Special fast diffusion with slow asymptotics. Entropy method and flow on a Riemannian manifold

Matteo Bonforte[1,3], Gabriele Grillo[2,4], Juan Luis Vázquez[1,5]

Abstract

We consider the asymptotic behaviour of positive solutions $u(t,x)$ of the fast diffusion equation $u_t = \Delta\,(u^m/m) = \mathrm{div}\,(u^{m-1}\nabla u)$ posed for $x \in \mathbb{R}^d$, $t>0$, with a precise value for the exponent $m=(d-4)/(d-2)$. The space dimension is $d \ge 3$ so that $m<1$, and even $m=-1$ for $d=3$. This case had been left open in the general study [7] since it requires quite different functional analytic methods, due in particular to the absence of a spectral gap for the operator generating the linearized evolution.

The linearization of this flow is interpreted here as the heat flow of the Laplace-Beltrami operator of a suitable Riemannian Manifold $(\mathbb{R}^d, \mathbf{g})$, with a metric $\mathbf{g}$ which is conformal to the standard $\mathbb{R}^d$ metric. Studying the pointwise heat kernel behaviour allows to prove suitable Gagliardo-Nirenberg inequalities associated to the generator. Such inequalities in turn allow to study the nonlinear evolution as well, and to determine its asymptotics, which is identical to the one satisfied by the linearization. In terms of the rescaled representation, which is a nonlinear Fokker–Planck equation, the convergence rate turns out to be polynomial in time. This result is in contrast with the known exponential decay of such representation for all other values of m.

Keywords. Nonlinear evolutions, singular parabolic equations, fast diffusion, Riemannian manifolds, asymptotics.

Mathematics Subject Classification. 35B45, 35B65, 35K55, 35K65, 58J35.

(1) Departamento de Matemáticas, Universidad Autónoma de Madrid, Campus de Cantoblanco, 28049 Madrid, Spain

(2) Dipartimento di Matematica, Politecnico di Torino, corso Duca degli Abruzzi 24, 10129 Torino, Italy

(3) e-mail address: matteo.bonforte@uam.es

(4) e-mail address: gabriele.grillo@polito.it

(5) e-mail address: juanluis.vazquez@uam.es

1 Introduction

In this paper we shall describe the asymptotic behaviour (as $t \to \infty$) of a class of solutions $u(t,x) \geq 0$ of the fast diffusion equation (FDE)

$$\partial_t u = \Delta\left(u^m/m\right) = \nabla \cdot (u^{m-1}\nabla u), \qquad m < 1, \tag{1.1}$$

posed[(a)] for $t > 0$ in the whole space, $x \in \mathbb{R}^d$, in dimensions $d \geq 3$, and taking initial data

$$u(0,x) = u_0(x) > 0, \tag{1.2}$$

where u_0 belongs to a class to be made precise below, in particular u_0 is bounded and decays at infinity like $c\,|x|^{2/(1-m)}$ with lower order terms. Actually, since $m < (d-2)/d$ it is well-known that for initial data of the above form the weak solution exists and is unique for small times, and then extinguishes completely after a finite time $T = T(m, d, u_0)$, [37]. We are interested in the behaviour of such solutions near extinction, as $t \nearrow T$. A detailed analysis of this question has been performed in a recent paper [7] for general $m < 1$ (even when $m \leq 0$), but rates of convergence could not be obtained for a special value of the diffusion exponent m, precisely[(b)] for $m_* = (d-4)/(d-2)$. We refer to that paper for further references to the abundant literature on the topics of entropy methods, rescaling and rates of convergence for this type of nonlinear diffusion equations, cf. also [10], [12], [17], [19], [21], [26], [28], [31], [36].

The present paper is devoted to settle the asymptotic behaviour in the special case $m = m_*$. We shall see that it falls out of the scope of asymptotic theory developed in the paper [7] for the rest of the values $m < 1$, both in the type of techniques and in the type of results. The clue to finding the stabilization rates of the rescaled orbits towards their equilibrium states in this special case relies on

(i) Realizing that a suitable linearization of the rescaled flow can be viewed as plain heat flow in a suitable Riemannian manifold. This allows us to use the very detailed theory that has been developed for studying (the long-time behaviour of) such flows, see [32, 18];

(ii) Performing a study of nonlinear stability based on an interesting modification of the entropy methods of [7].

The paper gives precise statements and proof of these assertions. It is organized as follows: in the next section we shall review the needed facts about the asymptotic behaviour of our problem in the more general setting of variable $m \in \mathbb{R}$. We also introduce the family of entropies that allow to prove the plain stabilization result of [7], as well as the linearization method that allows to obtain rates of convergence when $m \neq m_*$, when used in combination with the limit of the previous entropies. The failure of this approach in the special case m_* is identified in [7] as the lack of a suitable *spectral gap* in the operator analysis of the linearized problem.

We then focus on $m = m_*$ and address such an essential difficulty. The convergence results are carefully stated in Section 3. We start the new work in Section 4 by a detailed analysis of the linearized equation, identified as a heat flow on a cigar-like Riemannian manifold. This is followed by the results on linearized stability. Section 5 gathers all the results needed in the comparison of linear and nonlinear entropies. The proof of nonlinear stability is given in Section 6. In Section 7 we revise the convergence for the case $m \neq m_*$ and show that our method provides a shorter proof and also a small improvement with respect to [7].

(a) There is no restriction $m > 0$. The last expression represents a parabolic equation whenever $u > 0$ even if $m \leq 0$. For $m = 0$ the first expression must be replaced by $\Delta \log(u)$.

(b) In space dimension $d = 4$ we have $m_* = 0$, logarithmic diffusion. For $d = 3$ we deal with $m_* < 0$, a very singular case that was only briefly exposed in [7].

The main difference in the asymptotic results is that convergence to a selfsimilar profile takes place with a rate of approach that differs in a marked way from the power rate of all the cases $m \neq m_*$. The convergence is most clearly visualized below in the rescaled representation, a nonlinear Fokker–Planck equation, where it takes the form of stabilization towards equilibrium with a polynomial rate of approach in terms of the new time variable s. Specifically, the study is made in terms of rescaled variable

$$v(s,y) = (T-t)^{-d\beta}u(t,x), \quad y = ax(T-t)^{\beta}, \quad s = \gamma \log(T/(T-t)), \tag{1.3}$$

where $T > 0$ is the extinction time and the constants β, γ and a are precisely defined in Section 2[c]. This rescaled variable satisfies the nonlinear Fokker-Planck equation, see (2.4) or (2.6), which is better suited for the asymptotic analysis. The stationary profile for the latter version of the equation is given by the simple expression

$$V_D(y) = 1/(D + |y|^2)^{(d-2)/2}, \qquad D > 0, \tag{1.4}$$

for a suitable constant D determined by the initial data. This simple expression is handy since V_D and powers of it will appear as weights in some functional inequalities that are essential in our study. In terms of the new logarithmic time s (that goes to infinity as $t \to T$), the long time behaviour of the rescaled flow takes the form of stabilization towards the profile V_D with a power rate of convergence:

$$\|v(s,y) - V_D(y)\|_{L^\infty(\mathbb{R}^d)} = O(s^{-1/4}) \quad \text{as} \quad s \to +\infty. \tag{1.5}$$

This rate replaces the exponential decay formulas with respect to s of the cases $m \neq m_*$, that have been obtained in [7]. This polynomial rate in s is slower than the exponential rate in s that obtains in all other cases $m < 1$, $m \neq m_*$. Summing up, we are in a case of what is called *slow asymptotics*, or *critical slowing down*, in mechanical systems and statistical mechanic, and such cases need as a rule special analytical methods.

The needed assumption on the initial data is that $v(0,y)$ be a small perturbation of $V_D(y)$ in a sense made precise by assumptions (H1') and (H2') below. Let us stress that some kind of similar assumption on the data is needed to obtain the asymptotic result. Actually, for data that decay at infinity with a slower rate than $O(|y|^{2/(1-m)})$ (i.e., with a smaller power) solutions do not even extinguish in finite time. On the other hand, for data that decrease with a larger power, the behaviour near the extinction time follows a completely different pattern that is described in the monograph [37].

We complete this introduction with some comments on related topics. Let us first recall that there are two other known instances of interpretation of fast diffusions as geometrical flows. The first case is the evolution Yamabe flow, i.e., the fast diffusion with $m = (d-2)/(d+2)$, $d \geq 3$. It describes how a conformal Riemannian metric evolves by scalar curvature; in that case u is interpreted as the conformal factor of the metric raised to the power $(d+2)/4$. An asymptotic study of this problem is made by Del Pino and Sáez in [20] with exponential convergence to a separate variable solution, and the results are extended in [37]. In the second case we deal with Ricci flow in dimension $d = 2$, as proposed by Hamilton [25], and then $m = 0$ (logarithmic diffusion). The asymptotic behaviour in that case is rather complex, cf. [16] or the monograph [37]. Both models happen in a different context, since they consist of interpreting the variable u in the FDE as the evolving conformal factor of a conformal representation, while here we consider a heat flow on a fixed manifold as the linearization limit of a nonlinear fast diffusion flow. They have in common the property of extinction in finite time.

