Quantitative Bounds for Subcritical Semilinear Elliptic Equations

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Abstract

We prove a priori bounds for weak solutions of semilinear elliptic equations of the form $-\Delta u = cu^p$, with $0 < p < p_s = (d+2)/(d-2)$, $d \geq 3$, posed on a bounded domain Ω of $\mathbb{R}^d$ with boundary conditions $u = 0$. The bounds are quantitative and we give explicit expressions for all the involved constants. These estimates also allow to compare solutions corresponding to different values of p, an in particular take the limit $p \to 1$.

Besides their own interest, these results are useful in the study of the asymptotic convergence with rate of the solutions to the Cauchy-Dirichlet problem for the Fast Diffusion Equation.

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1 Introduction

In this paper we obtain upper and lower estimates for the weak solutions of semilinear elliptic equations of the form

$$-\Delta u = f(u) \tag{1.1}$$

posed in a bounded domain $\Omega \subset \mathbb{R}^d$ with homogeneous boundary conditions

$$u(x) = 0 \qquad \text{for all } x \in \partial\Omega\,. \tag{1.2}$$

For simplicity we assume that $\partial\Omega$ is $C^{2,\alpha}$ smooth. The choice of right-hand side we have in mind is $f(u) = \lambda\, u^p$ with $\lambda > 0$ and $0 < p < p_s := (d+2)/(d-2)$ if $d \ge 3$, or $p > 0$ if $d = 1, 2$. We shall restrict for notational simplicity to the case $d \ge 3$, without further comment throughout the paper, in order to use the usual Sobolev inequality valid in such case. This problem is one of the most popular problems in nonlinear elliptic theory and enjoys a large bibliography, see for instance [2, 8, 9, 13, 14, 15, 19, 20, 21, 22, 23, 24, 25, 27, 28, 33, 34, 35, 36, 37, 38] for different p, and [7, 11] for the limit case $p = p_s$.

We are interested in obtaining a priori estimates for nonnegative weak solutions of Problem (1.1)–(1.2) that either do not depend on the particular solution (then called absolute or universal bounds), or depend on the solution trough an explicit expression involving some norm.

We have devoted a recent paper [4] to obtaining such a priori estimates for general local weak solutions, i. e., weak solutions of equation (1.1) without any reference to the boundary conditions. The estimates obtained are quantitative upper bounds for solutions of any sign, quantitative lower bounds for positive solutions, and also local Harnack inequalities and gradient bounds. By quantitative estimates we mean keeping track of all the constants during the proofs.

It is purpose of this paper to complete the study by obtaining the improved quantitative estimates that involve knowledge of the boundary condition (1.2). As far as we know, there does not exist in literature a systematic set of quantitative local upper and lower bounds in the explicit form we provide here, though the qualitative statements of most of our results are known in the litarature. We recall that the quantitative control of the constants of such inequalities may be important in the applications; it is needed for instance in the results of [3] on the asymptotic properties of solutions of the fast diffusion equation in bounded domains. We obtain global upper and lower estimates in terms of the distance from the boundary $d(x, \partial\Omega)$, and Harnack inequalities up to the boundary. We will also study the limit $p \to 1$ to show how this problem approaches the linear eigenvalue problem, expanding on work we did in [3]. We devote some space to comparison on small sets, which is an important tool, see [34].

In fact, our estimates apply to a larger class of operators and nonlinearities. First of all, we can treat left-hand sides of the form

$$-\nabla \cdot A(x, u, \nabla u)\,, \tag{1.3}$$

where A is a Carathéodory function such that

$$\nu_1 |\xi|^2 \le A(x, u, \xi) \cdot \xi \qquad \text{and} \qquad |A(x, u, \xi)| \le \nu_2 |\xi|$$

for suitable positive constants. Secondly, we can easily change the right-hand side and consider supersolutions of the problem

$$-\nabla \cdot A(x, u, \nabla u) = f(x, u)\,, \tag{1.4}$$

as long as $f(u) \ge a_0\, u^p$ with $a_0 > 0$, since they are supersolutions of $-\nabla \cdot A(x, u, \nabla u) = a_0\, u^p$. We can consider subsolutions of (1.4) with $f(u) \le a_1 u^p$, and $a_1 \ge 0$. We have decided here to consider the model case, to simplify the presentation and to focus on the main ideas.

We will use the following standard definitions.

Definition 1.1 *A weak solution to problem* (1.1)–(1.2) *in* Ω *is a function* $u \in W_0^{1,2}(\Omega)$ *with* $f(u) \in L^1(\Omega)$ *which satisfies*

$$\int_\Omega [\nabla u \cdot \nabla \varphi - f(u)\varphi]\, \mathrm{d}x = 0 \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega)\,. \tag{1.5}$$

A weak subsolution is defined by changing equality into $\le$ *in formula* (1.5)*, that must be applied to test functions* $\varphi \in W_0^{1,2}(\Omega)$ *,* $\varphi \ge 0$*. A weak supersolution is defined in a similar way with equality replaced by the* $\ge$ *sign.*

2 Maximum and comparison principles on small sets

The maximum and comparison principle need not hold in general for solutions to nonlinear elliptic equations. This is an important feature of elliptic equations and it does not necessarily depend on the presence of a nonlinearity. Indeed, if we consider the linear eigenvalue Dirichlet problem for the equation $-\Delta u = \lambda u$ with $\lambda > \lambda_1$, it happens for instance that for $\lambda = \lambda_2 > \lambda_1 > 0$ the corresponding second eigenfunction Φ_2 has at least a change of sign, hence the standard comparison principle does not hold.

In any case, it is known that a (local) maximum and comparison principle holds on small sets. We are going to extend to our framework an idea originally due to Serrin, see for example Section 3.3 of the book [34]. These strong tools will allow us to construct explicit upper and lower barriers near the boundary, which are needed to obtain quantitative global Harnack estimates up to the boundary.

Throughout this section we will always assume $1 \le p < p_s$. We remark that when $0 < p < 1$ the standard strong maximum principle holds, and the comparison principle follows by standard methods.

Let $u \in W^{1,q}(B)$, where B is any bounded set of $\mathbb{R}^d$. Let $u_\varepsilon = (u-\varepsilon)_+ = \max\{u-\varepsilon, 0\}$. The support of u_ε is the closure of the set

$$\Gamma_\varepsilon = \Gamma_\varepsilon(u,B) = \big\{x \in B \,\big|\, u(x) > \varepsilon\big\}.$$

We can easily see that if $u \in W^{1,q}(B)$, and $u \le 0$ on ∂B then $u_\varepsilon \in W^{1,q}_0(B)$, for any $\varepsilon > 0$. Moreover,

$$\big|\nabla u_\varepsilon(x)\big| = \begin{cases} |\nabla u(x)| & \text{if } x \in \Gamma_\varepsilon \\ 0 & \text{if } x \in B \setminus \Gamma_\varepsilon, \end{cases}$$

∇u_ε is also supported in Γ_ε and $|\nabla u_\varepsilon| \le |\nabla u| \in \mathrm{L}^q(B)$. Notice that we consider the inequality $u \le 0$ on ∂B in the sense of the trace theorem (see for instance [19], Theorem 1, pg.272, or [1], Thm. 5.36): indeed the restriction to the boundary, the so called trace operator, is a continuous operator $T : W^{1,q}(\Omega) \to L^q(\partial\Omega)$ whenever Ω is a domain of class C^1 (or Lipschitz) and $1 \le q < \infty$. If moreover $1 < q < \infty$ the trace operator is continuous $T : W^{1,q}(\Omega) \to W^{1-1/q,q}(\partial\Omega)$ and is also a continuous operator $T : W^{1,q}(\Omega) \to L^r(\partial\Omega)$, for all $1 \le q \le d$, and all $q \le r \le (d-1)q/(d-q)$. Notice that the above discussion is indeed interesting only when $1 \le q \le d$, since when $q > d$ the Morrey imbedding guarantees that functions of $W^{1,q}(\Omega)$ are Hölder continuous of class $C^\alpha(\overline{\Omega})$, with $\alpha = 1 - d/q$.

We are now ready to state the first main result of this section, in which we relate the validity of an inverse Poincaré inequality for the truncated u_ε to the validity of a maximum principle for u. The proof uses the following version of the Poincaré inequality (see e.g. [34], Theorem 3.9.4), valid for functions $f \in W^{1,q}_0(B)$:

$$\|f\|_{\mathrm{L}^q(B)} \le \left(\frac{|B|}{\omega_d}\right)^{\frac{1}{d}} \|\nabla f\|_{\mathrm{L}^q(B)} \tag{2.1}$$

where ω_d is the volume of the unit ball. The multiplicative constant in (2.1) may not be sharp, but it has the advantage of not being dependent on $1 \le q < \infty$.

Theorem 2.1 (Reverse Poincaré implies maximum principle on small sets) *Let $B \subset \mathbb{R}^d$ be a bounded connected domain, and let $u \in W^{1,q}(B)$, with $1 \le q \le \infty$. Let u_ε and Γ_ε be as above and assume that there exists a constant k_q such that the following reverse Poincaré inequality holds for any $0 < \varepsilon < \overline{\varepsilon}$:*

$$\|\nabla u\|_{\mathrm{L}^q(\Gamma_\varepsilon)} \le k_q \, \|u\|_{\mathrm{L}^q(\Gamma_\varepsilon)} \tag{2.2}$$

Then, if

$$|B| < \frac{\omega_d}{k_q^d} \qquad \text{and} \qquad u \le 0 \text{ on } \partial B \tag{2.3}$$

we have that $u \le 0$ almost everywhere on B.

Proof. Let us calculate

$$\begin{aligned} \|u\|_{\mathrm{L}^q(\Gamma_\varepsilon)} &\le \|u-\varepsilon\|_{\mathrm{L}^q(\Gamma_\varepsilon)} + \|\varepsilon\|_{\mathrm{L}^q(\Gamma_\varepsilon)} = \|(u-\varepsilon)_+\|_{\mathrm{L}^q(\Gamma_\varepsilon)} + \varepsilon|\Gamma_\varepsilon|^{\frac{1}{q}} = \|u_\varepsilon\|_{\mathrm{L}^q(\Gamma_\varepsilon)} + \varepsilon|\Gamma_\varepsilon|^{\frac{1}{q}} \\ (b) &\le \left(\frac{|B|}{\omega_d}\right)^{\frac{1}{d}} \|\nabla u_\varepsilon\|_{\mathrm{L}^q(B)} + \varepsilon|B|^{\frac{1}{q}} \le_{(c)} \left(\frac{|B|}{\omega_d}\right)^{\frac{1}{d}} \|\nabla u\|_{\mathrm{L}^q(\Gamma_\varepsilon)} + \varepsilon|B|^{\frac{1}{q}} \\ (d) &\le k_q \left(\frac{|B|}{\omega_d}\right)^{\frac{1}{d}} \|u\|_{\mathrm{L}^q(\Gamma_\varepsilon)} + \varepsilon|B|^{\frac{1}{q}}. \end{aligned}$$

In (b) we have applied (2.1) to u_ε noticing that, since $u \in W^{1,q}(B)$, then $u_{|\partial\Omega} \in L^q(\partial\Omega)$ and $u_\varepsilon \in W_0^{1,q}(B)$. In (c) we have used that $\|\nabla u_\varepsilon\|_{\mathrm{L}^q(B)} = \|\nabla u\|_{\mathrm{L}^q(\Gamma_\varepsilon)}$. Finally in (d) we have used (2.2). Hence, for any $0 < \varepsilon \leq \overline{\varepsilon}$ we have

$$0 < \left[1 - k_q \left(\frac{|B|}{\omega_d}\right)^{\frac{1}{d}}\right] \|u\|_{\mathrm{L}^q(\Gamma_\varepsilon)} \leq \varepsilon |B|^{\frac{1}{q}} \tag{2.4}$$

where the first inequality follows from (2.3). To take the limit as $\varepsilon \to 0$ in the above inequality notice that $u_\varepsilon^q \to u_+^q = \max\{u^q, 0\}$ almost everywhere in B, and that $0 \leq u_\varepsilon^q \leq 2^{q-1}(|u|^q + \overline{\varepsilon}^q) \in \mathrm{L}^1(B)$, so that by dominated convergence we have

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{\mathrm{L}^q(B)}^q = \|u_+\|_{\mathrm{L}^q(B)}^q.$$

