Positivity, local smoothing and Harnack inequalities for very fast diffusion equations

Dedicated to Luis Caffarelli for his upcoming 60th birthday

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Abstract

We investigate qualitative properties of local solutions $u(t,x)\ge 0$ to the fast diffusion equation, $\partial_t u=\Delta(u^m)/m$ with $m<1$, corresponding to general nonnegative initial data. Our main results are quantitative positivity and boundedness estimates for locally defined solutions in domains of the form $[0,T]\times\mathbb{R}^d$. They combine into forms of new Harnack inequalities that are typical of fast diffusion equations. Such results are new for low m in the so-called very fast diffusion range, precisely for all $m\le m_c=(d-2)/d$. The boundedness statements are true even for $m\le 0$, while the positivity ones cannot be true in that range.

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Introduction

We study qualitative and quantitative properties of solutions $u = u(t, x)$ of the nonlinear diffusion equation

$$\partial_t u = \nabla \cdot (u^{m-1} \nabla u) = \Delta(u^m/m) \tag{0.1}$$

in the whole parameter range $-\infty < m < 1$, where it is called Fast Diffusion Equation (FDE). We consider local nonnegative weak solutions, defined in an open cylinder Q of space-time $\mathbb{R} \times \mathbb{R}^d$ with $d \geq 1$. Note that the factor $1/m$ in the last expression is inessential when $m > 0$ (up to a time rescaling, $t' = t/m$) but becomes essential for $m < 0$, in order to obtain a parabolic equation; for $m = 0$ the last expression has to be written as $\partial_t u = \Delta \log(u)$[1].

Assuming the basic existence and uniqueness theory, [14], [31], we are interested in the qualitative properties of the solutions such as boundedness, positivity, and Harnack inequalities. For the FDE these properties depart from the properties of the linear Heat Equation (case $m = 1$), [34], and even more from the Porous Medium Equation (case $m > 1$), [32]. Moreover, they are still partially understood when m is far from 1, precisely for $m \leq m_c$ where $m_c = (d-2)/d$ is called the *first critical fast diffusion exponent*. Our goal here is to obtain bounds from above and below for the solutions in that low range of exponents. We look for precise quantitative versions based on local estimates. Such estimates should be of interest in developing a general theory of this equation in the detail that is already known both for $m \geq 1$ and for $m_c < m < 1$.

Precedents and problems

The existence and uniqueness of weak solutions of the initial value problem and other standard initial and boundary value problems for the FDE, as well as the main qualitative properties of the solutions (such as the ones already mentioned, or the asymptotic behaviour), are by now well understood when m is close to one, more precisely in the so-called *good parameter range:* $m_c < m < 1$.[2] To be specific, when the problem is posed in the whole space, weak solutions are uniquely determined by their initial data if u_0 is a locally integrable nonnegative function, or even a locally finite Radon measure. In that case, the solution is C^∞ smooth and positive for all $x \in \mathbb{R}^d$ and $t > 0$, and the initial data are taken in the sense of initial trace, [24], [27], [14]. Solutions are bounded for data $u_0 \in L^p(\mathbb{R}^d)$ for any $p \geq 1$, and even for data in the Marcinkiewicz spaces $M^p(\mathbb{R}^d)$, $p > 1$, [31]. They are locally bounded under the very mild restriction that u_0 is Radon measure, even if it is not globally finite.

The theory of the FDE has been much less studied until recently in the *subcritical fast-diffusion range* $m < m_c$, even under the condition $m > 0$, since essential difficulties have been found in the different chapters of the theory, like existence, uniqueness, and regularity. Note that $0 < m < m_c$ is possible only if $d > 2$. We refer for background to the book [31] that discusses in some detail the range $m \leq m_c$, even for $m \leq 0$, along with the cases $m > m_c$. Let us give an idea of the difficulties that arise and that we address in our work below:

BOUNDEDNESS. Though weak solutions with data in the spaces $\mathrm{L}^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, exist and are unique for $0 < m < 1$, counterexamples show that for $m < m_c$ these weak solutions need not be

[1] We will always interpret u^m/m as $\log(u)$ when $m = 0$. In the whole paper, ∇ indicates the gradient operator, $\nabla\cdot$ the divergence operator, and Δ the Laplacian operator, all of them taken with respect to the space variables, $x \in \mathbb{R}^d$.

[2] With the extra restriction $m > 0$ if $d = 1$, the case $-1 < m < 0$ and $d = 1$ being somewhat different, cf. [31].

bounded, and as a consequence they are not smooth. The simplest such example seems to be the separate-variables function

$$U(t,x;T,x_0)=c\,\frac{(T-t)^{1/(1-m)}}{|x-x_0|^{2/(1-m)}} \tag{0.2}$$

For every $m<m_c$, even $m\le 0$, there exists a suitable constant $c(m,d)>0$ such that U is a weak solution of the FDE in the cylinder $Q=(0,T)\times\mathbb{R}^d$, cf. [31], page 80, but obviously the solution never improves its initial regularity until it extinguishes in finite time. The precise space regularity is $U(\cdot,t)\in \mathrm{L}^p_{loc}(\mathbb{R}^d)$ for all $p<p_c$, where the critical integrability exponent is $p_c=d(1-m)/2$, which is larger than 1 precisely for $m<m_c$, i.e., in the subcritical range.

There is a positive result concerning boundedness, that is also tied to the exponent p_c : solutions with initial data in $\mathrm{L}^p(\mathbb{R}^d)$ with $p>p_c$ become bounded and C^∞ smooth for all positive times as long as the solution does not disappear. This *smoothing effect* happens for all $p\ge 1$ if $m>m_c$, for $p>1$ if $m=m_c$ (in the last cases there is no problem of disappearance). The results are sharp, cf. [31].

Extinction in finite time, EFT. The above example exhibits another typical feature of the Cauchy problem for $m<m_c$, namely, the possible lack of positivity due to EFT. The occurrence of EFT depends on the type of problem we consider.

In the case of the Cauchy problem posed in $\mathbb{R}^d$ with $d\ge 3$, Bénilan and Crandall gave in [3] a proof of the extinction in finite time, EFT, of solutions of the FDE in the range $0<m<m_c$ when $u_0\in\mathrm{L}^p(\mathbb{R}^d)$ with $p=p_c$. It is proved in [31] that EFT occurs for the solutions with $m<m_c$ for all functions with initial data in the Marcinkiewicz space $M^{p_c}(\mathbb{R}^d)$, hence in $\mathrm{L}^{p_c}(\mathbb{R}^d)$. We recall the EFT does not happen for the Cauchy Problem when $m>m_c$.

In the case of the Cauchy-Dirichlet problem posed in a bounded domain with zero boundary data, EFT happens for all $0<m<1$. There is an interesting functional connection: we can show that EFT occurs if we have a global Poincaré and a Sobolev inequality, and this result can be extended to more general settings, such as Riemannian manifolds, as it has been done by the authors in [7]. On the other hand, Bénilan and Crandall's proof for the Cauchy problem is based only on the Sobolev inequality, but it holds only in the lower range $m<m_c$.

Harnack inequalities. Concerning finer regularity properties, the possible occurrence of EFT is compatible with the fact that nonnegative bounded solutions are positive, and consequently C^∞ smooth, as long as they are not identically zero, i.e., before extinction. However, the existence of EFT for low m is tied to the breakdown of the standard forms of Harnack inequalities, which are a strong tool in developing a regularity theory. Obtaining some kind of Harnack inequality is therefore a main research issue for $m\le m_c$ and has been an open problem for some years. More specifically, we concentrate on parabolic lower Harnack inequalities of the type called Aronson-Caffarelli estimates [1], and examine their consequences to obtain quantitative forms of positivity. An extension work has been done in [8] for $m_c<m<1$ but the method collapses for $m\le m_c$ due to the very different properties of the solutions. As a consequence of our local smoothing effect and of positivity estimates, we will obtain some intrinsic Harnack inequalities of forward, elliptic or backward type, which are new in this range.

In a recent preprint [18], DiBenedetto, Gianazza and Vespri study the validity of intrinsic Harnack inequalities in the good range $m>m_c$ and show, with an explicit counterexample, that any kind of Harnack Inequality, intrinsic, elliptic, backward and forward can not hold if $m<m_c$, for a fixed size of the intrinsic cylinder, that is, if we fix the size of the parabolic cylinder "a priori" in terms of the

value of u at the center of the cylinder (t_0, x_0). They leave as an open problem to find which kind of Harnack Inequalities, if any, are typical of the very fast diffusion range $0 < m < m_c$. In this paper we give an answer to this intriguing problem.

Very singular range. Most the literature has avoided the cases $m \le 0$, where the diffusivity $D(u) = u^{1-m}$ is very singular at $u = 0$. Recently, it has been shown that a large part of the theory of the subcritical range goes over to this very singular range, on the condition of working with solutions that "are not too small". See [30] among the older references, then [13], and the books [14], [31] for a more complete reference. Note that this recovers a subcritical range for dimensions $d = 1, 2$, and also that we can study the interesting log-diffusion problems where $m = 0$, cf. [23], [33] and the references.

More specifically, there is an extension of the results called smoothing effects, whereby data in $\mathrm{L}^p(\mathbb{R}^d)$ with $p > p_c$ imply bounded solutions for all $t > 0$, and also the extinction in finite time for data in $M^p(\mathbb{R}^d)$, $p = p_c$. But a very different situation happens for data in $\mathrm{L}^p(\mathbb{R}^d)$ with $1 \le p < p_c$, which is called *immediate extinction*, whereby the solutions obtained as limit of any reasonable approximation are identically zero for all $t > 0$. This makes it difficult to think of a general study of positivity. Immediate extinction happens for the Cauchy-Dirichlet problem posed in a bounded domain with zero boundary data for all $m \le 0$, $d \ge 1$. Our study of this range is confined therefore to upper estimates.

Comparison with elliptic problems. Part of the difficulties of the FDE in the lower range of m can be explained by the intimate relation of the equation with semilinear elliptic theory. This remarkable connection will be briefly explained in Subsection 4.3.

Results and organization

Our work focuses on a peculiar feature of the FDE, which is the existence of very strong local estimates. This was presumably first mentioned in the paper by Herrero and Pierre [24], 1985, who get solutions in the whole range $0 < m < 1$ under the sole condition on the initial data $u_0 \in \mathrm{L}^1_{loc}(\mathbb{R}^d)$. Much of the subsequent work has been influenced by the local character of the equation. Here, we want push this idea to its final consequence concerning two different areas: the question boundedness of local solutions, and the question of positivity of nonnegative solutions, measured quantitatively by so-called lower Harnack inequalities. We will then combine the local upper and lower estimates, into a full form of Harnack inequality. While the boundedness results hold for all $m < 1$, positivity estimates are confined to $0 < m < 1$ because of the possible occurrence of immediate extinction. As we have said, the main interest of our results lies in their application in the subcritical range, $m < m_c$. They are also new for the critical exponent $m = m_c$.

Let us be more specific about the contents of the paper. It is divided into three main parts.

(I) The study of positivity and *lower Harnack inequalities,* both of local and global type. The first main contribution of the paper is a *parabolic lower Harnack inequality of the Aronson-Caffarelli type* that is presented in Section 1, along with a detailed comparison with the forms available for other ranges of m. We devote Subsection 1.1 to prove the lower estimate, Theorem 1.1, for a minimal problem. This is extended in Subsection 1.2 to general solutions. We then show that in the range $m_c < m < 1$ we can further eliminate the presence of the extinction time and recover stronger estimates that are known in that range. Subsection 1.4 discusses upper bounds for the extinction time T in terms of L^p norms of the data, which give an alternative type of lower bound in the range where estimates depending only on L^1-norms of the data are not true.

(II) The study of *local upper bounds.* This takes two forms: the first is the control of the evolution in time of some spatial $\mathrm{L}^p_{\rm loc}$ norms, which is performed in Section 2.1. Then, we get a local in space-time version of the smoothing effect from $\mathrm{L}^p_{\rm loc}$ into $\mathrm{L}^\infty_{\rm loc}$, an important regularity result that opens the road to higher regularity and was known for $m > m_c$, and is false in general for $m \le m_c$. We show in this paper that the estimate holds $m < m_c$, on the condition that p must be large enough. We finally obtain the finest local upper estimates, called local smoothing effects, in the form given in Theorem 2.1 , just by combining the space-time smoothing effect and the $\mathrm{L}^p_{\rm loc}$ obtained in the first Section 2.1.

(III) Parabolic Harnack Inequalities. In Section 3, we combine the local upper and lower estimates obtained in Parts I and II in the form of parabolic Harnack inequalities of forward, backward and elliptic type, together with an alternative form.

To conclude, we sketch a panorama of the obtained local estimates depending on the ranges of m, together with general remarks, some related open problem and a short review on related works. A final Appendix contains some useful technical results.

Notations. We will work with weak solutions $u \ge 0$ of the FDE with $m < 1$, defined in a cylinder $Q = \Omega \times (T_0, T_1)$ for some domain $\Omega \subset \mathbb{R}^d$ and $T_0 < T_1$. Usually, we take $T_0 = 0, T_1 = T$. T_1 can be infinite and Ω can be the whole space. In view of existing theory we may assume that the solutions are positive and smooth as long as they do not extinguish identically. We will be mostly interested in the local theory where the space domain is bounded and the boundary conditions are not taken into account. In deriving local estimates it will be often sufficient to take as space domain a ball, which we will denote by $B = B_R(x_0)$ or $B = B_{\lambda R}(x_0)$ for some $\lambda > 1$. We will frequently consider the annulus region $A_{R,\lambda} = B_{\lambda R} \setminus B_R$. As indicated before, we put

$$m_c = \frac{d-2}{d}, \qquad p_c = \frac{d(1-m)}{2}\,.$$

We have pointed out that $p_c > 1$ if and only if $m < m_c$. We will take integrability exponents $p \ge 1$ if $m > m_c$, $p > p_c$ if $m \le m_c$. Moreover, for $p \ne p_c$ we set

$$\vartheta_p = \frac{1}{2p - d(1-m)}\,, \tag{0.3}$$

which is positive if and only if $p > p_c$.

1 Part I. Local lower bounds

The first part of the paper addresses the question of quantitative estimates of positivity. The exponent range in this part is $0 < m < 1$, since it is well known that the FDE does not admit solutions of the Dirichlet problem with zero boundary data when $m \le 0$, thus blocking any possibility of a general local positivity theory in that range [30, 31]. Our main contribution is a parabolic inequality in the spirit of the one obtained by Aronson and Caffarelli [1] in their path-breaking paper for $m > 1$, and the ones produced by the authors in [8] for $m_c < m < 1$. The purpose of such formulas is giving quantitative information on the positivity of solutions at later times in terms of information on L^p norms of u at a former time that we take as $t = 0$. This is why they are called parabolic lower Harnack formulas.

We take $0 < m < 1$ and consider a u be a local, nonnegative weak solution of the FDE defined in a cylinder $Q = (0,T) \times \Omega$, taking initial data $u(0,x) = u_0(x)$ in Ω and having finite extinction time T. We make no assumption on the boundary condition (apart from nonnegativity). For ease of proof we will assume that the solutions are smooth so that the different computations and comparison results are valid. This assumption is then eliminated by approximation, which is justified according to known theory.

Theorem 1.1 *Let $0 < m < 1$ and let u be the solution to the FDE under the above assumptions. Let x_0 be a point in Ω and let $d(x_0, \partial\Omega) \geq 3R$. Then the following inequality holds for all $0 < t < T$*

$$R^{-d} \int_{B_R(x_0)} u_0(x)\,\mathrm{d}x \leq C_1\, R^{-2/(1-m)}\, t^{\frac{1}{1-m}} + C_2\, T^{\frac{1}{1-m}} R^{-2}\, t^{-\frac{m}{1-m}}\, u^m(t, x_0). \tag{1.1}$$

with C_1 and C_2 given positive constants depending only on d. This implies that there exists a time t_* such that for all $t \in (0, t_*]$

$$u^m(t, x_0) \geq C_1'\, R^{2-d} \|u_0\|_{\mathrm{L}^1(B_R)} T^{-\frac{1}{1-m}}\, t^{\frac{m}{1-m}}. \tag{1.2}$$

where $C_1' > 0$ depends only on d; t_* depends on R and $\|u_0(x)\|_{\mathrm{L}^1(B_R)}$ but not on T.

Simplified version. The dependence on the parameters makes the formula apparently complicated. But it can be reduced to a simpler, equivalent one. Actually, we may assume that $x_0 = 0$ by translation. Given $R > 0$ and $M = \int_{B_R(0)} u_0(x)\,dx > 0$, we use the rescaling

$$u(t, x) = \frac{M}{R^d}\, \widehat{u}\left(\frac{t}{\tau}, \frac{x}{R}\right), \qquad \tau = R^{2-d(1-m)} M^{1-m}, \tag{1.3}$$

to pass from a solution with mass M in the ball of radius R to a solution $\widehat{u}$ with mass 1 in the ball of radius 1. So we only need to prove the version with $M = R = 1$ to get the full version. The scaling is simpler for $m = m_c$ where $\tau = M^{1-m}$. Of course, the extinction time has to be rescaled accordingly, $T = R^{2-d(1-m)} M^{1-m} \widehat{T}$.

Improvements. As stated, estimate (1.1) applies only to solutions with finite extinction time, and it involves the value of the extinction time T in an explicit way; both things can make it impractical. However, a simple comparison argument shows that we only need to estimate from below any subsolution. In particular, we may replace the solution under consideration by the solution of the problem with initial data $u_0(x)\chi_{B_R(x_0)}(x)$, and zero Dirichlet boundary conditions on $x \in \partial B_{3R}(x_0)$. Let us call this problem *minimal problem* for the given data. The extinction time of the corresponding solution will be called the *minimal life time* of such domain and data, $T_m(u_0, B)$. Clearly, $T_m(u_0, B) \leq T(u)$.

Corollary 1.2 *The above positivity result holds with $T(u)$ replaced by the minimal life time $T_m(u_0, B)$, u is defined in Q_T, and the estimate applies for $0 < t < T'$ with $T' = \min\{T, T_m\}$.*

This modified result is specially interesting in the range $1 > m > m_c$ where the solutions of the Cauchy Problem do not vanish. On the other hand, it is known that T_m is finite if u_0 satisfies some local integrability conditions [16, 32].

Comparison with the estimate for the PME and other FDE

THE PME. Let us write Aronson-Caffarelli's result [1] for $m > 1$ with a similar notation:

$$R^{-d} \int_{B_R(x_0)} u_0(x)\,\mathrm{d}x \le C_1\, R^{2/(m-1)}\, t^{-\frac{1}{m-1}} + C_2\, R^{-d} t^{d/2} u^{1+(d(m-1)/2)}(t, x_0). \tag{1.4}$$

We recall that this formula is valid for all nonnegative weak solutions of the PME defined in the whole space. The form of the first term in the right-hand side is the same in both results, (1.1) and (1.4). This term plays the role of blocking the positivity information when it is large relative to the left-hand side integral, and allowing for such information when it is small. The critical time at which we begin to get positivity information is obtained by making this term a fraction of the left-hand side, i.e., for

$$t_c = c(m,d) \|u_0\|^{1-m}_{\mathrm{L}^1(B_R(x_0))} R^{2+d(m-1)}. \tag{1.5}$$

But since the exponents have just the opposite sign in the above expressions for $m > 1$ and $m < 1$, the consequences are qualitatively very different: the information on positivity happens for us when t is smaller than t_*, while for the PME it happens when t is larger. This is in accord with the basic properties of these equations, which the present inequalities faithfully reproduce. Rescaling allows to check the inequality only at $t = 1$ for $R = 1$, and in that case we only have to prove that there are constants $M_0 = M_0(n, m)$ and $k = k(m, d)$ such that for $M \geqslant M_0$

$$u(0,1) \geqslant k\, M^{2/(d(m-1)+2)}. \tag{1.6}$$

As to the second term, it is different. We cannot expect to have the A-C term in the range $m < m_c$ since then the exponent of u would be negative. In fact, the proof of [1] uses conservation of mass that is not valid for the fast diffusion equation in the low m range.