[c] The exponent β is essential, whereas the values of $a, \gamma > 0$ are just convenient.

Finally, we mention that a number of formulas and ideas used in the theory of Ricci flows bear a close similarity with developments in linear and nonlinear diffusion theory. Thus, the use of entropies is prominent in Perelman's study of the Ricci flow, [35], where he introduces his functionals $\mathcal{F}$ and $\mathcal{W}$ which are extensions of the Einstein-Hilbert functional. He then writes the gradient flow for the functionals as a system of equations for the evolving metric g_{ij} and a scalar function f, which satisfies a backward heat equation. Strong connections exist with studies of entropies for heat equations on a static manifold, see for example Ni [34] and also the general references [14, 15, 29]. In a recent paper [33] Lu, Ni, Villani and one of the authors investigate Harnack inequalities and entropies for porous medium and fast diffusion equations on static manifolds that are closely related to Yau, Hamilton and Perelman's work, and on the other hand are close to the subject of this paper. The whole topic calls for further understanding.

List of notations

D_0, D_1, D_*: the constants involved in Assumptions (H1), (H1′), (H2),(H2′). See Section 2.2.

f, $\tilde{f}$: the functions involved in Assumptions (H2), (H2′). See Section 2.2.

$\mathcal{F}(w)$: the relative entropy. See Formula (2.10).

F: the linearized relative entropy. See Proposition 4.12 and Lemma 5.3. The argument of F can be both g and w (see below for the meaning of the latter quantities).

g: the (weighted) linearization of $w-1$. See Formulas (2.14) and (2.15).

$\mathbf{g}_\alpha$: the metric describing the geometric interpretation of the linearized operator. See Formula (4.3).

$\mathcal{I}(w)$: the relative Fisher information. See Formula (2.12).

I_m: the linearized Fisher information, see (4.2). The index m is dropped in Section 5 for brevity.

$K(t,x,y)$: the heat kernel of the Laplace–Beltrami associated to $\mathbf{g}_\alpha$. See Section 4.1.

L_m: the linearized generator. See Formula (4.1).

μ_*: the weighted measure $\mathrm{d}\mu_* = V_D^{2-m_*}\,\mathrm{d}x$. See just before Section 4.2. The L^p norms in Section 4 are taken w.r.t. μ_*.

T: the extinction time of the Barenblatt solutions and of the solutions considered. See Formula (2.3) and Assumption (H1).

$u(x,t)$: the solution to the fast diffusion equation. See Formula (1.1).

$U_D(t,x)$: the Barenblatt solutions for $m > m_c$. See Formulas (2.1) and (2.2).

$U_{D,T}(t,x)$: the pseudo–Barenblatt solutions for $m < m_c$. See Formula (2.3).

$v(y,s)$: the rescaled solution of the nonlinear Fokker–Planck equation. See Formula (2.6).

$V_D(y)$: the Barenblatt profiles in rescaled variables. See Formula (2.7). $V_*(y) := V_{D_*}(y)$ is defined in Section 2.3.

w: the ratio v/V_{D_*}. See Formula (2.9) for the equation satisfied by w.

W_0, W_1: lower and upper bounds for w. See Section 2.3.

The notation $\|\cdot\|_p$ denotes in principle the standard norm in $\mathrm{L}^p(\mathbb{R}^d)$, but starting at the end of Subsection 4.1 we will use weighted spaces and it will indicate $\mathrm{L}^p(\mathbb{R}^d,\,\mathrm{d}\mu)$ with a weight μ related to the Barenblatt solutions. The context will always make it clear.

2 Preliminaries: rescaling, stabilization and entropy

The fast diffusion equation with $0 < m < 1$ has attracted the attention of researchers in recent times, once the theory of the corresponding slow diffusion case $m > 1$ came to be well known. In the latter case the long-time behaviour of all solutions with nonnegative and L^1 data u_0 is given by a one-parameter family of explicit self-similar solutions of the form

$$U_D(t,x) = t^{-\alpha} B_D(x t^{-\beta}), \tag{2.1}$$

with $\beta = 1/(2+d(m-1))$, $\alpha = d\beta$ and profile $B_D = (D - k|\xi|^2)_+^{1/(m-1)}$ with a free constant $D > 0$, a fixed constant $k = \beta(m-1)/2$, and putting $\xi = xt^{-\beta}$. These solutions, usually called Barenblatt solutions, replace the Gaussian profiles found in the long time behaviour of the classical heat equation, which is the case $m = 1$. See the precise asymptotic result in [38, Chapter 18].

When going over to the fast diffusion equation, the situation has been well understood in a first range of exponents $1 > m > m_c = (d-2)/d$ (the 'good' fast diffusion range); indeed, solutions of the above initial value problem exist and are unique, they are positive and smooth for every choice of the initial data in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$, and even in more general cases, cf. [13]. In particular, Barenblatt solutions still exist, they have the same selfsimilar form though the profile looks a bit different

$$B_D(\xi) = (D + k|\xi|^2)^{-1/(1-m)} \tag{2.2}$$

now with $k = \beta(1-m)/2$. This is a positive function everywhere in $\mathbb{R}^d$ and decays at infinity like $O(|\xi|^{-2/(1-m)})$, so that $B_D \in L^1(\mathbb{R}^d)$ if $m > m_c$. The Barenblatt solutions still represent the asymptotic behaviour of all solutions with nonnegative and L^1 data u_0, with even better convergence result in relative error, cf. [8, 36]. Factors like B_D will appear in the sequel as weights in functional inequalities and measure spaces. We shall use below a proper scaling to get rid of the inessential constant k.

However, such a simple theory breaks down for $m < m_c$, even if $m > 0$ (which is possible if $d \geq 3$), due in particular to the phenomenon of extinction in finite time, cf. [37]. In particular, our model solutions cannot be continued in the same form because the similarity exponents α and β go to infinity as m goes down to m_c. But for $m < m_c$ a related family of extinction solutions is found of the backward self-similar form

$$U_{D,T}(t,x) = (T-t)^{\alpha} B_D(x(T-t)^{\beta}), \tag{2.3}$$

with $\beta = 1/(d(1-m)-2) > 0$ and $\alpha = d\beta > 0$ (just minus the formulas used before). Here, T and C are arbitrary positive constants and B_D is given just as in the case $m_c < m < 1$. It is to be noted that B_D is no more an integrable function in $\mathbb{R}^d$, so we are completely away from the functional setting we started from. These new solutions are sometimes called pseudo-Barenblatt solutions to distinguish them from the original Barenblatt family.

2.1 Rescaled flow equation

Actually, these solutions do not possess the strong attractivity properties of their relatives for $m > m_c$. In order to investigate their partial attractivity (more precisely, their rescaled stability), we have studied in the paper [7] the extinction behaviour of solutions with initial data close to a pseudo-Barenblatt solution. This is the situation in short terms: we can show that after a rescaling step we obtain the nonlinear Fokker-Planck equation

$$\partial_s v = \frac{a^2}{\gamma}\nabla_y(v^{m-1}\nabla_y v) + \frac{\beta}{\gamma}\nabla_y \cdot (yv) \tag{2.4}$$

in terms of the rescaled variable $v(s, y)$ defined as

$$v(s,y) = (T-t)^{-d\beta}u(t,x), \quad y = ax(T-t)^{\beta}, \quad s = \gamma\log(T/(T-t)). \tag{2.5}$$

Here $T = T(u_0)$ is the extinction time of the solution, $\beta = (d(1-m)-2)^{-1}$ and we will choose the free constants $a, \gamma > 0$ to be $a^2 = \gamma = (1-m)\beta/2$. Note that $s(0) = 0$ and $s(t) \to \infty$ as $t \to T$. This means that whenever we use as T the actual extinction time of the solution u, then v is globally defined, for $y \in \mathbb{R}^d$ and $0 \le s < \infty$. With such choices equation (2.4) takes the convenient form

$$\partial_s v = \nabla_y \cdot (v^{m-1}\nabla_y v) + \frac{2}{1-m}\nabla_y \cdot (yv) = \nabla_y \cdot \left[v\nabla_y\left(\frac{v^{m-1} - V_D^{m-1}}{m-1}\right)\right] \tag{2.6}$$

This is a convenient choice since the stationary states are now given by

$$V_D(y) = (D + |y|^2)^{-1/(1-m)}, \quad D > 0, \tag{2.7}$$

which is just the profile B_D of (2.2) without the undesired constant k. We end this paragraph by noting that for $m = m_*$ the exponent in the above stationary profile is $-1/(1-m) = -(d-2)/2$, so that $V_D(y)$ decays at infinity like $O(|y|^{-(d-2)})$, i.e., like the stationary fundamental solution or harmonic potential. This is one of the reasons that makes m_* special.