Moreover we have that

$$\|u\|_{\mathrm{L}^q(\Gamma_\varepsilon)}^q = \int_{\Gamma_\varepsilon} |u-\varepsilon+\varepsilon|^q \,\mathrm{d}x \geq_{(a)} \int_{\Gamma_\varepsilon} |u-\varepsilon|^q \,\mathrm{d}x + \int_{\Gamma_\varepsilon} |\varepsilon|^q \,\mathrm{d}x = \|u_\varepsilon\|_{\mathrm{L}^q(\Gamma_\varepsilon)}^q + \varepsilon^q |\Gamma_\varepsilon| =_{(b)} \|u_\varepsilon\|_{\mathrm{L}^q(B)}^q + \varepsilon^q |\Gamma_\varepsilon|$$

where in (a) we have used the inequality $(a+b)^q \geq a^q + b^q$ valid for any $a, b \geq 0$ and the fact that $u - \varepsilon \geq 0$ on Γ_ε. In (b) we have used that $\|u_\varepsilon\|_{\mathrm{L}^q(\Gamma_\varepsilon)} = \|u_\varepsilon\|_{\mathrm{L}^q(B)}$ since u_ε is supported in $\Gamma_\varepsilon \subseteq B$. Taking limits as $\varepsilon \to 0$ gives

$$\liminf_{\varepsilon\to 0} \|u\|_{\mathrm{L}^q(\Gamma_\varepsilon)}^q \geq \lim_{\varepsilon\to 0} \|u_\varepsilon\|_{\mathrm{L}^q(B)}^q + \varepsilon^q |\Gamma_\varepsilon| = \|u_+\|_{\mathrm{L}^q(B)}^q. \tag{2.5}$$

Joining inequalities (2.4) and (2.5) and taking the limits as $\varepsilon \to 0$, we get

$$0 \leq \|u_+\|_{\mathrm{L}^q(B)}^q \leq \liminf_{\varepsilon\to 0} \|u\|_{\mathrm{L}^q(\Gamma_\varepsilon)}^q \leq \lim_{\varepsilon\to 0} \frac{\varepsilon\, |B|^{\frac{1}{q}}}{1 - k_q \left(\frac{|B|}{\omega_d}\right)^{\frac{1}{d}}} = 0.$$

Hence $\|u_+\|_{\mathrm{L}^q(B)}^q = 0$, so that $u_+ = 0$ and $u \leq 0$ almost everywhere in B. □

Theorem 2.2 (Comparison with supersolutions on small sets) *Let $B \subset \mathbb{R}^d$ be a bounded connected domain, let $p \geq 1$, $\lambda > 0$ and let $u, \overline{u}$ be weak solution and supersolution respectively (in the sense of Definition 1.1) to*

$$\begin{cases} -\Delta u = \lambda\, u^p & \text{in } B \\ -\Delta \overline{u} \geq \lambda\, \overline{u}^p & \text{in } B \\ \overline{u} \geq u & \text{on } \partial B \\ 0 \leq u, \overline{u} \leq M & \text{in } \overline{B} \end{cases}$$

and assume that $|B| < \omega_d / \left(2p\, \lambda\, M^{p-1}\right)^d$. Then, we have that $\overline{u} \geq u$ in $\overline{B}$.

Proof. Let $v = u - \overline{u}$. We will prove a reverse Poincaré inequality for v on the sets Γ_ε relative to v for any $\varepsilon \in (0, 1]$. Notice that since $v \in W^{1,2}(B)$, then its truncated $v_\varepsilon \in W_0^{1,2}(B)$, so that we can use it as a test function in the weak formulation of the above equation.

We know that $v \in W^{1,2}(B)$, and that v satisfies the inequality

$$\Delta v = \Delta u - \Delta \overline{u} \geq -\lambda(u^p - \overline{u}^p).$$

Hence

$$-\int_B v_\varepsilon \Delta v \,\mathrm{d}x \geq \int_B \nabla v_\varepsilon \cdot \nabla v \,\mathrm{d}x = \int_{\Gamma_\varepsilon} |\nabla v|^2 \,\mathrm{d}x.$$

In the last formula the integration by parts holds since $v_\varepsilon = 0$ in a neighborhood of ∂B, and the second equality holds since the support of v_ε is the closure of $\Gamma_\varepsilon = \{x \in B \,|\, v > \varepsilon\}$ and since $|\nabla v_\varepsilon| = |\nabla v|$ on Γ_ε. On the other hand, using the inequality satisfied by $-\Delta v$, we get

$$\int_{\Gamma_\varepsilon} |\nabla v|^2 \,\mathrm{d}x = -\int_B v_\varepsilon \Delta v \,\mathrm{d}x \le \lambda \int_B v_\varepsilon (u^p - \overline{u}^p)\,\mathrm{d}x = \lambda \int_{\Gamma_\varepsilon} (u - \overline{u} - \varepsilon)(u^p - \overline{u}^p)\,\mathrm{d}x$$
$$= \lambda \int_{\Gamma_\varepsilon} (u - \overline{u})(u^p - \overline{u}^p)\,\mathrm{d}x - \lambda\,\varepsilon \int_{\Gamma_\varepsilon} (u^p - \overline{u}^p)\,\mathrm{d}x := (I) + (II).$$

We will treat the two integrals separately. The first integral can be estimated using the numerical inequality (7.10) with $a = u \ge 0$ $b = \overline{u} \ge 0$:

$$(u - \overline{u})(u^p - \overline{u}^p) \le p\big(u^{p-1} + \overline{u}^{p-1}\big)(u - \overline{u})^2 \le 2pM^{p-1}(u - \overline{u})^2$$

since $0 \le u, \overline{u} \le M$, so that

$$(I) \le 2M^{p-1}\lambda\, p \int_{\Gamma_\varepsilon} (u - \overline{u})^2 \,\mathrm{d}x = 2M^{p-1}\lambda\, p \|v\|^2_{\mathrm{L}^2(\Gamma_\varepsilon)}.$$

As for the second integral, we notice that on Γ_ε we have $u > \overline{u}$, so that $(II) = -\lambda\,\varepsilon \int_{\Gamma_\varepsilon} (u^p - \overline{u}^p)\,\mathrm{d}x < 0$. We have obtained the following reverse Poincaré inequality for v:

$$\|\nabla v\|^2_{\mathrm{L}^2(\Gamma_\varepsilon)} \le 2M^{p-1}\lambda\, p \|v\|^2_{\mathrm{L}^2(\Gamma_\varepsilon)} \tag{2.6}$$

for all $0 < \varepsilon < 1$, where $\Gamma_\varepsilon = \big\{x \in B \,\big|\, v > \varepsilon\big\}$. We are now in the conditions to apply the maximum principle of Theorem 2.1, with $q = 2$, $k_q = p\,\lambda\, M^{p-1}$, and since we know by hypothesis that $v = \overline{u} - u \le 0$ on ∂B, and that

$$|B| < \frac{\omega_d}{\left(2p\,\lambda\, M^{p-1}\right)^d}.$$

We conclude that $v \le 0$ a.e. in B, which means $u \le \overline{u}$ a.e. in B. □

A similar result holds for subsolutions, with an analogous proof.

Theorem 2.3 (Comparison with subsolutions on small sets) *Let $B \subset \mathbb{R}^d$ be a bounded connected domain, let $p \ge 1$, $\lambda > 0$ and let $u, \underline{u}$ be weak solution and subsolution respectively (in the sense of Definition* 1.1*) to*

$$\begin{cases} -\Delta u = \lambda\, u^p & \text{in } B \\ -\Delta \underline{u} \le \lambda\, \underline{u}^p & \text{in } B \\ \underline{u} \le u & \text{on } \partial B \\ 0 \le u, \overline{u} \le M & \text{in } \overline{B} \end{cases}$$

and assume that $|B| < \omega_d / \left(2p\,\lambda\, M^{p-1}\right)^d$. Then, we have that $\underline{u} \le u$ in B.

3 Global estimates I

We recall that we consider nonnegative weak solutions in the sense of Definition 1.1 of the homogeneous Dirichlet problem (1.1)–(1.2) posed in a smooth bounded domain $\Omega \subset \mathbb{R}^d$, and $f(u) = \lambda u^p \in L^1(\Omega)$.

3.1 Global upper bounds

In this section we will obtain global upper bounds for weak solutions of the Dirichlet problem (1.1)–(1.2). We will need the global Sobolev inequality on Ω, namely

$$\|v\|^2_{\mathrm{L}^{2^*}(\Omega)} \le \mathcal{S}_2^2(\Omega) \|\nabla v\|^2_{\mathrm{L}^2(\Omega)}\,, \qquad \forall v \in W_0^{1,2}(\Omega) \tag{3.1}$$

to prove global L^∞ bounds via Moser iteration.

Theorem 3.1 (Global upper bounds via Moser iteration) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $\lambda > 0$. Let u be a weak (sub-)solution in Ω to $-\Delta u = \lambda u^p$, subject to homogeneous Dirichlet conditions $u = 0$ on $\partial\Omega$, with $0 \le p < p_s = 2^* - 1 = (d+2)/(d-2)$. Then the following bound holds true:*

$$\|u\|_\infty \le I_{\infty,\overline{q}}(\Omega)\, \|u\|_{\overline{q}}^{2\overline{q}\theta_{\overline{q}}} \qquad \text{for any} \qquad \overline{q} > \frac{d(p-1)_+}{2}\,, \tag{3.2}$$

where $\theta_{\overline{q}} = 1/[2\overline{q} - d(p-1)_+]$ and, given $q_0 > \frac{d}{d-2}$,

(3.3)

$$I_{\infty,\overline{q}}(\Omega) = \begin{cases} I_1(\overline{q})\,, & \text{for } \overline{q} > \frac{d}{d-2} \\ 2^{2(q_0-\overline{q})\theta_{\overline{q}}} I_1(q_0)\,, & \text{for } \frac{d(p-1)_+}{2} < \overline{q} \le \frac{d}{d-2} < q_0 \end{cases} \qquad \text{with} \quad I_1(s) = \left[\lambda\, \frac{\mathcal{S}_2^2(\Omega) c_1 s\, d^{d+1}}{4(d-2)^{d+1}}\right]^{d\theta_{\overline{q}}}$$

and with c_1 as in (7.5).

Remarks. (i) Notice that taking $\overline{q} = p+1 > d(p-1)_+/2$, which will be necessary later (see Theorems 5.9, 5.10)is possible if and only if $p < p_s = (d+2)/(d-2)$.

(ii) This result has its local version, namely Theorem 3.1 of [4] , that differs from this one only in the more complicated expression for the constant, see (7.4).

Proof. We just sketch the proof. It is sufficient to prove the result for a nonnegative weak solution $u \in W_0^{1,2}(\Omega)$. We shall use the following choice of the test function:

$$\varphi_\varepsilon = u \left(\frac{\varepsilon + u}{1+\varepsilon u}\right)^{\alpha-1} \in W_0^{1,2}(\Omega)\,, \quad \text{so that} \quad \nabla\varphi_\varepsilon = \left(\frac{\varepsilon + u}{1+\varepsilon u}\right)^{\alpha-2} \frac{\alpha u + \varepsilon[u^2+1] + \varepsilon^2(2-\alpha)u}{(1+\varepsilon u)^2}\nabla u$$

for any $\alpha > 0$. We therefore obtain the energy identity $\lambda \int_\Omega u^p \varphi_\varepsilon \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla\varphi_\varepsilon \,\mathrm{d}x$, then letting $\varepsilon \to 0$ we get

$$\lambda \int_\Omega u^{p+\alpha} \,\mathrm{d}x = \alpha \int_\Omega u^{\alpha-1} |\nabla u|^2 \,\mathrm{d}x$$

that holds for any $\alpha > 0$; we can rewrite it in the form:

$$\int_\Omega \left|\nabla u^{\frac{\alpha+1}{2}}\right|^2 \mathrm{d}x = \frac{\lambda(\alpha+1)^2}{4\alpha} \int_\Omega u^{p+\alpha} \,\mathrm{d}x\,. \tag{3.4}$$

Using then the Sobolev inequality (3.1) on Ω, we obtain, letting $\beta = \alpha + 1 > 1$ and $v = u^{\beta/2}$:

$$\left[\int_\Omega u^{\frac{2^*}{2}\beta} \,\mathrm{d}x\right]^{\frac{2}{2^*}} \le \frac{\mathcal{S}_2^2 \lambda \beta^2}{4|\beta - 1|} \int_\Omega u^{p-1+\beta} \,\mathrm{d}x\,.$$