THE GOOD FDE. The validity of the Aronson-Caffarelli formula was extended by the authors in [8] to local solutions of the FDE in the good exponent range $m_c < m < 1$, and the already mentioned sign change in the exponents implies that we get good lower estimates for $0 < t \le t_*$. Moreover, we can continue these estimates thanks to the fortunate circumstance that we have further differential inequalities, like $\partial_t u \ge -Cu/t$ in the case of the Cauchy problem, which allow for a continuation of the lower bounds for $t \ge t_*$ with optimal decay rates in time. The final form is

$$u(t,x) \ge \overline{M}_R(x_0)\, H(t/t_c), \quad \overline{M}_R(x_0) = R^d \int_{B_R(x_0)} u_0\, dx. \tag{1.7}$$

The critical time is defined as in (1.5); the function $H(\eta)$ is defined as $K\eta^{1/(1-m)}$ for $\eta \le 1$ while $H(\eta) = K\eta^{-d\vartheta}$ for $\eta \ge 1$, with $K = K(m,d)$. Note that for $0 < t < t_c$ the lower bound means $u(t, x_0) \ge k(m,d)(t/R^2)^{1/(1-m)}$ which is independent of the initial mass.

ELIMINATING THE TIME T. A natural question is to try to recover this sharp results of the good fast diffusion range via the present methods. If one wants to do that, one needs upper estimates for the minimal life time, that is upper estimates for the extinction time for the MDP, in terms of the L^1-norm on the ball B_{R_0}. We prove the following result.

Theorem 1.3 *Let $m_c < m < 1$. Then,* (i) *We have sharp upper and lower estimates for the extinction time for the Dirichlet problem on any ball B_R of the form:*

$$c_1 \|u_0\|^{1-m}_{\mathrm{L}^1(B_{R/3})} R^{2-d(1-m)} \le T \le c_2 \|u_0\|^{1-m}_{\mathrm{L}^1(B_R)} R^{2-d(1-m)}. \tag{1.8}$$

(ii) *In that range of* m *the lower estimates of Theorem 1.1 imply the lower Harnack inequalities of* [8, 20, 21, 18], *in the form*

$$u(t,x_0) \geq c_{m,d}\left[\frac{t}{R^2}\right]^{\frac{1}{1-m}} \tag{1.9}$$

for any $0 < t < t^*$ *and any* $x \in B_R$*, where* t_* *is given by* (1.26).

This result shows that the form of the lower bounds given in Theorem 1.1 is sharp, since it allows to obtain sharp local lower bounds not only in the good fast diffusion range. And it also applies in the very fast diffusion range, that is the new interesting part of this paper. We are thus led to the question of eliminating all extinction times from the estimate, i.e., replacing T or T_m by some information on the initial data, also in the range $0 < m < m_c$.

Theorem 1.4 *Let* $0 < m < m_c$ *and let* u *be the solution to the FDE under the above assumption that* $u_0 \in \mathrm{L}^{p_c}_{loc}(\mathbb{R}^d)$*. Let* x_0 *be a point in* Ω *and let* $d(x_0, \partial\Omega) \geq 3R$*. Then, the following inequality holds for all* $0 < t < T$

$$R^{-d}\|u_0\|_{\mathrm{L}^1(B_R(x_0))} \leq C_1\, R^{-2/(1-m)}\, t^{\frac{1}{1-m}} + C_3\, \|u_0\|_{\mathrm{L}^{p_c}(B_R(x_0))} R^{-2}\, t^{-\frac{m}{1-m}}\, u^m(t,x_0). \tag{1.10}$$

with C_1 *and* C_3 *given positive constants depending on* d.

We can also obtain formulas in terms of the norms $\|u_0\|_{\mathrm{L}^p(B_R(x_0)}$ for all $p > p_c$, that can be seen below.

Obstruction to a simpler estimate with L^1 norm

The presence of the extinction time T in the lower estimates, or equivalently of some L^p norm of the initial data, is a drawback in the formulas that is not present in the original Aronson-Caffarelli estimate for $m > 1$, or in the version of the authors for $m \in (m_c, 1)$ in the whole space. But it is a consequence of the 'bad' behaviour of the fast diffusion equation for low values of m, a fact that can be seen in different ways.

Thus, we will show here that the local lower estimates cannot depend only on the local L^1 norm of the initial data when $0 < m \leq m_c$. We do it by means of a counterexample based on the behaviour of solutions with data that approximate a Dirac delta. We solve the FDE for smooth and positive initial data $\varphi(x) \in \mathrm{L}^1(\mathbb{R}^d)$ with integral equal to 1. We assume that φ is radially symmetric, compactly supported and decreasing with $|x|$. We obtain a smooth and positive solution $u(t,x)$ defined in a cylinder Q_{T_1} and vanishing identically at some $t = T_1$. The scale invariance of the equation implies that the solution corresponding to data $\varphi_k(x) = k^d\varphi(kx)$ is

$$u_k(x) = k^d u(k^{-\sigma}t, kx), \quad \sigma = d(1-m) - 2 > 0, \tag{1.11}$$

so that it has extinction time $T_k = T_1\, k^{\sigma}$. As $k \to \infty$ it is clear that $u_k(0,t)$ converges to the Dirac delta. We also observe that $T_k \to \infty$, so that we lose the previous estimates. On the other hand, we see that losing the estimates is inevitable. If we consider a point x_0 very close to $x = 0$ and take a radius $R > |x_0|$, then $\|u_k(0,x)\|_{\mathrm{L}^1(B_R(x_0))} = 1$. However, by continuity of u with respect to the initial data at $t = 0$, x large, we have

$$u_k(t,x_0) = k^d u(k^{-\sigma}t, kx_0) \to 0$$

(note that $\int_{\mathbb{R}^d} u_k(t,x)\,dx \le 1$ at all times). This means that no lower estimate could be uniformly valid for this sequence.

A scaling argument was used by Brezis and Friedman [11] to prove that there exist no weak solutions with initial data a Dirac delta.

1.1 Positivity for a "minimal" Dirichlet Problem

We will assume that $0 < m < 1$ in the study of positivity (cf. the comment in the Introduction). Since $m > 0$ we eliminate the factor $1/m$ from equation (0.1) for simplicity without loss of generality. As a preliminary step, we first prove positivity for a problem posed on a ball of radius R_0, zero boundary data and particular initial data. Since the problem of getting quantitative positivity estimates has been successfully studied in [8] in the range $m_c < m < 1$, the techniques we introduce are mainly aimed at producing positivity in the cases $0 < m \le m_c$, where previous methods failed.

Specifically, we shall consider the following Dirichlet problem on the ball $B_{R_0} \subset \mathbb{R}^d$:

$$\begin{cases} \partial_t u = \Delta(u^m) & \text{in } Q_{T,R_0} = (0,T) \times B_{R_0} \\ u(0,x) = u_0(x) & \text{in } B_{R_0}, \quad \text{and} \quad \text{supp}(u_0) \subseteq B_R \\ u(t,x) = 0 & \text{for } t > 0 \text{ and } x \in \partial B_{R_0}\,, \end{cases} \tag{1.12}$$

where $R_0 > 2R > 0$. We only consider nonnegative data and solutions. The problem admits a unique solution $u \in C([0,\infty) : \mathrm{L}^1(B_{R_0}))$ for every $u_0 \in \mathrm{L}^1\big(B_{R_0}\big)$, [4]. We will refer to this problem as the minimal Dirichlet problem, or more briefly, the *minimal problem*, because obtaining positivity for solutions to this problem implies in an easy way local positivity for any other problem, thanks to the comparison principle. The solution vanishes in finite time; let $T > 0$ be the finite extinction time, shortly FET. Later on we would like to eliminate the dependence of the results on T and make the estimates depend only on the initial data, see Section 1.4.

Our goal is to obtain positivity with a quantitative estimate for this "minimal" problem. Our most novel idea consists in passing the information on the initial data via the flux of the solution on the boundary of the ball B_{2R} into an averaged positivity result outside the ball for times that are not too small, more precisely on the annulus $A_0 := B_{R_0} \setminus B_{2R}$. This property can be interpreted as the expansion of positivity outside a ball in which the initial datum has nonzero mean. It is in some sense it is analogous to the expansion of positivity already introduced by DiBenedetto et al. , see e.g. [19, 18, 21] for the upper m-range . The expansion of positivity turns out to be a key tool in proving lower Harnack also in our case.

Once we have proved that positivity spreads out from a ball, then for suitable positive times the mean value of the solution on an annulus is positive. We then "fill the hole" in the middle using Aleksandrov's Reflection Principle, cf. the Appendix and [8]. In this way we arrive at the positivity result in the inner ball for any positive time.

1.1.1 Flux and transfer of positivity

We start the proof of the positivity results for the minimal problem by a result on mass transfer to an outside annulus based on the flux across an internal boundary. We recall that $R_0 > 2R$ and $A_0 := B_{R_0}(x_0) \setminus B_{2R}(x_0)$. In order to simplify the final formulas, we write $\lambda = R_0/2R > 1$ (we take for instance $R_0 = 3R$).

Lemma 1.5 (Flux Lemma) *If u is a positive smooth solution of the Minimal Problem* (1.12) *in Q_T with extinction time $T>0$. Then, the following estimate holds true*

$$k_0\,(R_0-2R)^2\int_{B_{R_0}} u(s,x)\,dx \le \int_s^T\int_{A_0} u^m\,\mathrm{d}x\,\mathrm{d}t, \tag{1.13}$$

for any $0\le s\le T$, and any $0<2R<R_0$, and for a suitable constant $k_0=k_0(d)$.

Proof. We shall use a C^∞ test function $\varphi(x)$ that is supported in the ball B_{R_0} and takes the value 1 in B_{2R}. It is clear that we can choose φ such that there exist a constant $k_0>0$ depending only on d such that

$$\big|\Delta\varphi(x)\big| \le \frac{k_0^{-1}}{(R_0-2R)^2}\,, \tag{1.14}$$

Let $0\le s<t\le T$. We compute

$$\int_s^t\int_{A_0}\partial_t u\,\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_s^t\int_{A_0}\Delta(u^m)\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_s^t\int_{A_0}u^m\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_s^t\int_{\partial B_{2R}}\partial_\nu(u^m)\varphi\mathrm{d}\sigma\,\mathrm{d}t$$
$$+\int_s^t\int_{\partial B_{R_0}}\big[\partial_\nu(u^m)\varphi - u^m\partial_\nu\varphi\big]\mathrm{d}\sigma\,\mathrm{d}t - \int_s^t\int_{\partial B_{2R}}u^m\partial_\nu\varphi\mathrm{d}\sigma\,\mathrm{d}t.$$

We remark that the last three integrals vanish since φ and $u\equiv$ vanishes identically near the boundary ∂B_{R_0}, and $\partial_\nu\varphi\equiv 0$ on ∂B_{2R}. We also have

$$\int_s^t\int_{A_0}\partial_t u\,\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_{A_0}u(t,\cdot)\varphi\,\mathrm{d}x - \int_{A_0}u(s,\cdot)\varphi\,\mathrm{d}x$$

Hence,

$$\int_{A_0}u(t,\cdot)\varphi\,\mathrm{d}x - \int_{A_0}u(s,\cdot)\varphi\,\mathrm{d}x = \int_s^t\int_{A_0}u^m\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_s^t\int_{\partial B_{2R}}\partial_\nu(u^m)\varphi\mathrm{d}\sigma\,\mathrm{d}t.$$

We will use this equality with $t=T$, $T=T(u_0)$ being the finite extinction time for the solution to Problem (1.12), so that we obtain

$$\int_{A_0}u(s,\cdot)\varphi\,\mathrm{d}x = -\int_s^T\int_{A_0}u^m\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_s^T\int_{\partial B_{2R}}\partial_{\nu^*}(u^m)\varphi\mathrm{d}\sigma\,\mathrm{d}t\,, \tag{1.15}$$

where ν^* is the exterior normal to B_{2R}, which is the opposite of ν which is the exterior normal to the inner boundary of A_0, so that $\partial_{\nu^*}(u^m) = -\partial_\nu(u^m)$.

On the other hand, a simple calculation shows that

$$\int_{B_{2R}}(u(t,x)-u(s,x))\,\mathrm{d}x = \int_s^t\int_{B_{2R}}\partial_t u\,\mathrm{d}x\,\mathrm{d}t = \int_s^t\int_{B_{2R}}\Delta(u^m)\,\mathrm{d}x\,\mathrm{d}t = \int_s^t\int_{\partial B_{2R}}\partial_{\nu^*}(u^m)\mathrm{d}\sigma\,\mathrm{d}t.$$

Letting $t=T$, with T as above, we obtain

$$-\int_{B_{2R}}u(s,x)\,\mathrm{d}x = \int_s^T\int_{\partial B_{2R}}\partial_{\nu^*}(u^m)\mathrm{d}\sigma\,\mathrm{d}t. \tag{1.16}$$

Joining equalities (1.15) and (1.16) we get

$$\int_{B_{R_0}} u(s,x)\,\mathrm{d}x = \int_{B_{2R}} u(s,x)\,\mathrm{d}x + \int_{A_0} u(s,x)\,\mathrm{d}x = -\int_s^T \int_{A_0} u^m\,\Delta\varphi\,\mathrm{d}x\,\mathrm{d}t$$

We conclude by using estimates (1.14) for $\Delta\varphi$: for any $0 < 2R < R_0$ we then get

$$\int_{B_{R_1}} u(s,x)\,\mathrm{d}x \le \int_{B_{R_0}} u(s,x)\,\mathrm{d}x = -\int_s^T \int_{A_0} u^m\,\Delta\varphi\;\mathrm{d}x\,\mathrm{d}t \le \frac{k_0^{-1}}{(R_0-2R)^2}\int_s^T \int_{A_0} u^m\;\mathrm{d}x\,\mathrm{d}t. \tag{1.17}$$

The proof is complete. □

Remark. *Lower Bound on the Extinction Time.* As a first consequence of this Lemma we can easily obtain useful lower estimates for the FET:

$$\begin{aligned}
k_0\,(R_0-2R)^2 \int_{B_{R_0}} u(s,x)\;\mathrm{d}x &\le \int_s^T \int_{A_0} u^m\,\mathrm{d}x\,\mathrm{d}t \le (T-s)\;\mathrm{Vol}(A_0)\;\int_{A_0} u^m(\overline{s},x)\frac{\mathrm{d}x}{\mathrm{Vol}(A_0)}\\
&\le (T-s)\;\mathrm{Vol}(A_0)\;\left[\int_{B_{R_0}} u(\overline{s},x)\frac{\mathrm{d}x}{\mathrm{Vol}(A_0)}\right]^m\\
&\le (T-s)\;\mathrm{Vol}(A_0)^{1-m}\;\left[\int_{B_{R_0}} u(s,x)\,\mathrm{d}x\right]^m
\end{aligned}$$

where in the first step we have used the mean value theorem for the time integral (see details in Step 2 of next section), with $\overline{s} \in (s,T)$, in the second step the Hölder inequality, and in the third step we used the contractivity of the global $\mathrm{L}^1(B_{R_0})$-norm. Letting then $s = 0$ gives the desired lower bound, once we notice that $\mathrm{supp}(u_0) \subseteq B_R$

$$k_0\,(R_0-2R)^2 \left[\frac{\int_{B_R} u_0\,\mathrm{d}x}{\mathrm{Vol}(A_0)}\right]^{1-m} \le T\,. \tag{1.18}$$

1.1.2 Pointwise lower estimate for initial times

We have just shown how positivity of the initial datum propagates on the annulus in the weak form of a positive space-time mean value. We will now see that this is sufficient to fill the hole inside the annulus. As in the study of the exponent range $m_c < m < 1$ performed in [8], the estimate uses a *critical time* that is defined in terms of the initial norms. In the present case it is given by

$$t_* := \frac{k_0}{2}\,(R_0-2R)^2 \left[\frac{\int_{B_R} u_0\,\mathrm{d}x}{\mathrm{Vol}(A_0)}\right]^{1-m} \tag{1.19}$$

where k_0 as in the Flux Lemma 1.5. Note in passing that the positivity result that follows, formula (1.25), implies that this quantity is less than T.

Obtaining the lower bound needs several steps.

• Step 1. *Time Integrals.* Hölder's inequality, together with the fact that the global $\mathrm{L}^1\big(B_{R_0}\big)$-norm decreases, gives

$$\int_{A_0} u(t,x)^m \,\mathrm{d}x \le \mathrm{Vol}\big(A_0\big)^{1-m}\left[\int_{A_0} u(t,x)\,\mathrm{d}x\right]^m \le \mathrm{Vol}\big(A_0\big)^{1-m}\left[\int_{B_{R_0}} u(t,x)\,\mathrm{d}x\right]^m$$
$$\le \mathrm{Vol}\big(A_0\big)^{1-m}\left[\int_{B_{R_0}} u(0,x)\,\mathrm{d}x\right]^m = \mathrm{Vol}\big(A_0\big)^{1-m}\left[\int_{B_R} u_0\,\mathrm{d}x\right]^m$$

since $\mathrm{supp}(u_0)\subseteq B_R$. For any $0\le s\le t$ we then have

$$\int_s^t\int_{A_0} u(\tau,x)^m\,\mathrm{d}x\mathrm{d}\tau \le \mathrm{Vol}\big(A_0\big)^{1-m}\;\left[\int_{B_R} u_0\,\mathrm{d}x\right]^m\;(t-s)$$

We use this estimate together with estimate (1.13) to get

$$\begin{aligned} k_0\,(R_0-2R)^2\int_{B_R} u_0(x)\,\mathrm{d}x &\le \int_0^T\int_{A_0} u^m\,\mathrm{d}x\,\mathrm{d}t = \int_0^{t_*}\int_{A_0} u^m\,\mathrm{d}x\,\mathrm{d}t + \int_{t_*}^T\int_{A_0} u^m\,\mathrm{d}x\,\mathrm{d}t \\ &\le \mathrm{Vol}\big(A_0\big)^{1-m}\;\left[\int_{B_R} u_0\,\mathrm{d}x\right]^m\; t_* + \int_{t_*}^T\int_{A_0} u^m\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{1.20}$$

In view of the definition of t_* we can eliminate one term and get

$$k_2\,(R_0-2R)^2\int_{B_R} u_0(x)\,\mathrm{d}x \;\le \int_{t_*}^T\int_{A_0} u^m\,\mathrm{d}x\,\mathrm{d}t\;. \tag{1.21}$$

with $k_2 = k_0/2$. In particular, this means that the left-hand side remains strictly positive.

• Step 2. We introduce the function

$$Y(t) = \int_{A_0} u^m(t,x)\,\mathrm{d}x,$$

and apply the mean value theorem -for the time integral- to prove that there exists $t_1\in[t_*,T]$ such that $\int_{t_*}^T Y(t)\,\mathrm{d}t = (T-t_*)\,Y\big(t_1\big)$. In other words,

$$\int_{t_*}^T\int_{A_0} u^m(t,x)\,\mathrm{d}x\,\mathrm{d}t = \big(T-t_*\big)\int_{A_0} u^m\big(t_1,x\big)\,\mathrm{d}x.$$

Using now the estimate obtained in the previous step, we conclude that there exists $t_1\in\big[t_*\,,\,T\big)$, such that

$$k_2\,(R_0-2R)^2\int_{B_R} u_0(x)\,\mathrm{d}x \le \int_{t_*}^T\int_{A_0} u^m(t,x)\,\mathrm{d}x\,\mathrm{d}t = \big(T-t_*\big)\int_{A_0} u^m\big(t_1,x\big)\,\mathrm{d}x,$$

and this implies that for some $t_1\in[t_*,T]$ we have

$$\frac{k_2\,(R_0-2R)^2}{T}\int_{B_R} u_0(x)\,\mathrm{d}x \le \frac{k_2\,(R_0-2R)^2}{(T-t_*)}\int_{B_R} u_0(x)\,\mathrm{d}x \le \int_{A_0} u^m\big(t_1,x\big)\,\mathrm{d}x. \tag{1.22}$$

• Step 3. *Aleksandrov Principle. Positivity at the critical time.* We can now use the Aleksandrov Principle to deduce positivity at x_0 from inequality (1.22). In fact,

$$\int_{A_0} u^m(t_1, x)\,\mathrm{d}x \le \mathrm{Vol}(A_0)\, u^m(t_1, x_0) \tag{1.23}$$

where $x_0 \in \mathbb{R}^d$ is the center of the ball B_{R_0}, since we know from Aleksandrov Principle, that $u(t,x) \le u(t, x_0)$ for any $x \in A_0$ and any $t > 0$ (see Appendix for details).
Joining inequality (1.22) and (1.23) we obtained that there exists a $t_1 \in \big[t_*\,,\,T)$ such that

$$\frac{k_2\,(R_0 - 2R)^2}{\mathrm{Vol}(A_0)\,T} \int_{B_R} u_0(x)\,\mathrm{d}x \le\, u^m(t_1, x_0) \tag{1.24}$$

• Step 4. *Positivity backward in time.* The last step consists in obtaining a lower estimate when $0 \le t \le t_1$. This argument is based on Bénilan-Crandall's differential estimate, cf. [4] :

$$\partial_t u(t,x) \le \frac{u(t,x)}{(1-m)t}$$

that is valid for all nonnegative solutions of this initial and boundary value problem. It easily implies that the function:

$$u(t,x)t^{-\frac{1}{1-m}}$$

is non-increasing in time, thus for any $t \in (0, t_1]$ we have that

$$u(t_1, x) \;\le\; t^{-\frac{1}{1-m}}\, t_1^{\frac{1}{1-m}} u(t,x) \;\le\; t^{-\frac{1}{1-m}}\, T^{\frac{1}{1-m}}\, u(t,x)$$

since $t_1 \le T$. It is now sufficient to apply inequality (1.24) to the l.h.s. in the above inequality to get

$$\frac{k_2\,(R_0 - 2R)^2}{\mathrm{Vol}(A_0)\,T} \int_{B_R} u_0(x)\,\mathrm{d}x \le\, u^m(t_1, x_0) \le\, t^{-\frac{m}{1-m}}\, T^{\frac{m}{1-m}}\, u^m(t, x_0) \tag{1.25}$$

This is the inequality we were looking for.
• Step 5. In order to simplify the final formulas, it is convenient to make a choice for the ratio $\lambda = R_0/2R > 1$ (for instance $R_0 = 3R$). The formula for t_* becomes

$$t_* = c_0'\, R^{2-d(1-m)}\, \|u_0\|_{\mathrm{L}^1(B_R)}^{1-m} \tag{1.26}$$

and $c_0' > 0$ depends only on d and λ. We have proved the following positivity result.

