (i.e. the constant does not depend on u); this happens in the subcritical range, i.e. when $0 < s \le p-1$, cf. Theorem 6.2, or in the supercritical range $p-1 < s < s_c^*$, cf. Theorem 6.3. In the range $s_c^* < s < r-1$, we are not able to prove such clean forms of Harnack inequalities, and we conjecture that the dependence on some L^q norm of the solution can not be avoided, as already mentioned in the introduction. The fact that the "constant" involved has to depend on u when $s_c^* \le s < r-1$ is confirmed by the results of [6, 7, 9, 8, 26, 27], [26] applied to separation of variable solutions of parabolic problems, see also [27]. This is also related to the fact that, in the range $s_c^* \le s < r-1$, there may exist (very weak, when $p = 2$) singular solutions, cf. [24, 40, 42, 43, 44, 45].

When $s < p^* - 1$, in the case $r = p^*$, so $p < d$, bounded weak solutions are known to be $C^{1,\alpha}$, see [25], and the $C^{1,\alpha}$ modulus of continuity depends on the local L^∞-norm of the solution or on the constant in the Harnack inequality. Therefore having absolute bounds (independent of u) for the solution or for the Harnack constant, allow to have absolute bounds for the $C^{1,\alpha}$ modulus of continuity. What we show here, is that the $C^{1,\alpha}$ modulus of continuity is independent on the solution when $s < s_c^* < p^* - 1$, while it depends on (some L^q-norms of) the solution when $s_c^* < s < p^* - 1$. If one wants to have a $C^{1,\alpha}$ modulus of continuity independent on u also when $s > s_c^*$, one has to add some extra hypothesis on the solution, and this will be done in the next section, for the special class of stable solutions.

Theorem 6.1 (Harnack inequality for $0 \le s < s_c$**)** *Let* $\Omega \subset \mathbb{R}^d$*. Let* u *be a nonnegative local weak solution to* $-\Delta_p u = \lambda u^s$ *in* Ω*, with* $p > 1$*,* $\lambda \ge 0$*,* $0 \le s < s_c = r-1$*,* r *as in* (3.2)*. Then for any* $R_\infty < R_0$ *and* $\varepsilon > 0$*, we assume*

$$0 < \underline{q} \le q_0 := \frac{(p-1)^{\frac{2}{p}}\, 2^{\frac{(d-1)(p-1)}{p}}}{p\,\omega_d^2\, d[e(d-1)+\varepsilon]}, \quad \overline{q} > \frac{r[s-(p-1)]_+}{r-p}.$$

Moreover, if $0 < \overline{q} < s_c^* = r(p-1)/p$ *we also assume*

$$\mathrm{i.\,p.}\left[\frac{\log\frac{r(p-1)}{\overline{q}p}}{\log\frac{r}{p}}\right] \neq \frac{\log\frac{r(p-1)}{\overline{q}p}}{\log\frac{r}{p}}.$$

Then the following bound holds true

$$\sup_{x\in B_{R_\infty}} u(x) \le \mathcal{H}_s[u] \inf_{x\in B_{R_\infty}} u(x) \tag{6.1}$$

where $\mathcal{H}_s[u]$ depends on u through some local norms as follows

$$\begin{aligned}
\mathcal{H}_s[u] &= \mathcal{H}_s[u](d,p,r,R_0,R_\infty,\overline{q},\underline{q},\varepsilon)\\
&= \frac{I_{\infty,\overline{q}}}{I_{-\infty,\underline{q}}}\,\frac{\left(\displaystyle⨍_{B_{R_0}} u^{\overline{q}}\,\mathrm{d}x\right)^{\frac{1}{\overline{q}}}}{\left(\displaystyle⨍_{B_{R_0}} u^{\underline{q}}\,\mathrm{d}x\right)^{\frac{1}{\underline{q}}}}\left[\frac{\left(\displaystyle⨍_{B_{R_0}} u^{\overline{q}}\,\mathrm{d}x\right)^{\frac{[s-(p-1)]_+}{\overline{q}}}}{\displaystyle⨍_{B_{R_\infty}} u^{[s-(p-1)]_+}\,\mathrm{d}x}\right]^{\frac{r}{(r-p)\overline{q}-r[s-(p-1)]_+}}
\end{aligned}$$

with $I_{\infty,\overline{q}}$ as in (4.4) and $I_{-\infty,\underline{q}}$ as in (5.1).

Proof. The local upper estimates of Theorems 4.1, give for any $B_{R_\infty}\subset B_{R_0}\subseteq\Omega$,

$$\sup_{x\in B_{R_\infty}} u(x)=\|u\|_{\infty,R_\infty}\le I_{\infty,\overline{q}}\left[\frac{\left(\displaystyle⨍_{B_{R_0}} u^{\overline{q}}\,\mathrm{d}x\right)^{\frac{[s-(p-1)]_+}{\overline{q}}}}{\displaystyle⨍_{B_{R_\infty}} u^{[s-(p-1)]_+}\,\mathrm{d}x}\right]^{\frac{r}{(r-p)\overline{q}-r[s-(p-1)]_+}}\frac{\|u\|_{\overline{q},R_0}}{|B_{R_0}|^{\frac{1}{\overline{q}}}},\tag{6.2}$$

for any $\overline{q}>r[s-(p-1)]_+/(r-p)$, $I_{\infty,\overline{q}}$ given by (4.4) and when $0<\overline{q}<s_c^*$ we require the additional condition (4.6). Moreover Theorem 5.1 states that, for any $\varepsilon>0$,

$$\inf_{x\in B_{R_\infty}} u(x)\,\frac{|B_{R_0}|^{\frac{1}{\underline{q}}}}{I_{-\infty,\underline{q}}\,\|u\|_{\underline{q},R_0}}\ge 1,\tag{6.3}$$

$I_{-\infty,\underline{q}}$ given by (5.1) and

$$0<\underline{q}\le\frac{(p-1)^{\frac{2}{p}}\,2^{\frac{(d-1)(p-1)}{p}}}{p\,\omega_d^2\,d\,[e(d-1)+\varepsilon]}=q_0.$$