2.2 Stabilization Result

In paper [7] we have shown stabilization of solutions of equation (2.6) towards one of the stationary profiles V_D for initial data that are not very far from V_D to start with. We can write the assumptions on the *initial conditions* in terms of either u_0 or v_0. The assumptions on u_0 are

(H1) u_0 is a non-negative function in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and that there exist positive constants T and $D_0 > D_1$ such that

$$U_{D_0,T}(0,x) \le u_0(x) \le U_{D_1,T}(0,x) \quad \forall\, x \in \mathbb{R}^d.$$

(H2) There exist $D_* \in [D_1, D_0]$ and $f(|\cdot|) \in \mathrm{L}^1(\mathbb{R}^d)$ such that

$$\left|u_0(x) - U_{D_*,T}(0,x)\right| \le f(|x|) \quad \forall\, x \in \mathbb{R}^d.$$

In the case $m < m_c$ under consideration here, (H1) implies in particular that the extinction occurs at time T. Moreover, when $m > m_*$ (H2) follows from (H1) since the difference of two pseudo-Barenblatt solutions is always integrable. For $m \le m_*$ this is no more true, and (H2) is an additional restriction.

In terms of v_0, conditions (H1) and (H2) can be rewritten as follows.

(H1') v_0 is a non-negative function in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$V_{D_0}(y) \le v_0(y) \le V_{D_1}(y) \quad \forall\, x \in \mathbb{R}^d.$$

(H2') There exist $D_* \in [D_1, D_0]$ and $\tilde{f}(|\cdot|) \in \mathrm{L}^1(\mathbb{R}^d)$ such that

$$\left|v_0(y) - V_{D_*}(y)\right| \le \tilde{f}(|y|) \quad \forall\, y \in \mathbb{R}^d.$$

We point out that condition (H1') means a decay for large y of the form

$$v_0(y) = |y|^{-2/(1-m)}(1 - c(y)|y|^{-2})$$

with $c(y)$ bounded above and below away from zero. Moreover, (H2') imposes a stronger decay condition for $m \le m_*$. Notice we can take $\tilde{f}(|y|) = T^{-d\beta} f(|y|/aT^\beta)$, so that they can be identified up to an elementary scaling.

As a starting point for our asymptotic study, we state the result of [7] about the convergence of $v(t)$ towards a unique Barenblatt profile.

Theorem 2.1 (Convergence to the asymptotic profile) *Let $d \ge 3$, $m < 1$. Consider the solution v of* (2.6) *with initial data satisfying* (H1')-(H2').

(i) *For any $m > m_*$, there exists a unique $D_* \in [D_1, D_0]$ such that $\int_{\mathbb{R}^d} (v(s) - V_{D_*})\,\mathrm{d}x = 0$ for any $t > 0$. Moreover, for any $p \in (q(m), \infty]$, $\lim_{t\to\infty} \int_{\mathbb{R}^d} |v(s) - V_{D_*}|^p dy = 0$.*

(ii) *For $m \le m_*$, $v(s) - V_{D_*}$ is integrable, $\int_{\mathbb{R}^d} (v(s) - V_{D_*})\mathrm{d}y = \int_{\mathbb{R}^d} (v(0) - V_{D_*})\mathrm{d}y$ and $v(s)$ converges to V_{D_*} in $\mathrm{L}^p(\mathbb{R}^d)$ as $t \to \infty$, for any $p \in (1, \infty]$.*

(iii) (Convergence in Relative Error) *For any $p \in (d/2, \infty]$,*

$$\lim_{t\to\infty} \|v(s)/V_{D_*} - 1\|_p = 0\,. \tag{2.8}$$

For simplicity, we write $v(s)$ instead of $y \mapsto v(s, y)$ whenever we want to emphasize the dependence on the time s. The exponent $q(m)$ is defined as the infimum of all positive real numbers p for which two Barenblatt profiles V_{D_1} and V_{D_2} are such that $|V_{D_1} - V_{D_2}|$ belongs to $\mathrm{L}^p(\mathbb{R}^d)$:

$$q(m) := \frac{d(1-m)}{2(2-m)}\,.$$

We see that $q(m) > 1$ if $m \in (0, m_*)$, $q(m_*) = 1$, and $q(m) < 1$ if $m > m_*$. In case $m > m_*$, the value of D_* can be computed at $s = 0$ as a consequence of the mass balance law $\int_{\mathbb{R}^d} (v_0 - V_{D_*})\,\mathrm{d}x = 0$, and then the conservation result holds for all $s > 0$ as is proved in the paper [7]. On the other hand, in the case $m \le m_*$ the mass balance does not make sense, but D_* is determined by Assumption (H2'). In this case, the presence of a perturbation of V_{D_*} with nonzero mass, does not affect the asymptotic behavior of the solution to first order.

2.3 Relative error, entropies, and linearization

The deeper stabilization analysis of equation (2.6) leads to an interesting connection with a family of Poincaré-Hardy functional inequalities. In this way, we obtain stabilization rates that are exponential in the new time s, which means that they are power-like in the original time. The exponent m_* appears precisely as the only exponent for which the linearized analysis based on Poincaré-Hardy inequalities fails and the corresponding rates are not obtained by that method. We shall prove below that the linearized analysis when m takes the special value m_* leads to a different functional framework and the actual rates are different, and actually slower.

In any case, the approach and the use of entropies starts in the same way. Let v be a solution to the rescaled Fokker-Plank equation (2.6), and let $V_* = V_{D_*}$ be the Barenblatt solution mentioned in Theorem 2.1. We pass to the quotient $w(s,y) = v(s,y)/V_*(y)$. Notice that $w-1 = (v-V_*)/V_*$ is the relative error of v with respect to V_*. Notice also that, by straightforward calculations, our running assumptions imply that $W_0 \le w \le W_1$, where $W_0 = (D_*/D_0)^{1/(1-m)} < 1$ and $W_1 = (D_*/D_1)^{1/(1-m)} > 1$.

The equation for w reads

$$\partial_s w = \frac{1}{V_*}\nabla\cdot\left[wV_*\nabla\left(\frac{w^{m-1}-1}{m-1}V_*^{m-1}\right)\right] \tag{2.9}$$

In terms of w, we define the *relative entropy*

$$\mathcal{F}[w] := \frac{1}{1-m}\int_{\mathbb{R}^d}\left[(w-1)-\frac{1}{m}(w^m-1)\right]V_*^m\,\mathrm{d}y \tag{2.10}$$

Strictly speaking, we are assuming that a time $s \ge 0$ is given and then we get $\mathcal{F}(w(s))$. In terms of v, when m is sufficiently close to one, it can be derived as $E(v) - E(V_*)$ where

$$E[v] =: \frac{1}{1-m}\int_{\mathbb{R}^d}\left[vV_*^{m-1}-\frac{1}{m}v^m\right]\mathrm{d}y \tag{2.11}$$

(for m farther away from 1 both $E[v]$ and $E[V_*]$ become infinite and only the expression for the difference, the relative entropy, makes sense). We also introduce the *relative Fisher information*

$$\mathcal{I}[w] = \int_{\mathbb{R}^d}\left|\nabla\left(\frac{w^{m-1}-1}{m-1}V_*^{m-1}\right)\right|^2 V_* w\,\mathrm{d}y = \int_{\mathbb{R}^d}\left|\nabla\left(\frac{v^{m-1}-V_*^{m-1}}{m-1}\right)\right|^2 v\,\mathrm{d}y \tag{2.12}$$

(again, we should have written $\mathcal{I}[w(s)]$). By differentiation in time and using the equation, we get

$$\frac{\mathrm{d}\mathcal{F}[w(s)]}{\mathrm{d}s} = -\mathcal{I}[w(s)]\,, \qquad \forall s > 0\,. \tag{2.13}$$

For a detailed proof of this time derivation, we refer to Proposition 2.6 of [7].