Theorem 1.6 *Let $0 < m < 1$ and let u be the solution to the Minimal Problem* (1.12) *and let $T = T(u_0)$ be the MET. Then $T \ge 2t_*$, and the following inequality holds true for all $t \in (0, t_*]$*

$$u^m(t, x_0) \ge c_1'\, R^{2-d} \|u_0\|_{\mathrm{L}^1(B_R)} T^{-\frac{1}{1-m}}\, t^{\frac{m}{1-m}}. \tag{1.27}$$

where $c_1' > 0$ depends only on d.

For the particular time $t = t_*$ we get

$$\left(\frac{u(t_*, x_0)}{\oint_{B_R} u(0,x)\,dx}\right)^m \ge c_2'\,(R^2/T)^{1/(1-m)} \oint_{B_R} u(0,x)\,dx.$$

1.1.3 Estimate of Aronson-Caffarelli type for Very Fast Diffusion

It is interesting to present the above result in the form that has been used by Aronson and Caffarelli in their work [1]. We have to argue as follows: we have arrived at the following alternative

$$\text{either} \qquad t^* < t \qquad \text{or} \qquad \frac{k_2(\lambda^2-1)}{\omega_d\,(\lambda^d-1)R^{d-2}\,T^{\frac{1}{1-m}}}\int_{B_R} u_0(x)\,\mathrm{d}x \le\; t^{-\frac{m}{1-m}}\;u^m(t,x_0)$$

Writing the expression of t^*, we either have

$$R^{-d}\int_{B_R} u_0(x)\,\mathrm{d}x \le\; C_1\,R^{-2/(1-m)}\,t^{\frac{1}{1-m}}, \quad C_1 = \frac{\omega_d(\lambda^d-1)}{k_1^{\frac{1}{1-m}}\,(\lambda-2)^{\frac{2}{1-m}}},$$

or

$$R^{-d}\int_{B_R} u_0(x)\,\mathrm{d}x \le C_2\,T^{\frac{1}{1-m}}R^{-2}\,t^{-\frac{m}{1-m}}\;u^m(t,x_0), \quad C_2 = \frac{\omega_d\,(\lambda^d-1)}{k_2\,(\lambda^2-1)}$$

We now sum up the two expressions to get

$$R^{-d}\int_{B_R} u_0(x)\,\mathrm{d}x \le C_1\,R^{-2/(1-m)}\,t^{\frac{1}{1-m}} + C_2\,T^{\frac{1}{1-m}}R^{-2}\,t^{-\frac{m}{1-m}}\;u^m(t,x_0). \tag{1.28}$$

with C_1 and C_2 given constants depending on d and $\lambda > 2$.

1.2 Proof of Theorem 1.1 and Corollary 1.2

The previous results will now be used to prove uniform positivity on balls for any local solution of the FDE problem. We proceed as follows: let u be a positive and continuous weak solution of the FDE defined in $Q = (0,T)\times\Omega$ taking initial data $u(0,x) = u_0(x)$ in Ω. We make no assumption on the boundary condition (apart from nonnegativity). Let us select a point $x_0 \in \Omega \subseteq \mathbb{R}^d$. Select two radii $R_0 \ge 3R > 0$ so that $B_{R_0}(x_0) \subseteq \Omega$, that is $R_0 \le \mathrm{dist}(x_0,\partial\Omega)$. In the case $\Omega = \mathbb{R}^d$ there is obviously no restriction on R_0. Let u_D be the solution to the corresponding MDP, as defined in (1.12). It has extinction time T_m. By parabolic comparison, it is then clear that $u_D \le u$ in $Q = (0,T')\times B_{R_0}(x_0)$, where $T' = \min\{T, T_m\}$, hence we can easily extend the positivity results for the MDP obtained in the previous section to any other local weak solution, either in the form given by Theorem 1.6, or in the Aronson-Caffarelli form (1.28). This concludes the proof of Corollary 1.2, and as a consequence we get Theorem 1.1. □

1.3 Lower estimates independent of the extinction time for $m_c < m < 1$. Proof of Theorem 1.3

The proof is divided in some short steps. Here, $m_c < m < 1$.

• Reduction. Let $u_R(t,x)$ be the solution to the homogeneous Dirichlet problem on the ball B_R, corresponding to the initial datum $u_0 \in \mathrm{L}^1(B_R)$ and having extinction time $T(R,u_0)$. The rescaled solution

$$u(t,x) = \frac{M}{R^d}\,\widehat{u}\left(\frac{t}{\tau},\frac{x-x_0}{R}\right), \qquad \tau = R^{2-d(1-m)}M^{1-m}, \qquad M = \int_{B_R} u_0(x)\,dx \tag{1.29}$$

allows us to pass from a solution with mass M defined in the ball of radius R centered at x_0 to a solution $\widehat{u}$ with mass 1 in the ball of radius 1 centered at 0. The extinction times have to be rescaled accordingly, $T(u) = R^{2-d(1-m)}M^{1-m}\overline{T}$, where $\overline{T} = T(\widehat{u})$. Therefore, we will work with rescaled problems and solutions.

• Barenblatt solutions. We now consider the solution $\mathcal{B}$ of the Dirichlet problem posed on B_1, and corresponding to the Dirac mass as initial data, $\mathcal{B}(0,\cdot) = \delta_0$, and zero boundary data, that we call Barenblatt solution. First we recall that by approximation with L^1-data, or by comparison with the solutions of the Cauchy problem, it is easy to check that the smoothing effect applies and

$$\mathcal{B}(t,x) \le c_{m,d}\, t^{-d\vartheta_1}, \qquad \text{for any } (t,x) \in [0,+\infty) \times B_1\,. \tag{1.30}$$

Moreover, it is known that this is the solution that extinguishes at the later time among all nonnegative solutions with the same mass of the initial data and same boundary data. Such comparison is a consequence of the concentration-comparison and symmetrization arguments developed in detail in [32, 31]. Thus, we need to prove that the Barenblatt solution extinguishes in finite time $\overline{T}$. The proof is based on the fact that it is bounded for $t \ge t_0 > 0$.

• Solution by separation of variables. Consider now the solution $\mathcal{U}_r(t,x) = S(x)(T_1 - t)^{1/(1-m)}$ of the Dirichlet problem on B_r, $r > 1$. It corresponds to the initial datum $\mathcal{U}_r(0,x) = S(x)T_1^{1/(1-m)}$, and extinguishes at a time T_1, to be chosen later. Here, S is the solution to the stationary elliptic Dirichlet problem $\Delta S^m + 1/(1-m)S = 0$ on B_{R_0}, and therefore it can be chosen radially symmetric, $S(x) = S(|x|)$. It will also be nonincreasing in $r = |x|$. By standard regularity theory $S(x)^m$ can be bounded from above an below by the distance to the boundary. The parameter T_1, extinction time of $\mathcal{U}_r$ can be chosen at will. To fix it, we pick a time $t_0 > 0$ and define the T_1 through the relation

$$S(1)(T_1 - t_0)^{\frac{1}{1-m}} = c_{m,d}\, t_0^{-d\vartheta_1}. \tag{1.31}$$

• Comparing the two solutions. We now consider the homogeneous Dirichlet problem on $[t_0, T] \times B_r$, and we compare the Barenblatt solution $\mathcal{B}$ with the solution $\mathcal{U}_r$ constructed above in the cylinder $Q = B_1(0) \times [t_0, T_1)$. At the initial time t_0 we know by construction that $\mathcal{B}(t_0,x) \le \mathcal{U}_r(t_0,x)$. The comparison of the boundary data is immediate. By well-known parabolic comparison results, this implies that $\mathcal{B}(t,x) \le \mathcal{U}_r(t,x)$, on $[t_0, T_1] \times B_r$, and hence the extinction times satisfy

$$\overline{T} \le T_1 = \left[\frac{c_{m,d}}{S(1)\, t_0^{d\vartheta_1}}\right]^{1-m} + t_0. \tag{1.32}$$

We only need to choose a $t_0 \in (0,T)$ to obtain an expression for the upper bound of $\overline{T}$ that depends only on m and d (the reader may choose to optimize the expression for T_1 with respect to t_0).

• Conclusion. As a consequence of the above upper bound, we know that any solution to the Dirichlet problem on the unitary ball and with unitary initial mass extinguish at a time $T \le \overline{T} \le \tau(m,d)$. Rescaling back to the original variables we have proved that any solution u_R to the Dirichlet problem on the ball B_R and with initial mass $M = \int_{B_R} u_0\,\mathrm{d}x$ extinguish at a time T_R that can be bounded above by

$$T(R,u_0) \le \tau_{m,d} R^{2-d(1-m)} \|u_0\|_{\mathrm{L}^1(B_R)}^{1-m}.$$

The lower bounds come from the fact that $t_* \le T$ and is given by (1.26).

• The lower Harnack inequality. Inequality (1.9) follows by plugging the upper bound (1.8) into the lower bound (1.27). □

The reader should notice that the properties that we have used are typical of the good fast diffusion range, $m_c < m < 1$, and cannot be extended to the very fast diffusion range, $m < m_c$.

1.4 Lower estimates independent of the extinction for $0 < m < m_c$.

The presence of the minimal extinction time $T_m = T$ in the formula for the lower Harnack inequality responds to an essential characteristic of the problem. Actually, lower estimates in terms of only L^1 norms cannot be true for $m \le m_c$ as we have shown at the beginning of this section: there is no positive lower bound at a time $t_0 > 0$ and a point x_0 that depends only on t_0, R and the mass of u_0 in $B_R(x_0)$. Similar examples can be constructed if $u_0 \in \mathrm{L}^p_{loc}(\mathbb{R}^d)$ with $p < p_c$, and we can be found in [31], Chapters 5 and 7.

Fortunately, controlling the local (or global) L^p norm gives a control on the MET T, and in this way we get valid lower estimate without T, as we explain next.

1.4.1 Estimates in terms of the L^{p_c} norm

In [3] Bénilan and Crandall prove that for any $0 \le s \le t \le T$, and for any $m < m_c$

$$\|u(t)\|_{p_c}^{1-m} \le \|u(s)\|_{p_c}^{1-m} - \mathcal{K}_{p_c}\,(t-s)\,, \qquad \text{with} \qquad \mathcal{K}_{p_c} = \frac{8\,[d(1-m)-2]\,\mathcal{S}_2^2}{(d-2)^2(1-m)}\,, \tag{1.33}$$

where $\mathcal{S}_2$ in the constant of the Sobolev inequality

$$\|f\|_{2^*} \le \mathcal{S}_2\,\|\nabla f\|_2 \tag{1.34}$$

and the above estimate holds for any solution with initial datum $u_0 \in \mathrm{L}^{p_c}$. We also stress on the fact that the constant $\mathcal{K}_{p_c}$ is universal in the sense that it only depends on m and d. As a consequence of (1.33) , we have the following *universal upper bound for the extinction time*

$$T(u_0) \le \mathcal{K}_{p_c}^{-1}\,\|u_0\|_{p_c}^{1-m}\,. \tag{1.35}$$

We remark that while for lower bounds on FET we only need local information on the initial datum, upper estimates for the FET require global information. Fortunately, in the minimal problem that we are considering, global and local are equivalent since $u_0(x) = 0$ for $|x - x_0| \ge R$.

Proof. We sketch here the proof for the reader's convenience. It is well known that the time derivative of the global L^p norm of the solution $u(t)$ of the MDP problem under consideration is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p &= -\frac{4p(p-1)}{(p+m-1)^2}\int\left|\nabla u^{\frac{p+m-1}{2}}\right|^2\mathrm{d}x\\ &\le -\frac{4p(p-1)\,\mathcal{S}_2^2}{(p+m-1)^2}\left[\int u^{\frac{(p+m-1)2^*}{2}}\,\mathrm{d}x\right]^{\frac{2}{2^*}} = -\frac{4p(p-1)\,\mathcal{S}_2^2}{(p+m-1)^2}\,\|u\|_{\frac{(p+m-1)2^*}{2}}^{p+m-1}\,,\end{aligned} \tag{1.36}$$

where in the last step we used the Sobolev inequality (1.34) applied to the function $f = u^{(p+m-1)/2}$, where $2^* = 2d/(d-2)$ and $\mathcal{S}_2$ is the Sobolev constant.

Notice that if $m > m_c$, then $p_c < 1$, so that the global L^p-norm increases, and this originates the so called Backward Effect, see e.g. [31]. This explains our assumption $m < m_c$. Moreover

$$p_c + m - 1 = p_c\left(1 - \frac{2}{d}\right), \qquad \frac{(p_c + m - 1)2^*}{2} = p_c\,, \qquad \frac{4p_c(p_c-1)\,\mathcal{S}_2^2}{(p_c+m-1)^2} = \frac{8\,[d(1-m)-2]\,\mathcal{S}_2^2}{(d-2)^2(1-m)} > 0,$$

so that (1.36) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{p_c}^{p_c} \le -\frac{8\,[d(1-m)-2]\,\mathcal{S}_2^2}{(d-2)^2(1-m)}\|u(t)\|_{p_c}^{p_c\left(1-\frac{2}{d}\right)}$$

integrating the differential inequality gives the bound (1.36) for any $0 \le s \le t$. Letting $s = 0$ and $t = T(u_0)$ in (1.36) finally gives (1.35). □

Application to Theorem 1.4. Using this bound, we can now formulate the second version of the lower Harnack estimate, reflected in the theorem. The proof is immediate when u_0 is as in the minimal problem, since in that case local and global norm is the same. Comparison as done in Subsection 1.2, allows to pass to the general solutions. Notice that in this way we use a local L^{p_c} norm, not the global one!

Remark. When we have not only the Sobolev inequality, but also the Poincaré, we can prove similar estimates for any $m \in (0,1)$. This happens for instance for problems posed in bounded domains, or for the minimal Dirichlet problem.

1.4.2 Estimates in terms of other L^p norms

Proposition 1.7 *Let $m < 1$, $\alpha \ge 1$, $R > 0$ and let u be the solution to the Dirichlet problem*

$$\begin{cases} u_t = \frac{1}{m}\Delta(u^m) & \text{in } (0,T)\times B_{\alpha R} \\ u(0,x) = u_0(x) & \text{in } B_{\alpha R}, \quad \text{and} \quad \mathrm{supp}(u_0) \subseteq B_R \\ u(t,x) = 0 & \text{for } t>0 \text{ and } x \in \partial B_{\alpha R} \end{cases}$$

with $u_0 \in \mathrm{L}^p\big(B_{\alpha R}\big)$, with $p > \max\{p_c, 1\} = \max\{d(1-m)/2, 1\}$. Then the following estimate

$$\big\|u(t)\big\|_p^{1-m} - \big\|u(s)\big\|_p^{1-m} \le -\mathcal{K}_p\,(t-s)\,. \tag{1.37}$$

hold for any $0 \le s \le t$, where

$$\mathcal{K}_p = 4\frac{(1-m)(p-1)}{\big[p+m-1\big]^2}\,[\mathcal{P}\,\alpha R]^{-2\left(1-\frac{p_c}{p}\right)}\;\mathcal{S}_2^{-\frac{2p_c}{p}} > 0$$

and where $\mathcal{S}_2$ is the Sobolev constant of $\mathbb{R}^d$ and $\mathcal{P}$ is the Poincaré constant on the unit ball.

Proof. First consider, for any $f \in W_0^{1,2}\big(B_{\alpha R}\big)$, the Sobolev and Poincaré inequalities:

$$\|f\|_{2^*} \le \mathcal{S}_2\,\|\nabla f\|_2\,, \qquad \text{and} \qquad \|f\|_2 \le \mathcal{P}\,\alpha\,R\,\|\nabla f\|_2\,,$$

where $2^* = 2d/(d-2)$, and where the constants $\mathcal{S}_2$, the optimal Sobolev constant on $\mathbb{R}^d$, and $\mathcal{P}$, the Poincaré constant on the unit ball, only depend on the dimension d. By combining them through the Hölder inequality, we then get for any $q \in (2, 2^*)$

$$\|f\|_q \le \|f\|_2^{1-\vartheta}\|f\|_{2^*}^{\vartheta} \le [\mathcal{P}\,\alpha R]^{1-\vartheta}\ \mathcal{S}_2^{\vartheta}\ \|\nabla f\|_2 .$$

Let now

$$f = u^{\frac{p+m-1}{2}}\,, \qquad q := \frac{2p}{p+m-1} \qquad \text{and} \qquad \vartheta = \frac{d(1-m)}{2p} = \frac{p_c}{p}\,.$$

We remark that $q < 2^*$ if and only if $p > p_c$, while $q > 2$ if and only if $q < \infty$. We obtain then

$$\|u\|_p^{p\left[1-\frac{1-m}{p}\right]} \le [\mathcal{P}\,\alpha R]^{2\left(1-\frac{p_c}{p}\right)}\ \mathcal{S}_2^{\frac{2p_c}{p}}\ \left\|\nabla u^{\frac{p+m-1}{2}}\right\|_2^2 := \mathcal{K}_0 \left\|\nabla u^{\frac{p+m-1}{2}}\right\|_2^2 \tag{1.38}$$

The derivative of the global L^p-norm then satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p = -\frac{4p(p-1)}{\left[p+m-1\right]^2}\left\|\nabla u^{\frac{p+m-1}{2}}\right\|_2^2 \le -\frac{4p(p-1)\mathcal{K}_0^{-1}}{\left[p+m-1\right]^2}\|u\|_p^{p\left[1-\frac{1-m}{p}\right]} \tag{1.39}$$

where in the last step we used (1.38). Integrating the differential inequality over $[s,t] \subseteq [0,T]$, gives

$$\|u(t)\|_p^{1-m} - \|u(s)\|_p^{1-m} \le -\frac{4(1-m)(p-1)}{\left[p+m-1\right]^2}\,[\mathcal{P}\,\alpha R]^{-2\left(1-\frac{p_c}{p}\right)}\ \mathcal{S}_2^{-\frac{2p_c}{p}}\ (t-s)\,. \quad \square$$

Upper Bounds on the Extinction Time. The above estimates (1.37), prove that any solution of the Dirichlet problem extinguish in finite time, and this is not surprising, but they also provide an *Upper Bound for the extinction time* T, indeed letting $s=0$ and $t=T$, we obtain

$$T \le \mathcal{K}_p^{-1}\|u_0\|_p^{1-m} = \frac{\left[p+m-1\right]^2}{4(1-m)(p-1)}\,[\mathcal{P}\,\alpha R]^{2\left(1-\frac{p_c}{p}\right)}\ \mathcal{S}_2^{\frac{2p_c}{p}}\ \|u_0\|_p^{1-m}$$

Notice that in the limit $p \to p_c$ we recover the previous result (1.33). Summing up, the above result proves that a *global Sobolev and Poincaré inequality provides that the solution extinguish in finite time* T *and an gives a quantitative upper bound for* T.

Remarks. These results can be extended to different domains or manifolds in a straightforward way, the only important thing is to have global Sobolev and Poincaré inequalities, as already studied by the authors in [7], in the case of Riemannian manifolds with nonpositive curvature

Using this bound, we can now formulate a version of the lower Harnack estimate similar to Theorem 1.4. We leave the easy details to the reader.