Combining (6.2) and (6.3) we obtain the desired result. □

Theorem 6.2 (Harnack inequality, $0\le s\le p-1$) *Let $\Omega\subset\mathbb{R}^d$ and let $\lambda>0$. Let u be a nonnegative local weak solution to $-\Delta_p u=\lambda u^s$ in Ω, with $0\le s\le p-1$. Then, for any $R_\infty<R_0$ the following bound holds true*

$$\sup_{x\in B_{R_\infty}} u(x)\le\mathcal{H}_s\inf_{x\in B_{R_\infty}} u(x)$$

where

$$\begin{aligned}
\mathcal{H}_s &= \left(\frac{r}{p}\right)^{\frac{p^2 r}{q_0(r-p)}}\left\{\frac{2^d\left[(p-1)^{\frac{2}{p}-1}\,2^{\frac{(d-1)(p-1)}{p}}\left(\frac{r}{p}\right)^{n_0-\frac{1}{p}}+e\,p\,\omega_d^2\,d\right]\sqrt{\omega_d}}{(p-1)^{\frac{2}{p}-1}\,2^{\frac{(d-1)(p-1)}{p}}\left(\frac{r}{p}\right)^{n_0-\frac{1}{p}}-e\,(d-1)\,p\,\omega_d^2\,d}\right\}^{\frac{2}{q_0}}\\
&\quad\times\frac{R_0^{\frac{d}{q_0}\left(\frac{r}{p}+1\right)+\frac{rd}{q_0(r-p)}}\,\omega_d^{\frac{p}{q_0}}}{R_\infty^{(d+p)\frac{r}{q_0(r-p)}}\,(R_0-R_\infty)^{\frac{2rp}{q_0(r-p)}}}\left\{S_p^{2p}[2^{p-1}R_\infty^p+(R_0-R_\infty)^p]2^{\frac{pr}{r-p}}\,q_0^p\,c_1\right\}^{\frac{r}{q_0(r-p)}}\\
&\quad\times\left[\frac{\Lambda_{s,0}\,r}{p\,q_0}\left(\frac{R_0-R_\infty}{R_\infty}\right)^p+c_2\left(\frac{R_0-R_\infty}{R_\infty}\right)^p+\frac{2^{p-1}c_1^{p-1}}{c_0^p}\right]^{\frac{r}{q_0(r-p)}}
\end{aligned}\tag{6.4}$$

with

$$q_0 = \left(\frac{p}{r}\right)^{n_0 - \frac{1}{p}} (p-1), \qquad n_0 = \left[\frac{\log \frac{p\,\omega_d^2\, d\, e\,(d-1)}{(p-1)^{\frac{2}{p}-1}\, 2^{\frac{(d-1)(p-1)}{p}}}}{\log \frac{r}{p}} + \frac{1}{p}\right] + 1,$$

$$\frac{1}{c_0^p} \le \left(\frac{r}{(r-p)q_0}\right)^p, \quad c_1 = \frac{r^{\frac{1}{p}}}{r^{\frac{1}{p}} - p^{\frac{1}{p}}}, \quad c_2 = \max\left\{\frac{|pq_0 - r(p-1)|\, r^p}{(p\,q_0)^{p+1}}, \left(\frac{p}{p-1}\right)^p \frac{1}{(p+1)^{p+1}}\right\}$$

and

$$\Lambda_{s,0} = \frac{\lambda}{p^{p-1}} R_0^p \quad \textit{if } s = p-1 \qquad \textit{and} \qquad \Lambda_{s,0} = 2^{p-1+d} \quad \textit{if } s \neq p-1.$$

Proof. The goal of the proof is to simplify the quotient of L^q-norms in the expression of the constant $\mathcal{H}_s[u]$ of the Harnack inequality (6.1). Since we are dealing with the range $0 \le s \le p-1$, we can choose any $\overline{q} > 0$, hence we can let

$$0 < \overline{q} = \underline{q} = q_0 = q_0(\varepsilon) = \frac{(p-1)^{\frac{2}{p}}\, 2^{\frac{(d-1)(p-1)}{p}}}{p\,\omega_d^2\, d[e(d-1)+\varepsilon]} \quad \text{with} \quad \text{i.p.}\left[\frac{\log \frac{r(p-1)}{\overline{q}p}}{\log \frac{r}{p}}\right] \neq \frac{\log \frac{r(p-1)}{\overline{q}p}}{\log \frac{r}{p}}.$$

In fact, we shall arrive, with a suitable choice of the parameter ε, at a value of q_0 smaller than $r(p-1)/p$, so that the request $\log \frac{r(p-1)}{\overline{q}p} / \log \frac{r}{p}$ not be integer is necessary. The last condition means $q_0(\varepsilon) \neq (p/r)^{n-1}(p-1)$ for all $n \in \mathbb{N}$ and this is possible since we can always choose ε

$$0 < \varepsilon = \frac{(p-1)^{\frac{2}{p}-1}\, 2^{\frac{(d-1)(p-1)}{p}}}{p\,\omega_d^2\, d} \left(\frac{r}{p}\right)^{n_0 - \frac{1}{p}} - e(d-1) \quad \text{so that} \quad q_0 = \left(\frac{p}{r}\right)^{n_0 - \frac{1}{p}} (p-1),$$

where n_0 is the first integer n such that $\varepsilon(n) > 0$, which is

$$n_0 = \left[\frac{\log \frac{p\,\omega_d^2\, d\, e\,(d-1)}{(p-1)^{\frac{2}{p}-1}\, 2^{\frac{(d-1)(p-1)}{p}}}}{\log \frac{r}{p}} + \frac{1}{p}\right] + 1.$$

The constants become in this case

$$\begin{aligned} I_{\infty,\overline{q}} = I_{\infty,q_0} \quad &= \quad \Bigg\{ \frac{|B_{R_0}|^{\frac{r-p}{r}+1}}{|B_{R_\infty}|} \frac{S_p^p\, q_0^p\, p^p\, c_1}{r^p\,(R_0 - R_\infty)^p} \left[\frac{\Lambda_{s,0}\, r}{p\, q_0} \left(\frac{R_0 - R_\infty}{R_\infty}\right)^p \right. \\ &\quad + \left. \left(\frac{R_0 - R_\infty}{R_\infty}\right)^p c_2 + \frac{c_1^{p-1}\, 2^{p-1}}{c_0^p}\right] \left(\frac{r}{p}\right)^{\frac{rp}{r-p}} \Bigg\}^{\frac{r}{q_0(r-p)}} \end{aligned}$$

where $\Lambda_{s,0}$ is given by (4.5),

$$\frac{1}{c_0^p} \le \left(\frac{r}{q_0(r-p)}\right)^p, \qquad c_2 = \max\left\{\frac{|pq_0 - r(p-1)|\, r^p}{(p\,q_0)^{p+1}}, \left(\frac{p}{p-1}\right)^p \frac{1}{(p+1)^{p+1}}\right\}$$

and since $q_0 < r(p-1)/p$

$$c_1 = \max_{i=0,1} \frac{\left(\frac{r}{p}\right)^{j_0+i-1} q_0}{\left|\left(\frac{r}{p}\right)^{j_0+i-1} q_0 - (p-1)\right|} = \max_{i=0,1} \frac{\left(\frac{r}{p}\right)^{i+\frac{1}{p}}}{\left(\frac{r}{p}\right)^{i+\frac{1}{p}} - 1} = \frac{r^{\frac{1}{p}}}{r^{\frac{1}{p}} - p^{\frac{1}{p}}}$$

since

$$j_0 = \text{i.p.}\left[\frac{\log \frac{r(p-1)}{q_0 p}}{\log \frac{r}{p}}\right] = \text{i.p.}\left[1 + \frac{\log \frac{p-1}{q_0}}{\log \frac{r}{p}}\right] = \text{i.p.}\left[n_0 + 1 - \frac{1}{p}\right] = n_0 + 1.$$