We now introduce the linearization idea in [7] that allows to treat the long-time behaviour of w. It consists in writing the relative error in the form

$$w(s,y) - 1 = \varepsilon g(s,y)V_*^{1-m}(y) \tag{2.14}$$

where the choice of weight V_*^{1-m} is crucial. After a brief formal computation we obtain the differential equation for g that is implied by (2.9) in the limit $\varepsilon \to 0$:

$$\partial_s g = V_*^{m-2}\nabla\cdot(V_*\nabla g)\,. \tag{2.15}$$

Actually, since Theorem 2.1, formula (2.8), implies that $w \to 1$ as $s \to \infty$, the factor ε will not be needed in the actual linearization step.

Our next task is to study this linear flow; then, we shall have to relate the actual nonlinear flow to its linearized approximation. But let us point out that we will not need to prove the convergence of solutions of the original problem to solutions of the linear problem, the analysis is rather based on the relationship between the two linear quantities, entropy and Fisher information, associated to equation (2.15), and the close similarity of these linear quantities and the previously defined nonlinear ones. These facts plus (2.13) produce the desired convergence result.

In the cases $m < 1$, $m \neq m_*$, a suitable functional setting was found where the functional inequalities of Hardy-Poincaré type corresponding to the linear flow implied the existence of a spectral gap. According to more or less standard theory, existence of such a gap implies exponential decay rates (in s) of the norms and entropy of the solutions of the linear flow. A delicate analysis of comparison of entropy and Fisher information between the linear and nonlinear flow allowed finally to transfer the result about decay rate to the original nonlinear flow. See full details in [7].

The problem arising when $m = m_*$ is the absence of spectral gap. We shall prove below that this is essential, in fact the actual rates are not exponential but power–like in s. This is related to the heat kernel behaviour of the operator appearing in (2.15), $V_*^{m-2}\nabla \cdot (V_* \nabla g)$, acting in a suitable weighted Hilbert space. Details will be given in Section 4, where a sharp power-like decay for the heat kernel is proved using a most fortunate coincidence, i. e., the representation of the linear semigroup as the heat flow on a conformally flat Riemannian manifold. It will also be proved that no Hardy–type inequality can hold for the quadratic form associated to the generator, so that it is hopeless to use the same line of reasoning of paper [7].

3 Statement of the main results for $m = m_*$

We are now ready to state our main results. We use the notations $v(s), w(s)$ instead of $v(s, y), w(s, y)$ and $u(t)$ instead of $u(t, x)$ when the dependence on time is stressed.

We prove convergence of $v(s)$ to the appropriate Barenblatt profile in several senses. More precisely we prove quantitative bounds on the convergence in suitable L^p norms, on the convergence of moments, and on the uniform convergence of all derivatives. Convergence takes place *with the same rate of the linearized case.*

Theorem 3.1 (Convergence with rate to the asymptotic profile) *Consider a solution v of the equation* (2.6) *such that v_0 satisfies* (H1')-(H2') *and fix some $s_0 > 0$. Then, the entropy of the quotient variable satisfies*

$$\mathcal{F}[w(s)] \leq K s^{-1/2} \quad \forall\, s \geq s_0\,. \tag{3.1}$$

for some $K = K(v_0, s_0)$. As a consequence, for any $\vartheta \in \left[0, \frac{d}{2}\right]$, there exists a positive constant K_ϑ such that

$$\left\| |y|^\vartheta (v(s) - V_{D_*}) \right\|_2 \leq K_\vartheta s^{-1/4} \quad \forall\, s \geq s_0\,. \tag{3.2}$$

The analysis of the linearized equation indicates that this rate should be optimal. We also have convergence without weights in suitable L^p and C^j spaces with the same rates, where we use interior regularity theory for parabolic equations:

Corollary 3.2 (i) *For any $q \in (1, \infty]$, there exists a positive constant $K(q)$ such that*

$$\|v(s) - V_{D_*}\|_q \leq K(q) s^{-1/4} \quad \forall\, s \geq s_0\,. \tag{3.3}$$

(ii) *For any $j \in \mathbb{N}$ there exists a positive constant H_j such that*

$$\|v(s) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \leq H_j s^{-1/4} \quad \forall\, s \geq s_0\,. \tag{3.4}$$

These power-decay results are in contrast with the exponential rates obtained in [7] for $-\infty < m < 1$ and $m \neq m_*$. Rescaling back to the original space–time variables one gets the following result which can be called *intermediate asymptotics*.

Corollary 3.3 *Consider a solution u of* (1.1) *with $m = m_*$, with initial data satisfying* (H1)-(H2), *and extinction time T. For t sufficiently close to T and for any $q \in (1, \infty]$, there exists a positive constant C such that:*

$$\|u(t) - U_{D_*}(t)\|_q \leq C\,(T-t)^{\sigma(q)}\,\log\left(T/(T-t)\right)^{-1/4}.$$

with $\sigma(\infty) = d(d-2)/4$, and $\sigma(q) = \sigma(\infty)(q-1)/q$ for $q < \infty$.

We also obtain a quantitative bound on the decay of the *relative error of $v(s)$ with respect to V_{D_*}*.

Corollary 3.4 (Decay of Relative Error) *Consider a solution v of* (2.6) *such that v_0 satisfies* (H1')-(H2') *and fix some $s_0 > 0$. Then for any $q \in (d/2, \infty]$ and all $\varepsilon > 0$ there exists a positive constant $\mathcal{C}_q$ such that*

$$\left\|v(s)/V_{D_*} - 1\right\|_q \leq \mathcal{C}_q s^{-\frac{1-\varepsilon}{d}} \quad \forall\, s \geq s_0\,. \tag{3.5}$$

If $q = d/2$ there is a positive constant $\mathcal{C}$ such that

$$\left\|v(s)/V_{D_*} - 1\right\|_{d/2} \leq \mathcal{C} s^{-\frac{1}{d}} \quad \forall\, s \geq s_0\,. \tag{3.6}$$

Finally we also have, for all $j \in \mathbb{N}$, that there exists a positive constant $\mathcal{C}_j$

$$\left\|v(s)/V_{D_*} - 1\right\|_{C^j(\mathbb{R}^d)} \leq \mathcal{C} s^{-\frac{1-\varepsilon}{d}} \quad \forall\, s \geq s_0\,. \tag{3.7}$$

Notice that, besides having a quantitative bound, we have some other improvements on Theorem 2.1 first because the value $q = d/2$ is now allowed and because convergence of C^j norms is also dealt with. The constants involved depend also on m, d, D_0, D_*, D_1, but also on the solution at time s_0 through the relative mass (conserved along the evolution) and through the uniform bound c_0 on the ratio $\int_{\mathbb{R}^d} |\nabla v(y)|^2 V_D(y)\,\mathrm{d}y / \|v\|_1^2 \leq c_0$.

4 Analysis of the linear case

We address now a central topic of the paper, i.e., establishing of the long-time behaviour of the linearized flow in the still open case with exponent $m_* = (d-4)/(d-2)$. The clue to our study of the linearized flow in this case is to interpret it as the heat flow of the Laplace-Beltrami operator of a suitable Riemannian manifold $(M, \mathbf{g})$, with a metric $\mathbf{g}$ which is conformal to the standard $\mathbb{R}^d$ metric. Studying the pointwise heat kernel behaviour allows to prove Nash and log-Sobolev inequalities associated to the generator. Such inequalities will later on allow us to study the nonlinear evolution as well, and to determine its asymptotics, which will be shown to proceed with the same rate of convergence as the linearized one. Since the study can have independent interest, we replace g by v, y by x, and s by t throughout the section to conform to more standard notations.

4.1 Linear equation and geometry

Given $m<1$ and $D>0$, we consider the operator given on $C_c^\infty(\mathbb{R}^d)$ $(d\geq 3)$ by

$$L_m v = (D+|x|^2)^{(2-m)/(1-m)}\nabla\cdot\left(\frac{\nabla v}{(D+|x|^2)^{1/(1-m)}}\right) = V_D^{m-2}\nabla\cdot(V_D\nabla v)\,. \tag{4.1}$$

We recall that for $m=m_*$ the following holds: $1/(1-m)=(d-2)/2$, and $V_D^{m-2}(x)=(1+|x|^2)^{d/2}$. We have dropped the index $*$ from D_* to simplify the notation, since the particular value of D has no role here. We shall think of this operator as acting on the Hilbert space $H_m=\mathrm{L}^2(\mathbb{R}^d,\,\mathrm{d}\mu)$ with $d\mu=V_D^{2-m}\,\mathrm{d}x$. To define it more precisely we construct the quadratic form

$$I_m[v]=\int_{\mathbb{R}^d}\frac{|\nabla v(x)|^2}{(D+|x|^2)^{1/(1-m)}}\mathrm{d}x=\int_{\mathbb{R}^d}|\nabla v(x)|^2V_D(x)\mathrm{d}x,\quad u\in C_c^\infty(\mathbb{R}^d). \tag{4.2}$$