2 Part II. Local upper bounds

In the second part of this work we turn our attention to the question of upper estimates for solutions with data in some L^p_{loc}, $p \ge 1$, and obtain quantitative forms of the bounds that are sharp in various respects. The range of application is all $m < 1$, even $m \le 0$. We assume moreover that $d \ge 3$, which is the interesting case also for the lower estimates, in order to avoid technical complications which break the flow of the proofs and results, but we remark that the qualitative fact, the existence of local upper bounds, is also true for $d = 1, 2$.

As a preliminary for the main result, we devote Section 2.1 to establish the conservation of the local L^p integrability of the solutions and the control of the evolution of the local L^p norm for suitable $p \ge 1$. Let $u = u(t,x)$ be a nonnegative weak solution of the FDE for $m < 1$ defined in a space-time cylinder $Q = (0,T] \times B_{R_0}$ for some $R_0, T > 0$. This is the form of the estimate we get:

$$\left[\int_{B_R(x_0)} |u(t,x)|^p \,\mathrm{d}x\right]^{(1-m)/p} \le \left[\int_{B_{R_0}(x_0)} |u(s,x)|^p \,\mathrm{d}x\right]^{(1-m)/p} + K\,(t-s),$$

for any $R_0 > R$ and $0 \le s \le t < T$. It is valid for all $m < 1$ if $p \ge 1$, $p > 1 - m$. The dependence of K on R and R_0 is explicitly given in Theorem 2.3 below. The estimate extends Herrero-Pierre's well-known estimate to $p > 1$ and is valid for $m \le 0$.

The main result of this part is the local upper bound that applies for the same type of solution and initial data, under different restrictions on p. Here is the precise formulation.

Theorem 2.1 *Let $p \ge 1$ if $m > m_c$ or $p > p_c$ if $m \le m_c$. Let u be a local weak solution to the FDE in the cylinder $(0,T) \times \Omega \subseteq (0,+\infty) \times \mathbb{R}^d$. Then there are positive constants $\mathcal{C}_1$, $\mathcal{C}_2$ such that we have*

$$u(t,x_0) \le \frac{\mathcal{C}_1}{t^{d\vartheta_p}} \left[\int_{B_{R_0}(x_0)} |u_0(x)|^p \,\mathrm{d}x\right]^{2\vartheta_p} + \mathcal{C}_2 \left[\frac{t}{R_0^2}\right]^{\frac{1}{1-m}}. \tag{2.1}$$

where $R_0 \le \operatorname{dist}(x_0, \partial\Omega)$ and the constants $\mathcal{C}_i$ depend on m, d and p.

We recall that $\vartheta_p = 1/(2p - d(1-m)) = 1/2(p - p_c)$. Note that the constants $\mathcal{C}_i$ do not depend on the radii, but only on m, d and p. An explicit formula for them is given at the end of the proof, but we point out that such values need not be optimal. The result is proved in Sections 2.2 and 2.3. A similar smoothing effect result has been proved for the first time by Herrero and Pierre in [24] in the good fast diffusion range $m_c < m < 1$ using $p = 1$, but it is new in the range $m \le m_c$ where HP's result cannot hold in view of solutions like (0.2). HP's technique relies on stronger differential estimates that do not hold in the subcritical fast diffusion case or on the local setting; our impression is that their techniques can not be adapted to the very fast diffusion range. Related estimates for $p > 1$ are due to DiBenedetto and Kwong, [19] , and Daskalopoulos and Kenig, [14], but as far as we know no results cover the very fast diffusion range. Finally, note that the smoothing effect L^p_{loc} into L^∞_{loc} is false for exponents $p < p_c$ as has been demonstrated in [31]. In fact, that monograph studies the existence of the so-called backward smoothing effects that go from $\mathrm{L}^p(\mathbb{R}^d)$ into $\mathrm{L}^1(\mathbb{R}^d)$ for $p < p_c$.

The local bound in (2.1) is expressed as the sum of two independent terms, one due to initial data, the other one due to effects near the boundary. The estimate is optimal in the following senses:

(i) The first term responds to the influence of the initial data and has the exact form that has been demonstrated to be exact for solutions that are defined in the whole space and have initial data in $\mathrm{L}^p(\mathbb{R}^d)$, see [31, Chapters 3,5]. By exact we mean that the integral is the same (but extended to the whole space) and the exponents are the same, only the constant $\mathcal{C}_1$ may differ. We can then recover the global smoothing effect on $\mathbb{R}^d$, just by letting $R \to \infty$ so that the second term disappears; as mentioned above the constant $\mathcal{C}_1$ is not the optimal one: the best constant for the global smoothing effect on $\mathbb{R}^d$ has been calculated by one of the authors in [31] .

(ii) The last term accounts for the influence of the boundary data and is special to fast diffusion in the sense that it does include any information on the precise boundary data, thus allowing for the so-called *large solutions* that take on the value $u = +\infty$ on ∂B_R. The term has the exact form prescribed by the explicit singular solutions (0.2). This last term has the meaning of an absolute bound for all solutions with zero or bounded initial data; thus it can also be interpreted as a *universal bound* for the influence of any boundary effects. In applications it is interpreted as an absolute damping of all external influences.

2.1 Evolution of Local L^p-norms

A basic question in the existence theory is obtaining a priori estimates of the solutions in terms of the data measured in some appropriate norm. The peculiar feature of the FDE is the local nature of the estimates. A fundamental result in this direction is the local $\mathrm{L}^1_{\rm loc}$-$\mathrm{L}^1_{\rm loc}$ estimate due to Herrero and Pierre (which is valid for $m > 0$):

Lemma 2.2 *Let $u, v \in C\left([0,+\infty)\,;\,\mathrm{L}^1_{\rm loc}(\mathbb{R}^d)\right)$ be weak solutions of*

$$\partial_t u = \Delta(u^m/m), \quad 0 < m < 1.$$

Let $R > 0$, $R_0 = \lambda R$ with $\lambda > 1$, and $x_0 \in \mathbb{R}^d$ be such that $B_1 = B_{R_0}(x_0) \subset \mathbb{R}^d$. Let moreover $v \le u$ a.e. Then, the following inequalities hold true:

$$\left[\int_{B_R} \big[u(t,x) - v(t,x)\big]\,\mathrm{d}x\right]^{1-m} \le \left[\int_{B_1} \big[u(s,x) - v(s,x)\big]\,\mathrm{d}x\right]^{1-m} + K_{R,R_0,1}\,|t-s|, \tag{2.2}$$

for any $t, s \ge 0$, where

$$K_{R,R_0,1} = \frac{c_1}{(R_0 - R)^2}\mathrm{Vol}\left(B_{R_0}\setminus B_R\right)^{(1-m)} > 0 \tag{2.3}$$

and the constant $c_1 > 0$ depends only on m, d.

This result was proven in Prop. 3.1 of [24] and has been generalized to the case of fast diffusion on a Riemannian manifold by the authors in [7]. Our goal here is to extend such a result into and $\mathrm{L}^p_{\rm loc}$-$\mathrm{L}^p_{\rm loc}$ estimate for suitable $p > 1$. This estimate has two merits: first, it is valid for all $-\infty < m < 1$; second, it is needed for some values $p > p_c$ for the proof of boundedness estimates.

Theorem 2.3 *Let $u \in C\left((0,T)\,;\,\mathrm{L}^1_{\rm loc}(\Omega)\right)$ be a nonnegative weak solution of*

$$\partial_t u = \Delta(u^m/m), \tag{2.4}$$

and assume that $u(t,\cdot) \in \mathrm{L}^p_{\rm loc}(\Omega)$ for some $p \ge 1$, $p > 1-m$, and for all $0 < t < T$. Here, Ω is a domain in $\mathbb{R}^d$ that contains the ball $B_1 = B_{R_0}(x_0)$. Then, the following inequality holds true:

$$\left[\int_{B_R(x_0)} |u(t,x)|^p\,\mathrm{d}x\right]^{(1-m)/p} \le \left[\int_{B_{R_0}(x_0)} |u(s,x)|^p\,\mathrm{d}x\right]^{(1-m)/p} + K_{R,R_0,p}\,(t-s), \tag{2.5}$$

for any $0 \le s \le t < T$, *where*

$$K_{R,R_0,p} = \frac{p\, c_{m,d}}{\big(R_0 - R\big)^2} \,\mathrm{Vol}\,(B_{R_0} \setminus B_R)^{(1-m)/p} > 0, \tag{2.6}$$

and the constant $c_{m,d} > 0$ *depends only on* m, d.

Remarks. (i) The result implies for those values of p that whenever $u(s,\cdot) \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$ for some $s > 0$, then $u(t,\cdot) \in \mathrm{L}^p_{\mathrm{loc}}(\Omega)$ for all $t > s$. Note that the dependence of the local L^p norm is again expressed as the sum of two independent terms, one due to the initial data, the other one due to effects near the boundary.

(ii) Note that the times t and s must be ordered in this result, a condition that is not required in Lemma 2.2.

(iii) The last term may blow up as we approach the boundary of Ω (where no information on the data is used). Indeed, the constant can be written in the form

$$K_{R,R_0,p} = p c'_{m,d} R_0^{2(p-p_c)/p} F(R/R_0), \quad F(s) = \frac{(1-s^d)^{(1-m)/p}}{(1-s)^2}.$$

Now, if $x_0 \in \Omega$ we may take $R_0 = d(x_0, \partial\Omega)$ and $R = R_0(1-\varepsilon)$. In that case the constant in the last term behaves as $\varepsilon \to 0$ in the form

$$K_{R,R_0,p} \sim R_0^{2(p-p_c)/p} \varepsilon^{-\beta}, \quad \beta = 2 - (1-m)/p = (2p + m - 1)/p.$$

(iv) The constant blows up in the limit $m \to 1^-$, and this is perfectly coherent, since a similar estimate is false for the Heat Equation.

Moreover, the constant $K_{R,R_0,p}$, blows up when $p \to \infty$, thus it does not provide L^∞ local stability, while it provides local L^p stability, for $p > p_c$.

Proof of Theorem 2.3. (i) Let $u \ge 0$ and take a test function $\psi \in C_c^\infty(\Omega)$ and $\psi \ge 0$. We can compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \psi\, u^p \,\mathrm{d}x &= p \int_\Omega \psi\, u^{p-1} \partial_t u \,\mathrm{d}x = -p \int_\Omega \nabla\left(\psi\, u^{p-1}\right) \cdot \nabla\left(\frac{u^m}{m}\right) \mathrm{d}x \\
&= -p \left[\int_\Omega \nabla\psi \cdot (u^{p+m-2} \nabla u) \,\mathrm{d}x + (p-1) \int_\Omega \psi\, u^{p+m-3} |\nabla u|^2 \,\mathrm{d}x \right] \\
&= -p \left[\frac{1}{p+m-1} \int_\Omega \nabla\psi \cdot \nabla(u^{p+m-1}) \,\mathrm{d}x + \frac{4(p-1)}{(p+m-1)^2} \int_\Omega \psi \left| \nabla u^{\frac{p+m-1}{2}} \right|^2 \mathrm{d}x \right] \\
&= \frac{p}{p+m-1} \int_\Omega (\Delta\psi)\, u^{p+m-1} \,\mathrm{d}x - \frac{4p\,(p-1)}{(p+m-1)^2} \int_\Omega \psi \left| \nabla u^{\frac{p+m-1}{2}} \right|^2 \mathrm{d}x \\
&\le \frac{p}{p+m-1} \int_\Omega \left| \Delta\psi \right| u^{p+m-1} \,\mathrm{d}x.
\end{aligned} \tag{2.7}$$

This computation holds true for any $p \ge 1$, and any $m \in \mathbb{R}$, when one replaces, in the limit $m \to 0$, the quantity $(u^m)/m$ with $\log u$; we also have to replace $u^{p+m-1}/(p+m-1)$ by $\log(u)$ if $p + m - 1 = 0$. Of course, when $p + m - 1 \le 0$ the last term may be infinite, since it contains u^{p+m-1} so we make the assumption $p > 1 - m$.

(ii) Under such assumptions, inequality (2.7) implies that for any solution $u \geq 0$ we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \psi(x)u(t,x)^p\,\mathrm{d}x &\leq \frac{p}{p+m-1}\int_\Omega |\Delta(\psi(x))|\,|u(t,x)|^{p+m-1}\,\mathrm{d}x \\
&\leq \frac{p}{|p+m-1|}\left[\int_\Omega |\Delta(\psi(x))|^{\frac{p}{(1-m)}}\,\psi(x)^{1-p/(1-m)}\,\mathrm{d}x\right]^{\frac{1-m}{p}}\left[\int_\Omega \psi(x)\,u(t,x)^p\,\mathrm{d}x\right]^{\frac{p+m-1}{p}} \\
&= C(\psi)\left[\int_\Omega \psi(x)\,u(t,x)^p\,\mathrm{d}x\right]^{1-\frac{1-m}{p}}
\end{aligned}
\tag{2.8}
$$

where in the second step we have used Hölder's inequality with conjugate exponents $p/(1-m)$ and $p/(p+m-1)$, and where

$$
C(\psi) = \frac{p}{p+m-1}\left[\int_\Omega |\Delta(\psi(x))|^{\frac{p}{(1-m)}}\,\psi^{1-\frac{p}{(1-m)}}\,\mathrm{d}x\right]^{\frac{1-m}{p}}. \tag{2.9}
$$

We will check below that this quantity can be made finite by a proper choice of ψ. Assuming this for the moment, formula (2.8) can be expressed as a differential inequality of the form

$$
y'(\tau) \leq C y^{1-\varepsilon}(\tau)
$$

where $y(\tau) = \int_\Omega \psi(x)|u(\tau,x)|^p\,\mathrm{d}x$, $C = C(\psi)$ and $\varepsilon = (1-m)/p \in (0,1)$. Integrating such differential inequality over (s,t) lead to

$$
y^\varepsilon(t) - y^\varepsilon(s) \leq C\,\varepsilon\,(t-s)
$$

that is

$$
\left[\int_\Omega \psi(x)|u(t,x)|^p\,\mathrm{d}x\right]^{(1-m)/p} \leq \left[\int_\Omega \psi(x)|u(s,x)|^p\,\mathrm{d}x\right]^{(1-m)/p} + \frac{(1-m)}{p}\,C(\psi)(t-s)
$$

for any $0 \leq s \leq t$. This will immediately imply the statement, once we prove the bounds

$$
\frac{(1-m)}{p}\,C(\psi) = K_{R,\lambda,p} < +\infty. \tag{2.10}
$$

(iii) We only have to verify the form of the last bound. To this end we consider a function $\psi = \varphi^b \in C_c^\infty(M)$, with

$$
0 \leq \varphi \leq 1\,, \quad \varphi \equiv 1 \text{ in } B_R\,, \quad \varphi \equiv 0 \text{ outside } B_{\lambda R} \tag{2.11}
$$

with $\lambda = R_0/R > 1$. Moreover, we will assume that φ is radially symmetric and $\varphi(x) = \overline{\varphi}\,(|x|/R)$, where $\overline{\varphi}:\mathbb{R}\to\mathbb{R}$ is a $C_c^\infty(\mathbb{R})$ function such that:

$$
0 \leq \overline{\varphi}(s) \leq 1\,, \quad \overline{\varphi}(s) \equiv 1\,, \text{ for } 0 \leq s \leq 1\,, \quad \overline{\varphi} \equiv 0\,, \text{ for } s \geq \lambda
$$

where $\lambda > 1$ and $|x|$ is the distance from a fixed point. We then have

$$
\begin{aligned}
|\Delta\,(\psi(x))|^{\frac{p}{1-m}}\,\psi(x)^{1-\frac{p}{1-m}} &= \varphi(x)^{b\left[1-\frac{p}{1-m}\right]}\left|b(b-1)\,\varphi^{b-2}\,|\nabla\,\varphi|^2 + b\,\varphi^{b-1}\,\Delta\varphi\right|^{\frac{p}{1-m}} \\
&\leq [b(b-1)]^{\frac{p}{1-m}}\,\varphi^{b\left[1-\frac{p}{1-m}\right]+\frac{(b-2)p}{1-m}}\left|\,|\nabla\,\varphi|^2 + |\Delta\varphi|\right|^{\frac{p}{1-m}}
\end{aligned}
\tag{2.12}
$$

the last inequality follows from the fact that we are considering a radial function $0 \le \varphi(x) = \overline{\varphi}(|x|/R) \le 1$, with $b > \frac{2p}{1-m}$. Then, we compute

$$\begin{aligned}
|\nabla\varphi(x)|^2 &= R^{-2}|\overline{\varphi}'(|x|/R)|^2\,|\nabla|x||^2 \le R^{-2}\,|\overline{\varphi}'(|x|/R)|^2 \le c_\lambda'^2\,R^{-2}\\
|\Delta\varphi(x)| &= \left|R^{-2}\overline{\varphi}''(|x|/R)\,|\nabla(|x|)|^2 + R^{-1}\overline{\varphi}'(|x|/R)\Delta|x|\right|\\
&\le \frac{1}{R}\left[\frac{|\overline{\varphi}''(|x|/R)|}{R} + |\overline{\varphi}'(|x|/R)|\,\frac{d-1}{|x|}\right] \le \frac{(d-1)c_\lambda''}{R^2},
\end{aligned}$$

where in the last step we used the fact that $\Delta\varphi$ is supported in $A_{R,\lambda} = B_{\lambda R}\setminus B_R$ and that the smooth function $\overline{\varphi}$ has bounded derivatives in $A_{R,\lambda}$

$$\left|\overline{\varphi}(|x|/R)\right| \le \frac{c_0'}{\lambda-1} = c_\lambda'\,, \qquad \left|\overline{\varphi}'(|x|/R)\right| + \left|\overline{\varphi}''(|x|/R)\right| \le \frac{c_0''}{(\lambda-1)^2} = c_\lambda''\,, \tag{2.13}$$

we just remark that this last estimate depend on an explicit choice of the test function $\overline{\varphi}$.
Inequality (2.12) together with (2.13) gives

$$\begin{aligned}
|\Delta\left(\psi(x)\right)|^{\frac{p}{1-m}}\,\psi(x)^{1-\frac{p}{1-m}} &\le [b(b-1)]^{\frac{p}{1-m}}\varphi^{b\left[1-\frac{p}{1-m}\right]+\frac{(b-2)p}{1-m}}\left|\,|\nabla\varphi|^2+|\Delta\varphi|\right|^{\frac{p}{1-m}}\\
&\le [b(b-1)]^{\frac{p}{1-m}}\left[\frac{c_0'^2+(d-1)c_0''}{\left[(\lambda-1)R\right]^2}\right]^{\frac{p}{1-m}} := \frac{c_p'^{\frac{p}{1-m}}}{\left[(\lambda-1)R\right]^{\frac{2p}{1-m}}}
\end{aligned}$$

if $b > \frac{2p}{1-m}$, $c_p' = b(b-1)(c_0'^2 + (d-1)c_0'')$. An integration over $A_{R,\lambda}$ gives:

$$\begin{aligned}
\frac{(1-m)}{p}\,C(\psi) &= \frac{1-m}{p+m-1}\left[\int_{A_{R,\lambda}}|\Delta(\psi(x))|^{\frac{p}{(1-m)}}\psi^{1-\frac{p}{(1-m)}}\,\mathrm{d}x\right]^{\frac{1-m}{p}}\\
&\le \frac{1-m}{p+m-1}\frac{c_p'}{\left[(\lambda-1)R\right]^2}\mathrm{Vol}(A_{R,\lambda})^{\frac{1-m}{p}} = \frac{c_p}{\left[(\lambda-1)R\right]^2}\mathrm{Vol}(A_{R,\lambda})^{\frac{1-m}{p}} := K_{R,\lambda,p} < +\infty
\end{aligned}$$

where $c_p = b(b-1)(c_0'^2 + (d-1)c_0'')(1-m)/(p+m-1)$, and $b > 2p/(1-m)$, we can choose $b = 3p/(1-m)$ to get

$$c_p \le c_{m,d}\,p$$

where $c_{m,d}$ is independent of p. The proof is thus complete. □

2.2 Smoothing effect in terms of space-time integrals

In this section we are going to prove a first version of the Local Smoothing Effect for the FDE. More precisely, we prove that $\mathrm{L}^p_{\mathrm{loc}}$ regularity in space-time implies $\mathrm{L}^\infty_{\mathrm{loc}}$ estimates, even when $m < m_c$, on the condition that p must be large enough. The estimates are local, both in space and in time, but uniform on balls and the dependence is quantitative. We consider a nonnegative weak solution of the FDE for $m < 1$ defined in a space-time cylinder $Q = (0,T]\times B_R$ for some $R, T > 0$.