We have obtained the iterative inequality:

$$\|u\|_{\frac{2^*}{2}\beta_n} \le I_n^{\frac{1}{\beta_n}} \|u\|_{\frac{2^*}{2}\beta_{n-1}}^{\frac{2^*}{2}\frac{\beta_{n-1}}{\beta_n}}\,, \qquad \text{with } \beta_n = \left[\frac{2^*}{2}\right]^n \left[\beta_0 - (p-1)_+ \frac{d-2}{2}\right] + (p-1)_+ \frac{d-2}{2} \tag{3.5}$$

we require moreover that $\beta_0 > (p-1)_+(d-2)/2$, which will be assumed from now on, so that $\beta_n \to +\infty$ as $n \to +\infty$. Moreover

$$I_n = \frac{\mathcal{S}_2^2 \lambda \beta_n^2}{4|\beta_n - 1|} \le \frac{\mathcal{S}_2^2 \lambda c_1}{4}\beta_n \le \frac{\mathcal{S}_2^2 \lambda c_1 \beta_0}{4}\left(\frac{2^*}{2}\right)^n := I_0 \left(\frac{2^*}{2}\right)^n$$

where we have used that $\beta_n \le \left[\frac{2^*}{2}\right]^n \beta_0$. We have also required that $\beta_n \ne 1$ for all n: see the discussion in item *ii*) after Theorem 3.1 of [4] and we have estimated $\beta_n/|\beta_n - 1| \le c_1$, as in Step 4 of the proof of Theorem 3.1 of [4]: we recall the value of c_1 in formula (7.5) in Appendix 7.1. Iterating the above inequality (3.5) yields

$$
\begin{aligned}
\|u_n\|_{\frac{2^*}{2}\beta_n} &\le I_n^{\frac{1}{\beta_n}}\,\|u_n\|_{\frac{2^*}{2}\beta_{n-1}}^{\frac{2^*}{2}\frac{\beta_{n-1}}{\beta_n}} \le I_n^{\frac{1}{\beta_n}}\,I_{n-1}^{\frac{2^*}{2}\frac{1}{\beta_n}}\,\|u_n\|_{\frac{2^*}{2}\beta_{n-2}}^{\left(\frac{2^*}{2}\right)^2\frac{\beta_{n-2}}{\beta_n}} \\
&\le I_n^{\frac{1}{\beta_n}}\,I_{n-1}^{\frac{2^*}{2}\frac{1}{\beta_n}}\,\ldots\,I_1^{\left(\frac{2^*}{2}\right)^{n-1}\frac{1}{\beta_n}}\,\|u_n\|_{\frac{2^*}{2}\beta_0}^{\left(\frac{2^*}{2}\right)^n\frac{\beta_0}{\beta_n}} \le \prod_{j=1}^{n} I_j^{\left(\frac{2^*}{2}\right)^{n-j}\frac{1}{\beta_n}}\,\|u_n\|_{\frac{2^*}{2}\beta_0}^{\left(\frac{2^*}{2}\right)^n\frac{\beta_0}{\beta_n}}
\end{aligned}
\tag{3.6}
$$

Taking the limit as $n \to \infty$ we obtain

$$
\begin{aligned}
\|u\|_\infty &= \lim_{n\to\infty}\|u\|_{\frac{2^*}{2}\beta_n} \le \lim_{n\to\infty}\prod_{k=1}^{n} I_k^{\left(\frac{2^*}{2}\right)^{n-k}\frac{1}{\beta_n}}\,\|u\|_{\frac{2^*}{2}\beta_0}^{\frac{\beta_0}{\beta_0-\frac{d-2}{2}(p-1)_+}} \\
&\le \lim_{n\to\infty}\prod_{k=1}^{n} I_k^{\left(\frac{2^*}{2}\right)^{n-k}\frac{1}{\beta_n}}\,\|u\|_{\frac{2^*}{2}\beta_0}^{\frac{\beta_0}{\beta_0-\frac{d-2}{2}(p-1)_+}} \le I_\infty \|u\|_{\overline{q}}^{\frac{2\overline{q}}{2\overline{q}-d(p-1)_+}}
\end{aligned}
\tag{3.7}
$$

In fact, the penultimate passage follows because $\prod_{k=1}^{n} I_k^{\left(\frac{2^*}{2}\right)^{n-k}\frac{1}{\beta_n}}$ has a limit as $n \to +\infty$, which can be bounded as follows (for the details see Step 4 of the proof of Theorem 3.1 of [4].)

$$
\lim_{n\to\infty}\prod_{k=1}^{n} I_k^{\left(\frac{2^*}{2}\right)^{n-k}\frac{1}{\beta_n}} \le I_\infty := \left[I_0\,\left(\frac{2^*}{2}\right)^d\right]^{\frac{d-2}{2\beta_0-(d-2)(p-1)_+}}
$$

Finally, letting $\overline{q} = \beta_0 2^*/2$, we have obtained

$$
\|u\|_\infty \le I_\infty\,\|u\|_{\overline{q}}^{\frac{2\overline{q}}{2\overline{q}-d(p-1)_+}} \qquad \text{for any} \qquad \frac{d(p-1)_+}{2} < \overline{q}\,,
\tag{3.8}
$$

which is exactly (3.2) with I_∞ given by

$$
I_\infty \le \left[I_0\,\left(\frac{2^*}{2}\right)^d\right]^{\frac{d}{2\overline{q}-d(p-1)_+}} = \left[\lambda\,\frac{\mathcal{S}_2^2 c_1 \overline{q}\, d^{d+1}}{4(d-2)^{d+1}}\right]^{\frac{d}{2\overline{q}-d(p-1)_+}} := I_1^{\frac{d}{2\overline{q}-d(p-1)_+}}
$$

provided $\beta_0 > \max\{1\,,\,(p-1)_+(d-2)/2\}$, that is for any $\overline{q} > \max\{d/(d-2)\,,\,d(p-1)_+/2\}$. It remains to extend the upper bound to all $d(p-1)_+/2 < \overline{q}$ in the case when $d/(d-2) > d(p-1)_+/2$ that is when $0 < p < p_c = d/(d-2)$. To this end we recall Young's inequality, valid for any $\nu > 1$, $a, b \ge 0$, $\varepsilon > 0$:

$$
ab \le \frac{\varepsilon}{\nu}a^\nu + \frac{\nu-1}{\nu}\frac{b^{\frac{\nu}{\nu-1}}}{\varepsilon^{\frac{1}{\nu-1}}} \le \varepsilon a^\nu + \frac{b^{\frac{\nu}{\nu-1}}}{\varepsilon^{\frac{1}{\nu-1}}}
$$

with the choices $\varepsilon = 1/2$ and

$$
\nu = \frac{2q_0 - d(p-1)_+}{2(q_0-\overline{q})} > 1 \iff q_0 > \frac{d(p-1)_+}{2} \qquad \text{and} \qquad \frac{\nu}{\nu-1} = \frac{2q_0-d(p-1)_+}{2\overline{q}-d(p-1)_+}\,.
$$

We apply it to (3.8) with $q_0 > d/(d-2)$ to get, for all $d(p-1)_+ < \overline{q} \le d/(d-2)$:

$$
\begin{aligned}
\|u\|_\infty &\le I_\infty\,\|u\|_{q_0}^{\frac{2q_0}{2q_0-d(p-1)_+}} \le I_1^{\frac{d}{2q_0-d(p-1)_+}}\,\|u\|_\infty^{\frac{2(q_0-\overline{q})}{2q_0-d(p-1)_+}}\,\|u\|_{\overline{q}}^{\frac{2\overline{q}}{2q_0-d(p-1)_+}} \\
&\le \frac{1}{2}\|u\|_\infty + 2^{\frac{2(q_0-\overline{q})}{2\overline{q}-d(p-1)_+}}\,I_1^{\frac{d}{2\overline{q}-d(p-1)_+}}\,\|u\|_{\overline{q}}^{\frac{2\overline{q}}{2\overline{q}-d(p-1)_+}}\,.
\end{aligned}
$$

This concludes the proof. $\square$

3.2 Quantitative global absolute bounds

When we consider the homogeneous Dirichlet problem, we can obtain global upper and lower estimates, but only for some global L^p-norm. Obviously, the lower bound for the $\mathrm{L}^{-\infty}(\Omega)$-norm is zero. Improved global absolute bounds are given in Section 5.3 in the range of exponents $0 < p < 1$ and $1 < p < p_c = d/(d-2)$, using the global Harnack estimates of Section 5. Local absolute bounds have been obtained by the authors in [4] and will be recalled in Section 4.2.

Theorem 3.2 (Global absolute lower bounds when $1 < p < p_s$**)** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $\lambda > 0$. Let u be a weak solution in Ω to $-\Delta u = \lambda u^p$, subject to homogeneous Dirichlet conditions $u = 0$ on $\partial\Omega$, with $1 < p < p_s = 2^* - 1 = (d+2)/(d-2)$. Then the following bound holds true:*

$$\text{(3.9)} \qquad \frac{4\big(q_0 - p\big)}{\mathcal{S}_2^2(\Omega)\lambda\big(q_0 - (p-1)\big)^2}|\Omega|^{\frac{2}{2^*} - \frac{q_0 - (p-1)}{q_0}} \le \|u\|_{q_0}^{p-1}, \qquad \text{for any } q_0 > \frac{d(p-1)}{2}.$$

Moreover, when $p_c < p < p_s$, we have

$$\text{(3.10)} \qquad \frac{8\,|(d-2)(p-1)-2|}{\mathcal{S}_2^2(\Omega)\lambda(d-2)^2(p-1)^2} \le \|u\|_{\frac{d(p-1)}{2}}^{p-1}.$$

Note that the lower bound for q_0 tends to 0^+ as $p \to 1^+$.

Proof. Consider the global energy inequality (3.4) valid for $\alpha > 0$

$$\int_\Omega \left|\nabla u^{\frac{\alpha+1}{2}}\right|^2 \mathrm{d}x = \frac{\lambda(\alpha+1)^2}{4\alpha}\int_\Omega u^{p+\alpha}\,\mathrm{d}x$$

Using then the Sobolev inequality (3.1) on Ω, valid since $u \in W_0^{1,2}(\Omega)$, we obtain, letting $\beta = \alpha + 1 > 1$:

$$|\Omega|^{\frac{2}{2^*} - \frac{\beta}{\beta+p-1}}\left[\int_\Omega u^{\beta+p-1}\,\mathrm{d}x\right]^{\frac{\beta}{\beta+p-1}} \le \left[\int_\Omega u^{\frac{2^*}{2}\beta}\,\mathrm{d}x\right]^{\frac{2}{2^*}} \le \frac{\mathcal{S}_2^2\lambda\beta^2}{4|\beta-1|}\int_\Omega u^{\beta+p-1}\,\mathrm{d}x$$

where in the first step we have used Hölder inequality, that holds since $2 * \beta/2 > \beta + p - 1$ whenever $\beta > (d-2)(p-1)/2$. We have proved (3.9) when $q_0 = \beta + p - 1 > d(p-1)/2$.

Finally, we prove (3.10), by letting $\beta = \frac{d-2}{2}(p-1)$ in the above inequality, so that $q_0 = \frac{2^*}{2}\beta = \beta + p - 1 = d(p-1)/2$; we can do this only when $\beta > 1$ and this is possible only when $p_c < p < p_s$, in which case we obtain directly (3.10).

Remarks. (i) This absolute lower bound is a typical feature of the nonlinear equation, which does not hold in the linear case $p = 1$. On the other hand, in the case $1 < p < p_s$ absolute upper bounds are difficult to prove and we will discuss this issue in Section 5.3.

(ii) When $0 < p < 1$, the lower bound (3.10) of Theorem 3.2 formally transforms into an absolute upper bound:

$$\frac{8\,|(d-2)(p-1)-2|}{\mathcal{S}_2^2\lambda(d-2)^2(p-1)^2} \le \|u\|_{\frac{d(p-1)}{2}}^{p-1} \qquad \Longleftrightarrow \qquad \|u\|_{\frac{d(p-1)}{2}}^{1-p} \le \frac{\mathcal{S}_2^2\lambda(d-2)^2(p-1)^2}{8\,|(d-2)(p-1)-2|}$$

Actually we can do better, indeed the above result is not satisfactory since it involves a negative L^q norm when $0 < p < 1$, namely $q = d(p-1)/2 < 0$.

Theorem 3.3 (Global absolute upper bounds when $0 < p < 1$**)** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let $\lambda > 0$. Let u be a weak solution in Ω to $-\Delta u = \lambda u^p$, subject to homogeneous Dirichlet conditions $u = 0$ on $\partial\Omega$, with $0 \le p < 1$. Then the following bound holds true:*

$$\text{(3.11)} \qquad \|u\|_\infty \le \left[\lambda\,\frac{\mathcal{S}_2^2(\Omega)c_1 q\,d^{d+1}}{4(d-2)^{d+1}}\right]^{\frac{d}{2q}}\left(\frac{\mathcal{S}_2^2(\Omega)\lambda\,q^2(d-2)}{4d\left|q - \frac{d}{2-2}\right|}\right)^{\frac{1}{1-p}}|\Omega|^{\frac{2q-d(1-p)}{dq(1-p)}}$$

with c_1 as in (7.5) and $q > d/(d-2)$.