Then, I_m is closable in H_m (for a quite general result implying the validity of the above assertion see e.g. [18], Section 4.7). We denote again by $-L_m$ the unique nonnegative self–adjoint operator in H_m associated with its closure. In fact L_m has the above explicit expression (4.1) on smooth compactly supported functions. There is a particular value of m for which the above operator can be seen as the Laplace-Beltrami operator of a certain Riemannian manifold $(M,\mathbf{g})$, as we shall show. This in particular will imply (since M turns out to be complete) that L_m is essentially self–adjoint on $C_c^\infty(M)$ by a result of Calabi (see e.g. [18], Theorem 5.2.3). Consider indeed the following manifold, denoted by M, given by $\mathbb{R}^d$ endowed with the Riemannian, conformally flat metric defined, in Euclidean (global) coordinates, by

$$\mathbf{g}_\alpha(x)=(D+|x|^2)^{-\alpha}\mathbf{I}, \tag{4.3}$$

where $\mathbf{I}$ is the Euclidean metric and $|\cdot|$ is the Euclidean norm. We denote by $\mu_{\mathbf{g}_\alpha}$ the Riemannian measure, by $|\mathbf{g}_\alpha|=\det(\mathbf{g}_\alpha)$ the determinant of the metric tensor, by ∇_α the Riemannian gradient and by Δ_α the Laplace-Beltrami operator, defined on $\mathrm{L}^2(\mu_{\mathbf{g}_\alpha})$, associated to the given metric.

Lemma 4.1 *The Laplace-Beltrami operator Δ_α coincides with L_m, precisely when $\alpha=1$ and $m=m_*:=(d-4)/(d-2)$, both as concerns its explicit expression (in Euclidean coordinates) and as concerns the Hilbert space it acts on.*

Proof. We notice that for the above choice of metric we have

$$\sqrt{|\mathbf{g}_\alpha|(x)}=(D+|x|^2)^{-\alpha d/2},\quad g_\alpha^{ij}(x)=(D+|x|^2)^\alpha\delta^{ij}.$$

Then we have that the Dirichlet form associated to Δ_α is given, on test functions, by

$$\begin{aligned}J_\alpha(v)&:=\int_M\mathbf{g}_\alpha(\nabla_\alpha v,\nabla_\alpha v)\,\mathrm{d}\mu_{\mathbf{g}_\alpha}=\int_{\mathbb{R}^d}\sqrt{|\mathbf{g}_\alpha|(x)}g_\alpha^{ij}(x)\frac{\partial v}{\partial x^i}\frac{\partial v}{\partial x^j}\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}(D+|x|^2)^{(-d\alpha/2)+\alpha}|\nabla_e v(x)|^2\,\mathrm{d}x\end{aligned} \tag{4.4}$$

where ∇_e is the Euclidean gradient and the summation convention is used. Then we notice that the conditions that identify Δ_α with L_m:

$$\sqrt{|\mathbf{g}_\alpha|(x)}=(D+|x|^2)^{-(2-m)/(1-m)}$$
$$\sqrt{|\mathbf{g}_\alpha|(x)}g_\alpha^{ij}(x)=(D+|x|^2)^{-1/(1-m)}\delta^{ij}$$

force α, m to be related by $(d\alpha/2) - \alpha = 1/(1-m)$ and $d\alpha/2 = (2-m)/(1-m)$. This is equivalent to $\alpha = 1$, $m = (d-4)/(d-2) = m_*$ as claimed. $\square$

We shall now compute, in the case discussed in the above Lemma, the Ricci curvature of $(M, \mathbf{g}_\alpha)$. Hereafter we shall drop the index α, since we always choose $\alpha = 1$. We put $D = 1$ for simplicity without loss of generality.

Lemma 4.2 *Then the Ricci curvature of* $(M, \mathbf{g}_{\alpha=1})$ *is given, in Euclidean coordinates, by*

$$R_{ij} = -\frac{(d-2)x_i x_j}{(1+|x|^2)^2} + \left[\frac{(d-2)|x|^2 + 2(d-1)}{(1+|x|^2)^2}\right]\delta_{ij}, \tag{4.5}$$

where we write $\mathrm{Ric} = (R_{ij})$*. In particular* $\mathrm{Ric} > 0$ *on* M*, such lower bound cannot be improved, and* Ric *is bounded on* M*. Actually,* $R_{ij}(x) = O(|x|^{-2})$ *as* $|x| \to \infty$ *in the transversal directions and it behaves as* $O(|x|^{-4})$ *in the radial directions. Finally, the scalar curvature is given by*

$$R = (d-1)\frac{2d + (d-2)|x|^2}{1+|x|^2}. \tag{4.6}$$

We will postpone the proof of these formulas to appendix A1 not to break the flow of the exposition. It immediately follows that the symmetric tensor Ric is positive; indeed, given $\xi \in \mathbb{R}^d$, we have

$$R_{ij}(x)\xi_i\xi_j \geq \frac{2(d-1)}{(1+|x|^2)^2}|\xi|^2 > 0.$$

The boundedness of Ric is clear from its explicit expression. Note that for $d = 2$ we are dealing with an Einstein metric, $\mathrm{Ric} = k\,\mathbf{g}$ (actually, it is Hamilton's cigar soliton to the Ricci flow, [25, 14]), but for $d \geq 3$ it is not.

Let us continue with the asymptotic analysis of the flow. By a celebrated result of Li and Yau [32], the heat kernel $K(s, x, y)$ of the Laplace–Beltrami operator of a complete Riemannian manifold $(M, \mathbf{g})$ with nonnegative Ricci curvature is pointwise comparable with the quantity

$$\frac{1}{\mathrm{Vol}[B(x, \sqrt{t})]} e^{-c\frac{d^2(x,y)}{t}}$$

where $d(\cdot\,,\cdot)$ is the Riemannian distance in $(M, \mathbf{g})$, $B(x, r)$ is the Riemannian ball centered at x and of radius r and Vol is the Riemannian volume. More precisely,

Corollary 4.3 *For all small positive* ε *there exists positive constants* c_1, c_2 *such that*

$$\frac{c_1(\varepsilon)}{\mathrm{Vol}[B(x, \sqrt{t})]} e^{-\frac{d^2(x,y)}{(4-\varepsilon)t}} \leq K(t, x, y) \leq \frac{c_2(\varepsilon)}{\mathrm{Vol}[B(x, \sqrt{t})]} e^{-\frac{d^2(x,y)}{(4+\varepsilon)t}}$$

for all $x, y \in M$, $t > 0$.

We recall that the Li–Yau bounds require completeness, a property which clearly holds for the manifold we are considering. We use the notation $a \wedge b = \min\{a, b\}$.

Corollary 4.4 *The heat kernel satisfies the following properties:*

$$
\begin{aligned}
&K(t,x,x) \underset{t\to 0}{\approx} \left(1 \wedge \frac{1}{|x|}\right)\frac{1}{t^{\frac{d}{2}}},\\
&K(t,x,x) \le \frac{C}{t^{\frac{1}{2}}} \quad \forall t \ge 1, \forall x \in \mathbb{R}^d,
\end{aligned}
\tag{4.7}
$$

where $f_1 \underset{t\to t_0}{\approx} f_2$ means that there exists two constants $c_1, c_2 > 0$ such that $c_1 f_1 \le f \le c_2 f_2$ near t_0.

Proof. First notice that

$$d(0,x) = \int_0^{|x|} \frac{1}{\sqrt{1+t^2}}\,\mathrm{d}t$$

where $|x|$ is the Euclidean length, so that $d(0,x) \sim \log|x|$ for large $|x|$. Hence,

$$
\begin{aligned}
\mathrm{Vol}(B(0,R)) &= \int_{B(0,R)} \sqrt{|\mathbf{g}|}\,\mathrm{d}x = \int_{d(0,x)<R} \frac{1}{(D+|x|^2)^{d/2}}\,\mathrm{d}x\\
&\underset{R\to+\infty}{\sim} c\int_{r<e^R} \frac{r^{d-1}}{(D+r^2)^{d/2}}\,\mathrm{d}r \underset{R\to+\infty}{\sim} cR.
\end{aligned}
$$

Proceeding similarly, one shows that $d(x_0,x) \approx \log\frac{|x|}{|x_0|}$ for large $|x|$ and hence that $\mathrm{Vol}(B(x_0,R)) \sim c(R+\log|x_0|)$ for large R and, say, $|x_0| \ge 2$. The short time behaviour is clearly locally Euclidean, with a weight depending on x given by definition by $1/\sqrt{D+|x|^2}$. □

Remark. The above corollary extends, for the present choice of the parameter m, the result of [18], Th. 4.7.5, in several respects. In fact, in the quoted Theorem the bounds on the heat kernel are from above and for short time only. Notice that the short time bound in the following results matches with the one of [18]. One may notice that, in fact, we have proved the bound

$$K(t,x,x) \approx \frac{1}{t^{\frac{1}{2}} + \log(1+|x|)} \quad \forall t \ge 1, \forall x \in \mathbb{R}^d,$$

although we shall make no further use of it.