Moreover

$$
\begin{aligned}
I_{-\infty,\underline{q}} = I_{-\infty,q_0} \quad &= \quad \left[\frac{(R_0-R_\infty)R_\infty}{R_0^{\frac{d(r-p)}{rp}}}\right]^{\frac{pr}{q_0(r-p)}} \left\{S_p^p[2^{p-1}\,R_\infty^p+(R_0-R_\infty)^p]2^{\frac{pr}{r-p}}\right\}^{-\frac{r}{q_0(r-p)}}\\
&\times\left\{\frac{(p-1)^{\frac{2}{p}-1}\,2^{\frac{(d-1)(p-1)}{p}}\,\left(\frac{r}{p}\right)^{n_0-\frac{1}{p}}-e\,(d-1)\,p\,\omega_d^2\,d}{2^d\left[(p-1)^{\frac{2}{p}-1}\,2^{\frac{(d-1)(p-1)}{p}}\,\left(\frac{r}{p}\right)^{n_0-\frac{1}{p}}+e\,p\,\omega_d^2\,d\right]\sqrt{\omega_d}}\right\}^{\frac{2}{q_0}}.
\end{aligned}
$$

Hence we get the expression of $\mathcal{H}_s = I_{\infty,q_0}/I_{-\infty,q_0}$ as in (6.4). □

Unfortunately, when $s > p-1$ we can not join the upper and the lower bound so easily, we need a further iteration.

Theorem 6.3 (Harnack Inequalities when $p-1 < s < s_c^*$) *Let $\Omega \subset \mathbb{R}^d$. Let u be a nonnegative local weak solution to $-\Delta_p u = \lambda u^s$ in Ω, with $p>1$, $\lambda>0$ and $p-1<s<s_c^*=r(p-1)/p$. Then for any $0<R_\infty<\bar{R}<R_0$ there exists an explicit constant $\mathcal{H}_s>0$ such that*

$$\sup_{x\in B_{R_\infty}} u(x) \le \mathcal{H}_s \inf_{x\in B_{R_\infty}} u(x) \tag{6.5}$$

where $\mathcal{H}_s$ does not depend on u, and is given by

$$\mathcal{H}_s = I_{\infty,\overline{q}}\left(\frac{I_{\overline{q},\underline{q}}}{I_{-\infty,\underline{q}}}\right)^{\frac{\overline{q}(r-p)}{\overline{q}(r-p)-r[s-(p-1)]}} \tag{6.6}$$

with $\underline{q}\in(0,q_0\wedge\overline{q}]$, q_0 and $I_{-\infty,\underline{q}}$ are given in (5.9) and (5.1) respectively, $I_{\overline{q},\underline{q}}$ in (5.5) and (5.6) and $I_{\infty,\overline{q}}$ in (4.4); moreovere, since $\overline{q}<s_c^*$ we require the additional condition (4.6).

Proof. Let $B_{R_\infty}\subset B_{\bar{R}}\subset B_{R_0}\subseteq\Omega$, then by Theorem 5.4 we have

$$\frac{\|u\|_{\overline{q},\bar{R}}}{|B_{\bar{R}}|^{\frac{1}{\overline{q}}}} \le \frac{I_{\overline{q},\underline{q}}}{I_{-\infty,\underline{q}}}\inf_{x\in B_{R_\infty}} u(x), \tag{6.7}$$

with $r[s-(p-1)]/(r-p)<\overline{q}<r(p-1)/p$, $\underline{q}\in(0,q_0\wedge\overline{q}]$, q_0 as in (5.9), $I_{\overline{q},\underline{q}}$ as in (5.5) and (5.6) and $I_{-\infty,\underline{q}}$ as in (5.1). Moreover Theorem 4.1, applied with $R_0=\bar{R}$, gives

$$
\begin{aligned}
\sup_{x\in B_{R_\infty}} u(x) = \|u\|_{\infty,R_\infty} \quad &\le \quad I_{\infty,\overline{q}}\,\frac{\|u\|_{\overline{q},\bar{R}}}{|B_{\bar{R}}|^{\frac{1}{\overline{q}}}}\left(\frac{\|u\|_{\overline{q},\bar{R}}^{[s-(p-1)]}}{|B_{\bar{R}}|^{\frac{[s-(p-1)]}{\overline{q}}}}\,\frac{|B_{R_\infty}|}{\int_{B_{R_\infty}}u^{[s-(p-1)]}\,\mathrm{d}x}\right)^{\frac{r}{(r-p)\overline{q}-r[s-(p-1)]}}\\
&\le \quad I_{\infty,\overline{q}}\,\frac{\|u\|_{\overline{q},\bar{R}}}{|B_{\bar{R}}|^{\frac{1}{\overline{q}}}}\left(\frac{\|u\|_{\overline{q},\bar{R}}}{|B_{\bar{R}}|^{\frac{1}{\overline{q}}}}\,\frac{1}{\inf_{x\in B_{R_\infty}}u(x)}\right)^{\frac{r[s-(p-1)]}{(r-p)\overline{q}-r[s-(p-1)]}},
\end{aligned}
$$

for any $\overline{q}>r[s-(p-1)]/(r-p)$ and $I_{\infty,\overline{q}}$ as in (4.4). Therefore, using twice the lower bound (6.7) in the previous inequality, we conclude the proof of the theorem. □

7 Local absolute bounds

The interest of having absolute upper bounds for solutions of nonlinear elliptic equations is related to several aspects of the theory of such equations. If we have at our disposal at local absolute upper bounds, then the constant in the general Harnack inequality of Theorem 6.1 can be independent of u and also the $C^{1,\alpha}$ modulus of continuity will be independent of u, as already discussed at the beginning of Section 6. The absolute estimates that we present here have a local nature, which means that they are independent of the boundary conditions, which can be of Dirichlet, Neumann, Robin, or also "large", i.e. $u = +\infty$ on the boundary. Such absolute bounds have many more applications, for example, they may imply Liouville-type Theorems on $\mathbb{R}^d$ [32, 52], or they imply existence of large solutions, and the fact that the constant is explicit is really useful although not always indispensable. For the homogeneous Dirichlet problem for semilinear equations (namely for $p = 2$), absolute upper bounds, sometimes called universal bounds, have been proved by many authors, [11, 23, 31, 32, 46, 52], but in that papers the constant was not quantitative, and to our knowledge it can not be made quantitative with the proofs presented there. An effort to provide quantitative global absolute bounds for this Dirichlet problem has been done in [5].