Throughout this section T will not denote extinction time.

Theorem 2.4 *Let u and m be as above, and let $p \ge 1$ if $m > m_c$ or $p > p_c$ if $m \le m_c$. For any two finite cylinders $Q_1 \subset Q_0$, $Q_i = (T_i, T] \times B_{R_i}$, with $0 < R_1 < R_0$, and $0 \le T_0 < T_1 < T$, we have*

$$\sup_{Q_1} |u| \le \mathcal{C}_{\rm loc} \left[\frac{1}{(R_0 - R_1)^2} + \frac{1}{T_1 - T_0}\right]^{\frac{d+2}{2p+d(m-1)}} \left[\iint_{Q_0} u^p \,\mathrm{d}x\,\mathrm{d}t + \mathrm{Vol}(Q_0)\right]^{\frac{2}{2p+d(m-1)}} . \tag{2.14}$$

Moreover, the constant $\mathcal{C}_{\rm loc}$ depends only on m, d, p.

The proof presented here uses Moser's iteration process, and borrows some ideas of [14] and [21]. We will consider nested space-time cylinders, in order to obtain the first estimates needed to prove local SE. The proof will consist of the combination of several partial results, which maybe of independent interest, and will be split into several steps. Note that by scaling the proof of this kind of result need only to be done for a unit cylinder Q_0 where $R_0 = 1$ and $T_1 - T_0 = 1$, and this is the case that will be needed in the sequel.

Step 1. Space-Time Energy Inequality

Now we consider a solution u defined in a parabolic cylinder $Q = (T_0, T] \times B_R$ for some $R > R_1 > 0$, $T > 0$ and consider another parabolic cylinder $Q_1 = (T_1, T] \times B_{R_1}$, contained in Q, since we also let $T_0 < T_1 < T$. Then

Lemma 2.5 *Under these assumptions, for every $m < 1$ and $p > \max\{1, 1-m\}$, we have*

$$\int_{B_{R_1}} u^p(T,x)\,\mathrm{d}x + \iint_{Q_1} \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 \,\mathrm{d}x\,\mathrm{d}t \le \mathcal{C}(m,p) \left[\frac{1}{(R-R_1)^2} + \frac{1}{T_1 - T_0}\right] \left[\iint_Q \left(u^{p+m-1} + u^p\right) \mathrm{d}x\,\mathrm{d}t\right]. \tag{2.15}$$

The result also holds when u is a sub-solution, i.e., $u_t \le \Delta u^m$.

Proof. (i) We multiply the equation $\partial_t u = \frac{1}{m}\Delta u^m$ by $\psi^2 u^{p-1}$, with $p > 1$ to be chosen later in a suitable way, we take $\psi = \psi(t,x)$ any smooth compactly supported test function, and we integrate on the cylinder $Q = (0, T] \times B_R$. By definition of local weak solution, we obtain

$$\iint_Q \left[u^{p-1}\partial_t u + \frac{4(p-1)}{(p+m-1)^2}\left|\nabla u^{\frac{p+m-1}{2}}\right|^2\right] \psi^2 \,\mathrm{d}x\,\mathrm{d}t = -2\iint_Q u^{p+m-2}\nabla\psi\cdot\psi\nabla u\,\mathrm{d}x\,\mathrm{d}t. \tag{2.16}$$

We now use Young's inequality: for any $\overrightarrow{a}$, $\overrightarrow{b} \in \mathbb{R}^d$, and any $\delta > 0$ we have

$$|\overrightarrow{a}\cdot\overrightarrow{b}| \le \frac{\delta}{2}|\overrightarrow{a}|^2 + \frac{1}{2\delta}|\overrightarrow{b}|^2$$

Together with Hölder's inequality, this allows to estimate the right-hand side of (2.16):

$$\begin{aligned}
-2\iint_Q u^{p+m-2}\nabla\psi\cdot\psi\nabla u\,\mathrm{d}x\,\mathrm{d}t &= -\frac{4}{p+m-1}\iint_Q u^{\frac{p+m-1}{2}}\nabla\psi\cdot\psi\nabla u^{\frac{p+m-1}{2}}\,\mathrm{d}x\,\mathrm{d}t \\
&\le \frac{4}{p+m-1}\left[\frac{1}{2\delta}\iint_Q u^{p+m-1}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\delta}{2}\iint_Q \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 \psi^2\,\mathrm{d}x\,\mathrm{d}t\right] \\
&= \frac{2}{p-1}\iint_Q u^{p+m-1}|\nabla\psi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{2(p-1)}{p+m-1}\iint_Q \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 \psi^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

where in the last step we have chosen $\delta = \frac{p-1}{p+m-1} > 0$. Putting this calculation into (2.16), we obtain

$$\iint_Q u^{p-1}\partial_t u\, \psi^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{2(p-1)}{(p+m-1)^2}\iint_Q \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 \psi^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{2}{p-1}\iint_Q u^{p+m-1}\left|\nabla\psi\right|^2 \mathrm{d}x\,\mathrm{d}t$$

Now, we integrate the first term by parts (with respect to the time variable)

$$\begin{aligned}\iint_Q u^{p-1}\partial_t u\, \psi^2 \,\mathrm{d}x\,\mathrm{d}t &= \frac{1}{p}\int_{B_R}\int_0^T \partial_t(u^p)\,\psi^2\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{p}\left[\int_{B_R} u^p(T,x)\psi^2(T,x)\,\mathrm{d}x - \int_{B_R} u^p(0,x)\psi^2(0,x)\,\mathrm{d}x\right] - \frac{1}{p}\iint_Q u^p\,\partial_t(\psi^2)\,\mathrm{d}x\,\mathrm{d}t\\ &= \frac{1}{p}\left[\int_{B_R} u^p(T,x)\psi^2(T,x)\,\mathrm{d}x - \int_{B_R} u^p(0,x)\psi^2(0,x)\,\mathrm{d}x\right] - \frac{2}{p}\iint_Q u^p\,\psi\,\partial_t(\psi)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Collecting all the previous calculations, we obtain the first basic inequality:

$$\begin{aligned}&\frac{1}{p}\left[\int_{B_R} u^p(T,x)\psi^2(T,x)\,\mathrm{d}x - \int_{B_R} u^p(0,x)\psi^2(0,x)\,\mathrm{d}x\right] - \frac{2}{p}\iint_Q u^p\,\psi\,\partial_t(\psi)\,\mathrm{d}x\,\mathrm{d}t\\ &+\frac{2(p-1)}{(p+m-1)^2}\iint_Q \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 \psi^2 \,\mathrm{d}x\,\mathrm{d}t \le \frac{2}{p-1}\iint_Q u^{p+m-1}\left|\nabla\psi\right|^2 \mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{2.17}$$

(ii) In order to continue, we assume that the test function ψ satisfies

- $0 \le \psi(t,x) \le 1$, for any $(t,x) \in Q$, and $\psi(0,x) = 0$, for any $x \in B_R$
- $\psi \equiv 1$ on $Q_1 = [T_1, T] \times B_{R_1} \subset Q$ and $\psi \equiv 0$ outside Q. Of course, we take $0 \le R_1 < R$ and $0 \le T_0 < T_1 \le T$.
- Moreover, on $R \setminus R_1$, we assume that

$$|\nabla\psi| \le \frac{c_\psi}{R-R_1} \qquad \text{and} \qquad |\partial_t\psi| \le \frac{c_\psi^2}{T_1-T_0}\,.$$

We may then write (2.17) in the form

$$\begin{aligned}&\frac{p-1}{p}\int_{B_R} u^p(0,x)\psi^2(T,x)\,\mathrm{d}x + \frac{2(p-1)^2}{(p+m-1)^2}\iint_Q \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 \psi^2\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\le\ 2\left[\iint_Q u^{p+m-1}\left|\nabla\psi\right|^2\mathrm{d}x\,\mathrm{d}t + \frac{p-1}{p}\iint_Q u^p\,\psi\,\left|\partial_t(\psi)\right|\,\mathrm{d}x\,\mathrm{d}t\right]\\ &\qquad\le 2c_\psi^2\left[\frac{1}{(R-R_1)^2} + \frac{1}{T_1-T_0}\right]\left[\iint_Q \left(u^{p+m-1} + \, u^p\right)\mathrm{d}x\,\mathrm{d}t\right]\end{aligned}$$

We observe that

$$\iint_{Q_1}\left|\nabla u^{\frac{p+m-1}{2}}\right|^2 \mathrm{d}x\,\mathrm{d}t \le \iint_Q \left|\nabla u^{\frac{p+m-1}{2}}\right|^2 \psi^2\,\mathrm{d}x\,\mathrm{d}t$$

since $Q_1 \subset Q$ and $\psi \equiv 1$ on Q_1, so that we finally obtain

$$\mathcal{C}_{m,p}\left[\int_{B_{R_1}} u^p(0,x)\,\mathrm{d}x + \iint_{Q_1}\left|\nabla u^{\frac{p+m-1}{2}}\right|^2\,\mathrm{d}x\,\mathrm{d}t\right] \le 2c_\psi^2\left[\frac{1}{(R-R_1)^2}+\frac{1}{T_1-T_0}\right]\left[\iint_Q \left(u^{p+m-1}+u^p\right)\,\mathrm{d}x\,\mathrm{d}t\right],$$

where

$$\mathcal{C}_{m,p} = \min\left\{\frac{p-1}{p}\,,\,\frac{2(p-1)^2}{(p+m-1)^2}\right\}. \tag{2.18}$$

As a conclusion, we have obtained (2.15) with precise constant

$$\mathcal{C}(m,p) = 2c_\psi^2\,\mathcal{C}_{m,p}^{-1}\,. \tag{2.19}$$

Note that $\mathcal{C}_{m,p}$ depends also on d though we are not indicating it.

We conclude by noticing that the proof can be repeated for u sub-solution (with the same regularity), that means, when $u_t \le \Delta u^m$ the above estimate continues to hold. □

Improving the constant. We would like to eliminate the dependence of $\mathcal{C}(m,p)$ on p in what follows since p will vary (in an increasing way). This dependence takes place through $\mathcal{C}_{m,p}$. Now, for $m \ge 0$ it is easy to see that $\mathcal{C}_{m,p} \ge (p-1)/p$ and we have to assume that $p \ge p_0 > 1$, so that $\mathcal{C}(m,p)$ is bounded by an expression that depends only on p_0 and d.

For $m < 0$, since we have $p > 1-m$ we get $(p-1)/p > |m|/(1-m)$. A lower bound for $\mathcal{C}_{m,p}$ needs the last term to be bounded above, and this implies that p must be away from $1-m$, so that we assume that $p \ge p_0' = (1+\alpha)(1-m)$ for some $\alpha > 0$ in which case we get

$$\mathcal{C}_{m,p} \ge \min\left\{\frac{|m|}{1-m},\ 2\left(1+\frac{|m|}{\alpha(1-m)}\right)^2\right\} := \mathcal{C}(m). \tag{2.20}$$

In any case we may write $\mathcal{C}(m)$ instead of $\mathcal{C}_{m,p}$ if the family of p's fulfills the stated conditions.

Final result of Step 1. We need to improve Lemma 2.5 in the following way

Corollary 2.6 *Under the running assumptions, for every $m < 1$ and $p > \max\{1, 1-m\}$, then for any $T_0 < T_1 < T$, $0 < R_1 < R$ we have*

$$\sup_{s\in(T_1,T)}\int_{B_{R_1}} u^p(s,x)\,\mathrm{d}x + \int_{T_1}^T\int_{B_{R_1}}\left|\nabla u^{\frac{p+m-1}{2}}\right|^2\,\mathrm{d}x\,\mathrm{d}t \le \mathcal{C}(m)\left[\frac{1}{(R-R_1)^2}+\frac{1}{T_1-T_0}\right]\left[\int_{T_0}^T\int_{B_R}\left(u^{p+m-1}+u^p\right)\,\mathrm{d}x\,\mathrm{d}t\right]. \tag{2.21}$$

Moreover, if u is a sub-solution, and $u \ge 1$, we have

$$\sup_{s\in(T_1,T)}\int_{B_{R_1}} u^p(s,x)\,\mathrm{d}x + \int_{T_1}^T\int_{B_{R_1}}\left|\nabla u^{\frac{p+m-1}{2}}\right|^2\,\mathrm{d}x\,\mathrm{d}t \le \mathcal{C}(m)\left[\frac{1}{(R-R_1)^2}+\frac{1}{T_1-T_0}\right]\left[\int_{T_0}^T\int_{B_R} u^p\,\mathrm{d}x\,\mathrm{d}t\right] \tag{2.22}$$

with $\mathcal{C}(m)$ as in (2.20).

Proof. First we recall a property of the supremum: there exists a $t_0 \in (T_1, T]$ such that

$$\frac{1}{2}\sup_{s\in(T_1,T)}\int_{B_{R_1}} u^p(s,x)\,\mathrm{d}x \le \int_{B_{R_1}} u^p(t_0,x)\,\mathrm{d}x$$

We use this observation together to the result of Lemma 2.5, in two different ways:

(i) We use Lemma 2.5 with the T replaced by t_0 and still keeping $0 \le T_0 < T_1 < t_0$. We get

$$\begin{aligned}\frac{1}{2}\sup_{s\in(T_1,T)}\int_{B_{R_1}} u^p(s,x)\,\mathrm{d}x &\le \int_{B_{R_1}} u^p(t_0,x)\,\mathrm{d}x \\ &\le \mathcal{C}(m)\left[\frac{1}{(R-R_1)^2}+\frac{1}{T_1-T_0}\right]\left[\int_{T_0}^{t_0}\int_{B_R}\left(u^{p+m-1}+\,u^p\,\right)\,\mathrm{d}x\,\mathrm{d}t\right]\end{aligned}\tag{2.23}$$

In the sequel recall that $t_0 \le T$.

(ii) Next, we choose the same T_1 and we apply Lemma 2.5, to get

$$\int_{T_1}^{T}\int_{B_{R_1}}\left|\nabla u^{\frac{p+m-1}{2}}\right|^2\,\mathrm{d}x\,\mathrm{d}t \le \mathcal{C}(m)\left[\frac{1}{(R-R_1)^2}+\frac{1}{T_1-T_0}\right]\left[\int_{T_0}^{T}\int_{B_R}\left(u^{p+m-1}+\,u^p\,\right)\,\mathrm{d}x\,\mathrm{d}t\right].\tag{2.24}$$

Summing up the two inequalities (2.23) and (2.24) gives the desired inequality (2.21) .

For the last part, we remark that if we apply inequality (2.21) to a sub-solution $u \ge 1$, then $u^{p+m-1} \le u^p$ so that we obtain (2.22) , and the proof is thus concluded. □

Step 2. Iterative form of the Sobolev Inequality

The next lemma is just a different form of the usual Sobolev inequality, adapted to our aims.

Lemma 2.7 *Let $f \in \mathrm{L}^2(Q)$ with $\nabla f \in \mathrm{L}^2(Q)$. We then have*

$$\int_{T_1}^{T}\int_{B_R} f^{2\sigma}\,\mathrm{d}x\,\mathrm{d}t \le\, 2\mathcal{S}_2^2\left[\int_{T_1}^{T}\int_{B_R}\left(f^2+R^2\left|\nabla f\right|^2\right)\mathrm{d}x\,\mathrm{d}t\right]\sup_{s\in(T_1,T)}\left[\frac{1}{R^d}\int_{B_R} f^{2(\sigma-1)q}(s,x)\,\mathrm{d}x\right]^{\frac{1}{q}}\tag{2.25}$$

for any $\sigma \in \left(1,\sigma^*\right)$, and for any $0 \le T_1 < T$ and $R > 0$, where

$$\sigma^* = \frac{2^*}{2} = \begin{cases}\frac{d}{d-2}=\frac{1}{m_c}\,, & \text{if } d\ge 3\\ 2\,, & \text{if } d=1,2\end{cases}\qquad \text{and}\qquad q=\frac{\sigma^*}{\sigma^*-1}=\begin{cases}\frac{d}{2}\,, & \text{if } d\ge 3\\ 2\,, & \text{if } d=1,2.\end{cases}\tag{2.26}$$

Here, $\mathcal{S}_2$ is the constant of the classical Sobolev inequality $\|f\|_{2^*} \le \mathcal{S}_2\left(\|\nabla f\|_2+\|f\|_2\right)$, with $2^* = 2d/(d-2)$ for $d \ge 3$ and $2^* = 4$ for $d = 1, 2$.

Proof. Since the estimate (2.25) is scaling invariant, it is sufficient to prove it for $R = T - T_1 = 1$, and we denote by $B = B_1$ the unit ball of $\mathbb{R}^d$. By Sobolev and Hölder inequalities we then get

$$\begin{aligned}\int_B f^{2\sigma}\,\mathrm{d}x = \int_B f^2 f^{2(\sigma-1)}\,\mathrm{d}x &\le \left[\int_B f^{2^*}\,\mathrm{d}x\right]^{\frac{2}{2^*}}\left[\int_B f^{2(\sigma-1)q}\,\mathrm{d}x\right]^{\frac{1}{q}} \\ &\le 2\mathcal{S}_2^2\left[\int_B\left|\nabla f\right|^2\mathrm{d}x+\int_B f^2\,\mathrm{d}x\right]\sup_{s\in(0,1)}\left[\int_{B_1} f^{2(\sigma-1)q}(s,x)\,\mathrm{d}x\right]^{\frac{1}{q}}\end{aligned}$$

Integrating in time over $(0, 1)$ and rescaling back, gives inequality (2.25). □

Step 3. The Iteration

In this step we use the inequalities of the preceding steps to start the iteration in the Moser style. We first define $v(t,x) = \max\{u(t,x),1\}$. Then we observe that when u is a local weak solution to $u_t = \Delta u^m$, then v is a local weak *sub-solution* to $v_t = \Delta v^m$. It is clear that $u \le v \le 1+u$ for almost any $(t,x) \in Q$.

Preparation of the iteration step. Letting $f^2 = v^{p+m-1}$ in the modified Sobolev inequality (2.25) gives

$$\iint_{Q_1} v^{\sigma(p+m-1)}\,\mathrm{d}x\,\mathrm{d}t \le 2\mathcal{S}_2^2 \left[\iint_{Q_1} \left(v^{p+m-1} + R_1^2 \left|\nabla v^{\frac{p+m-1}{2}}\right|^2\right)\mathrm{d}x\,\mathrm{d}t\right] \left[\sup_{t\in(T_1,T)} \frac{1}{R_1^d}\int_{B_{R_1}} v^{(p+m-1)(\sigma-1)q}\,\mathrm{d}x\right]^{\frac{1}{q}} \tag{2.27}$$

where $Q_1 = (T_1,T]\times B_{R_1} \subset Q_0 = (T_0,T]\times B_R$.

Since we have assumed that $v \ge 1$, then $v^{p+m-1} \le v^p$ and we can use the energy inequality (2.22) to estimate the two terms of the right hand side of the above inequality (2.27), in terms of the same quantity. First we estimate

$$\begin{aligned}\iint_{Q_1} \left(v^{p+m-1} + R_1^2 \left|\nabla v^{\frac{p+m-1}{2}}\right|^2\right)\mathrm{d}x\,\mathrm{d}t &\le \iint_{Q_1} v^p\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + R_1^2\,\mathcal{C}(m)\left[\frac{1}{(R_0-R_1)^2} + \frac{1}{T_1-T_0}\right]\left[\iint_{Q_0} v^p\,\mathrm{d}x\,\mathrm{d}t\right] \\ &\le 2\,R_1^2\,\mathcal{C}(m)\left[\frac{1}{(R_0-R_1)^2} + \frac{1}{T_1-T_0}\right]\left[\iint_{Q_0} v^p\,\mathrm{d}x\,\mathrm{d}t\right].\end{aligned}$$

In the last step we use the fact that $R_1^2\,\mathcal{C}(m)\left[\frac{1}{(R_0-R_1)^2} + \frac{1}{T_1-T_0}\right] \ge 1$, which is not restrictive.