Corollary 4.5 *Each solution to the linear evolution equation $\partial_t v = L_{m_*} v$ corresponding to an initial datum in $\mathrm{L}^1(\mathbb{R}^d, (D+|x|^2)^{(m-2)/(1-m)})$ satisfies the bound*

$$
\|v(t)\|_\infty \le H(t)\|v_0\|_1 = \begin{cases} c_1 \dfrac{\|v_0\|_1}{t^{d/2}} & \text{for any } 0 < t \le 1\\[2mm] c_2 \dfrac{\|v_0\|_1}{t^{1/2}} & \text{for any } t > 1 \end{cases}
\tag{4.8}
$$

where c_i are positive constants. The power of t cannot be improved for such general initial data, as can be seen by considering the time evolution of a Dirac delta.

Warning: Here, the symbol $\|\cdot\|_p$ denotes the norm in $\mathrm{L}^p(\mathbb{R}^d, \mathrm{d}\mu_*)$, where $\mathrm{d}\mu_* = V_D^{2-m_*}(x)\,\mathrm{d}x$, and we know that $V_D^{2-m_*}(x) = (D+|x|^2)^{-d/2}$. This notation will be kept in the next three sections.

4.2 Functional Inequalities

We recall that $I_{m_*}[v] = \int_{\mathbb{R}^d} |\nabla v(x)|^2 V_D \, \mathrm{d}x$ on smooth compactly supported functions. The domain of its closure will be indicated by $\mathrm{Dom}\,(I_{m_*})$.

Corollary 4.6 *There is a family of logarithmic Sobolev inequalities*

$$\int_{\mathbb{R}^d} v^2 \log\left(\frac{v}{\|v\|_2}\right) \mathrm{d}\mu_* \le \varepsilon I_{m_*}[v] + \beta(\varepsilon)\|v\|_2^2 \tag{4.9}$$

valid for all $v \in \mathrm{Dom}(I_{m_*}) \cap \mathrm{L}^1(\mathbb{R}^d, \mathrm{d}\mu_*) \cap \mathrm{L}^\infty(\mathbb{R}^d, \mathrm{d}\mu_*)$ *and all positive* ε*, where* $\beta(\varepsilon) = c - \frac{d}{4}\log\varepsilon$ *for* $\varepsilon < 1$*,* $\beta(\varepsilon) = c - \frac{1}{4}\log\varepsilon$ *for* $\varepsilon \ge 1$*, and* c *is a suitable positive constant.*

Proof. We have $\|v(s)\|_\infty \le Cs^{-1/2}\|v_0\|_1$ for large s. Interpolating between such bound and the L^∞ contractivity property (valid since I_{m_*} is a Dirichlet form) shows that $\|v(s)\|_\infty \le Cs^{-1/4}\|v_0\|_2$ for large s. Similarly, $\|v(s)\|_\infty \le Cs^{-d/4}\|v_0\|_2$ for small s. The validity of such ultra-contractive bounds for the solution of the linear evolution considered is known to be equivalent, by [18], Example 2.3.2, to the stated logarithmic Sobolev inequalities for the initial datum u_0 if it belongs to $\mathrm{Dom}(I_{m_*}) \cap \mathrm{L}^1(\mathbb{R}^d, \mathrm{d}\mu_*) \cap \mathrm{L}^\infty(\mathbb{R}^d, \mathrm{d}\mu_*)$. At this point the evolution has no role anymore and to avoid confusions we choose to write v instead of u_0 in the statement. □

The next consequences we draw involve the recurrence of the semigroup considered.

Corollary 4.7 *The semigroup* $\{T_s\}_{s\ge 0}$ *associated to* L_{m_*} *is recurrent. In particular,* L_{m_*} *does not admit a (minimal) positive Green function and the manifold* $(\mathbb{R}^d, \mathbf{g}_{\alpha=-1})$ *is parabolic.*

Proof. It suffices to note that a semigroup $\{T_s\}_{s\ge 0}$ is, by definition, transient, iff $\int_0^\infty T_s v \, \mathrm{d}s$ is a.e. finite for all $v \in \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}\mu_*)$. This of course does not hold in the present case because of the $s^{-1/2}$ behaviour for long times of the heat kernel. □

Corollary 4.8 *There is no bounded, strictly positive,* μ_**–integrable function* h *such that*

$$\int_{\mathbb{R}^d} |v| h \, \mathrm{d}\mu_* \le I_{m_*}[v]^{1/2}$$

for all $v \in \mathrm{Dom}\,(I_{m_*})$*.*

Proof. The existence of a function h with the stated properties is equivalent to the transience of the semigroup at hand, by [23], Th. 1.5.1. □

Corollary 4.9 *There is no bounded, strictly positive,* μ_**-integrable function* h *such that for all* $v \in$ $\mathrm{Dom}\,(I_{m_*})$

$$\int_{\mathbb{R}^d} v^2 h \, \mathrm{d}\mu_* \le I_{m_*}[v].$$

Proof. Since h is assumed to be integrable so that $h \, \mathrm{d}\mu_*$ is a finite measure, that we can normalize to 1, one would have by Hölder inequality that

$$\int_{\mathbb{R}^d} |v| h \, \mathrm{d}\mu_* \le \left(\int_{\mathbb{R}^d} v^2 h \, \mathrm{d}\mu_*\right)^{1/2} \le (I_{m_*}[v])^{1/2}$$

for all $v \in \mathrm{Dom}(I_{m_*})$, contradicting the above result. $\square$

Remark. The above results prove that Hardy–type inequalities relative to the Dirichlet form I_{m_*} and to a strictly positive integrable weight h cannot hold, even if h is required to be bounded. This shows that the strategy of [7], which relied heavily on the validity of Hardy–type inequalities and allowed to deal with the case $m \neq m_*$ cannot be adapted to the present situation.

The ultra-contractive bounds discussed above can also be related to the validity of Nash inequalities for I_{m_*}. In fact we prove now some inequalities of that type in weighted Sobolev spaces which will be very important when dealing with the nonlinear evolution. Such inequalities play here the role that Hardy–type inequalities played in the case $m \neq m_*$ studied in [7], cf. also Section 7. The following crucial result is a purely functional inequality which is proved using the linear evolution only, but will turn out to be the key point for the study of the nonlinear evolution as well.

Proposition 4.10 *For all v such that $I_{m_*}[v]/\|v\|_1^2 \leq c_0$ for some $c_0 > 0$, the following Gagliardo–Nirenberg inequality holds true:*

$$\|v\|_2^2 \leq K I_{m_*}[v]^{1/3} \|v\|_1^{4/3}, \tag{4.10}$$

for all $v \in \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}\mu_*) \cap \mathrm{Dom}(I_{m_*})$, where the positive constant K depends on c_0, and diverges as $c_0 \to +\infty$.

Proof. To get the claim, first interpolate between the bound $\|v(s)\|_\infty \leq H(s)\|v_0\|_1$ and the L^1 contraction property to get $\|v(s)\|_2 \leq H(s)^{1/2}\|v_0\|_1$. From this starting point we can use a known argument, cf. [18], and we briefly recall it for the sake of completeness. In fact, use the semigroup property and the fact that $I_{m_*}[v(s)]$ is nonincreasing as a function of s to write

$$\begin{aligned} H(s)\|v_0\|_1^2 &\geq (v(s), v(s)) = (v(2s), v_0) \\ &= (v_0, v_0) - \int_0^{2s} I_{m_*}\left[v\left(\frac{\lambda}{2}\right)\right] \mathrm{d}\lambda \\ &\geq (v_0, v_0) - 2s I_{m_*}[v_0]. \end{aligned}$$