In this section we first prove absolute upper and lower bounds for weak solutions, in the range $p-1 < s < s_c^*$ and $0 < s \le p-1$ respectively. Next we want to obtain quantitative absolute upper bounds for $s > s_c^*$, which are known to be false in the whole class of weak (or very weak when $p = 2$) solutions, in view of the existence of singular solutions, as already mentioned in the Introduction, cf. [24, 40, 42, 43, 44, 45]; therefore we have to pass to a special class of solutions, the so-called stable solutions [13, 14, 16, 22, 28, 29], for which we can bound absolutely from above the $\mathrm{L}^{\overline{m}}$-norm of the solution, for $\overline{m}$ sufficiently large, and we combine such bounds with the upper bounds of type II of Theorem 4.5 to get our quantitative absolute upper bounds for stable solutions. We can cover the whole range of $s > 0$ only for small spatial dimensions, namely $d \le \frac{p(p+3)}{p-1}$; for larger dimensions, it appears a new exponent $r - 1 < s_{JL} < \infty$, the so-called Joseph-Lundgren exponent, and the absolute bounds holds only until that exponent.

7.1 Local absolute bounds for $s < s_c^*$

In this section we will prove local absolute lower bounds when $0 < s < p-1$ and a local absolute upper bounds when $p-1 < s < s_c^*$ as a consequence of the Harnack inequalities of the previous section together with the Caccioppoli estimate (2.11).

Theorem 7.1 (Local absolute bounds) *Let $\Omega \subset \mathbb{R}^d$. Let u be a nonnegative local weak solution to $-\Delta_p u = \lambda u^s$ in Ω, with $p > 1$, $\lambda \ge 0$ and $0 \le s < s_c^* = r(p-1)/p$, r as in (3.2). Then for any $0 < R_\infty < \bar{R} < R_0$ there exists a constant $\mathcal{H}_s$ that does not depend on u, such that*

$$\sup_{x\in B_{\bar{R}}} u(x) \le \mathcal{H}_s \left(\frac{p^{p-1}\,2^{p-1}\,R_0^d}{\lambda\,(R_0-\bar{R})^p\,\bar{R}^d}\right)^{\frac{1}{s-(p-1)}} \qquad \text{if } \; p-1 < s < s_c^*$$

with $\mathcal{H}_s$ given by (6.6), and, if $u \neq 0$ on B_{R_0}

$$\inf_{x\in B_{\bar{R}}} u(x) \ge \mathcal{H}_s^{-1} \left(\frac{\lambda\,(R_0-\bar{R})^p\,\bar{R}^d}{p^{p-1}\,2^{p-1}\,R_0^d}\right)^{\frac{1}{(p-1)-s}} \qquad \text{if } \; 0 \le s < p-1,$$

with $\mathcal{H}_s$ given by (6.4).

Proof. First, we note that the Caccioppoli estimate (2.11), with $R_1 = \overline{R}$, implies when $s > p-1$

$$\inf_{x\in B_{\bar{R}}} u(x) \le \left(\frac{1}{|B_{\bar{R}}|}\int_{B_{\bar{R}}} u^{s-(p-1)}\,\mathrm{d}x\right)^{\frac{1}{s-(p-1)}} \le \left(\frac{p^{p-1}\,2^{p-1}\,R_0^d}{\lambda(R_0-\bar{R})^p\,\bar{R}^d}\right)^{\frac{1}{s-(p-1)}}.$$

Moreover, since $u \neq 0$ on B_{R_0}, if $0 \le s < p-1$, (2.11), applied always with $R_1 = \bar{R}$, gives

$$\left(\frac{\lambda(R_0-\bar{R})^p\,\bar{R}^d}{p^{p-1}\,2^{p-1}\,R_0^d}\right)^{\frac{1}{(p-1)-s}} \le \left(\frac{|B_{\bar{R}}|}{\int_{B_{\bar{R}}} u^{s-(p-1)}\,\mathrm{d}x}\right)^{\frac{1}{(p-1)-s}} \le \sup_{x\in B_{\bar{R}}} u(x).$$

The above estimates can be now combined with the corresponding Harnack inequalities (6.5) and (6.2) to obtain the desired bounds in both cases. □

7.2 Local absolute bounds for stable solutions. The supercritical case

In this section we establish local upper bounds for stable solutions. From now on, we assume $p \geq 2$. The results can be proved also in the case $1 < p < 2$, but we need some modifications in the definitions of stable solutions and in the proofs. We have decided to deal with $p \geq 2$ in order to simplify the exposition. When $1 < p < 2$ we refer to [14] and references therein. Let us mention that the proof that we give here is a modification of an idea originally due to A. Farina, see [16, 28, 29]; see also [13, 14] for an alternative approach. Our proof is slightly different from [16, 28, 29] and provides explicit constants.

Definition 7.2 *A function u is a local stable solution to $-\Delta_p u = \lambda u^s$, if and only if $0 \leq u \in W^{1,p}_{\rm loc}(\Omega)$ and satisfies*

$$\int_\Omega \left\{ |\nabla u|^{p-2} \left[|\nabla \varphi|^2 + (p-2) \left(\nabla \varphi \cdot \frac{\nabla u}{|\nabla u|} \right)^2 \right] - \lambda s u^{s-1} \varphi^2 \right\} {\rm d}x \geq 0 \tag{7.1}$$

for all bounded $\varphi \in C^1_0(K)$ and for any compact $K \subset \Omega$.

We recall that the stability condition translates into the fact that the second variation of the energy functional is non-negative, see [13, 14, 16, 22, 28, 29] for more a more detailed study of stable solutions related to this kind of problem.

Remark 7.3 *From the stability condition* (7.1) *we immediately obtain*

$$\lambda s \int_\Omega u^{s-1} \varphi^2 \,{\rm d}x \leq (p-1) \int_\Omega |\nabla u|^{p-2} |\nabla \varphi|^2 \,{\rm d}x, \tag{7.2}$$

for any compact $K \subset \Omega$ and for all bounded $\varphi \in C^1_0(K)$.

Now we have the following estimate for nonnegative stable solutions.