We next estimate the sup term, again using the energy inequality (2.22), but we replace p with $(p+m-1)(\sigma-1)q$. If the exponent is larger than $\max\{1, 1-m\}$ we obtain

$$\sup_{t\in(T_1,T)} \frac{1}{R_1^d}\int_{B_{R_1}} v^{(p+m-1)(\sigma-1)q}\,\mathrm{d}x \le \frac{\mathcal{C}(m)}{R_1^d}\left[\frac{1}{(R-R_1)^2} + \frac{1}{T_1-T_0}\right]\left[\iint_{Q_0} v^{(p+m-1)(\sigma-1)q}\,\mathrm{d}x\,\mathrm{d}t\right]$$

Summing up, we have estimated (2.27) as follows

$$\begin{aligned}\iint_{Q_1} v^{\sigma(p+m-1)}\,\mathrm{d}x\,\mathrm{d}t &\le 4\mathcal{S}_2^2\; R_1^{2-\frac{d}{q}}\,\mathcal{C}(m)^{1+\frac{1}{q}}\left[\frac{1}{(R_0-R_1)^2} + \frac{1}{T_1-T_0}\right]^{1+\frac{1}{q}} \\ &\quad \times\left[\iint_{Q_0} v^p\,\mathrm{d}x\,\mathrm{d}t\right]\left[\iint_{Q_0} v^{(p+m-1)(\sigma-1)q}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{q}}\end{aligned} \tag{2.28}$$

Finally, we remark that $R_1^{2-\frac{d}{q}} = 1$, since $\frac{d}{q} = 2$, q being defined as in Lemma 2.7.

The First Iteration Step. We now use (2.28) in the following way: first we choose $\sigma \in (1,\sigma^*)$, where σ^* is as in Lemma 2.7, in such a way that

$$(p+m-1)(\sigma-1)q = p \qquad \text{that is} \qquad \sigma = 1 + \frac{p}{q(p+m-1)}.$$

A straightforward calculation shows that $\sigma \in (1,\sigma^*)$ if and only if $p > p_c$. This is where the restriction on p appears for the first time.

We are now ready to begin with the first iterative step, by letting

$$p_0 = p = (p+m-1)(\sigma-1)q\,, \qquad \text{and} \qquad p_1 = (p_0+m-1)\sigma = p_0\left(1+\frac{1}{q}\right)+m-1\,.$$

We remark that

$$p_1 > p_0 \iff p_0 > p_c = \frac{d(1-m)}{2}\,.$$

Estimate (2.28) now becomes

$$\begin{aligned}\iint_{Q_1} v^{p_1}\,\mathrm{d}x\,\mathrm{d}t &\le 4\mathcal{S}_2^2\ \mathcal{C}(m)^{1+\frac{1}{q}}\left[\frac{1}{(R_0-R_1)^2}+\frac{1}{T_1-T_0}\right]^{1+\frac{1}{q}}\left[\iint_{Q_0} v^{p_0}\ \mathrm{d}x\,\mathrm{d}t\right]^{1+\frac{1}{q}}\\ &= I_{0,1}\left[\iint_{Q_0} v^{p_0}\ \mathrm{d}x\,\mathrm{d}t\right]^{1+\frac{1}{q}}\end{aligned} \tag{2.29}$$

which is the first iterative step.

The k-th Iteration Step. Letting

$$p_{k+1} = p_k\left(1+\frac{1}{q}\right)+m-1\,, \qquad \text{with} \qquad p_{k+1} > p_k \iff p_k \ge p_0 > p_c,$$

we get the iterative inequality

$$\left[\iint_{Q_{k+1}} v^{p_{k+1}}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_{k+1}}} \le I_{k,k+1}^{\frac{1}{p_{k+1}}}\left[\iint_{Q_k} v^{p_k}\ \mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_k}\left(1+\frac{1}{q}\right)\frac{p_k}{p_{k+1}}}. \tag{2.30}$$

In order to find a convenient value for $I_{k,k+1}$ we choose a decreasing sequence of radii $R_\infty \longleftarrow R_{k+1} < R_k < R_0$ such that $0 < R_k - R_{k+1} = \rho/k^2$, and a sequence of times $0 \le T_0 \le T_k \le T_{k+1} \to T_\infty < T$ such that $T_{k+1}-T_k = \tau/k^4$. This means taking

$$\rho = c_1(R_0-R_\infty), \qquad \tau = c_2(T_\infty - T_0), \tag{2.31}$$

with $c_1 = 1/\left(\sum_{k=0}^{+\infty} k^{-2}\right) > 0$, and $c_2 = 1/\left(\sum_{k=0}^{+\infty} k^{-4}\right) > 0$. Then,

$$\begin{aligned}I_{k,k+1} &= 4\mathcal{S}_2^2\,\mathcal{C}(m)^{1+\frac{1}{q}}\left[\frac{1}{(R_k-R_{k+1})^2}+\frac{1}{T_{k+1}-T_k}\right]^{1+\frac{1}{q}} \le 4\mathcal{S}_2^2\,\mathcal{C}(m)^{1+\frac{1}{q}}\ \left(2\left(\rho^{-2}+\tau^{-1}\right)\,k^4\right)^{1+\frac{1}{q}}\\ &\le [2\mathcal{S}_2]^2\ \left[2\left(\rho^{-2}+\tau^{-1}\right)\mathcal{C}(m)\right]^{1+\frac{1}{q}}\ \left(k^4\right)^{1+\frac{1}{q}} = J_0\,J_1^{1+\frac{1}{q}}\ \left(k^4\right)^{1+\frac{1}{q}}\end{aligned} \tag{2.32}$$

We now calculate the exponents p_k:

$$\begin{aligned}p_{k+1} &= p_k\left(1+\frac{1}{q}\right)+m-1 = \left[1+\frac{1}{q}\right]^{k+1} p_0 + (m-1)\sum_{n=0}^{k}\left[1+\frac{1}{q}\right]^n\\ &= \left[1+\frac{1}{q}\right]^{k+1}\left[p_0+(m-1)\sum_{j=1}^{k+1}\left[1+\frac{1}{q}\right]^{-j}\right] = \left[p_0-q(1-m)\right]\left[1+\frac{1}{q}\right]^{k+1}+q(1-m)\,.\end{aligned} \tag{2.33}$$

notice that

$$\lim_{k\to\infty} \frac{\left[1+\frac{1}{q}\right]^{k+1}}{p_{k+1}} = \frac{1}{p_0+q(m-1)} \qquad \text{and} \qquad \lim_{k\to\infty} \frac{1}{p_{k+1}}\sum_{j=0}^{k}\left[1+\frac{1}{q}\right]^j = \frac{q}{p_0+q(m-1)}\,.$$

The iterative step now reads

$$\left[\iint_{Q_{k+1}} v^{p_{k+1}}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_{k+1}}} \le I_{k,k+1}^{\frac{1}{p_{k+1}}}\; I_{k-1,k}^{\left[1+\frac{1}{q}\right]\frac{1}{p_{k+1}}} \;\ldots\; I_{0,1}^{\left[1+\frac{1}{q}\right]^k\frac{1}{p_{k+1}}} \left[\iint_{Q_0} v^{p_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{\left[1+\frac{1}{q}\right]^{k+1}}{p_{k+1}}} \tag{2.34}$$

Now we use (2.32) to estimate

$$\begin{aligned} I_{k,k+1}^{\frac{1}{p_{k+1}}}\; I_{k-1,k}^{\left[1+\frac{1}{q}\right]\frac{1}{p_{k+1}}} \;\ldots\; I_{0,1}^{\left[1+\frac{1}{q}\right]^k\frac{1}{p_{k+1}}} &\le \left[J_0\, J_1^{1+\frac{1}{q}}\right]^{\frac{1}{p_{k+1}}\sum_{j=0}^{k}\left[1+\frac{1}{q}\right]^j} \\ &\quad\times\; k^{4^{\frac{1}{p_{k+1}}}}\,(k-1)^{4^{\frac{1+\frac{1}{q}}{p_{k+1}}}}\,(k-2)^{4^{\frac{(1+\frac{1}{q})^2}{p_{k+1}}}} \;\ldots\; 2^{4^{\frac{(1+\frac{1}{q})^{k-2}}{p_{k+1}}}}\;1 \\ &= \left[J_0\, J_1^{1+\frac{1}{q}}\right]^{\frac{1}{p_{k+1}}\sum_{j=0}^{k}\left[1+\frac{1}{q}\right]^j} \prod_{j=1}^{k} j^{4^{\frac{(1+\frac{1}{q})^{k-j}}{p_{k+1}}}} \end{aligned} \tag{2.35}$$

Moreover, passing to the limit in (2.34) when $k\to\infty$, we get (we refer to Appendix A3 for further details)

$$\sup_{Q_\infty}|v| \le J_0^{\frac{q}{p_0+q(m-1)}}\, J_1^{\frac{q+1}{p_0+q(m-1)}}\, s_1 \mathrm{e}^{4(q+1)} \left[\iint_{Q_0} v^{p_0}\,\mathrm{d}x dt\right]^{\frac{1}{p_0+q(m-1)}}. \tag{2.36}$$

We have estimated the constants to ensure that they remain bounded in the limit $k\to+\infty$, see the Appendix for the details. Moreover, these constants blow up when $R_\infty\to R_0$ or $T_\infty\to T_0$: indeed, while J_0 and s_1 only depend on m,d and p, the constant J_1 depends on ρ and τ: and blows up as $T_0-T_\infty\to 0$ or $R_0-R_\infty\to 0$ since $J_1\sim\left(\rho^{-2}+\tau^{-1}\right)$. We have finished this part of the iteration since $q=d/2$ which gives in the sequel the correct exponent in the last integral. In case $d=1,2$ we have to observe that $q=2$ so that the exponent is $s=1/(p_0+2(m-1)$.

This fact forces the final cylinder $Q_\infty=(T_\infty,T]\times B_{R_\infty}\subset Q_0=(T_0,T]\times B_{R_0}$ to be strictly contained in the initial one. We obtain

$$\sup_{Q_\infty}|v| \le \mathcal{C}_{\mathrm{loc}}\left[\frac{1}{(R_0-R_\infty)^2}+\frac{1}{T_\infty-T_0}\right]^{\frac{q+1}{p_0+q(m-1)}} \left[\iint_{Q_0} v^{p_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_0+q(m-1)}}. \tag{2.37}$$

Notice that $\mathcal{C}_{\mathrm{loc}}$ only depends on m,d and p. We conclude the proof by going back from v to u, using the fact that $u\le v\le u+1$, by definition of v. ¿From (2.37) we easily get

$$\begin{aligned} \sup_{Q_\infty}|u| \le \sup_{Q_\infty}|v| &\le \mathcal{C}_{\mathrm{loc}}\left[\frac{1}{(R_0-R_\infty)^2}+\frac{1}{T_\infty-T_0}\right]^{\frac{q+1}{p_0+q(m-1)}} \left[\iint_{Q_0} v^{p_0}\,\mathrm{d}x\,\mathrm{d}t\right]^{\frac{1}{p_0+q(m-1)}} \\ &\le \mathcal{C}_{\mathrm{loc}}\left[\frac{1}{(R_0-R_\infty)^2}+\frac{1}{T_\infty-T_0}\right]^{\frac{q+1}{p_0+q(m-1)}} \left[\iint_{Q_0} u^{p_0}\,\mathrm{d}x\,\mathrm{d}t+\mathrm{Vol}\big(Q_0\big)\right]^{\frac{1}{p_0+q(m-1)}}. \end{aligned}$$

This concludes the proof, after changing the notation, putting $p=p_0$, $R_\infty=R_1<R_0$, $T_\infty=T_1>T_0$, and $q=d/2$ since we are dealing with $d\ge 3$. □

2.3 Local Smoothing effect. Proof of Theorem 2.1

We now combine the results of Theorem 2.3 and of Theorem 2.4, to prove the local smoothing effect in the form described in Theorem 2.1. We consider u defined in $(0,T)\times B_{R_0}(x_0)$, then take a smaller radius R_1 and write $R_1=(1-\varepsilon)\varrho$ and $R_0=(1+\varepsilon)\varrho$: this defines ρ and ε. We then consider the rescaled solution

$$\widehat{u}(t,x)=K\,u\left(\tau\,t\,,\varrho\,x+x_0\right)\,,\qquad K=\left(\frac{\varrho^2}{\tau}\right)^{\frac{1}{1-m}} \tag{2.38}$$

with $0<\tau<T$. Then, we apply to $\widehat{u}$ the result of Theorem 2.4 over the cylinders $Q_0=(0,1]\times B_1$ and $Q_1=(\varepsilon^2,1]\times B_{1-\varepsilon}$, for some $\varepsilon\in(0,1)$, so that the bound reads

$$\sup_{Q_1}|\widehat{u}|\le\frac{\mathcal{C}_{\rm loc}}{\varepsilon^{2\frac{q+1}{p+q(m-1)}}}\left[\iint_{Q_0}\widehat{u}^p\,{\rm d}x\,{\rm d}t+{\rm Vol}\big(Q_0\big)\right]^{\frac{1}{p+q(m-1)}}\le\frac{\mathcal{C}_{\rm loc}}{\varepsilon^{2\frac{q+1}{p+q(m-1)}}}\left[\iint_{Q_0}\widehat{u}^p\,{\rm d}x\,{\rm d}t+\omega_d\right]^{\frac{1}{p+q(m-1)}}$$

since ${\rm Vol}\big(Q_0\big)=\omega_d$. Moreover, we know that $\mathcal{C}_{\rm loc}$ only depends on m,d,p.

Next, we estimate the time integral, using Theorem 2.3 applied to the rescaled solution $\widehat{u}$ on the balls $B_1\subset B_{1+\varepsilon}$ and for times $t\in[0,1]$ and for $p>p_c>1-m$:

$$\int_{B_1}|\widehat{u}(t,x)|^p\,{\rm d}x\le 2^{\frac{p}{1-m}-1}\int_{B_{1+\varepsilon}}|\widehat{u}(0,x)|^p\,{\rm d}x+2^{\frac{p}{1-m}-1}\left[K_{\varepsilon,p}\,t\right]^{\frac{p}{1-m}}\,, \tag{2.39}$$

where

$$K_{\varepsilon,p}=\frac{p\,c_{m,d}}{\varepsilon^2}\,{\rm Vol}\left(B_{1+\varepsilon}\setminus B_1\right)^{(1-m)/p}\le p\,c'_{m,d}\varepsilon^{\frac{1-m}{p}-2}$$

An integration in time over $(0,1)$ gives

$$\begin{aligned}\iint_{Q_0}\widehat{u}^{p_0}\,{\rm d}x\,{\rm d}t=\int_0^1\int_{B_1}\widehat{u}^{p_0}\,{\rm d}x\,{\rm d}t&\le 2^{\frac{p}{1-m}-1}\int_{B_{1+\varepsilon}}|\widehat{u}(0,x)|^p\,{\rm d}x+\frac{2^{\frac{p}{1-m}-1}}{\frac{p}{1-m}+1}\left[p\,c'_{m,d}\varepsilon^{\frac{1-m}{p}-2}\right]^{\frac{p}{1-m}}\\&=\mathcal{S}_0\int_{B_\lambda}|\widehat{u}(0,x)|^p\,{\rm d}x+\frac{\mathcal{K}_{m,d,p}}{\varepsilon^{\frac{2p}{(1-m)}-1}}\end{aligned}$$

and we remark that $\mathcal{K}_{m,d,p}$ and $\mathcal{S}_0$ only depend on m,p.

Now we put together the above estimates, and we rescale back from $\widehat{u}$ to u, also changing variable $\varrho x=y$ in the integrals, and we obtain (using $K^{1-m}=\varrho^2/\tau$)

$$\begin{aligned}\sup_{(s,y)\in(\varepsilon^2\tau,\tau]\times B_{(1-\varepsilon)\varrho}}u(s,y)&\le K^{-1}\frac{\mathcal{C}_{\rm loc}}{\varepsilon^{2\frac{q+1}{p+q(m-1)}}}\left[\mathcal{S}_0\int_{B_{1+\varepsilon}}|\widehat{u}(0,x)|^p\,{\rm d}x+\frac{\mathcal{K}_{m,d,p}}{\varepsilon^{\frac{2p}{(1-m)}-1}}+\omega_d\right]^{\frac{1}{p+q(m-1)}}\\&\le\varrho^{\frac{2q-d}{p+q(m-1)}}\frac{\mathcal{C}_{\rm loc}}{\varepsilon^{2\frac{q+1}{p+q(m-1)}}}\frac{\kappa_1\,\mathcal{S}_0^{\frac{1}{p+q(m-1)}}}{\tau^{\frac{q}{p+q(m-1)}}}\left[\int_{B_{(1+\varepsilon)\varrho}}|u(0,y)|^p{\rm d}y\right]^{\frac{1}{p+q(m-1)}}\\&+\kappa_2\,\frac{\mathcal{C}_{\rm loc}}{\varepsilon^{2\frac{q+1}{p+q(m-1)}}}\left[\frac{\mathcal{K}_{m,d,p}}{\varepsilon^{\frac{2p}{(1-m)}-1}}+\omega_d\Big)\right]^{\frac{1}{p+q(m-1)}}\left[\frac{\tau}{\varrho^2}\right]^{\frac{1}{1-m}}.\end{aligned} \tag{2.40}$$

In dimension $d\ge3$ we have taken $q=d/2$ which allows to cancel the appearance of ϱ in the first term of the right-hand side and simplify the dependence on τ. We then have

$$\sup_{(s,y)\in(\varepsilon^2\tau,\tau]\times B_{R_1}}u(s,y)\le\frac{\overline{\mathcal{C}}_1}{\tau^{d\vartheta_p}}\left[\int_{B_{R_0}}|u_0(x)|^p\,{\rm d}x\right]^{2\vartheta_p}+\overline{\mathcal{C}}_2\left[\frac{\tau}{\varrho^2}\right]^{\frac{1}{1-m}} \tag{2.41}$$

where we have also used $(a+b)^\sigma \le \kappa_1 a^\sigma + \kappa_2 b^\sigma$. Putting $\tau = t$ we have obtained in particular the desired formula (2.1) . We last remark that ε must be strictly positive even if it can be chosen arbitrarily small; in the limit $\varepsilon \to 0$, the constants $\overline{\mathcal{C}}_i$ blow up, since

$$\overline{\mathcal{C}}_1 = \kappa_1\, \mathcal{S}_0^{\frac{1}{p+q(m-1)}}\, \frac{\overline{\mathcal{C}}_{\rm loc}}{\varepsilon^{2\frac{q+1}{p+q(m-1)}}} \qquad \text{and} \qquad \overline{\mathcal{C}}_2 = \kappa_2\, \frac{\overline{\mathcal{C}}_{\rm loc}}{\varepsilon^{2\frac{q+1}{p+q(m-1)}}} \left[\frac{\mathcal{K}_{m,d,p}}{\varepsilon^{\frac{2p}{(1-m)}-1}} + \omega_d\right]^{\frac{1}{p+q(m-1)}} \tag{2.42}$$

We conclude the proof by switching to the same notations as in the statement of Theorem 2.1, just by substituting $R_1 = (1-\varepsilon)\varrho$ and $R_0 = (1+\varepsilon)\varrho$, it is clear that the result holds for any $R_1 < R_0$, and that the constants $\overline{\mathcal{C}}_i$ blow up when $R_1 \to R_0$. We recall that $\mathcal{C}_{\rm loc}$ only depends on m, d, p.

We have thus proved the following result:

Theorem 2.8 *Let $p \ge 1$ if $m > m_c$ or $p > p_c$ if $m \le m_c$. Then there are positive constants $\overline{\mathcal{C}}_1$, $\overline{\mathcal{C}}_2$ such that for any $0 < R_1 < R_0$ we have*

$$\sup_{(s,y)\in(t_0,t]\times B_{R_1}} u(s,y) \le \frac{\overline{\mathcal{C}}_1}{t^{d\vartheta_p}} \left[\int_{B_{R_0}} |u_0(x)|^p \,{\rm d}x\right]^{2\vartheta_p} + \overline{\mathcal{C}}_2 \left[\frac{t}{R_0^2}\right]^{\frac{1}{1-m}}. \tag{2.43}$$

where $t_0 = \big[(R_0 - R_1)/(2R_0)\big]^2\, t$ and the constants $\overline{\mathcal{C}}_i$ depend on m, d and p, R_1 and R_0 and blow up when $R_1/R_0 \to 1$; an explicit formula for $\overline{\mathcal{C}}_i$ is given by (2.42).