Proof. Consider the global energy equality (3.4) valid for $\alpha > 0$:

$$\int_\Omega \left|\nabla u^{\frac{\alpha+1}{2}}\right|^2 \mathrm{d}x = \frac{\lambda(\alpha+1)^2}{4\alpha}\int_\Omega u^{p+\alpha}\,\mathrm{d}x$$

Using then the Sobolev inequality (3.1) on Ω, valid since $u \in W_0^{1,2}(\Omega)$, we obtain, letting $\beta = \alpha + 1 > 1$ and using Hölder inequality:

$$\left[\int_\Omega u^{\frac{2^*}{2}\beta}\,\mathrm{d}x\right]^{\frac{2}{2^*}} \le \frac{\mathcal{S}_2^2\lambda\beta^2}{4|\beta-1|}\int_\Omega u^{p-1+\beta}\,\mathrm{d}x \le \frac{\mathcal{S}_2^2\lambda\beta^2}{4|\beta-1|}|\Omega|^{1-\frac{2}{2^*}\frac{\beta+p-1}{\beta}}\left[\int_\Omega u^{\frac{2^*}{2}\beta}\,\mathrm{d}x\right]^{\frac{2(\beta+p-1)}{2^*\beta}}$$

Setting $2^*\beta/2 = q > 0$ we have obtained so far

$$\|u\|_q \le \left(\frac{\mathcal{S}_2^2\lambda\, q^2(d-2)}{4d\left|q-\frac{d}{2-2}\right|}\right)^{\frac{1}{1-p}}|\Omega|^{\frac{2q+d(1-p)}{dq(1-p)}} \qquad \text{for all } q > 0.$$

Combining the above upper bounds with the upper bounds (3.2) of Theorem 3.1

$$\|u\|_\infty \le \left[\lambda\,\frac{\mathcal{S}_2^2(\Omega)c_1 q\, d^{d+1}}{4(d-2)^{d+1}}\right]^{\frac{d}{2q}}\|u\|_q \le \left[\lambda\,\frac{\mathcal{S}_2^2(\Omega)c_1 q\, d^{d+1}}{4(d-2)^{d+1}}\right]^{\frac{d}{2q}}\left(\frac{\mathcal{S}_2^2\lambda\, q^2(d-2)}{4d\left|q-\frac{d}{2-2}\right|}\right)^{\frac{1}{1-p}}|\Omega|^{\frac{2q+d(1-p)}{dq(1-p)}}$$

with c_1 as in (7.5) and $q > d/(d-2)$. We have obtained the absolute bound (3.11). $\square$

4 Reminder on quantitative local bounds

We now recall for completeness the results of the companion paper [4] concerning quantitative bounds of local type. The explicit expression of all the constants is given in Appendix 7.1.

4.1 Harnack inequalities

We recall here the quantitative Harnack inequalities of [4] which beside a general form given in Theorem 4.1, have different explicit constants in the two ranges $0 \le p \le 1$ and $1 < p < p_c$.

Theorem 4.1 (Harnack inequality for $0 \le p < p_s$) *Let $\Omega \subseteq \mathbb{R}^d$ and let $\lambda > 0$. Let u be a nonnegative local weak solution in $B_{R_0} \subseteq \Omega$ to $-\Delta u = \lambda u^p$, with $0 \le p < p_s = (d+2)/(d-2)$. Given $R_\infty < R_0$ and $\varepsilon > 0$ we assume*

(4.1) $$0 < \underline{q} \le q_0 := \frac{2^{\frac{d-3}{2}}}{d\omega_d^2[\mathrm{e}(d-1)+\varepsilon]}, \quad \overline{q} > \frac{d(p-1)_+}{2}.$$

If $0 < \overline{q} < d/(d-2)$ we also assume

$$\left[\frac{\log\frac{2^*-d(p-1)_+}{2\overline{q}-d(p-1)_+}}{\log\frac{d}{d-2}}\right] \text{ not integer.}$$

Then, the following bound holds true

(4.2) $$\sup_{x\in B_{R_\infty}} u(x) \le \mathcal{H}_p[u]\inf_{x\in B_{R_\infty}} u(x)$$

where $\mathcal{H}_p[u]$ depends on u through some local norms as follows

$$\mathcal{H}_p[u] = \mathcal{H}_p[u](d,\overline{q},\underline{q},\varepsilon,R_0,R_\infty) = \frac{I_{\infty,\overline{q}}}{I_{-\infty,\underline{q}}}\left(\frac{\left(⨍_{B_{R_0}} u^{\overline{q}}\,\mathrm{d}x\right)^{\frac{(p-1)_+}{\overline{q}}}}{⨍_{B_{R_\infty}} u^{(p-1)_+}\,\mathrm{d}x}\right)^{\frac{d}{2\overline{q}-d(p-1)_+}} \frac{\left(⨍_{B_{R_0}} u^{\overline{q}}\,\mathrm{d}x\right)^{\frac{1}{\overline{q}}}}{\left(⨍_{B_{R_0}} u^{\underline{q}}\,\mathrm{d}x\right)^{\frac{1}{\underline{q}}}}\,. \tag{4.3}$$

with $I_{\infty,\overline{q}}$ given by (7.4), $I_{-\infty,\underline{q}}$ is given by (7.7).

Theorem 4.2 (Harnack inequality, $0 \le p \le 1$) *Let $\Omega \subseteq \mathbb{R}^d$ and let $\lambda > 0$. Let u be a nonnegative local weak solution in $B_{R_0} \subseteq \Omega$ to $-\Delta u = \lambda u^p$, with $0 \le p \le 1$. For all $R_\infty < R_0$ the following bound holds true*

$$\sup_{x\in B_{R_\infty}} u(x) \le \mathcal{H}_p \inf_{x\in B_{R_\infty}} u(x)$$

where $\mathcal{H}_p$ does not depend on u, and is given by (7.1).

Theorem 4.3 (Harnack Inequalities when $1 < p < p_c$) *Let $\Omega \subseteq \mathbb{R}^d$ and let $\lambda > 0$. Let u be a nonnegative local weak solution to $-\Delta u = \lambda u^p$ in $B_{R_0} \subseteq \Omega$, with $1 < p < p_c = d/(d-2)$. Then for any $0 < R_\infty < \overline{R} < R_0$ there exists an explicit constant $\mathcal{H}_p > 0$ such that*

$$\sup_{x\in B_{R_\infty}} u(x) \le \mathcal{H}_p \inf_{x\in B_{R_\infty}} u(x) \tag{4.4}$$

where $\mathcal{H}_p$ does not depend on u, and is given by (7.3).

Remark. Notice that the constant $\mathcal{H}_p$ does not depend on u in the range $0 \le p < p_c$, and it does not depend on $\lambda > 0$ when moreover $p \ne 1$.

4.2 Local Absolute bounds when $0 < p < 1$ and when $1 < p < p_c$

Theorem 4.4 (Local Absolute bounds) *Let $\Omega \subseteq \mathbb{R}^d$ and let $\lambda > 0$. Let u be a local nonnegative weak solution to $-\Delta u = \lambda u^p$ in $B_{R_0} \subseteq \Omega$, with $0 < p < p_c = d/(d-2)$. Then for any $0 < R_\infty < \overline{R} < R_0$ there exists a constant $\mathcal{H}_p > 0$ that does not depend on u, such that*

$$\sup_{x\in B_R(x_0)} u(x) \le \mathcal{H}_p \left(\frac{8R_0^d}{\lambda(R_0-R)^2R^d}\right)^{\frac{1}{p-1}} \qquad \text{when } 1 < p < p_c = \frac{d}{d-2}\,, \tag{4.5}$$

and, if $u \not\equiv 0$ on B_{R_0}

$$\inf_{x\in B_R(x_0)} u(x) \ge \mathcal{H}_p^{-1} \left(\frac{\lambda(R_0-R)^2R^d}{8R_0^d}\right)^{\frac{1}{1-p}} \qquad \text{when } 0 < p < 1\,. \tag{4.6}$$

The constant $\mathcal{H}_p$ is given by (7.1) when $0 < p < 1$ and by (7.3) when $1 < p < p_c$.

Remark. The way the estimate blows up as $R \to R_0$ is $(R_0-R)^{-2/(p-1)}$ which is natural from scaling considerations and is predicted by Dancer in the papers [12, 13].

5 Global estimates II. Boundary estimates and global Harnack inequalities

In this section we establish quantitative boundary estimates by means of suitable explicit lower and upper barriers that describe the behaviour near the boundary. We first prove two lemmata.

5.1 Preliminaries. Explicit sub- and super-solutions on annuli

Lemma 5.1 (Supersolutions on an annulus) *Let $x_0 \in \mathbb{R}^d$, $M > 0$ and $0 < R_1 < R$. The function*

$$0 \le \overline{u}(x) = M \left[2\frac{|x-x_0| - R_1}{R - R_1} - \left(\frac{|x-x_0| - R_1}{R-R_1} \right)^2 \right] \le M \tag{5.1}$$

is a supersolution of the Dirichlet problem

$$\begin{cases} -\Delta u \ge \lambda\, u^p & \text{in } B_R(x_0) \setminus \overline{B_{R_1}}(x_0) \\ u = 0 & \text{on } \partial B_{R_1}(x_0) \\ u = M & \text{on } \partial B_R(x_0) \end{cases} \tag{5.2}$$

whenever $0 < R_1 < R$ satisfies the bounds

$$R \le \min \left\{ \left(1 + \frac{1}{2(d-1)} \right) R_1 \,,\ \frac{1}{\sqrt{\lambda M^{p-1}}} + R_1 \right\}. \tag{5.3}$$

Proof. Set $r = |x - x_0|$. The function $\overline{u}$ is the parabola $M(2s - s^2)$ respect to the variable

$$s = \frac{|x-x_0| - R_1}{R - R_1} = \frac{r - R_1}{R - R_1}$$

for $s \in [0,1]$ which corresponds to $R_1 \le |x - x_0| = r \le R$. This parabola has its vertex at $s = 1$ which is a maximum, corresponding to the condition $\overline{u}(x) = M$ when $|x| = R$, and is zero at $s = 0$, corresponding to the condition $\overline{u}(x) = 0$ when $|x| = R_1$. Its derivative is nonnegative on the interval $0 \le s \le 1$, and less or equal than $2M/(R - R_1)$. Hence for all $R_1 \le r \le R$

$$\begin{aligned} &0 = \overline{u}(R_1) \le \overline{u}(r) \le \overline{u}(R) = M \\ &0 \le \overline{u}'(r) = \frac{2M(R-r)}{(R-R_1)^2} \le \frac{2M}{R-R_1} \\ &\overline{u}''(r) = -\frac{2M}{(R-R_1)^2} \le 0 \end{aligned} \tag{5.4}$$

Having seen that the boundary conditions are satisfied, it remains to check that $-\Delta u - \lambda\, u^p \ge 0$. using formulae (5.4), we see that

$$-\overline{u}''(r) - \frac{d-1}{r}\overline{u}'(r) - \lambda \overline{u}^p(r) \ge \frac{2M}{(R-R_1)^2} - \frac{d-1}{R_1}\frac{2M}{R-R_1} - \lambda M^p$$

and a sufficient condition for the positivity of such quantity is that both

$$\frac{M}{(R-R_1)^2} \ge \frac{d-1}{R_1}\frac{2M}{R-R_1} \qquad \text{that is} \qquad R - R_1 \le \frac{R_1}{2(d-1)}$$

and

$$\frac{M}{(R-R_1)^2} \ge \lambda M^p \qquad \text{that is} \qquad R - R_1 \le \frac{1}{\sqrt{\lambda M^{p-1}}}.$$

Both conditions are satisfied in view of hypothesis (5.3). □

Lemma 5.2 (Subsolutions on an annulus) *Let $x_0 \in \mathbb{R}^d$, $\varepsilon > 0$ and $0 < R_1 < R$. The function*

$$\underline{u}(x) = \frac{\varepsilon\, R_1^{d-2}}{R^{d-2} - R_1^{d-2}} \left[\frac{R^{d-2}}{|x-x_0|^{d-2}} - 1 \right] \tag{5.5}$$

is a subsolution of the Dirichlet problem

$$(5.6) \qquad \begin{cases} -\Delta u = \lambda\, u^p & \text{in } B_R(x_0) \setminus \overline{B_{R_1}}(x_0) \\ u = \varepsilon & \text{on } \partial B_{R_1}(x_0) \\ u = 0 & \text{on } \partial B_R(x_0) \end{cases}$$

for any $0 < R_1 < R$.

Proof. Set $r = |x - x_0|$. The boundary conditions are satisfied, indeed $\underline{u}(R) = 0$ and $\underline{u}(R_1) = \varepsilon$. To show that $-\Delta \underline{u} - \lambda\, \underline{u}^p \leq 0$ it is sufficient to notice that $\underline{u}$ is harmonic on $\mathbb{R}^d \setminus \{0\}$ and positive on $B_R(x_0) \setminus \overline{B_{R_1}}(x_0)$. □

We finally collect some properties of the function“distance to the boundary”. It is defined as usual:

$$\mathrm{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^d$.

Lemma 5.3 (Properties of the distance to the boundary) *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain with boundary* $\partial\Omega$ *of class* C^2. *Let for* $\delta > 0$

$$\Sigma_\delta := \{x \in \Omega \,:\, d(x) < \delta\}$$

be the open strip of width δ *near the boundary. Then,*

(a) *there exist a constant* $\delta_0 > 0$ *such that for every* $x \in \Sigma_{\delta_0}$, *there is a unique* $h(x) \in \partial\Omega$ *which realizes the distance:*

$$\mathrm{dist}(x, \partial\Omega) = |x - h(x)|.$$

Moreover, $d(x) \in C^2(\Sigma_{\delta_0})$ *and for all* $r \in [0, \delta_0)$ *the function* $H_r : \partial(\overline{\Sigma_r}) \cap \Omega \to \partial\Omega$ *defined by* $H_r(x) = h(x)$ *is a homeomorphism.*

(b) *The function* $\mathrm{dist}(\cdot, \partial\Omega)$ *is Lipschitz with constant* 1, *i.e.*

$$|\mathrm{dist}(x, \partial\Omega) - \mathrm{dist}(y, \partial\Omega)| \leq |x - y|.$$

Moreover,

$$0 < c \leq |\nabla d(x)| \leq 1, \qquad \text{for any } x \in \Sigma_{\delta_0}$$

and there exist a constant $K > 0$ *such that:*

$$(5.7) \qquad -K \leq \Delta \mathrm{dist}(x, \partial\Omega) \leq K, \qquad \text{for any } x \in \Sigma_{\delta_0}$$

We refer to [23] for the proof of this lemma. Part (a) is due to Serrin.