Lemma 7.4 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and let u be a local nonnegative stable weak solution to $-\Delta_p u = \lambda u^s$ in Ω, $\lambda > 0$ and $s > p-1$. Then the following estimate holds true for any $\alpha > -1$, δ, $\varepsilon \in (0, 1]$ and any test function $\phi \in C^2(\Omega) \cap C^1_0(\overline{\Omega})$, $\phi > 0$*

$$\begin{aligned} \lambda s \int_\Omega u^{s+\alpha} \phi^{\frac{s+\alpha}{s-(p-1)}} \,{\rm d}x \quad &\leq \quad \left[(p-1) \left(1 + \frac{\varepsilon}{2}\right) \left(\frac{\alpha+1}{2}\right)^2 + \frac{p-1}{4} \left(1 + \frac{1}{2\varepsilon}\right) \frac{(s+\alpha)^2}{[s-(p-1)]^2} \frac{\delta(p-2)}{p} \right] \\ &\qquad \times \frac{p^p}{[\alpha+(p-1)]^p} \int_\Omega \left| \nabla \left(u^{\frac{\alpha+(p-1)}{p}} \right) \right|^p \phi^{\frac{s+\alpha}{s-(p-1)}} \,{\rm d}x \\ &\qquad + \frac{p-1}{4} \left(1 + \frac{1}{2\varepsilon}\right) \frac{(s+\alpha)^2}{[s-(p-1)]^2} \frac{2}{p\,\delta^{\frac{p-2}{p}}} \int_\Omega u^{\alpha+(p-1)} |\nabla \phi|^p \phi^{\frac{s+\alpha}{s-(p-1)}-p} \,{\rm d}x. \end{aligned} \tag{7.3}$$

Proof. Let $0 < \phi \in C^2(\Omega) \cap C^1_0(\overline{\Omega})$. Using as test function $\varphi^2 := u^{\alpha+1}\phi^\gamma$,

$$\gamma := \frac{s+\alpha}{s-(p-1)} > 0 \quad \text{and} \quad \alpha > -1$$

in (7.2) and Young inequality with $\varepsilon > 0$, we get

$$\begin{aligned} \lambda s \int_\Omega u^{s+\alpha} \phi^\gamma \,{\rm d}x \quad &\leq \quad (p-1) \left(1 + \frac{\varepsilon}{2}\right) \left(\frac{\alpha+1}{2}\right)^2 \int_\Omega u^{\alpha-1} |\nabla u|^p \phi^\gamma \,{\rm d}x \\ &\qquad + \frac{p-1}{4} \left(1 + \frac{1}{2\varepsilon}\right) \gamma^2 \int_\Omega |\nabla u|^{p-2}\, u^{\alpha+1} |\nabla \phi|^2 \phi^{\gamma-2} \,{\rm d}x. \end{aligned}$$

Again using Young inequality with $\delta > 0$ and exponents $p/2$, $p/(p-2)$ we obtain, for the second terms in the right hand side of the previous inequality, the following estimate

$$\int_\Omega |\nabla u|^{p-2}\, u^{\alpha+1} |\nabla\phi|^2 \phi^{\gamma-2}\,\mathrm{d}x \le \frac{\delta(p-2)}{p}\int_\Omega |\nabla u|^p u^{\alpha-1}\phi^\gamma\,\mathrm{d}x + \frac{2}{p\,\delta^{\frac{p-2}{2}}}\int_\Omega u^{\alpha+(p-1)}|\nabla\phi|^p\phi^{\gamma-p}\,\mathrm{d}x.$$

Combining the previous estimates and noticing that

$$\int_\Omega |\nabla u|^p u^{\alpha-1}\phi^\gamma\,\mathrm{d}x = \frac{p^p}{[\alpha+(p-1)]^p}\int_\Omega \left|\nabla\left(u^{\frac{\alpha+(p-1)}{p}}\right)\right|^p \phi^\gamma\,\mathrm{d}x$$

we arrive at the desired result. □

Combining the previous estimate, coming from the stability condition (7.2), and the following form of the energy estimate (2.4),

$$\begin{aligned}\frac{p^p}{[\alpha+(p-1)]^p}\int_\Omega |\nabla(u^{\frac{\alpha+(p-1)}{p}})|^p\phi^{\frac{s+\alpha}{s-(p-1)}}\,\mathrm{d}x \quad\le\quad & \frac{\lambda\,p\,[s-(p-1)]}{p\,\alpha[s-(p-1)]-(s+\alpha)\bar\varepsilon(p-1)}\int_\Omega u^{\alpha+s}\phi^{\frac{s+\alpha}{s-(p-1)}}\,\mathrm{d}x\\ &+\frac{s+\alpha}{\bar\varepsilon^{p-1}\{p\,\alpha[s-(p-1)]-(s+\alpha)\bar\varepsilon(p-1)\}}\\ &\times\int_\Omega u^{\alpha+(p-1)}\,|\nabla\phi|^p\phi^{\frac{s+\alpha}{s-(p-1)}-p}\,\mathrm{d}x\end{aligned}\tag{7.4}$$

for any

$$0<\bar\varepsilon<\frac{p\,\alpha[s-(p-1)]}{(s+\alpha)(p-1)},\quad \alpha>0 \quad\text{and}\quad s>p-1.$$

For the proof of the above inequality we have to follow the proof of Lemma 2.1 and change slightly the test function (we have to use $u^\alpha\phi^{\frac{s+\alpha}{s-(p-1)}}$ instead of $(u+\delta)^\alpha\phi$). We get the following.

Lemma 7.5 *Let $\Omega\subset\mathbb{R}^d$ be a bounded domain and let u be a local nonnegative stable weak solution to $-\Delta_p u=\lambda u^s$ in Ω, $\lambda>0$ and $s>p-1$. Then the following estimate holds true for any*

$$0<\alpha<\bar\alpha:=\frac{2s-(p-1)+2\sqrt{s^2-s(p-1)}}{p-1},\tag{7.5}$$

and any test function $\phi\in C^2(\Omega)\cap C_0^1(\overline{\Omega})$, $\phi>0$

$$\int_\Omega u^{s+\alpha}\phi^{\frac{s+\alpha}{s-(p-1)}}\,\mathrm{d}x\le c_4\int_\Omega u^{\alpha+(p-1)}|\nabla\phi|^p\,\phi^{\frac{s+\alpha}{s-(p-1)}-p}\,\mathrm{d}x,\tag{7.6}$$

where c_4 is a positive constant that depends on s, p, λ and α.