The above theorem is nothing but a slightly stronger form for Theorem 2.1: we just take the limit $R_1 \to 0$ in inequality (2.43) to obtain (2.1). The final expression for the constants $\mathcal{C}_i$ in (2.1) corresponds to the limit of $\overline{\mathcal{C}}_i$ in (2.42) as $\varepsilon \to 1$ (i.e. $R_1 \to 0$) and do not depend on the radii. This concludes the proof of Theorem 2.1. □

3 Part III. Harnack Inequalities

By joining together the local upper and lower estimates obtained in Parts I and II we can draw interesting conclusions in terms of special forms of Harnack Inequalities. These are expressions relating the maximum and minimum of a solution inside certain cylinders. In the standard case one has

$$\sup_{Q_1} u(t,x) \le C \inf_{Q_2} u(t,x), \tag{3.1}$$

see [26] and [28]. The main idea is that the formula applies for a large class of solutions and the constant C that enters the relation does not depend on the particular solution, but only on the data like m, d and the size of the cylinder R, but not on time. The cylinders in the standard case are supposed to be ordered in time, $Q_1 = [t_1, t_2] \times B_R(x_0)$, $Q_2 = [t_3, t_4] \times B_R(x_0)$, with $t_1 \le t_2 < t_3 \le t_4$.

It is well-known that in the degenerate nonlinear elliptic or parabolic problems a plain form of the inequality does not hold. In the work of DiBenedetto and collaborators, see the book [17] or the recent work [18], versions are obtained where some information of the solution is used to define so-called intrinsic sizes, like the size of the parabolic cylinder(s), that usually depends on $u(t_0, x_0)$. They are called *intrinsic Harnack inequalities*. The authors of [18] show that the size of a convenient cylinder for the Harnack inequality to hold has the form

$$I_R(t_0,x_0) = \big(t_0 - c\,u(t_0,x_0)^{1-m}R^2,\, t_0 + c\,u(t_0,x_0)^{1-m}R^2\big) \times B_R(x_0)$$

with a fixed constant $c>0$ which depends only on m,d, that can be chosen "a priori", but only in the good range $m_c<m<1$. This cylinder is called intrinsic because it depends on the value of the solution u at a given point (t_0,x_0).

The Harnack Inequalities of [20, 18], in the supercritical range then read: *There exist positive constants $\overline{c}$ and $\overline{\delta}$ depending only on m,d, such that for all $(t_0,x_0)\in Q=(0,T)\times\Omega$ and all cylinders of the type $I_{8R}\subset Q$, we have*

$$\overline{c}\,u(t_0,x_0)\le \inf_{x\in B_R(x_0)} u(t,x)$$

for all times $t_0-\overline{\delta}\,u(t_0,x_0)^{1-m}R^2<t<t_0+\overline{\delta}\,u(t_0,x_0)^{1-m}R^2$. The constants $\overline{\delta}$ and $\overline{c}$ tend to zero as $m\to 1$ or as $m\to m_c$.

They also give a counter-example in the lower range $m<m_c$, by producing an explicit local solution that does not satisfy any kind of Harnack inequality (neither of the types called intrinsic, elliptic, forward, backward) if one fixes "a priori" the constant c. At this point a natural question is posed:

What form, if any, the Harnack estimate might take for m in the subcritical range $0<m\le m_c$?

The following is an answer this question.

New approach. After the introduction of the lower bounds of the Aronson-Caffarelli type, it became clear that the size of the initial L^1 or L^p norm in a certain ball can be used in a natural way to define intrinsic quantities for later times, and this is the approach the authors followed in [8] for the easier range $m_c<m<1$. The Harnack inequalities we derive below are based on such an idea and apply also for $0<m\le m_c$. Indeed, if one wants to apply the result of DiBenedetto et. al. [17, 18] mentioned above, to a local weak solution defined on $[0,T]\times\Omega$, where T is possibly the extinction time, the Harnack inequality of [20, 18] reads:

There exists positive constants $\overline{\delta}<\overline{c}$ depending only on m,d such that if

$$\overline{c}\,u(t_0,x_0)\le\left[\frac{\min\{t_0,T-t_0\}}{(8R)^2}\right]^{\frac{1}{1-m}}\qquad \textit{and}\qquad \mathrm{dist}(x_0,\partial\Omega)<\frac{R}{8}\,, \tag{3.2}$$

we then have that

$$\overline{c}\,u(t_0,x_0)\le \inf_{x\in B_R(x_0)} u(t,x)\,,$$

for all times $t_0-\overline{\delta}\,u(t_0,x_0)^{1-m}R^2<t<t_0+\overline{\delta}\,u(t_0,x_0)^{1-m}R^2$. The constants $\overline{\delta}$ and $\overline{c}$ tend to zero as $m\to 1$ or as $m\to m_c$. The intrinsic hypothesis (3.2) is guaranteed in the good range by the fact that solutions with initial data in $\mathrm{L}^1_{\mathrm{loc}}$ are bounded, while in the very fast diffusion range hypothesis (3.2) fails, and should be replaced by :

$$u(t,x_0)\le\frac{c_{m,d}}{\varepsilon^{\frac{2p\vartheta_p}{1-m}}}\left[\frac{\|u(t_0)\|_{\mathrm{L}^p(B_R)}\,R^d}{\|u(t_0)\|_{\mathrm{L}^1(B_R)}R^{\frac{d}{p}}}\right]^{2p\vartheta_p}\left[\frac{t_0}{R^2}\right]^{\frac{1}{1-m}}.$$

This local upper bound can be derived by the smoothing effect of Theorem 2.1, whenever $t_0+\varepsilon t_*(t_0)<t<t_0+t_*(t_0)$, see full details in the proof of Theorem 3.2.

The Size of Intrinsic Cylinders. We will show that the new critical time

$$t_*(s)=c_{m,d}\,R^{2-d(1-m)}\|u(s)\|^{1-m}_{\mathrm{L}^1(B_R(x_0))} \tag{3.3}$$

introduced in Part I, gives the size of the intrinsic cylinders: in the supercritical fast diffusion range this time can be chosen a priori just in terms of the initial datum, but in the subcritical range its size changes with time; roughly speaking the diffusion is so fast that the initial local information is not relevant after some time, which is represented by t^*. We must bear in mind that a large class of solutions completely extinguish in finite time. We proceed next with the new results.

Inequalities of Forward, Backward and Elliptic Type. For small times cf. Theorem 3.1, or for suitable intrinsic cylinders, cf. Theorem 3.2, we obtain inequalities where the infimum is taken at a later time than the supremum (forward Harnack inequalities), or at the same time (elliptic Harnack inequalities), or even at an earlier time (backward Harnack inequalities).

Throughout this section we take $0 < m < 1$ and consider a local nonnegative weak solution u of the FDE defined in a cylinder $Q = (0,T)\times\Omega$, taking initial data $u(0,x) = u_0(x)$ in $\mathrm{L}^p_{\mathrm{loc}}\big(\Omega\big)$, with $p = 1$ if $m_c < m < 1$ or $p > p_c$ if $0 < m \le m_c$. We make no assumption on the boundary condition (apart from nonnegativity). Also, let x_0 be a point in Ω and let $6R \le \mathrm{dist}(x_0, \partial\Omega)$. As before, we let T_m be the so-called minimal life time, corresponding to data u_0 and ball $B_R(x_0)$, and we define $t_*(s)$ as in (3.3) and $t_* = t_*(0)$, which is equal or less than T_m. First we prove Harnack inequalities for initial times

Theorem 3.1 *Under the above conditions, for any $t_0 \in (0, t_*]$ and $0 \le \vartheta \le \min\{t_* - t_0, t_0/2\}$ the following inequality holds*

$$\inf_{x\in B_R(x_0)} u\big(t_0 \pm \theta, x\big) \ge \mathcal{H}\, u\big(t_0, x_0\big) \tag{3.4}$$

where

$$\mathcal{H} = \mathcal{C}_6\, R^{\frac{2-d}{m}} \left[\frac{\|u_0\|_{\mathrm{L}^1(B_R)}}{T_m^{\frac{1}{(1-m)}}}\right]^{\frac{1}{m}} \left[\frac{\|u_0\|^{2p\vartheta_p}_{\mathrm{L}^p(B_R)}}{t_0^{\frac{2p\vartheta_p}{1-m}}} + \frac{1}{R^{\frac{2}{1-m}}}\right]^{-1} \tag{3.5}$$

and $\mathcal{C}_6$ depends only on m, d, p. $\mathcal{H}$ goes to zero when $t_0 \to 0$.

This form of Harnack estimate we propose must be called generalized, since the constant depends on the solution through certain norms of the data. But we remind the reader that a proper restriction of the class of initial data allows to control $\mathcal{H}$ at any time $0 < t < t_*$.

Proof. The proof consists of two steps.

From center to infimum. First of all we have to pass from the center to the minimum in the positivity estimates of Theorem 1.1. Fix a point $z \in \overline{B_R(x_0)}$, ad consider the following MDP, centered at z:

$$\begin{cases} \partial_t u = \Delta(u^m) & \text{in } Q_{T,R_0} = (0,T)\times B_{9R/2}(z) \\ u(0,x) = u_0(x)\chi_{B_R(x_0)} & \text{in } B_R(x_0), \quad \text{and} \quad \mathrm{supp}(u_0) \subseteq B_{2R}(z) \\ u(t,x) = 0 & \text{for } t>0 \text{ and } x \in \partial B_{9R/2}(z)\,, \end{cases} \tag{3.6}$$

it is then clear that $B_R(x_0) \subset B_{2R}(z) \subset B_{9R/2}(z) \subset B_{6R}(x_0)$. Applying the result of Theorem 1.6 to the solution u to the above minimal problem with minimal life time $T_m = T_m(u_0)$, we get

$$t_* := \frac{k_0}{2}\left(\frac{9}{2}R - 2R\right)^2 \left[\frac{\int_{B_{2R}(z)} u_0\,\mathrm{d}x}{\mathrm{Vol}\big(B_{9R/2}(z)\setminus B_{2R}(z)\big)}\right]^{1-m} = k_{m,d}\, R^{2-d(1-m)} \left[\int_{B_R(x_0)} u_0\,\mathrm{d}x\right]^{1-m} \le T_m\,, \tag{3.7}$$

which does not depend on $z(t)$. We then obtain:

$$u^m(t,z) \ge c_1'\,(2R)^{2-d} t^{\frac{m}{1-m}} T_m^{-\frac{1}{1-m}} \int_{B_{2R}(z)} u_0(x)\,\mathrm{d}x = c_1\,R^{2-d} t^{\frac{m}{1-m}} T_m^{-\frac{1}{1-m}} \int_{B_R(x_0)} u_0(x)\,\mathrm{d}x.$$

for any $t \in [0, t_*]$. Since $z \in B_R(x_0)$ is arbitrary and does not enter neither in the above lower bounds, neither in the formula (3.3) for $t_* = t_*(0)$, we can set $z = z(t)$ as the point such that:

$$\inf_{x\in B_R} u(t,x) = u\big(t, z(t)\big) \ge c_1 \left[\frac{\int_{B_R(x_0)} u_0(x)\,\mathrm{d}x}{T_m^{\frac{1}{(1-m)}}\,R^{d-2}}\right]^{\frac{1}{m}} t^{\frac{1}{1-m}} \tag{3.8}$$

for any $0 \le t \le t_*$, where $t_* = t_*(0)$ is given by (3.3). Once we have obtained the result for the minimal Dirichlet problem we pass to a general weak solution as it has been done in Subsection 1.2, hence estimate (3.8) holds for any weak solution.

Joining upper and lower estimates. Let now $t_0 \in (0, t_*]$ and choose $\theta > 0$ small so that $t_0/2 < t_0 - \theta$, and $t_0 + \theta \le t_*$. By the lower estimate (3.8) we know that $u(t_0, x_0)$ is positive for any $t_0 \in (0, t_*]$ and

$$\begin{aligned}\inf_{x\in B_R(x_0)} u\big(t_0 \pm \theta, x\big) &\ge c_1'^{\frac{1}{m}}\, R^{\frac{2-d}{m}} \|u_0\|_{\mathrm{L}^1(B_R)}^{\frac{1}{m}} T_m^{-\frac{1}{m(1-m)}}\, (t_0 \pm \theta)^{\frac{1}{1-m}} \\ &\ge c_1'' 2^{-\frac{1}{1-m}} \|u_0\|_{\mathrm{L}^1(B_R)}^{\frac{1}{m}} T_m^{-\frac{1}{m(1-m)}}\, R^{\frac{2-d}{m}}\, t_0^{\frac{1}{1-m}}\end{aligned} \tag{3.9}$$

We now use the local smoothing effect of Theorem 2.1 for $t = t_0 \in (0, t_*)$:

$$u(t_0, x_0) \le \mathcal{C}_3 \left[\frac{\|u_0\|_{\mathrm{L}^p(B_R)}^{2p\vartheta_p}}{t_0^{\frac{2p\vartheta_p}{1-m}}} R^{\frac{2}{1-m}} + 1\right] \left[\frac{t_0}{R^2}\right]^{\frac{1}{1-m}}. \tag{3.10}$$

We have thus proved that for $t_0 \in (0, t_*]$ and $\theta \le \min\{t_* - t_0, t_0/2\}$ we have

$$\inf_{x\in B_R(x_0)} u\big(t_0 \pm \theta, x\big) \ge c_2\, \|u_0\|_{\mathrm{L}^1(B_R)}^{\frac{1}{m}} T_m^{-\frac{1}{m(1-m)}}\, R^{\frac{2-d}{m}} \left[\frac{\|u_0\|_{\mathrm{L}^p(B_R)}^{2p\vartheta_p}}{t_0^{\frac{2p\vartheta_p}{1-m}}} + \frac{1}{R^{\frac{2}{1-m}}}\right]^{-1} u(t_0, x_0)\,. \tag{3.11}$$

This concludes the proof. □

By shifting the interval $[0, t_*]$ to $[t_0, t_0 + t_*(t_0)] \subseteq [0, T]$ we can prove a more intrinsic flavored version of backward-forward-elliptic Harnack inequality in the following

Theorem 3.2 *Under the above conditions, there exists constants h_1, h_2 depending only on m, d, p, such that, for any $\varepsilon \in [0,1]$ the following inequality holds*

$$\inf_{x\in B_R(x_0)} u\big(t \pm \vartheta, x\big) \ge h_1\, \varepsilon^{\frac{2p\vartheta_p}{1-m}} \left[\frac{\|u(t_0)\|_{\mathrm{L}^1(B_R)} R^{\frac{d}{p}}}{\|u(t_0)\|_{\mathrm{L}^p(B_R)} R^d}\right]^{2p\vartheta_p + \frac{1}{m}} u\big(t, x_0\big) \tag{3.12}$$

for any

$$t_0 + \varepsilon t_*(t_0) < t \pm \vartheta < t_0 + t_*(t_0)\,, \qquad t_*(t_0) = h_2\, R^{2-d(1-m)} \|u(t_0)\|_{\mathrm{L}^1(B_R(x_0))}^{1-m}$$

Proof of Theorem 3.2. Assume $t_0 = 0$, the result will follow by translation in time. We continue the proof of Theorem 3.1: we further estimate the local smoothing effect of Theorem 2.1 for $t = t_0 \in (0, t_*)$:

$$\begin{aligned} u(t_0, x_0) &\le \mathcal{C}_3 \left[\frac{\|u_0\|_{\mathrm{L}^p(B_R)}^{2p\vartheta_p}}{t_0^{\frac{2p\vartheta_p}{1-m}}} R^{\frac{2}{1-m}} + 1 \right] \left[\frac{t_0}{R^2} \right]^{\frac{1}{1-m}} \le \mathcal{C}_4 \left[\frac{\|u_0\|_{\mathrm{L}^p(B_R)}^{2p\vartheta_p}}{\varepsilon t_*^{\frac{2p\vartheta_p}{1-m}}} R^{\frac{2}{1-m}} \right] \left[\frac{t_0}{R^2} \right]^{\frac{1}{1-m}} \\ &\le \frac{\mathcal{C}_5}{\varepsilon^{\frac{2p\vartheta_p}{1-m}}} \left[\frac{\|u_0\|_{\mathrm{L}^p(B_R)} R^d}{\|u_0\|_{\mathrm{L}^1(B_R)} R^{\frac{d}{p}}} \right]^{2p\vartheta_p} \left[\frac{t_0}{R^2} \right]^{\frac{1}{1-m}} \end{aligned} \tag{3.13}$$

since we have put $t_0 \ge \varepsilon t_*$ and $t_* = t_*(0)$ as in (3.3). Next we use the estimate for the extinction time proved in Sections 1.4.1 and 1.4.2 , that can be rewritten as

$$T_m^{\frac{1}{1-m}} \le k_{m,p,d} R^{\frac{2}{1-m} - \frac{d}{p}} \|u_0\|_{\mathrm{L}^p(B_R)} \qquad \text{for any } p \ge \max\{p_c, 1\}.$$

The lower estimates (3.9) becomes

$$\inf_{x \in B_R(x_0)} u\big(t_0 \pm \theta, x\big) \ge c_1 \left[\frac{\|u_0\|_{\mathrm{L}^1(B_R)} R^{\frac{2}{1-m} - d}}{T_m^{\frac{1}{(1-m)}}} \right]^{\frac{1}{m}} \left[\frac{t_0}{R^2} \right]^{\frac{1}{1-m}} \ge c_2 \left[\frac{\|u_0\|_{\mathrm{L}^1(B_R)} R^{\frac{d}{p}}}{\|u_0\|_{\mathrm{L}^p(B_R)} R^d} \right]^{\frac{1}{m}} \left[\frac{t_0}{R^2} \right]^{\frac{1}{1-m}} \tag{3.14}$$

Joining now inequalities (3.13) and (3.14) we obtain that there exists constants h_1 , h_2 depending only on m, d, p, such that, for any $\varepsilon \in [0, 1]$ the following inequality holds

$$\inf_{x \in B_R(x_0)} u\big(t \pm \vartheta, x\big) \ge h_1\, \varepsilon^{\frac{2p\vartheta_p}{1-m}} \left[\frac{\|u(0)\|_{\mathrm{L}^1(B_R)} R^{\frac{d}{p}}}{\|u(0)\|_{\mathrm{L}^p(B_R)} R^d} \right]^{2p\vartheta_p + \frac{1}{m}} u\big(t, x_0\big) \tag{3.15}$$

for any

$$\varepsilon t_*(0) < t \pm \vartheta < t_*(0)\,, \qquad t_*(0) = h_2\, R^{2-d(1-m)} \|u(0)\|_{\mathrm{L}^1(B_R(x_0))}^{1-m}\,.$$

We conclude the proof by translating the result from $[0, t_*(0)]$ to $[t_0, t_0 + t_*(t_0)]$, that we know to be included in $[0, T]$ as explained in Part I . □

Remarks. (i) Estimate (3.12) is completely of local type, since it involves only local quantities. In the supercritical range $m_c < m < 1$, we can let $p = 1$ in (3.12) to get

$$\inf_{x \in B_R(x_0)} u\big(t \pm \vartheta, x\big) \ge h_1\, \varepsilon^{\frac{2p\vartheta_p}{1-m}} u\big(t, x_0\big)$$

for any $t_0 + \varepsilon t_*(t_0) < t \pm \vartheta < t_0 + t_*(t_0)$. In this way we recover the above mentioned results of DiBenedetto et al., cf. [20, 18]. This theorem complements and supports the lower Harnack inequality (1.9) of Part I, in which the constant does not depend on u_0.

Joining the upper and lower estimates for the Cauchy Problem, we obtain the Global Harnack principle as the authors did in [8].

(ii) In the subcritical range $0 < m \le m_c$, the Harnack estimates cannot have a universal constant independent of u_0, as already mentioned, cf. also in [18] for a counterexample. We have proved that if one allows the constant to depend on the initial data, then it is possible to obtain intrinsic Harnack inequalities, and the price we pay is having the minimal life time in the constant, as in Theorem 3.1, but this information is a bit unpractical and we replace it with some local L^p-norm with $p>p_c$ of the initial datum.

(iii) We have shown that the size of the intrinsic cylinders is always proportional to a ratio of local L^p norms. Note that in the supercritical range it simplifies and only depends on the local L^1 norm.

(iv) The quantity ε represents an arbitrary small waiting time, that is needed in order for the regularization to take place and to allow quantitative intrinsic Harnack inequalities.

(v) Backward Harnack inequalities are a bit surprising, but they reflect a typical feature of the fast diffusion processes, that is the extinction phenomena, namely

$$\inf_{x\in B_R(x_0)} u\big(t-\vartheta,x\big)\ge h_1\,\varepsilon^{\frac{2p\vartheta_p}{1-m}}\left[\frac{\|u(t_0)\|_{\mathrm{L}^1(B_R)}R^{\frac{d}{p}}}{\|u(t_0)\|_{\mathrm{L}^p(B_R)}R^d}\right]^{2p\vartheta_p+\frac{1}{m}} u\big(t,x_0\big) \tag{3.16}$$

for any

$$t_0+\varepsilon t_*(t_0)<t-\vartheta<t_0+t_*(t_0)\,,\qquad t_*(t_0)=h_2\,R^{2-d(1-m)}\|u(t_0)\|_{\mathrm{L}^1(B_R(x_0))}^{1-m}\,.$$

This inequality is compatible with the fact that the solution extinguish at some later time, remaining strictly positive before. This backward inequality is typical of singular equation and can not hold for the degenerate -porous media- case $m>1$, neither for the linear heat equation case, $m=1$.