5.2 Global Harnack estimates

The above lemmata will be needed for the barrier argument that we will prove later. This will prove general boundary estimates that can be combined with the local estimates of the previous sections in the form of *global Harnack estimates*. The proof presented here allows to obtain quantitative global absolute upper and lower bounds in the form of global Harnack estimates, but not in the whole range $1 < p < p_s$. More specifically we will obtain quantitative global Harnack estimates when $0 < p < 1$ and when $1 < p < p_c$.

Lemma 5.4 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and δ_0 as in Lemma 5.3. Let u be a weak solution to $-\Delta u = \lambda u^p$, with $0 < p < p_s = 2^* - 1 = (d+2)/(d-2)$ with $u = 0$ on $\partial\Omega$. Assume that there exist $\varepsilon, \delta, M > 0$ such that*

$$u(x) \ge \varepsilon \qquad \text{for all } x \in \Omega_\delta = \{x \in \Omega \,:\, \mathrm{dist}(x, \partial\Omega) \ge \delta\}\,, \qquad \text{and} \qquad \sup_\Omega u(x) \le M\,,$$

provided $\delta \le \delta_1 := \min\left(\frac{\delta_0}{2(d-1)}, \frac{1}{\sqrt{\lambda M^{p-1}}}\right)$ and in addition, if $p \ge 1$, $\delta \le \min\left(\delta_1, \frac{1}{(2\delta_0)^{d-1}(2p\lambda M^{p-1})^d}\right)$. Then:

$$\frac{\varepsilon}{2^{d-2}\delta}\min\left\{\mathrm{dist}(x,\partial\Omega)\,,\, 2^{d-2}\delta\right\} \le u(x) \le \frac{2M}{\delta}\min\left\{\mathrm{dist}(x,\partial\Omega)\,,\, \delta/2\right\} \qquad \text{for all } x \in \overline{\Omega}\,. \tag{5.8}$$

Proof. The proof is divided into two steps. Note that when $0 < p < 1$ the standard comparison principle holds, while for $p \ge 1$ we can compare only on small sets.

Upper boundary estimates. Fix a point $\overline{x} \in \partial\Omega$ and consider an exterior tangent ball at $\overline{x}$, centered at x_0, then fix $R_1 = |\overline{x} - x_0|$ and $R = R_1 + \delta$. We can and shall always choose $R_0 \le \delta_0$. Consider $\overline{u}$, the supersolution of Lemma 5.1 defined in the annulus $A = B_{R_1} \setminus B_R$. We can compare u and $\overline{u}$ on the region $A \cap \Omega$, since on the boundary $\partial B_{R_1} \cap \partial\Omega$ we have that $u = 0 \le \overline{u}$, while on $\partial B_R \cap \overline{\Omega}$ we have that $\overline{u} = M \ge u$; consequently we obtain that, for any $x \in \Sigma_\delta$ lying on the line joining $\overline{x}$ and x_0:

$$u(x) \le \overline{u}(x) = M\left[2\frac{|x-x_0| - R_1}{R - R_1} - \left(\frac{|x-x_0| - R_1}{R - R_1}\right)^2\right] \le M\,\min\left\{\frac{2\mathrm{dist}(x,\partial\Omega)}{\delta}\,,\,1\right\}$$

since $R - R_1 = \delta$ by construction and it is clear that $\mathrm{dist}(x,\partial\Omega) = |x - x_0| - R_1$; we remark that the condition (5.3) on the smallness of $R - R_1$, required for $\overline{u}$ to be supersolution on A, are:

$$\delta = R - R_1 \le \min\left\{\frac{R_1}{2(d-1)}\,,\,\frac{1}{\sqrt{\lambda M^{p-1}}}\right\}.$$

and the condition on the smallness of the set needed for comparison to hold when $p \ge 1$, reads $|B_R(x_0) \setminus \overline{B_{R_1}}(x_0)| < \omega_d / \left(2p\,\lambda\,M^{p-1}\right)^d$, and it is sufficient to take $R^d \le R_1^d + 1/\left(2p\,\lambda\,M^{p-1}\right)^d$. In view of the fact that $R_1 \le \delta_0$ and of the numerical inequality (7.10) we get the claim, under the stated conditions, for x as above. We can repeat this uniformly for all points of the boundary $\overline{x} \in \partial\Omega$, to obtain

$$u(x) \le M\,\min\left\{\frac{2\mathrm{dist}(x,\partial\Omega)}{\delta}\,,\,1\right\} \qquad \text{for all} \qquad x \in \Sigma_\delta$$

and, since $u \le M$ in Ω_δ, we can conclude that the above upper bound extend to the whole $\overline{\Omega}$.

Lower boundary estimates. Fix a point $\underline{x} \in \partial\Omega_\delta$ and consider an inner tangent ball at $\underline{x}$, say $B_{R_1}(x_0) \subseteq \Omega_\delta$. Consider now a bigger ball $B_R(x_0)$, with $R = R_1 + \delta$, and consider the annulus $A = B_R \setminus B_{R_1}$. Note that we can always choose $\delta = R_1$. Consider the subsolution $\underline{u}$ on the annulus A of Lemma 5.2. We will compare u and $\underline{u}$ on $A \cap \Omega$: on the inner boundary $\partial B_{R_1} \cap \partial\Omega_\delta$ we have that $u \ge \varepsilon = \underline{u}$ while on the outer boundary $\partial B_R \cap \overline{\Omega}$ we have $u \ge 0 = \underline{u}$. As a consequence on $A \cap \Omega$ we have that, for any $x \in \Sigma_\delta$ lying on the line joining $\overline{x}$ and x_0:

$$\begin{aligned} u(x) \ge \underline{u}(x) &= \frac{\varepsilon}{(R/R_1)^{d-2} - 1}\left[\frac{R^{d-2}}{|x-x_0|^{d-2}} - 1\right] \\ &\ge^{(a)} \varepsilon\,\frac{R - |x-x_0|}{|x-x_0|}\,\frac{R_1}{(R/R_1)^{d-3}(R-R_1)} \ge^{(b)} \frac{\varepsilon}{\delta}\left[\frac{R_1}{R}\right]^{d-2}\mathrm{dist}(x,\partial\Omega) = \frac{\varepsilon}{2^{d-2}\delta}\,\mathrm{dist}(x,\partial\Omega) \end{aligned}$$

where in (a) we have used the inequality (recall that $d \ge 3$) $t - 1 \le t^{d-2} - 1$ valid for all $t \ge 1$. and in (b) the fact that $R_1 \le |x - x_0| \le R$, $R - R_1 = \delta$ and that $\mathrm{dist}(x,\partial\Omega) = R - |x - x_0|$. The condition on the smallness of the set needed for comparison to hold when $p \geq 1$ is identical the one studied in the previous step. We can repeat this for all points of the boundary $\overline{x} \in \partial\Omega_\delta$, to obtain

$$u(x) \geq \varepsilon \, \frac{\mathrm{dist}(x, \partial\Omega)}{2^{d-2}\delta} \qquad \text{for all} \qquad x \in \Sigma_\delta$$

and, since $u \geq \varepsilon$ in Ω_δ, we can conclude that the above bound extend to the whole $\overline{\Omega}$ in the desired form (5.8). □

The above lemma combined with the local Harnack inequalities provides a first form for the global Harnack inequalities.

Theorem 5.5 (Global Harnack inequality) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let u be a weak solution to $-\Delta u = \lambda u^p$, with $u = 0$ on $\partial\Omega$. Then the following inequality holds true*

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\Omega_{\delta/2}} u(x) \leq \mathcal{H}_p^{N(\Omega,\delta)} \inf_{x\in\Omega_\delta} u(x)\,, \qquad \text{for any } 0 < \delta \leq \delta_0\,, \tag{5.9}$$

where $N(\Omega, \delta) = 20$ is given by (5.14) and $\mathcal{H}_p$ is given by (7.1) or (7.3) when $0 \leq p \leq 1$ or when $1 < p < p_c$ respectively. On the other hand, when $p_c \leq p < p_s$, the following inequality holds true

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\Omega_{\delta/2}} u(x) \leq \mathcal{H}_p(\Omega, \delta)[u] \inf_{x\in\Omega_\delta} u(x)\,, \qquad \text{for any } 0 < \delta \leq \delta_0\,, \tag{5.10}$$

and the constant $\mathcal{H}_p(\Omega, \delta)$ may also depend on u through some local L^q-norms.

Remarks. (i) Note that $1 < \mathcal{H}_p(\Omega, \delta) \to +\infty$ as $\delta \to 0$.

(ii)When $p_c \leq p < p_s$, the constant $\mathcal{H}_p(\Omega, \delta)$ has the form

$$\mathcal{H}_p(\Omega, \delta) = \mathcal{H}_{p,1}\mathcal{H}_{p,2}\ldots\mathcal{H}_{p,N} \tag{5.11}$$

where $\mathcal{H}_{p,k}[u]$ depends on u through some local norms as the constants $\mathcal{H}_p[u]$ in Theorem 4.1, namely

$$\mathcal{H}_{p,k}[u] = \frac{I_{\infty,\overline{q}}}{I_{-\infty,\underline{q}}} \left(\frac{\left(⨍_{B_{\delta/2}} u^{\overline{q}}\,\mathrm{d}x \right)^{\frac{(p-1)_+}{\overline{q}}}}{⨍_{B_{\delta/4}} u^{(p-1)_+}\,\mathrm{d}x} \right)^{\frac{d}{2\overline{q}-d(p-1)_+}} \frac{\left(⨍_{B_{\delta/2}} u^{\overline{q}}\,\mathrm{d}x \right)^{\frac{1}{\overline{q}}}}{\left(⨍_{B_{\delta/2}} u^{\underline{q}}\,\mathrm{d}x \right)^{\frac{1}{\underline{q}}}}\,. \tag{5.12}$$

with $I_{\infty,\overline{q}}$ given by (7.4), $I_{-\infty,\underline{q}}$ is given by (7.7). In the previous formula, balls are centered at suitable points x_k, see the proof below for details.

Proof. The proof is divided into several steps. We will fix $\delta_0 = \delta_0(\Omega)$ as in Lemma 5.3, so that $\mathrm{dist}(\cdot, \partial\Omega) \in C^2(\Sigma_{\delta_0})$, where $\Sigma_{\delta_0} = \Omega \setminus \overline{\Omega_{\delta_0}}$.

• Step 1. *The maximum on Ω is attained in $\Omega_{\delta/2}$.* Fix a $\delta \leq \delta_0$. As a consequence of Lemma 5.4, we have that, letting $M = \sup_{x\in\Omega} u(x)$, then $u(x) \leq M \min\{2\mathrm{dist}(x, \partial\Omega)/\delta\,,\, 1\}$ for all $x \in \overline{\Omega}$, therefore $u(x) < M$ when $\mathrm{dist}(x, \partial\Omega) < \delta/2$, so that the supremum of u in Ω is attained at some point in $\Omega_{\delta/2}$.

• Step 2. *A global Harnack inequality on Ω_δ.* Let now $m := \inf_{x\in\Omega_{\delta/2}} u(x)$ and let $M = \sup_{x\in\Omega_{\delta/2}} u(x)$, the latter equality being proved in Step 1. Since u is continuous on $\Omega_{\delta/2}$ as a consequence of Harnack inequalities, m, M are attained, say at $x_0, \overline{x} \in \Omega_{\delta/2}$ respectively. We recall here the form of local Harnack inequality that we will use

$$\sup_{x\in B_{\delta/4}(x_k)} u(x) \leq \mathcal{H}_p \inf_{x\in B_{\delta/4}(x_k)} u(x) \tag{5.13}$$

where the constant $\mathcal{H}_p$ always depend on δ, but when $0 < p < p_c$ it does not depend neither on $x_k \in \Omega_\delta$ neither on u (see Theorems 4.2 and 4.3 for an explicit expression); on the other hand, when

$p_c \le p < p_s$ $\mathcal{H}_p$ may also depend on u, through some $\mathrm{L}^q(B_{\delta/2}(x_k))$-norm (see Theorem 4.1 for an explicit expression), in which case we will denote it by $\mathcal{H}_{p,k}$.