Proof. Using (7.4) to estimate the second term in the right hand side of (7.3) and operate some simple manipulations, we arrive at

$$\begin{aligned}&\left\{\lambda s-\left[(p-1)\left(1+\frac{\varepsilon}{2}\right)\left(\frac{\alpha+1}{2}\right)^2+\frac{p-1}{4}\left(1+\frac{1}{2\varepsilon}\right)\frac{(s+\alpha)^2}{[s-(p-1)]^2}\frac{\delta(p-2)}{p}\right]\right.\\ &\left.\quad\times\frac{\lambda p[s-(p-1)]}{p\alpha[s-(p-1)]-(s+\alpha)\bar\varepsilon(p-1)}\right\}\int_\Omega u^{\alpha+s}\phi^{\frac{s+\alpha}{s-(p-1)}}\,\mathrm{d}x\le c\int_\Omega u^{\alpha+(p-1)}|\nabla\phi|^p\,\phi^{\frac{s+\alpha}{s-(p-1)}-p}\,\mathrm{d}x,\end{aligned}$$

where $c=c(\alpha,\delta,\varepsilon,\bar\varepsilon,p,s)$. Now, since $\alpha<\bar\alpha$, we can always choose ε, δ and $\bar\varepsilon$ such that $\alpha<\alpha_{\varepsilon,\delta,\bar\varepsilon}<\bar\alpha$ and this fact assures that the constant that appears in the left hand side of the previous inequality is strictly positive. The lemma is proved. □

Theorem 7.6 *Let $\Omega \subset \mathbb{R}^d$. Let u be a local nonnegative stable weak solution to $-\Delta_p u = \lambda u^s$ in Ω, with $\lambda > 0$ and $s > p-1$. Then for any $B_{R_1} \subset B_{R_0} \subset\subset \Omega$ there exists a constant that does not depend on u such that*

$$\|u\|_{s+\alpha, R_1} \le c_5 \tag{7.7}$$

for any

$$0 < \alpha < \bar{\alpha} := \frac{2s - (p-1) + 2\sqrt{s^2 - s(p-1)}}{p-1}, \tag{7.8}$$

and

$$c_5 = c_4^{\frac{1}{s-(p-1)}} \left[\frac{2^{p-1}\, p^p}{(R_0 - R_1)^p}\right]^{\frac{1}{s-(p-1)}} R_0^{\frac{d}{s+\alpha}} \omega_d^{\frac{1}{s+\alpha}}. \tag{7.9}$$

Proof. The result follows from the previous Lemma, Hölder inequality and using the test function defined in Lemma 2.3. Indeed, by Hölder inequality with exponents $(s+\alpha)/[\alpha + (p-1)]$ and $(s+\alpha)/[s-(p-1)]$, applied to the right hand side of (7.6), we have

$$\int_\Omega u^{s+\alpha} \phi^{\frac{s+\alpha}{s-(p-1)}}\, \mathrm{d}x \le c_4 \left[\int_\Omega u^{s+\alpha}\, \phi^{\frac{s+\alpha}{s-(p-1)}}\, \mathrm{d}x\right]^{\frac{\alpha+(p-1)}{\alpha+s}} \left[\int_\Omega \left(\frac{|\nabla \phi|^p}{\phi^{p-1}}\right)^{\frac{s+\alpha}{s-(p-1)}} \mathrm{d}x\right]^{\frac{s-(p-1)}{s+\alpha}}.$$

Hence, we arrive to the desired results simplifying and choosing ϕ as in in Lemma 2.3. $\square$

The Joseph-Lundgren exponent s_{JL}. The above Theorem proves absolute bounds for some local $\mathrm{L}^{\overline{m}}$-norm, and we would like to have $\overline{m}$ sufficiently large, namely

$$\overline{m} > \frac{r[s-(p-1)]}{r-p}$$

to be able to combine the above absolute bounds (7.7) with the upper bounds of type II of Theorem 4.5. Letting then $\overline{m} = s + \alpha$, with α satisfying the condition (7.8), we have that

$$\frac{r[s-(p-1)]}{r-p} < \overline{m} = s + \alpha < s + \overline{\alpha} = s + \frac{2s - (p-1) + 2\sqrt{s^2 - s(p-1)}}{p-1}.$$

where we take $r = p^* = pd/(d-p)$, the Sobolev exponent, i.e. we are in the case $p < d$. Notice that when $p \ge d$, we can take $r \to \infty$ and the above condition is always satisfied. In the case under consideration, namely $1 < p < d$, the above condition is satisfied by all the s in some interval, more precisely, there exists an exponent s_{JL} such that for all $s \in (0, s_{JL})$ we have

$$\|u\|_{\overline{m}, R_1} \le c_5\,, \qquad \text{with } \overline{m} > \frac{r[s-(p-1)]}{r-p}$$

where c_5 is given in (7.9). Moreover, we call the exponent s_{JL} the Joseph-Lundgren exponent and it has the explicit form

$$s_{JL} := \begin{cases} +\infty & \text{if } d \le \frac{p(p+3)}{p-1} \\[2ex] \dfrac{[(p-1)d-p]^2 + p^2(p-2) - p^2(p-1)d + 2p^2\sqrt{(p-1)(d-1)}}{(d-p)[(p-1)d - p(p+3)]} & \text{if } d > \frac{p(p+3)}{p-1} \end{cases} \tag{7.10}$$

See [29, 30, 35] for more details on the derivation of the Joseph-Lundgren exponent.

All the above discussion can be summarized in the following:

Theorem 7.7 (Local absolute bounds for stable solutions) *Let $\Omega \subset \mathbb{R}^d$. Let u be a local non-negative stable weak solution to $-\Delta_p u = \lambda u^s$ in Ω, with $\lambda > 0$ and $p-1 < s < s_{JL}$. Then for any $B_{R_\infty} \subset B_{R_0} \subset\subset \Omega$ there exists a constant that does not depend on u such that*

$$\|u\|_{\infty,R_\infty} \le c_6 \tag{7.11}$$

where

$$c_6 = \frac{A^{(1)}_{\overline{m}}}{(R_0-R_\infty)^{\frac{pr}{\overline{m}(r-p)}}} \left[A^{(2)}_{\overline{m}} + A^{(3)}_{\overline{m}}\,\lambda^{\frac{\overline{m}r}{\overline{m}(r-p)-r[s-(p-1)]}}\, c_5^{\frac{\overline{m}r[s-(p-1)]}{\overline{m}(r-p)-r[s-(p-1)]}}\right]^{\frac{r}{\overline{m}(r-p)}} c_5 \tag{7.12}$$

where $A^{(1)}_{q_0}$, $A^{(2)}_{q_0}$ and $A^{(3)}_{q_0}$ are as in Theorem 4.4 and c_5 is given in (7.9).