Therefore,

$$\|v_0\|_2^2 \leq 2s I_{m_*}[v_0] + H(s)\|v_0\|_1^2. \tag{4.11}$$

It would then be easy to minimize the r.h.s. of the latter formula should one have $H(s) = cs^{-\alpha}$ for all $s > 0$. The fact that $H(s)$ has such form with different powers of time when s is small and when s is large forces us to proceed as follows. Assuming that $I_{m_*}[v_0]$ and $\|v_0\|_1$ are not zero, the right hand side takes the value infinity both as $s \to 0$ and $s \to \infty$ hence there is a minimum for one or several intermediate values of s. We want to take a particular value of s that almost minimizes the above formula, and we want that value to correspond to the range of not small s where $H(s) = c_2 s^{-1/2}$. Since we assumed that $I_{m_*}[v_0]/\|v_0\|_1^2 \leq c_0$ for some $c_0 > 0$, we consider the 1-parameter quantity

$$s_\alpha = \alpha \left[\frac{\|v_0\|_1^2}{I_{m_*}[v_0]}\right]^{2/3}$$

and observe that, trivially,

$$s_\alpha > 1 \qquad \Longleftrightarrow \qquad \alpha > c_0^{2/3}.$$

We choose α accordingly (so that it is bounded away from zero) and plug the corresponding s_α into (4.11), noticing that for $s > 1$ we have $H(s) = c_2 s^{-1/2}$; with these choices (4.11) becomes

$$\|v_0\|_2^2 \le K \|v_0\|_1^{4/3} I_{m_*}[v_0]^{1/3} \tag{4.12}$$

with $K = K(\alpha) = \alpha + c_2\alpha^{-1/2}$. This concludes the proof. $\square$

4.3 Mass conservation for the linear flow

We introduce here the calculation of "conservation of mass" for the linear semigroup. As usual we put $V_D^{2-m_*}\,\mathrm{d}x = \,\mathrm{d}\mu_*$.

Lemma 4.11 *The following property of mass conservation holds true for every nonnegative $v \in \mathrm{L}^1(\,\mathrm{d}\mu_*)$:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int v\,\mathrm{d}\mu_* = 0. \tag{4.13}$$

We give two proofs of the result, first a quite direct one and then a proof relying on the special geometric nature of the linear flow.

First proof. We use the specific form of the weights involved for a direct calculation, first for an initial datum belonging to $\mathrm{L}^1(\mathrm{d}\mu_*) \cap \mathrm{L}^2(\mathrm{d}\mu_*)$. Choose a test function φ_R supported in the Euclidean ball B_{2R} with $\varphi_R = 1$ on $B_R(0)$. Let $t \ge t_0 > 0$ and compute, for any such t:

$$\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} v\varphi_R\,\mathrm{d}\mu_*\right| = \left|-\int_{R\le|x|\le 2R} \nabla v\cdot\nabla\varphi_R V_D\,\mathrm{d}x\right| \\
&\le \int_{R\le|x|\le 2R} |\nabla V_D|\,|\nabla\varphi_R| v\,\mathrm{d}x + \int_{R\le|x|\le 2R} |\Delta\varphi_R|\,v\,V_D\,\mathrm{d}x \le \frac{c(m,d)}{R^2}\int_{R\le|x|\le 2R} v\,V_D\,\mathrm{d}x
\end{aligned} \tag{4.14}$$

because $|\nabla V_D| \le c_0(m,d) V_D/R$ since it is not restrictive to assume that $|\nabla\varphi_R| \le c_1/R$ and $|\Delta\varphi_R| \le c_2/R^2$ whenever $R \le |x| \le 2R$. By Hölder inequality we obtain that

$$\int_{R\le|x|\le 2R} v\,V_D\,\mathrm{d}x \le \left(\int_{R\le|x|\le 2R} v^2\,V_D^{2-m_*}\,\mathrm{d}x\right)^{1/2}\left(\int_{R\le|x|\le 2R} V_D^{m_*}\,\mathrm{d}x\right)^{1/2} \le \varepsilon_R\,R^2$$

since we let $\varepsilon_R := \left(\int_{R\le|x|\le 2R} v^2\,V_D^{2-m_*}\,\mathrm{d}x\right)^{1/2}$ and it is easy to check that $\int_{R\le|x|\le 2R} V_D^{m_*}\,\mathrm{d}x \le c_1 R^4$. We obtained that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} v\varphi_R\,\mathrm{d}\mu_*\right| \le c_1\varepsilon_R$$

and we notice that $\varepsilon_R \to 0$ as $R \to \infty$, a fact which holds because $v \in \mathrm{L}^2(\,\mathrm{d}\mu_*)$. This proves that

$$\lim_{R\to\infty}\left|\int_{\mathbb{R}^d} v(t_1)\varphi_R\,\mathrm{d}\mu_* - \int_{\mathbb{R}^d} v(t_0)\varphi_R\,\mathrm{d}\mu_*\right| \le c_1 \lim_{R\to\infty}\varepsilon_R(t_1 - t_0) = 0 \tag{4.15}$$

for any $0 \le t_0 \le t_1$. We can use dominated convergence in the left-hand side, since the Markov property implies $v(t) \in \mathrm{L}^1(\,\mathrm{d}\mu_*)$ for all $t \ge 0$. This yields the claim for strictly positive times and for initial data belonging to $\mathrm{L}^1(\mathrm{d}\mu_*) \cap \mathrm{L}^2(\mathrm{d}\mu_*)$. We can then reach $t = 0$ using the strong continuity in $\mathrm{L}^1(\mathrm{d}\mu_*)$ of the evolution, and consider general data in $\mathrm{L}^1(\mathrm{d}\mu_*)$ by approximation.

Second proof. We can also use a general argument involving conservation of probability on manifolds with curvature bounded below. Let $\{T_t\}_{t\geq 0}$ be the semigroup associated to the Laplace–Beltrami operator of the manifold considered. Then $\{T_t\}_{t\geq 0}$ is a Markov semigroup and in particular it acts on all L^p spaces ($p \in [1, +\infty]$), it is contractive on any such space and it preserves positivity. We have shown that the Ricci curvature of M is bounded. An application of [18], Theorem 5.2.6 then shows that $\{T_t\}_{t\geq 0}$ preserves the identity: $T_t 1 = 1$. From this, conservation of the L^1 norm for data $v \geq 0$ follows. In fact, with the notation $v(t) = T_t v$ and using the fact that the adjoint of T_t when seen as acting on L^1 is T_t itself but seen as acting on L^∞, we have:

$$\begin{aligned}\|v(t)\|_1 &= \sup_{h\in \mathrm{L}^\infty;|h|\leq 1}\left|\int_{\mathbb{R}^d}(T_t v)h\,\mathrm{d}\mu_*\right| = \sup_{h\in \mathrm{L}^\infty;h\in[0,1]}\int_{\mathbb{R}^d}(T_t v)h\,\mathrm{d}\mu_* \\ &= \sup_{h\in \mathrm{L}^\infty;h\in[0,1]}\int_{\mathbb{R}^d}(T_t h)v\,\mathrm{d}\mu_* = \int_{\mathbb{R}^d}(T_t 1)v\,\mathrm{d}\mu_* = \int_{\mathbb{R}^d} v\,\mathrm{d}\mu_* = \|v\|_1. \ \square\end{aligned}$$

4.4 Linear case. Entropy Method

The behaviour of the heat kernel of the linear evolution considered and the L^1 contraction property allow to notice that for all $t \geq t_0$

$$\|u(t)\|_2^2 \leq \|u(t)\|_1 \|u(t)\|_\infty \leq C\frac{\|u_0\|_1}{t^{1/2}}.$$

Notice that the above bound is sharp. In fact, consider the solution corresponding to the Dirac delta at x_0, namely $v(t,x) = K(t,x,x_0)$. Its L^2 norm then satisfies, using the symmetry of the heat kernel and the semigroup property:

$$\begin{aligned}\|v(t)\|_2^2 &= \int_{\mathbb{R}^d} K(t,x,x_0)^2\,\mathrm{d}\mu_* = \int_{\mathbb{R}^d} K(t,x,x_0)K(t,x_0,x)\,\mathrm{d}\mu_* \\ &= K(2t,x_0,x_0) \sim ct^{-1/2} \quad \text{for large } t\end{aligned}$$

It is easy to get the same result by *entropy methods*. Although this is not necessary in the present case due to the previous calculations, this will serve as a model for the strategy of proof used in the nonlinear setting, and will make already apparent the role of the Nash inequalities proved before.