Now we will choose a finite number of balls $B_{\delta/4}(x_k)$, such that $B_{\delta/4}(x_k) \cap B_{\delta/4}(x_{k+1}) \neq \emptyset$ for all k, and such that $x_0 \in B_{\delta/4}(x_1)$ and $\overline{x} \in B_{\delta/4}(x_{\overline{N}})$. The number of such balls in not greater than

$$N = N(\Omega,\delta) := \left(1 + \left[\frac{\mathrm{diam}(\Omega)}{\delta}\right]\right)^d \left(1 + 2^d\right) \tag{5.14}$$

(clearly the above bound is not optimal). We will choose $x_{i,j} \in B_{\delta/4}(x_i) \cap B_{\delta/4}(x_j) \neq \emptyset$. Now choose the sequence of points $x_1, \ldots x_{\overline{N}}$ such that $x_0 \in B_1(x_1)$ and $\overline{x} \in B_{\delta/4}(x_{\overline{N}})$, and such that $x_k' = x_{k,k-1} \in B_{\delta/4}(x_k) \cap B_{\delta/4}(x_{k+1}) \neq \emptyset$, obviously $\overline{N} \le N(\Omega,\delta)$. Then we use the above Harnack inequalities (5.13) in the iterative form

$$u(x_k') \ge \inf_{x \in B_{\delta/4}(x_k)} u(x) \ge \mathcal{H}_{p,k} \sup_{x \in B_\delta/4(x_k)} u(x) \ge \mathcal{H}_{p,k} u(x_{k+1}') \tag{5.15}$$

to get

$$\begin{aligned} u(x_0) &\ge \mathcal{H}_{p,1}^{-1} \sup_{x \in B_\delta/4(x_1)} u(x) \ge \mathcal{H}_{p,1}^{-1} u(x_1') \ge \mathcal{H}_{p,1}^{-1}\mathcal{H}_{p,2}^{-1} u(x_2') \\ &\ge \ldots \ge \mathcal{H}_{p,1}^{-1}\mathcal{H}_{p,2}^{-1} \ldots \mathcal{H}_{p,\overline{N}}^{-1} \sup_{x \in B_\delta/4(x_{\overline{N}})} u(x) = \mathcal{H}_{p,1}^{-1}\mathcal{H}_{p,2}^{-1} \ldots \mathcal{H}_{p,\overline{N}}^{-1} u(\overline{x}) \,. \end{aligned}$$

Recalling that $u(x_0) = \inf_{x \in \Omega_\delta} u(x)$ and $u(\overline{x}) = \sup_{x \in \Omega_\delta} u(x) = \sup_{x \in \Omega} u(x)$ (the latter equality follows by Step 1), we have obtained

$$\inf_{x \in \Omega_\delta} u(x) \ge \mathcal{H}_p^{-1}(\Omega,\delta) \sup_{x \in \Omega} u(x) \,, \qquad \text{for any } 0 < \delta \le \delta_0 \,,$$

where

$$\mathcal{H}_p^{-1}(\Omega,\delta) := \mathcal{H}_{p,1}^{-1}\mathcal{H}_{p,2}^{-1} \ldots \mathcal{H}_{p,\overline{N}}^{-1} \ge \mathcal{H}_{p,1}^{-1}\mathcal{H}_{p,2}^{-1} \ldots \mathcal{H}_{p,N}^{-1}$$

since $\overline{N} \le N(\Omega,\delta)$.

• STEP 3. *Absolute constant when* $0 \le p < 1$ *and* $1 < p < p_c$. In this case the constants $\mathcal{H}_{p,k} = \mathcal{H}_p$ are uniform and do not depend on δ, cf. Theorem 4.2 for $0 \le p \le 1$ and Theorem 4.3 when $1 < p < p_c$. Therefore

$$\inf_{x \in \Omega_\delta} u(x) \ge \mathcal{H}_p^{-N} \sup_{x \in \Omega} u(x) \,, \qquad \text{for any } 0 < \delta \le \delta_0 \,, \tag{5.16}$$

where $\mathcal{H}_p$ is given by (7.1) or (7.3) when $0 \le p \le 1$ or when $1 < p < p_c$ respectively, with the choices $R_\infty = \delta/4$ and $R_0 = \delta/2$, so that $\mathcal{H}_p$ do not depend on δ. □

5.3 Additional global absolute bounds when $0 \le p < 1$ and $1 < p < p_c$

In this section we show how the Global Harnack inequalities of the previous section allow to prove absolute bounds when $0 < p < 1$ and $1 < p < p_c$, when combined with the absolute bounds of Section 3.2. We recall that qualitative global absolute upper bounds are difficult to prove and have been proven in [8, 15, 12, 13, 21, 22]. Such upper absolute bounds are qualitative (i.e. the expression of the constant is not explicit), but cover the whole range $1 < p < p_s$; as far as we understand the techniques used in [15, 12, 13, 21, 22] that holds also in the range $p_c \le p < p_s$ can not be made quantitative.

- $1 < p < \frac{d+1}{d-1} < p_c$ Brezis and Turner [8] have proven absolute upper bounds using the Hardy inequalities of Proposition 6.2. The constants in this upper bound can be quantitatively estimated, but the method used does not allow to treat the case of larger exponents.

- If one wants to deal with the full range of exponents $1 \le p < p_s$, one has to proceed as Gidas-Ni-Nirenberg [21] when the domain is convex, or as DeFigueredo-Lions-Nussbaum [15] which extend the ideas of [21] to more general domains. We refer also to the paper by Gidas-Spruck [22] for a proof of qualitative absolute upper bounds for any $1 < p < p_s$. Unfortunately we are not able to provide a quantitative version of the proofs of the above mentioned absolute bounds. Similar remarks apply to the upper bounds given in Dancer [12, 13].

Theorem 5.6 (Global upper bounds when $0 \le p < 1$ **and** $1 < p < p_c$**)** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let u be a weak solution to $-\Delta u = \lambda u^p$, with $0 < p < p_c$, $p \neq 1$ and with $u = 0$ on $\partial\Omega$. Then*

$$\sup_{x\in\Omega} u(x) \le \mathcal{M}_{p,\delta} < +\infty\,,$$

where, for $0 \le p < 1$ $\mathcal{M}_{p,\delta}$ does not depend on δ and has the explicit form

$$\mathcal{M}_{p,\delta} := \left[\lambda\,\frac{\mathcal{S}_2^2(\Omega) c_1 q\, d^{d+1}}{4(d-2)^{d+1}}\right]^{\frac{d}{2q}} \left(\frac{\mathcal{S}_2^2(\Omega)\lambda\, q^2(d-2)}{4d\left|q-\frac{d}{2-2}\right|}\right)^{\frac{1}{1-p}} |\Omega|^{\frac{2q-d(1-p)}{dq(1-p)}}\,, \tag{5.17}$$

with c_1 as in (7.5) *and $q > d/(d-2)$. Moreover, for $1 < p < p_c$ we have that*

$$\mathcal{M}_{p,\delta} := \mathcal{H}_p^{N(\Omega,\delta)+1} \left(\frac{2^{d+3}}{\lambda\delta^2}\right)^{\frac{1}{p-1}}\,, \qquad \textit{for any } 0 < \delta \le \delta_0\,, \tag{5.18}$$

where $N(\Omega,\delta)$ is given in (5.14) *and $\mathcal{H}_p$ given by* (7.3)*.*

Proof. Now we split two cases, namely when $0 \le p < 1$ and $1 < p < p_c$. We keep the notations of the proof of Theorem 5.5.

• Step 1. *The case* $0 \le p < 1$. We recall the absolute lower bounds of Theorem 3.3, which give an explicit formula for the constant.

• Step 2. *The case* $1 < p < p_c$. We recall the absolute local upper bounds (4.5) which read in this context, and we let $R = \delta/4 < R_0 = \delta/2$. Hence

$$\sup_{x\in B_R(\overline{x})} u(x) \le \mathcal{H}_p \left(\frac{8R_0^d}{\lambda(R_0-R)^2R^d}\right)^{\frac{1}{p-1}} = \mathcal{H}_p\left(\frac{2^{d+3}}{\lambda\delta^2}\right)^{\frac{1}{p-1}} \qquad \text{when } 1 < p < p_c = \frac{d}{d-2}\,, \tag{5.19}$$

where the constant $\mathcal{H}_p$ is given by (7.3). Joining this inequality with (5.9) gives

$$\sup_{x\in\Omega} u(x) = \sup_{x\in\Omega_{\delta/2}} u(x) \le \mathcal{H}_p^N \inf_{x\in\Omega_{\delta/2}} u(x) \le \mathcal{H}_p^{N+1}\left(\frac{2^{d+3}}{\lambda\delta^2}\right)^{\frac{1}{p-1}}\,, \qquad \text{for any } 0 < \delta \le \delta_0\,. \ \square \tag{5.20}$$

Theorem 5.7 (Global lower bounds when $0 \le p < 1$ **and** $1 < p < p_c$**)** *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let u be a weak solution to $-\Delta u = \lambda u^p$, with $0 \le p < 1$ and $1 < p < p_c$ with $u = 0$ on $\partial\Omega$. Then for any $0 < \delta \le \delta_0$, we have*

$$\inf_{x\in\Omega_\delta} u(x) \ge \mathcal{L}_{p,\delta} > 0 \tag{5.21}$$

where, for $0 \le p < 1$

$$\mathcal{L}_{p,\delta} = \mathcal{H}_p^{-N(\Omega,\delta)-1}\left(\frac{\lambda\,\delta^2}{2^{d+7}}\right)^{\frac{1}{1-p}}\,, \tag{5.22}$$

while for $1 < p < p_c$ *and* $q_0 \ge \frac{d(p-1)}{2}$

$$\mathcal{L}_{p,\delta} = \mathcal{H}_p^{-N(\Omega,\delta)} \left[\frac{|\Omega|^{\frac{2}{2^*}-1} 4\big(q_0 - p\big)}{\mathcal{S}_2^2(\Omega)\lambda\big(q_0 - (p-1)\big)^2}\right]^{\frac{1}{p-1}} \tag{5.23}$$

where $N(\Omega, \delta)$ *is given in* (5.14) *and* $\mathcal{H}_p$ *is given by* (7.3)*.*

Proof. We split the cases $0 \le p < 1$ and $1 < p < p_c$. We keep the notations of the proof of Theorem 5.5.

• Step 1. *The case* $0 \le p < 1$*.* We recall the absolute local lower bounds (4.6) which read in this context, we let $R = \delta/4 < R_0 = \delta/2$

$$\inf_{x\in B_{\delta/4}(x_k)} u(x) \ge \mathcal{H}_p^{-1} \left(\frac{\lambda (R_0 - R)^2 R^d}{8R_0^d}\right)^{\frac{1}{1-p}} = \mathcal{H}_p^{-1} \left(\frac{\lambda\, \delta^2}{2^{d+7}}\right)^{\frac{1}{1-p}} \tag{5.24}$$

where the constant $\mathcal{H}_p$ is given by (7.1). We join this inequality with the global Harnack inequality (5.9) so that we obtain

$$\inf_{x\in\Omega_\delta} u(x) \ge \mathcal{H}_p^{-N} \sup_{x\in\Omega} u(x) \ge \mathcal{H}_p^{-N} \inf_{x\in B_{\delta/4}(x_k)} u(x) \ge \mathcal{H}_p^{-N-1} \left(\frac{\lambda\, \delta^2}{2^{d+7}}\right)^{\frac{1}{1-p}} \quad \forall\ 0 < \delta \le \delta_0 \tag{5.25}$$

• Step 2. *The case* $1 < p < p_c$*.* We recall the absolute lower bounds of Theorem 3.2

$$\frac{4\big(q_0 - p\big)}{\mathcal{S}_2^2(\Omega)\lambda\big(q_0 - (p-1)\big)^2} |\Omega|^{\frac{2}{2^*} - \frac{q_0-(p-1)}{q_0}} \le \|u\|_{q_0}^{p-1}\,, \qquad \text{for any } q_0 > \frac{d(p-1)}{2}\,. \tag{5.26}$$

We join this inequality with the global Harnack inequality (5.9), recalling that

$$\|u\|_{q_0}^{p-1} \le |\Omega|^{(p-1)/q_0} \left[\sup_{x\in\Omega} u(x)\right]^{p-1}\,,$$

so that we obtain

$$\inf_{x\in\Omega_\delta} u(x) \ge \mathcal{H}_p^{-N} \sup_{x\in\Omega} u(x) \ge \mathcal{H}_p^{-N} \left[\frac{|\Omega|^{\frac{2}{2^*}-1} 4\big(q_0 - p\big)}{\mathcal{S}_2^2(\Omega)\lambda\big(q_0 - (p-1)\big)^2}\right]^{\frac{1}{p-1}}\,, \qquad \text{for any } 0 < \delta \le \delta_0\,. \ \square \tag{5.27}$$

Joining the above upper and lower global bounds we can finally prove the global Harnack inequalities, as follows.