Proof. Combine the upper bounds (4.5) (with the choice $q_0 = \overline{m}$ with the absolute upper bounds (7.7) to get

$$\begin{aligned}
\|u\|_{\infty,R_\infty} &\frac{A^{(1)}_{q_0}}{(R_0-R_\infty)^{\frac{pr}{q_0(r-p)}}} \left[A^{(2)}_{q_0} + A^{(3)}_{q_0}\,\lambda^{\frac{\overline{m}r}{\overline{m}(r-p)-r[s-(p-1)]}}\, \|u\|_{\overline{m},R_0}^{\frac{\overline{m}r[s-(p-1)]}{\overline{m}(r-p)-r[s-(p-1)]}}\right]^{\frac{r}{q_0(r-p)}} \|u\|_{q_0,R_0}\\
&\le \frac{A^{(1)}_{\overline{m}}}{(R_0-R_\infty)^{\frac{pr}{\overline{m}(r-p)}}} \left[A^{(2)}_{\overline{m}} + A^{(3)}_{\overline{m}}\,\lambda^{\frac{\overline{m}r}{\overline{m}(r-p)-r[s-(p-1)]}}\, c_5^{\frac{\overline{m}r[s-(p-1)]}{\overline{m}(r-p)-r[s-(p-1)]}}\right]^{\frac{r}{\overline{m}(r-p)}} c_5
\end{aligned}$$

where $A^{(1)}_{q_0}$, $A^{(2)}_{q_0}$ and $A^{(3)}_{q_0}$ are as in Theorem 4.4 and c_5 is given in (7.9). $\square$

8 Appendix

8.1 The John-Nirenberg Lemma and reverse Hölder inequalities

First of all we recall a quantitative version of Lemma 7.20 of [33], proved in [4] (see Lemma 4.2 there). From now on we denote, as usual, by $M^m(\Omega)$ the Marcinkiewicz spaces for any $m > 1$ and by $\mathcal{V}_\mu[g]$ the Riesz potential of a function g, that is

$$\mathcal{V}_\mu[g](x) = \int_\Omega \frac{g(y)}{|x-y|^{d(1-\mu)}}\,\mathrm{d}y, \quad \mu \in (0,1].$$

Lemma 8.1 (A "potential" version of the Moser-Trudinger imbedding) *Let $g \in M^\sigma(\Omega)$ with $\sigma > 1$ and let us suppose $\|g\|_{M^\sigma(\Omega)} \le K$. Then there exist two constants k_2 and k_3 such that*

$$\int_\Omega \exp\left[\frac{\left|\mathcal{V}_{\frac{1}{\sigma}}[g](x)\right|}{k_2 K}\right] \mathrm{d}x \le k_3.$$

One can take

$$k_2 > (\sigma-1)e \quad \textit{and} \quad k_3 = |\Omega| + \frac{\mathrm{diam}(\Omega)^d}{\sqrt{2\pi}} \frac{s\,e\,\omega_d}{k_2-(\sigma-1)\,e}.$$

Now a quantitative version of Jonh-Nirenberg lemma for convex domains; for the proof see Lemma 4.3 in [4].

Lemma 8.2 (Jonh-Nirenberg) *Let $g \in W^{1,1}(\Omega)$ where Ω is convex, and suppose there exists a constant K such that*

$$\int_{B_R\cap\Omega} |\nabla g| \, \mathrm{d}x \le K\, R^{d-1}, \quad \textit{for all balls } B_R.$$

Then the following inequality holds

$$\int_\Omega \exp\left[\frac{|g-g_\Omega|}{k_0\,K}\right] \mathrm{d}x \le k_1,$$

where for any $k_2 > (d-1)e$

$$k_0 = \frac{d|\Omega|}{\mathrm{diam}(\Omega)^d}\, k_2, \quad k_1 = \frac{\omega_d\, \mathrm{diam}(\Omega)^d (k_2+e)}{k_2-(d-1)e} \quad \textit{and} \quad g_\Omega = \frac{1}{|\Omega|}\int_\Omega g \, \mathrm{d}x.$$

The John-Nirenberg Lemma has an important consequence when applied to $g = \log(u+\delta)$, $\delta > 0$.

Proposition 8.3 (Reverse Hölder inequalities) *Let $\delta \ge 0$ and let u be a positive measurable function such that $\log(u+\delta) \in W^{1,1}(\Omega)$, where Ω is convex, and suppose there exists a constant K (independent of δ) such that*

$$\int_{B_R\cap\Omega} |\nabla \log(u+\delta)| \, \mathrm{d}x \le K\, R^{d-1}, \quad \textit{for all balls } B_R.$$

Then the following inequality

$$\frac{\|u+\delta\|_{q,\Omega}}{\|u+\delta\|_{-q,\Omega}} \le k_1^{2/q}$$

holds for any

$$0 < q \le \frac{1}{k_0\,K},$$

where the constants k_i are given in Lemma 8.2.

Proof. See Proposition 4.4 in [4]. □

We conclude this section by showing that reverse Hölder inequalities hold for local solutions to our problem, as a consequence of Caccioppoli estimates (see Corollary 2.5).

Proposition 8.4 (Reverse Hölder inequalities) *Let $\Omega \subset \mathbb{R}^d$ and let $\lambda > 0$. Let u be a local weak solution to $-\Delta_p u = \lambda u^s$, with $0 \le s < s_c = r-1$, r as in (3.2). Then for any $\varepsilon > 0$, the following inequality holds true for any $\delta \ge 0$*

$$\left[\frac{\varepsilon}{2^d\,(e\,d+\varepsilon)}\right]^{2/q} \frac{\|u+\delta\|_{q,R_0}}{|B_{R_0}|^{\frac{1}{q}}} \le \frac{\|u+\delta\|_{-q,R_0}}{|B_{R_0}|^{-\frac{1}{q}}}, \quad \textit{for all} \quad 0 < q \le \frac{(p-1)^{\frac{2}{p}}\, 2^{\frac{(d-1)(p-1)}{p}}}{p\,\omega_d^2\, d\,[e(d-1)+\varepsilon]}. \tag{8.1}$$

Proof. The Caccioppoli estimate (2.10), with $R_1 = r$ and $R_0 = 2r$, implies that

$$\int_{B_r} |\nabla \log(u+\delta)|^p \, \mathrm{d}x \le \frac{2^{d+p-1}\, p^p\, r^{d-p}\, \omega_d}{(p-1)^2},$$

hence the hypothesis of the previous proposition are satisfied, more precisely

$$\begin{aligned}\int_{B_{R_0}\cap B_r} |\nabla \log(u+\delta)| \, \mathrm{d}x &\le |B_r|^{1-\frac{1}{p}} \left[\int_{B_r} |\nabla \log(u+\delta)|^p \, \mathrm{d}x\right]^{\frac{1}{p}} \\ &\le r^{d-1}\omega_d\, \frac{2^{\frac{p+d-1}{p}}\, p}{(p-1)^{\frac{2}{p}}} =: K\, r^{d-1}.\end{aligned}$$

Therefore putting $K = \omega_d\, \frac{2^{\frac{p+d-1}{p}}\, p}{(p-1)^{\frac{2}{p}}}$, taking an $\varepsilon > 0$ and choosing $k_2 = e\,(d-1)+\varepsilon$, by the Proposition 8.3, we get the desired result. □

8.2 Technical tools

In this section we recall, in order to be complete, some tools that we use in this paper. The first lemma concerns the geometric convergence of some sequences of real numbers.