The same remark applies for the Elliptic Harnack inequality, that is when $\vartheta=0$.

An alternative form of Harnack Inequalities. We provide a form of Harnack inequalities of forward, backward and elliptic type, avoiding the intrinsic framework, and the waiting time $\varepsilon\in[0,1]$.

Theorem 3.3 *Under the above conditions, there exists positive constants $\mathcal{C}_1$, $\mathcal{C}_2$ and h_2 depending only on m,d and p such that*

$$\sup_{x\in B_R} u(t,x)\le\frac{\mathcal{C}_1}{t^{d\vartheta_p}}\,\|u(t_0)\|_p^{2p\vartheta_p}+\mathcal{C}_2\left[\frac{\|u(t_0)\|_{\mathrm{L}^p(B_R)}R^d}{\|u(t_0)\|_{\mathrm{L}^1(B_R)}R^{\frac{d}{p}}}\right]^{\frac{1}{m}}\inf_{x\in B_R} u(t\pm\vartheta,x) \tag{3.17}$$

for any

$$0\le t_0<t\pm\vartheta<t_0+t_*(t_0)\le T\,,\qquad t_*(t_0)=h_2\,R^{2-d(1-m)}\|u(t_0)\|_{\mathrm{L}^1(B_R(x_0))}^{1-m}\,,$$

where $\vartheta_p=1/(2p-d(1-m))$.

Proof. First we observe that we can pass from the center x_0 to the supremum in the upper estimate of Theorem 2.1 by doubling the radius of the ball on the right hand side, namely

There exist positive constants $\mathcal{C}_1$, $\mathcal{C}_2'$ depending only on m,d, such that for any $t,R>0$ we have

$$\sup_{x\in B_R(x_0)} u(t,x)\le\frac{\mathcal{C}_1}{t^{d\vartheta_p}}\|u(t_0)\|_{\mathrm{L}^p(B_{2R}(x_0))}^{2p\vartheta_p}+\mathcal{C}_2'\left[\frac{t}{R^2}\right]^{\frac{1}{1-m}}\,.$$

Joining the above inequality with the lower bound of Theorem 1.1 in the form of (3.14), we obtain the inequality (3.17) for $t_0 = 0$. We conclude the proof by shifting the interval $[0, t_*(0)]$ to $[t_0, t_0 + t_*(t_0)]$, that we know to be included in $[0, T]$ as explained in Part I. □

Remark. In the good fast diffusion range we can let $p = 1$, so that inequality (3.17) reads

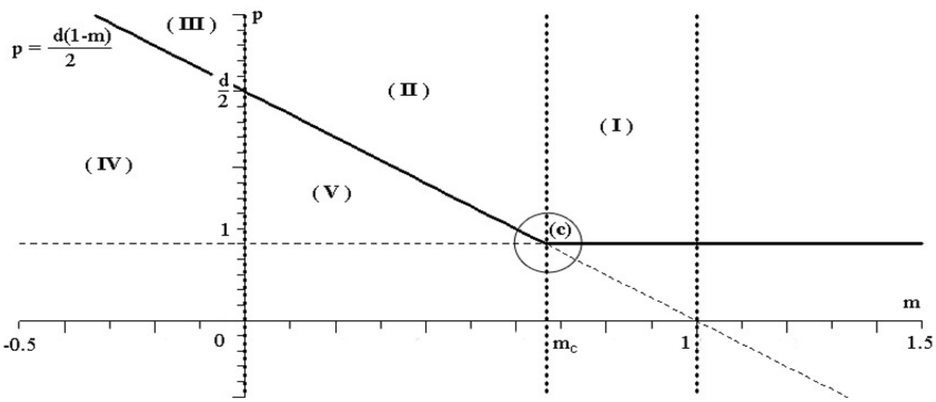
$$\sup_{x\in B_R} u(t,x) \le \frac{\mathcal{C}_1}{t^{d\vartheta_p}} \|u(t_0)\|_1^{2\vartheta_1} + \mathcal{C}_2 \inf_{x\in B_R} u(t\pm\vartheta, x)\,.$$

4 Concluding section

First, we sketch a panorama of the local estimates in the different exponent ranges $m < 1$ and for different integrability exponents $p \ge 1$ that may serve as further orientation for the reader. Then we make a series of general comments, and finally we review related works.

4.1 Panorama and open problems

The values of m_c and p_c are defined in the Introduction.



(I) *Good Fast Diffusion Range:* $m \in (m_c, 1)$ and $p \ge 1$. Here, the local smoothing effect holds, cf. Theorem 2.1, as well as the Reverse smoothing effect and the global Harnack principle, proved in [8], see also [9]. As for the result of the present paper, we have provided a different proof of the positivity result in Theorem 1.3. We have also proved intrinsic Harnack inequalities of forward, elliptic, or backward type, cf. Theorems 3.1, 3.2 and 3.3. recovering the existing results. Finally, for times close to the extinction time, in case extinction occurs, the authors show in [10, 9], that Elliptic Harnack inequalities hold up to extinction.

(II) *Very Fast Diffusion Range:* $m \in (0, m_c)$ and $p \ge p_c > 1$. The local smoothing effect of Theorem 2.1 holds, as well as the lower estimates of Theorem 1.1. These are the only known local positivity and smoothing results in this range. For any positive time we have the Aronson-Caffarelli type estimates of Theorem 1.1. We have also proved intrinsic Harnack inequalities of forward, elliptic, or backward type, cf. Theorems 3.1, 3.2 and 3.3. These are the only parabolic Harnack inequalities known in this whole range.

An open problem is to pass from local to global estimates in this very fast diffusion range. For $m > m_c$ this is done in [8] in the form of a global Harnack principle.

Another open problem is to find the rate of convergence to an appropriate extinction profile for the Dirichlet problem on domains. A minimal rate of extinction can be obtained by our intrinsic Harnack estimates, of Theorem 3.2, details will appear separately.

(c) *Critical case:* $m = m_c$ and $p > p_c = 1$. The local upper and lower estimates of zone (II) apply, as well as the Harnack inequalities. In an upcoming paper we will show how to pass from local to global estimates, obtaining global lower bounds with super-exponential time decay.

(III) *Negative exponent range:* $m \le 0$ with $p > p_c$. No positivity result is known in this range, and the technique developed in this paper does not allow to treat this case. Recall that solutions of the homogeneous Dirichlet problem on bounded domains vanish instantaneously (hence, there is no actual solution). The local upper bound of Theorem 2.1 is still valid, and this is the only known local upper estimate in this range. *It is an open problem in this case to find positivity and a posteriori Harnack inequalities, if any.*

(IV) *Negative range:* $m \le 0$ with $p < p_c$. The smoothing effect is not true, since initial data are not in L^p with $p > p_c$, cf. [31], and the solutions of the Cauchy problem with data in $\mathrm{L}^p(\mathbb{R}^d)$ will vanish instantaneously, setting a strong limitation to positivity results, which must be based of course on local bounds. Again, solutions to the homogeneous Dirichlet problem on bounded domains vanish instantaneously. *It is an open problem to find positivity estimates in this range, if any.* In general, Harnack inequalities are not possible in this range since solution may not be (neither locally) bounded.

(V) *Very Fast Diffusion Range,* $m \in (0, m_c)$ with small integrability exponent $p \in [1, p_c]$. It is well known that the smoothing effect is not true in general, since initial data are not in L^p with $p > p_c$, cf. [8, 12, 31] . The lower estimates in this case are given by Theorem 1.1. These are the only known local positivity results in this whole range. In general, Harnack inequalities are not possible in this case since solution may not be (neither locally) bounded.

4.2 Some general remarks

- We stress the fact that our results are completely local, and they apply to any kind of initial-boundary value problem, in any Euclidean domain: Dirichlet, Neumann, Cauchy, or problem for large solutions, namely when $u = +\infty$ on the boundary, etc. Natural extensions are fast diffusion problems with variable coefficients and fast diffusion problems on manifolds.
- We calculate (almost) explicitly all the constants, through all the paper.
- Our positivity and Harnack inequalities generalize the results of [19, 20, 18], valid only in the good fast diffusion range, in the sense that we recover their result with a different proof, and we extend it quantitatively to the very fast diffusion range.
- More specific information can be obtained when we restrict our attention to particular classes of solutions, like the solutions of the Cauchy Problem, the solutions of the initial and boundary value problem on a bounded domain with $u = 0$ on the boundary, or with $u = +\infty$ on the boundary. All these solutions have additional properties that can be exploited. We will not enter into those topics for reasons of space.

- We will not enter either into the derivation of Hölder continuity and further regularity from the Harnack inequalities. This is a subject extensively treated in the works of DiBenedetto et al., see [21, 17, 18] and references therein.

- Actually, the fast diffusion equation can be well-posed even for Borel measures (i.e., not locally finite measures) as initial data. This is done for the Cauchy problem in the whole space in [12], but in that case the smoothness of the constructed solutions is lost and the concept of extended continuous solution has to be introduced. The problem turns out to be well-posed in that extended class in the good range $m_c < m < 1$.

- The main ideas of this paper can be extended to related equations, such as the p-Laplacian, but the technical details may not be immediate.

4.3 Elliptic connection

Part of the difficulties of the FDE in the lower range of m can be explained by the intimate relation of the equation with semilinear elliptic theory. Indeed, if we try the separation of variables ansatz $u(t,x) = H(t)F(x)$ on equation (0.1) we easily get the possible formulas $H_1(t) = t^{1/(1-m)}$ or $H_2(t) = (T-t)^{1/(1-m)}$, and then $G = F^m$ satisfies the elliptic equation

$$-\Delta G \pm c\, G^q = 0, \quad \text{with} \ \ q = \frac{1}{m}\,, \tag{4.1}$$

so that $q > 1$ if $0 < m < 1$. The constant is $c = m/(1-m)$, and the sign $\pm$ corresponds to the choices H_1 or H_2 respectively. It is well-known that the theory of equation (4.1) is difficult for large values of q, notably for $q \ge q_s = (d+2)/(d-2)$. The exponent corresponding to m_c is $q_c = d/(d-2)$, a lower exponent that appears sometimes in the study of singularities. Note that when m is negative G is an inverse power of F; moreover, q is negative.[3] The FDE-elliptic connection extends to the study of self-similar solutions of different types, which is a fundamental tool of the FDE theory and is described in [25] and [31] among other references.

It is interesting to check the correspondence of our time evolving results with the theory of elliptic equations. An easy way of doing that it to apply the results to special solutions. The simplest case, i.e., stationary solutions, is too simple, hence we prefer to try the separate variable solutions

$$u(t,x) = (T-t)^{1/(1-m)}F(x).$$

Putting $U = F^m$ and $q = 1/m$ we get the elliptic equation $\Delta U + cU^q = 0$, as in (4.1). It can also be written as

$$\Delta U + a(x)U = 0, \quad a(x) = cU^{q-1} = cF^{1-m}.$$

Let us check the upper estimate (2.1) of Theorem 2.1. Using the separate variable form of u we immediately see that the time dependencies disappear (a confirmation of the correct scaling of the formula) and the we obtain a local boundedness result for U of the form

$$U(x) \le C\|U\|_q^{\theta(q)} + C_2 R^{-2p/(1-p)},$$

on the condition that $F \in \mathrm{L}^p_{loc}(\Omega)$ with $p \ge p_c$ which means that $a(x) \in \mathrm{L}^r_{loc}(\Omega)$ with $r \ge d/2$, a classical condition. This is for us another way of checking that p_c is a natural exponent.

[3] We leave to the reader the exponential formulas that are obtained for $m = 0$.

A similar conclusion can be derived from application of the lower estimate (1.1) of Theorem 1.1. We leave the details to the reader. The elliptic conclusions can also be checked on selfsimilar solutions of the type $u(t,x) = (T-t)^\alpha F(x\,(T-t)^\beta)$, as the ones considered in [31].

4.4 Short review of related works

The range $m \le m_c$ has remained outside of most of the publications on the questions of positivity and Harnack estimates. For positivity and boundedness, let us first mention the works of Bertsch and collaborators [5, 6] who treat the equation satisfied by the so-called pressure variable $v = c/u^{1-m}$, i.e., an inverse power of u. It covers the equivalent to the whole fast diffusion range $m < 1$ in terms of viscosity solutions; the questions are somewhat different from our program.

We list next and comment on a number of previous results on the subject of Harnack inequalities for the fast diffusion equation.

- DiBenedetto and Kwong proved in [19] that in the good fast diffusion range $m_c < m < 1$ intrinsic Harnack Inequalities of forward type do hold, under the positivity assumption that there exists a point (t_0, x_0) such that $u(t_0, x_0) > 0$. This value controls from below the infimum in a small ball at a later time, with sizes depending on $u(x_0, t_0)$.

- Later on DiBenedetto, Kwong and Vespri [20] improved on the previous result proving the Global Harnack Principle in the wider range $m_s = \frac{d-2}{d+2} < m < 1$, by means of comparison with the separation of variable solution, always under the assumption of positive solutions and a stronger assumption on the initial data, namely $u_0^m \in W_0^{1,2}(\Omega)$. These estimates are global in space but not in time since the constants blow up as $t \to 0$. Hence the interest in combining them with information we provide for all small times in direct dependence of the local L^p norms of the initial data.

- More recently, DiBenedetto, Gianazza and Vespri, [18], extended the results to the variable coefficient case in the form Harnack inequalities which are of Forward, Elliptic and Backward type; it applies in the good FDE range, and always under the positivity assumption for some (x_0, t_0). Some of these estimates had been proved by the authors in the constant coefficient case in [8] , see also [9].

- The above mentioned Harnack inequalities imply Hölder continuity of the solution and sometimes analyticity, cf. [20, 18].

- The power $m_s = (d-2)/(d+2)$, has been studied by Del Pino and Saez [15], as part of the study of the asymptotics of the evolutionary Yamabe problem. They perform the transformation into a fast diffusion problem posed on the sphere via stereographic projection, which is possible for this exponent. They get an elliptic Harnack inequality which holds for a good class of solutions, but they do not prove a parabolic Harnack inequality.

- None of the above quoted papers considers the problem of finding Harnack inequalities when the time approaches the finite extinction time (if there is one). This has been done by the authors in [10, 9], showing that Elliptic Harnack inequalities hold up to the extinction time. The proof is completely different, we draw fine asymptotic properties, by a careful analysis of the extinction profile.

Summing up, two results are known in the lower range $m \le m_c$: [15] that applies for $m = m_s$, and [20] that applies for $m_s < m < 1$. They both refer to a different point of view.

Appendixes

A1 Aleksandrov's Reflection Principle

Here we state the Reflection Principle of Aleksandrov in a slightly different form, more useful to our purposes. We already used this proposition, in [8]. Other forms of the same principle, in different settings can be found, for example in [22]), Proposition 2.24 (pg. 51) or in [2], Lemma 2.2.

Proposition 5.1 (Local Aleksandrov's Reflection Principle, [8]) *Let $B_{\lambda R_0}(x_0) \subset \mathbb{R}^d$ be an open ball with center in $x_0 \in \mathbb{R}^d$ of radius λR_0 with $R_0 > 0$ and $\lambda > 2$. Let u be a solution to problem*

$$\begin{cases} \partial_t u = \Delta(u^m) & \text{in } (0,+\infty) \times B_{\lambda R_0}(x_0) \\ u(0,x) = u_0(x) & \text{in } B_{\lambda R_0}(x_0) \\ u(t,x) = 0 & \text{for } t > 0 \text{ and } x \in \partial B_{\lambda R_0}(x_0) \end{cases} \tag{5.2}$$

with $\mathrm{supp}(u_0) \subset B_{R_0}(x_0)$*. Then, for any $t > 0$ one has:*

$$u(t,x_0) \ge u(t,x_2)$$

for any $t > 0$ and for any $x_2 \in A_{\lambda,R_0}(x_0) = B_{\lambda R_0}(x_0) \setminus B_{2R_0}(x_0)$. Hence,

$$u(t,x_0) \ge |A_{\lambda,R_0}(x_0)|^{-1} \int_{A_{\lambda,R_0}(x_0)} u(t,x)\,\mathrm{d}x = \fint_{A_{\lambda,R_0}(x_0)} u(t,x)\,\mathrm{d}x \tag{5.3}$$

The proof of this result can be found in the Appendix of [8].

A2 On the extinction time

The extinction time plays a role in our study of positivity in Part I. We devote some paragraphs to comment on its occurrence in FDE. It has been observed in the literature, cf. [7, 8, 31], that Lemma 2.2 can be used to obtain lower bounds on the finite extinction time $T = T(u_0)$. Indeed, just let $t = 0$, $s = T$ and $v \equiv 0$, in (2.2), to get

$$c_1(m,d,\lambda) R^{d(1-m)-2} \|u_0\|^{1-m}_{\mathrm{L}^1(B_R)} \le T \tag{5.4}$$

Suppose that $\Omega = \mathbb{R}^d$. When $m_c < m < 1$, it is easy to see that $2 - d(1-m) > 0$, so that, simply by letting $R \to \infty$, we see that $T(u_0) \to +\infty$ if $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, that means that solutions corresponding to initial data $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ do not extinguish in finite time, i.e., there is global positivity.

Such a situation does not occur for solutions defined in all of $\mathbb{R}^d$ when $0 < m < m_c$. Indeed, in that fast diffusion range solutions with initial data in $\mathrm{L}^1(\mathbb{R}^d)$ may extinguish. The question is studied in Chapters 5–7 of [31] where upper and lower estimates for T in terms of the data are obtained for the problem posed in the whole space $\mathbb{R}^d$ with $m < m_c$.

On the other hand, solutions extinguish in finite time for the Cauchy-Dirichlet problem posed in a bounded domain with zero boundary conditions for all $0 < m < 1$, cf. subsections 1.4.1, 1.4.2 or [**?**, 16].

An estimate similar to (5.4) cannot be valid for $m < 0$ since it is known that the Cauchy problem does not admit solutions with data $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ [30], or even in $\mathrm{L}^p(\mathbb{R}^d)$ with $p < p_c$ [13] for $m \le 0$. We may say in these cases that $T = 0$. Actually, when one tries to solve approximate problems with, say, strictly positive and bounded boundary data and pass to the limit, the approximate solutions converge to zero uniformly in cylinders of the form $Q_\tau = (\tau, \infty) \times \mathbb{R}^d$. This can be summed up as the formation of an initial discontinuity layer, cf. [31]. No solutions exist either for the Cauchy-Dirichlet problem posed in a bounded domain with zero boundary conditions when $m \le 0$. There is also a peculiar case, $m = 0$ and $d = 2$, where there exist solutions but the waiting time can be fixed a priori independently of the initial data [31], hence no estimate as above is possible either.

Note finally that the L^p estimate of Theorem 2.3 for $p > 1$ cannot be used to obtain lower estimates on FET, since they hold only for $t \ge s \ge 0$.

A3 Details of the iterative calculations of Subsection 2.2, Step 3

We show here in detail of a calculation used to pass to the limit when $k \to \infty$ in the inequality (2.34), in the proof of Theorem 2.4. We adopt the notations used there.

$$
\begin{aligned}
\prod_{j=1}^{k} j^{2\frac{(1+\frac{1}{q})^{k-j}}{p_{k+1}}} &\le \exp\left[2\sum_{j=1}^{k} \frac{(1+\frac{1}{q})^{k-j}}{p_{k+1}} \log(j)\right] \le \exp\left[2\sum_{j=1}^{k} \frac{(1+\frac{1}{q})^{k-j}\log(j)}{\left[p_0 - q(1-m)\right]\left[1+\frac{1}{q}\right]^{k+1} + (q+1)(1-m)}\right] \\
&\le s_0 \exp\left[2\sum_{j=1}^{k}(1+\frac{1}{q})^{-j-1}\log(j)\right] \le s_1 \exp\left[2\sum_{j=1}^{k}(1+\frac{1}{q})^{-j}\right] \\
&\le s_1 \exp\left[2\sum_{j=1}^{\infty}(1+\frac{1}{q})^{-j}\right] = s_1 \mathrm{e}^{2(q+1)}
\end{aligned}
$$

where s_i, $i = 0, 1$ are positive numerical constant. $\square$

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