5.4 Additional global Harnack inequalities when $0 < p < p_c$

Theorem 5.8 (Global Harnack inequalities when $0 \le p < 1$ and $1 < p < p_c$) *Let* $\Omega \subset \mathbb{R}^d$ *be a bounded domain, and let* u *be a weak solution to* $-\Delta u = \lambda u^p$*, with* $0 \le p < p_c$ *with* $u = 0$ *on* $\partial\Omega$*. Assume*

$$\delta \le \delta_1 := \min\left(\frac{\delta_0}{2(d-1)}, \frac{1}{\sqrt{\lambda \mathcal{M}_{p,\delta}^{p-1}}}\right) \qquad \textit{and, if } p \ge 1, \qquad \delta \le \min\left(\delta_1, \frac{1}{(2\delta_0)^{d-1}(2p\lambda\mathcal{M}_{p,\delta}^{p-1})^d}\right).$$

Then

$$\frac{\mathcal{L}_{p,\delta}}{2^{d-2}\delta} \min\big\{\mathrm{dist}(x, \partial\Omega)\,,\, 2^{d-2}\delta\big\} \le u(x) \le \frac{\mathcal{M}_{p,\delta}}{\delta} \min\left\{2\mathrm{dist}(x, \partial\Omega)\,,\, \delta\right\} \qquad \textit{for all } x \in \overline{\Omega}\,. \tag{5.28}$$

The constant $\mathcal{M}_{p,\delta}$ *is given by* (5.17) *(when* $0 \le p < 1$*) or* (5.18) *(when* $1 < p < p_c$*), and* $\mathcal{L}_{p,\delta}$ *is given by* (5.22) *(when* $0 \le p < 1$*) or* (5.23) *(when* $1 < p < p_c$*).*

Proof. Just combine the absolute upper and lower bounds of Theorems 5.6 and 5.7 with Lemma 5.4 . □

• We now give another version of the global Harnack inequality, that holds uniformly for all p and is needed in the next section when studying the limit $p \to 1$, when we will deal with normalized solutions, namely $\|u\|_{\mathrm{L}^{p+1}(\Omega)} = 1$.

Theorem 5.9 (Global Harnack inequalities when $0 \le p < p_c$) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let u be a weak solution to $-\Delta u = \lambda u^p$, with $0 \le p < p_c$ with $u = 0$ on $\partial\Omega$. Let*

$$\mathcal{M}_p[u] = I_{\infty,p+1}(\Omega)\ \|u\|_{p+1}^{\frac{2(p+1)}{2(p+1)-d(p-1)_+}}$$

where $I_{\infty,p+1}(\Omega)$ is given by (3.3) , and assume

$$\delta \le \delta_1 := \min\left(\frac{\delta_0}{2(d-1)}, \frac{1}{\sqrt{\lambda \mathcal{M}_p^{p-1}[u]}}\right) \qquad \text{and, if } p \ge 1, \qquad \delta \le \min\left(\delta_1, \frac{1}{(2\delta_0)^{d-1}(2p\lambda\mathcal{M}_p^{p-1}[u])^d}\right).$$

Then

$$\frac{\mathcal{M}_p[u]}{\mathcal{H}_p^N 2^{d-2}\delta}\,\min\left\{\mathrm{dist}(x,\partial\Omega)\,,\, 2^{d-2}\delta\right\} \le u(x) \le \frac{2\mathcal{M}_p[u]}{\delta}\,\min\left\{\mathrm{dist}(x,\partial\Omega)\,,\,\frac{\delta}{2}\right\}\ \forall\ x \in \overline{\Omega}\,. \tag{5.29}$$

where $N = N(\Omega,\delta)$ is given in (5.14) while $\mathcal{H}_p$ is given by (7.1) or (7.3) when $0 \le p \le 1$ or when $1 < p < p_c$ respectively.

Proof. We recall the global upper bound (3.2), namely

$$\|u\|_\infty \le I_{\infty,p+1}(\Omega)\ \|u\|_{p+1}^{\frac{2(p+1)}{2(p+1)-d(p-1)_+}} = \mathcal{M}_p[u]\,, \tag{5.30}$$

where $I_{\infty,p+1}(\Omega)$ is given by (3.3). Then we recall the global Harnack inequality (5.9)

$$\mathcal{M}_p[u] = \sup_{x\in\Omega} u(x) = \sup_{x\in\Omega_{\delta/2}} u(x) \le \mathcal{H}_p^N \inf_{x\in\Omega_\delta} u(x) = \mathcal{H}_p^N \varepsilon\,, \qquad \text{for any } 0 < \delta \le \delta_0\,, \tag{5.31}$$

and $\mathcal{H}_p$ is given by (7.1) or (7.3) when $0 \le p \le 1$ or when $1 < p < p_c$ respectively. Just combine the bounds with inequality (5.8) of Lemma 5.4 to get (5.29). Recall that moreover we require $\delta < 1/\left(2p\,\lambda\, M^{(p-1)}\right)$ if $p \ge 1$. □

Theorem 5.10 (Comparing solutions for different values of p) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain, and let U_p be a weak solution to $-\Delta U_p = \lambda U_p^p$, with $0 < p < p_c$ with $u = 0$ on $\partial\Omega$. Let $\|U_p\|_{p+1} = 1$ and assume δ satisfies the conditions of Theorem 5.9. Then*

$$\frac{I_{\infty,p+1}(\Omega)}{\mathcal{H}_p^N I_{\infty,2}(\Omega)} \le \frac{U_p(x)}{U_1(x)} \le \frac{I_{\infty,p+1}(\Omega)}{I_{\infty,2}(\Omega)}\mathcal{H}_1^N \qquad \textit{for all } x \in \overline{\Omega}\,. \tag{5.32}$$

where $N = N(\Omega,\delta)$ is given in (5.14) , $I_{\infty,q}(\Omega)$ is given by (3.3) and $\mathcal{H}_p$ is given by (7.1) or (7.3) when $0 \le p \le 1$ or when $1 < p < p_c$ respectively.

Proof. Under the running assumptions, inequality (5.29) of Theorem 5.9 gives for any $0 < p < p_c$:

$$\frac{I_{\infty,p+1}(\Omega)}{\mathcal{H}_p^N}\,\min\left\{\frac{\mathrm{dist}(x,\partial\Omega)}{2^{d-2}\delta}\,,\,1\right\} \le u(x) \le I_{\infty,p+1}(\Omega)\,\min\left\{\frac{2\mathrm{dist}(x,\partial\Omega)}{\delta}\,,\,1\right\} \qquad \text{for all } x \in \overline{\Omega}\,. \tag{5.33}$$

from which (5.32) follows easily . □

Remark. Note that the constant $\mathcal{H}_p$ has two different expressions when $p \le 1$ and $p > 1$, but both expression are stable in p, in the sense that have finite limits when $p \to 1$, even if they can be different:

$$0 < \lim_{p \to 1^\pm} \mathcal{H}_p = \mathcal{H}_{1,\pm} < +\infty\,,$$

therefore we can assure that taking the limit as $p \to 1$ in inequality (5.32) give

$$\mathcal{H}_{1,\pm}^{-N} \le \lim_{p \to 1^\pm} \frac{U_p(x)}{U_1(x)} \le \mathcal{H}_{1,\pm}^{N} \qquad \text{for all } x \in \overline{\Omega}\,. \tag{5.34}$$

the constant $\mathcal{H}_{1,\pm}$ is not necessarily 1, but has an explicit expression given by (7.1) or (7.3). In any case this stability in p is needed in the next section in which we will prove that $U_p/U_1 \to 1$ uniformly in $\overline{\Omega}$.

6 Comparing solutions for different values of p and the limit as $p \to 1$

This section contains results that we needed and proved in [3]. We are not giving the proofs, that can be found in that reference with sufficient detail.

Let $1 \le p < p_s$ and let U_p be a weak solution to the elliptic problem

$$\begin{cases} -\Delta U = \lambda_p\, U^p & \text{in } \Omega \\ U > 0 & \text{in } \Omega \\ U = 0 & \text{on } \partial\Omega \end{cases} \tag{6.1}$$

where $\lambda_p > 0$ if $1 < p < p_s$ and $\lambda_p = \lambda_1$ for $p = 1$. We are interested in the relation between solutions of the elliptic equation for different values of $p \in [1, p_s)$, in particular we would like to see whether the limit $V := \lim_{p \to 1} U_p$ exists and under which conditions it is the ground state of the Dirichlet Laplacian Φ_1 on Ω. The existence of a limit depends on a normalization that we will discuss below.

It is well understood by subcritical semilinear theory that positive weak solutions of the above elliptic problem are indeed classical solutions up to the boundary. Notice that when $p = 1$ there is a positive solution, unique up to a multiplicative constant, while when $p > 1$ uniqueness is not always true, it depends on the geometry of the domain. The difficulty in understanding the limit of U_p as $p \to 1^+$, relies indeed in the lack of uniqueness and on a scaling property typical of the nonlinear problem. In the case of uniqueness, for example in the case when Ω is a ball, solutions are variational, in the sense that they are minima of a the functional $\|\nabla U\|_2^2$ under the restriction $\|U\|_{p+1} = 1$, but when the uniqueness is not guaranteed, solutions are just critical points of such functional.

One can also easily see that the constant $\lambda_p > 0$ in the nonlinear problem can be manipulated by rescaling, because if $U_{p,(1)}(x)$ is a solution with parameter $\lambda_{p,(1)}$, then $U_{p,(2)}(x) = \mu^{1/(p-1)}\, U_{p,(2)}(x)$ is a solution with parameter $\lambda_{p,(2)} = \mu \lambda_{p,(1)}$. In any normed space $\|U_{p,(2)}\| = \mu^{1/(p-1)} \|U_{p,(1)}\|$. This means that scaling allows to fix the norm of a solution: changing the norm by a factor $\mu^{1/(p-1)}$ by scaling is equivalent to changing λ_p in the equation by a factor μ^{-1}.

Assumption. Let us fix λ_p as the factor for which $\|U_p\|_{p+1} = 1$, so that, using U_p as test function, we obtain the following identity

$$\|\nabla U_p\|_2^2 = \lambda_p \|U_p\|_{p+1}^{p+1} = \lambda_p\,, \tag{6.2}$$

so that it is equivalent to prove that $\lambda_p \to \lambda_1$ or to prove that $\|\nabla U_p\|_2 \to \|\nabla \Phi_1\|_2$, when $p \to 1$. Recall that Φ_1 has unit L^2-norm.

We state now the main result of this section.

Theorem 6.1 ([3]). *Let U_p be a family of solutions of Problem 6.1 with $p \in [1, p_s)$, $\|U_p\|_{p+1} = 1$ and let $\lambda_p > 0$ be chosen according to (6.2). Then as $p \to 1$, $\lambda_p \to \lambda_1$, $U_p \to \Phi_1$ in $\mathrm{L}^\infty(\Omega)$, $\nabla U_p \to \nabla \Phi_1$ in $\left(\mathrm{L}^2(\Omega)\right)^d$. Besides, there exist two explicit constants $0 < c_0 < c_1$ such that*

$$c_0^{p-1}\lambda_1 \le \lambda_p \le c_1^{p-1}\lambda_1 \,. \tag{6.3}$$

Moreover, there exists constants $0 < \widetilde{k}_0(p) \le \widetilde{k}_1(p)$ such that $\widetilde{k}_i(p) \to 1$ as $p \to 1^+$, such that

$$\widetilde{k}_0(p) \le \frac{U_p(x)}{\Phi_1(x)} \le \widetilde{k}_1(p), \qquad \text{for all } x \in \overline{\Omega}\,. \tag{6.4}$$

The delicate proof of such a result can be found in [3]. We just recall that a crucial ingredient in such a proof is a delicate comparison argument joined with following well-known result:

Proposition 6.2 *The following Hardy-type inequality holds true whenever Ω has a finite inradius and satisfies a uniform exterior ball condition*

$$\left\|\frac{f}{\Phi_1^r}\right\|_q \le H_{r,d}\|\nabla f\|_2 \qquad \text{if } f \in W_0^{1,2}(\Omega),\ 0 < q \le \frac{2d}{d-2+2r}\,, \text{ and } 0 \le r \le 1\,. \tag{6.5}$$

where Φ_1 is the unique positive ground state of the Dirichlet Laplacian on Ω, and $H_{r,d}$ is a suitable positive constant that depends only on r, d and $|\Omega|$.

6.1 Additional bounds on λ_p

We shall also prove suitable lower bounds for λ_p. These bounds are easier to obtain than the upper bounds.

(i) Using U_p as test function, we obtain the global energy equality $\lambda_p\|U_p\|_{p+1}^{p+1} = \|\nabla U_p\|_2^2$, that combined with the Sobolev inequality

$$\|f\|_{p+1} \le |\Omega|^{\frac{1}{p+1}-\frac{1}{2^*}}\|f\|_{2^*} \le |\Omega|^{\frac{1}{p+1}-\frac{1}{2^*}}\mathcal{S}_2\|\nabla f\|_2$$

gives, recalling that we have chosen λ_p in such a way that $\|U_p\|_{p+1} = 1$,

$$\frac{1}{|\Omega|^{\frac{2}{p+1}-\frac{2}{2^*}}} = \frac{\|U_p\|_{p+1}^2}{|\Omega|^{\frac{2}{p+1}-\frac{2}{2^*}}} \le \left[\int_\Omega U_p^{2^*}\,\mathrm{d}x\right]^{\frac{2}{2^*}} \le \mathcal{S}_2^2\|\nabla U_p\|_2^2 = \mathcal{S}_2^2\lambda_p\|U_p\|_{p+1}^{p+1} = \mathcal{S}_2^2\lambda_p\,.$$

We can rewrite the lower bound as follows

$$\frac{1}{\mathcal{S}_2^2\,|\Omega|^{\frac{2}{p+1}-\frac{2}{2^*}}} \le \lambda_p \qquad \text{and for } p \to 1 \qquad \frac{1}{\mathcal{S}_2^2\,|\Omega|^{1-\frac{2}{2^*}}} \le \lambda_1\,. \tag{6.6}$$

(ii) Other lower bounds can be obtained by combining Hölder, Poincaré and Sobolev inequalities:

$$\|U_p\|_{p+1}^2 \le \|U_p\|_{2^*}^{2\vartheta}\|U_p\|_2^{2(1-\vartheta)} \le \frac{\left(\lambda_1\mathcal{S}_2^2\right)^\vartheta}{\lambda_1}\|\nabla U_p\|_2^2 \qquad \text{with} \qquad \vartheta = \frac{d(p-1)}{2(p+1)}$$

which gives

$$\lambda_p = \int_\Omega |\nabla U_p|^2\,\mathrm{d}x \ge \frac{\lambda_1}{\left(\lambda_1\mathcal{S}_2^2\right)^\vartheta}\|U_p\|_{p+1}^2 = \lambda_1\left(\lambda_1\mathcal{S}_2^2\right)^{-\frac{d(p-1)}{2(p+1)}} \tag{6.7}$$

since we have chosen λ_p in such a way that $\|U_p\|_{p+1} = 1$.