Lemma 8.5 (Numerical Iteration) *Let $Y_n \geq 0$ be a sequence of numbers such that*

$$Y_n \leq I_{n-1}^{\sigma\,\theta^{n-1}}\,Y_{n-1} \quad \textit{with} \quad I_{n-1} \leq I_0\,C^{n-1}$$

for some σ, I_0, $C > 0$, $\theta \in (0,1)$. Then $\{Y_n\}$ is a bounded sequence and one has

$$Y_\infty := \limsup_{n\to+\infty} Y_n \leq I_0^{\frac{\sigma}{1-\theta}}\,C^{\frac{\sigma\,\theta}{(1-\theta)^2}}\,Y_0.$$

Proof. See for example Lemma 7.1 of [34]. □

The following lemma is due to E. De Giorgi and its proof is contained in several books and papers, see for example [34], Lemma 6.1.

Lemma 8.6 (De Giorgi) *Let $Z(t)$ be a bounded non-negative function in the interval $[t_0, t_1]$. Assume that for $t_0 \leq t < s \leq t_1$ we have*

$$Z(t) \leq \theta\,Z(s) + \frac{A}{(s-t)^\alpha},$$

with $A \geq 0$, $\alpha > 0$ and $0 \leq \theta < 1$. Then

$$Z(t_0) \leq \frac{A\,c(\alpha,\lambda,\theta)}{(t_1-t_0)^\alpha}$$

where

$$c(\alpha,\lambda,\theta) = \frac{1}{(1-\lambda)^\alpha\left(1-\frac{\theta}{\lambda^\alpha}\right)} \quad \textit{for any} \quad \lambda \in (\theta^{\frac{1}{\alpha}}, 1).$$

This lemma has an important consequence, indeed it is necessary to obtain extending local upper bounds (see Section 4). More precisely, it allows to prove that if a reverse Hölder inequality holds for some $0 < q < \overline{q} \leq \infty$, then it holds for any $0 < q_0 < \overline{q} \leq \infty$.

Lemma 8.7 *Assume that the following bound holds true for some $0 < \underline{q} < \overline{q} < \infty$ and for any $R_\infty \leq \rho < R \leq R_0$,*

$$\|u\|_{\overline{q},r} \leq \frac{K}{(R-\rho)^\gamma}\,\|u\|_{\underline{q},R}.$$

Then we have that for all $0 < q_0 \leq \underline{q} < \overline{q} < \infty$

$$\|u\|_{\overline{q},R_\infty} \leq 3\cdot 2^{\frac{\overline{q}(\underline{q}-q_0)}{q_0(\overline{q}-\underline{q})}}\left[\left(4\,\gamma\,\frac{\underline{q}(\overline{q}-q_0)}{q_0(\overline{q}-\underline{q})}\right)^\gamma \frac{K}{(R_0-R_\infty)^\gamma}\right]^{\frac{\underline{q}(\overline{q}-q_0)}{q_0(\overline{q}-\underline{q})}} \|u\|_{q_0,R_0}.$$

Moreover if $\overline{q} = \infty$,

$$\|u\|_{\infty,R_\infty} \leq 3\cdot 2^{\frac{\underline{q}-q_0}{q_0}}\left[\left(4\,\gamma\,\frac{\underline{q}}{q_0}\right)^\gamma \frac{K}{(R_0-R_\infty)^\gamma}\right]^{\frac{\underline{q}}{q_0}} \|u\|_{q_0,R_0}.$$

Proof. See Lemma 3.7 (Extending Local Upper Bounds) of [4]. □

8.3 Numerical identities and inequalities

Now, in order to be complete and to simplify the reading of this paper, we recall some numerical identities and inequalities that we will use in the following.

$$\sum_{j=1}^{\infty} s^j = \frac{s}{1-s}, \ \forall\, 0 \le s < 1 \quad \Longrightarrow \quad \sum_{j=1}^{\infty} \left(\frac{p}{r}\right)^j = \frac{p}{r-p}, \ \text{ for } r > p. \tag{8.2}$$

$$\sum_{j=1}^{\infty} j^N s^j = \left[s \frac{\mathrm{d}}{\mathrm{d}s}\right]^{(N)} \left(\frac{1}{1-s}\right), \ \forall\, 0 \le s < 1, \ N \in \mathbb{N}$$

and so

$$\sum_{j=1}^{\infty} j\, s^j = \frac{s}{(1-s)^2}, \ \forall\, 0 \le s < 1 \quad \Longrightarrow \quad \sum_{j=1}^{\infty} j\left(\frac{p}{r}\right)^j = \frac{pr}{(r-p)^2}, \ \text{ for } r > p. \tag{8.3}$$

$$\sum_{j=1}^{k} s^j = \frac{s(1-s^k)}{1-s}, \ \forall\, 0 \le s < 1 \quad \Longrightarrow \quad \sum_{j=1}^{k} \left(\frac{p}{r}\right)^j = \frac{p}{r-p}\left[1 - \left(\frac{p}{r}\right)^k\right], \ \text{ for } r > p. \tag{8.4}$$

$$\sum_{j=k+1}^{\infty} s^j = \frac{s}{1-s}\, s^k \ \forall\, 0 \le s < 1 \quad \Longrightarrow \quad \sum_{j=k+1}^{\infty} \left(\frac{p}{r}\right)^j = \frac{p}{r-p}\left(\frac{p}{r}\right)^k, \ \text{ for } r > p.$$

Stirling's formula:

$$n! = \sqrt{2\,\pi\,n}\left(\frac{n}{e}\right)^n e^{\alpha_n} \qquad \text{with} \qquad \frac{1}{12n+1} \le \alpha_n \le \frac{1}{12n}.$$

ε-version of Young's inequality:

$$a \cdot b \le \frac{\varepsilon}{\sigma}\, a^\sigma + \frac{\sigma-1}{\sigma}\, \frac{b^{\frac{\sigma}{\sigma-1}}}{\varepsilon^{\frac{1}{\sigma-1}}}, \tag{8.5}$$

for any $\varepsilon > 0$, a, $b \ge 0$ and $\sigma > 1$.

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