The case of variational solutions. Other estimates for λ_p can be easily obtained in the case in which solutions are minima of a suitable functional, this happens for instance in the case of domains Ω for which the solution is unique, hence they are minima, since a solution which is a minima always exists as a consequence of Kondrachov's compactness theorem.

When the solution of the Elliptic problem 6.1 are minima of a suitable functional, namely when we consider the homogeneous functional

$$J_p[u] = \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\left(\int_\Omega u^{p+1} \, \mathrm{d}x\right)^{\frac{2}{p+1}}}$$

defined on $W_0^{1,2}(\Omega)$, and we seek for its minimum under the restriction $\|u\|_{p+1} = 1$, we can define

$$\lambda_p = \inf_{u \in X_p} J_p[u] = \inf_{u \in X_p} \int_\Omega |\nabla u|^2 \, \mathrm{d}x \qquad \text{where} \qquad X_p = \left\{ u \in W_0^{1,2}(\Omega) \;\big|\; \|u\|_{p+1} = 1 \right\} .$$

Let $U_p \in X_p$ be a solution to the elliptic problem 6.1 with λ_p defined as above. Estimates in this case are simpler and hold for any $1 \le p < p_s$.

Proposition 6.3 ([3]). *Under the above assumptions, if U_p is a minimum for the functional J_p on the set X_p, then it is a positive weak (hence classical) solution to the elliptic Problem* 6.1*. Moreover the following estimates hold*

$$(\mathcal{S}_2 \lambda_1)^{-\frac{d(p-1)}{2(p+1)}} \le \frac{\lambda_p}{\lambda_1} = \frac{\inf\limits_{u \in X_p} J_p[u]}{\inf\limits_{u \in X_1} J_1[u]} \le |\Omega|^{\frac{p-1}{p+1}} \tag{6.8}$$

where λ_1 is the first eigenvalue of the Dirichlet Laplacian on Ω, and $\mathcal{S}_2$ is the constant on the Sobolev imbedding from $W_0^{1,2}(\Omega)$. As a consequence, $\lambda_p \to \lambda_1$ as $p \to 1^+$.

Proof. It is a standard fact in calculus of variations to see that a minimum of J_p is a weak solution to the elliptic problem under consideration. We can now prove the upper estimate:

$$\lambda_p = \inf_{u \in X_p} \int_\Omega |\nabla u|^2 \, \mathrm{d}x = \inf_{u \in W_0^{1,2}(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}x}{\left(\int_\Omega u^{p+1} \, \mathrm{d}x\right)^{\frac{2}{p+1}}} \le \frac{\int_\Omega |\nabla \Phi_1|^2 \, \mathrm{d}x}{\left(\int_\Omega \Phi_1^{p+1} \, \mathrm{d}x\right)^{\frac{2}{p+1}}} \le \lambda_1 \, |\Omega|^{\frac{p-1}{p+1}}$$

if we moreover assume $\|\Phi_1\|_2 = 1$ (not restrictive). We have just used the fact that $\Delta\Phi_1 = \lambda_1 \Phi_1$ together with Hölder inequality $\|\Phi_1\|_2^2 \le |\Omega|^{\frac{p-1}{p+1}} \|\Phi_1\|_{p+1}^2$. The lower estimates are exactly the same as (6.7). $\square$

Remark. The above considerations and quantitative estimates can be easily extended to the case when $0 < p < 1$, in which case the solution is well known to be unique, hence variational.

7 Appendix

7.1 Values of the constants of the local bounds

We present there the values of the constants in the local bounds of Section 4 , which have been calculated in [4]. In the proofs of the results of this paper, we have always taken R_∞ as a multiple of R_0, therefore eliminating the dependence on R_0 and R_∞ in the constants listed below.

The Harnack constant when $0 \le p \le 1$

$$
\begin{aligned}
\mathcal{H}_p = &\left[\frac{2^d \mathcal{S}_2^4 R_0^2}{(R_0 - R_\infty)^2}\left(\frac{dR_0^2}{(R_0-R_\infty)^2} + \frac{R_0^2}{R_\infty^2}\right)\right]^{\frac{d}{2q_0}} \\
&\times \left[\frac{2^d\left(\left(\frac{d}{d-2}\right)^{n_0-\frac12}\frac{2^{\frac{d-3}{2}}}{d\omega_d^2} + \mathrm{e}\right)\sqrt{\omega_d}}{\left(\frac{d}{d-2}\right)^{n_0-\frac12}\frac{2^{\frac{d-3}{2}}}{d\omega_d^2} - e(d-1)}\right]^{\frac{2}{q_0}} \times \left\{\left(\frac{d}{d-2}\right)^d \frac{2(d-2)\sqrt{d}}{\left(\sqrt{d}-\sqrt{d-2}\right)^3}\right. \\
&\times \left.\left[\Lambda_p + \frac{d-2}{q_0} + \frac{(R_0-R_\infty)^2}{R_\infty^2}\max\left\{\frac{d-2}{(dq_0)^2}|dq_0 - (d-2)|,\, \frac14\right\}\right]\right\}^{\frac{d}{2q_0}}
\end{aligned}
\tag{7.1}
$$

with

$$
q_0 = \left(\frac{d-2}{d}\right)^{n_0-\frac12} \qquad \text{and} \qquad n_0 = i.p.\left[\frac{\log\left(\mathrm{e}(d-1)\frac{d\omega_d^2}{2^{\frac{d-3}{2}}}\right)}{\log\frac{d}{d-2}} + \frac32\right]
\tag{7.2}
$$

The Harnack constant when $1 < p < p_c$

$$
\mathcal{H}_p = I_{\infty,\overline{q}}\left(\frac{I_{\overline{q},\underline{q}}}{I_{-\infty,\underline{q}}}\right)^{\frac{2\overline{q}}{2\overline{q}-d(p-1)}}, \qquad \text{with} \qquad \frac{d(p-1)}{2} < \overline{q} < \frac{d}{d-2}
\tag{7.3}
$$

where the constants $\underline{q} \in (0, q_0 \wedge \overline{q}]$, q_0 and $I_{-\infty,\underline{q}}$ are given in (7.7), $I_{\overline{q},\underline{q}}$ is given by (7.8), (7.9), $I_{\infty,\overline{q}}$ is given by (7.4); moreover, since $\overline{q} < d/(d-2)$ we require the additional condition (7.6).

The upper bound constant in the local case.

$$
\begin{aligned}
I_{\infty,q} = &\left[\frac{c_1 \mathcal{S}_2^2 \omega_d^{\frac{2(p-1)_+}{d(p-1)}}}{(1-\rho)^2}\right]^{\frac{d}{2q-d(p-1)_+}} \left\{\left(\frac{d}{d-2}\right)^d \frac{2(d-2)}{\left(\sqrt{d}-\sqrt{d-2}\right)^2}\right. \\
&\times\left.\left[\Lambda_p + \frac{d-2}{q} + (1-\rho)^2\max\left\{\frac{d-2}{(dq)^2}|dq-(d-2)|,\, \frac14\right\}\right]\right\}^{\frac{d}{2q-d(p-1)_+}}
\end{aligned}
\tag{7.4}
$$

where $\rho = R_\infty/R_0 < 1$ and we have used the convention $x_+/x = 0$ when $x = 0$ and, moreover, we have set $\Lambda_p = 2$ if $p \neq 1$, $\Lambda_p = \lambda/4$ if $p = 1$, with

$$
c_1 := \begin{cases}
\frac{(d-2)q}{(d-2)q-d} & \text{if } q > \frac{d}{d-2} \\
\max\limits_{i=0,1} \frac{\left(\frac{d}{d-2}\right)^{k_0-1+i}\left[q-\frac{d(p-1)_+}{2}\right]+(p-1)_+\frac{d-2}{2}}{\left|\left(\frac{d}{d-2}\right)^{k_0-1+i}\left[q-\frac{d(p-1)_+}{2}\right]+(p-1)_+\frac{d-2}{2}-1\right|} & \text{if } 0 < q < \frac{d}{d-2}.
\end{cases}
\tag{7.5}
$$

(iii) When q also satisfies $0 < q < d/(d-2)$, we will require in the proof the additional condition

$$
\frac{\log\frac{2^*-d(p-1)_+}{2q-d(p-1)_+}}{\log\frac{d}{d-2}} \quad \text{is not an integer, and} \quad k_0 := i.p.\left[\frac{\log\frac{2^*-d(p-1)_+}{2q-d(p-1)_+}}{\log\frac{d}{d-2}}\right],
\tag{7.6}
$$

($i.p.$ is the integer part of a real number). Notice that taking $q = p+1 > d(p-1)/2$ is possible if and only if $p < p_s = (d+2)/(d-2)$. In any case this condition is not essential as explained in [4], but it is needed to get a clean expression of the constant.

The lower bound constants. We let

$$(7.7)\qquad \begin{aligned} &0\le \underline{q}\le q_0 := \frac{2^{\frac{d-3}{2}}}{d^2\omega_d^2 \mathrm{e}}\\ &I_{-\infty,\underline{q}} = \left[2^d \mathcal{S}_2^2\left(\frac{dR_0^2}{(R_0-R_\infty)^2}+\frac{R_0^2}{R_\infty^2}\right)\right]^{-\frac{d}{2\underline{q}}}\left[\frac{\mathrm{e}}{2^d\,\mathrm{e}\,(d+1)\sqrt{\omega_d}}\right]^{\frac{2}{\underline{q}}}.\end{aligned}$$

Moreover, if $\frac{d-2}{d}\overline{q}\le \underline{q}\le\overline{q}$ we let

$$(7.8)\qquad I_{\overline{q},\underline{q}} := \left[\frac{2d\,\overline{q}\,\mathcal{S}_2^2}{(2^*-2\overline{q})}+\mathcal{S}_2^2\frac{(R_0-\overline{R})^2}{\overline{R}^2}\right]^{\frac{2^*}{2\overline{q}}}\left[\frac{\omega_d^{1/d}R_0}{R_0-\overline{R}}\right]^{\frac{2^*}{\overline{q}}}\left[\frac{R_0}{\overline{R}}\right]^{\frac{d}{\overline{q}}},$$

while when $0<\underline{q}<\frac{d-2}{d}\overline{q}$, with q_0 as in (7.7), we let

$$(7.9)\qquad I_{\overline{q},\underline{q}} = 3\cdot 2^{\frac{(d-2)\overline{q}}{2\underline{q}}-\frac{d}{2}}\left[\frac{2d\,\overline{q}\,\mathcal{S}_2^2}{(2^*-2\overline{q})}\frac{\overline{R}^2}{(R_0-\overline{R})^2}+\mathcal{S}_2^2\right]^{\frac{\overline{q}-\underline{q}}{\overline{q}\,\underline{q}}\frac{d}{2}}\left(4\omega_d^{\frac{1}{d}}\frac{\overline{q}-\underline{q}}{\underline{q}\overline{q}}\right)^{\frac{d}{\underline{q}}-\frac{d}{\overline{q}}}\left[\frac{R_0}{\overline{R}}\right]^{\frac{d}{\overline{q}}}.$$

7.2 A numerical Lemma

Lemma 7.1 *The following inequality holds for any $a,b\ge 0$ and for any $p\ge 1$:*

$$(7.10)\qquad (a-b)(a^p-b^p)\le p\,\max\left\{a^{p-1},b^{p-1}\right\}(a-b)^2$$

Moreover the following inequality holds for any $a,b\ge 0$ and $p\ge 1$:

$$(7.11)\qquad a^p-b^p\ge p\,b^{p-1}(a-b).$$

Proof. If $a\ge b$ the validity of (7.10) is equivalent, setting $x=\frac{b}{a}$, to the validity of $(1-x)(1-x^p)\le p(1-x)^2$ for all $x\in[0,1]$, that is to $1-x^p\le p(1-x)$ for all $x\in[0,1]$, which does in fact hold by the concavity of $g(x):=1-x^p$, since the line $h(x):=p(1-x)$ is the tangent to g at $x=1$. The case $a<b$ follows as well by interchanging the role of a and b.

The second inequality (7.11) follows by the inequality $x^p-1\ge p(x-1)$ for all $x\ge 0$ which is valid since x^p-1 is convex so that its graph lies above its tangent at $x=1$. □

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