Proposition 4.12 *Let $F(t) = \|v(t)\|_2^2$. Then $F(t) \leq ct^{-1/2}$ for all $t > t_0$.*

Proof. First consider nonnegative data. Having shown that the L^1 norm of such solutions is conserved and, moreover, using the fact that $I_{m_*}[v(t)]$ is decreasing as a function of t, we get that $I_{m_*}[v(t)]/\|v(t)\|_1^2 \leq c_0$ for all positive t and for some $c_0 > 0$. We are then allowed to use (4.10) with $r = 2, s = 1$, so that we have

$$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = -I_{m_*}[v(t)] \leq -c\frac{F^3}{\|v(t)\|_1^4} = -c\frac{F^3}{\|v_0\|_1^4}$$

Thus we get, integrating the above differential inequality:

$$F(t) \leq \tilde{c}\frac{\|v_0\|_1^2}{t^{1/2}}.$$

The same decay holds true for all L^1 data, since we may write $-(v_0)_- \leq v_0 \leq (v_0)_+$ and use the order preserving property of the evolution and the decay bound already proved for nonnegative (or nonpositive) solutions. In fact, denoting by $v_\pm(s)$ the time evolved of $(v_0)_\pm$, we have first that, by comparison, $-v_-(s) \leq v(s) \leq v_+(s)$ and $v^2(s) = v_+^2(s) + v_-^2(s)$. This, together with the above decay property for nonnegative solutions $\|v_\pm(s)\|_2^2 \leq \tilde{c}\|(v_0)_\pm\|_1 s^{-1/2}$, implies that

$$F(t) = \|v(t)\|_2^2 = \|v_-(t)\|_2^2 + \|v_+(t)\|_2^2 \leq \frac{\tilde{c}\left(\|(v_0)_-\|_1^2 + \|(v_0)_+\|_1^2\right)}{t^{1/2}} \leq \tilde{c}\|v_0\|_1^2 t^{-1/2}. \quad \square$$

5 Nonlinear Entropy Method

Once the linear flow has been examined and its behaviour described, we prepare the way for the proof of convergence with rate of the nonlinear flow via a new version of the entropy-entropy dissipation method. We shall use the entropy and Fisher information introduced at the end of Section 2. The results of this section hold for any $m < 1$, but the main interest is in employing them for the case $m = m_*$ as is done in the subsequent section. From now on we revert to the notations for space, time and flow variables introduced in sections 1 and 2. Thus, $w = w(s, y)$.

5.1 Comparing linear and nonlinear quantities. The Fisher information

We have to prove the basic inequalities that relate the linear and the nonlinear quantities of the entropy method. We start the analysis by a new inequality between linear and nonlinear Fisher information, then we recall a Lemma of [7] which compares linear and nonlinear entropy. We shall write V_* instead of V_{D_*}. We put

$$\mathcal{I}_m[w] = \int_{\mathbb{R}^d} \left| \nabla \left(\frac{w^{m-1} - 1}{m-1} V_*^{m-1} \right) \right|^2 V_* w \, \mathrm{d}y, \tag{5.1}$$

which is the (nonlinear) Fisher information. It can be linearized, as done in [7], by letting $w = 1 + \varepsilon g V_*^{1-m}$ and taking the limit as $\varepsilon \to 0$. We obtain the linearized form of the Fisher information, that takes the expression of the Dirichlet form typical of the linearized equation

$$I_m[w] = \int_{\mathbb{R}^d} \left|\nabla (w-1) V_*^{m-1}\right|^2 V_* \, \mathrm{d}y = \int_{\mathbb{R}^d} |\nabla g|^2 \, V_* \, \mathrm{d}y \tag{5.2}$$

the relation between g and w is $g = (w-1)V_*^{m-1}$; it is not restrictive to let $\varepsilon = 1$ in the sequel. The next Lemma compares in a quantitative way I_m and $\mathcal{I}_m$. This is a first attempt that will be improved subsequently for $m = m_*$ and $m \neq m_*$. We drop the subindex m from both quantities for brevity.

Lemma 5.1 *Let $0 < W_0 \leq w \leq W_1 < +\infty$, be a measurable function on $\mathbb{R}^d$, with $W_0 < 1$ and $W_1 > 1$, and assume that $\mathcal{I}(w) < +\infty$. Then the following inequality holds true*

$$I[w] \leq k_1 \mathcal{I}[w] + k_2 \int_{\mathbb{R}^d} g^4 V_*^{4-3m} \, \mathrm{d}y \tag{5.3}$$

for any $m < 1$, where $g = (w-1)V_*^{m-1}$, $k_1 = 2W_1^{3-2m}$ and k_2 depends only on W_1, W_0, m and d.

Proof. We have $w-1 = gV_*^{1-m}$. We first re-write the Fisher information (5.1) in the following way:

$$\begin{aligned}\mathcal{I}[w] &= \int_{\mathbb{R}^d} \left|\nabla\left(\frac{w^{m-1}-1}{m-1}V_*^{m-1}\right)\right|^2 V_* w\,\mathrm{d}y \\ &:= \int_{\mathbb{R}^d} \left|\nabla\left(A(w)(w-1)V_*^{m-1}\right)\right|^2 V_* w\,\mathrm{d}y,\end{aligned} \tag{5.4}$$

where we have defined

$$A(w) := \frac{w^{m-1}-1}{(m-1)(w-1)} = \frac{a(w)}{w-1}.$$

It is easy to check that $A(1)=1$, $A(w)>0$, and that $A(w)\to 0$, when $w\to\infty$. Moreover,

$$A'(w) = \frac{w^{m-2}-A(w)}{w-1} \le 0 \tag{5.5}$$

since the function $a(w) = (w^{m-1}-1)/(m-1)$ is concave in w, so that its incremental quotient $A(w)$ (taken in $w=1$) is a non-increasing function of w. If we let $W_0 \le w \le W_1$, with $0 < W_0 \le 1$ and $1 \le W_1 < +\infty$, we then have the bounds

$$W_1^{m-2} = a'(W_1) \le |A(w)| \le a'(W_0) = W_0^{m-2}. \tag{5.6}$$

We shall also need estimates for $|A'(w)|$ for $W_0 \le w \le W_1$ as above, and in fact it is easy to check that $A'(1) = (m-2)/2$ and that A' is bounded away from zero. Letting now w be a function, noticing that $w-1 = gV_*^{1-m}$ and that (5.5) can be rewritten as $(w-1)A'(w) + A(w) = w^{m-2}$, we get

$$\begin{aligned}\nabla\left[A(w)(w-1)V_*^{m-1}\right] &= \nabla\big[A(w)g\big] = A(w)\nabla g + A'(w)\big[\nabla w\big]g \\ &= A(w)\nabla g + A'(w)\big[\nabla(1+gV_*^{1-m})\big]g \\ &= A(w)\nabla g + A'(w)gV_*^{1-m}\big[\nabla g\big] + A'(w)g^2\big[\nabla V_*^{1-m}\big] \\ &= \big[A(w) + A'(w)(w-1)\big]\nabla g + A'(w)g^2\big[\nabla V_*^{1-m}\big] \\ &= w^{m-2}\nabla g + A'(w)\big[\nabla V_*^{1-m}\big]g^2.\end{aligned} \tag{5.7}$$

Now we can use this equality in (5.1) to get:

$$\begin{aligned}\mathcal{I}[w] &= \int_{\mathbb{R}^d} \big|A(w)\nabla g + \big[\nabla A(w)\big]g\big|^2 V_* w\,\mathrm{d}y \\ &= \int_{\mathbb{R}^d} \big|w^{m-2}\nabla g + A'(w)\big[\nabla V_*^{1-m}\big]g^2\big|^2 V_* w\,\mathrm{d}y \\ &\ge \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g|^2 V_* w^{2(m-2)+1}\,\mathrm{d}y - \int_{\mathbb{R}^d} g^4\big|A'(w)\big|^2\left|\nabla V_*^{1-m}\right|^2 V_* w\,\mathrm{d}y \\ &\ge \frac{1}{2}W_1^{2m-3}\int_{\mathbb{R}^d} |\nabla g|^2 V_*\,\mathrm{d}y - \int_{\mathbb{R}^d} g^4\big|A'(w)\big|^2\left|\nabla V_*^{1-m}\right|^2 V_* w\,\mathrm{d}y,\end{aligned}$$

where we have used the inequality $|a+b|^2 + |b|^2 \ge (1/2)|a|^2$ valid for any $a,b\in\mathbb{R}$, and the bounds $W_0 \le w \le W_1$. Thus, we have

$$I[g] = \int_{\mathbb{R}^d} |\nabla g|^2 V_*\,\mathrm{d}y \le \frac{2}{W_1^{2m-3}}\mathcal{I}[w] + \frac{1}{W_1^{2m-3}}\int_{\mathbb{R}^d} g^4\big|A'(w)\big|^2\left|\nabla V_*^{1-m}\right|^2 V_* w\,\mathrm{d}y$$

We next remark that the weight

$$\left|\nabla V_*^{1-m}(y)\right|^2 V_*(y) = \frac{4|y|^2}{\left(D+|y|^2\right)^4}\frac{1}{\left(D+|y|^2\right)^{\frac{1}{1-m}}}$$
$$\leq \frac{4}{\left(D+|y|^2\right)^3}\frac{1}{\left(D+|y|^2\right)^{\frac{1}{1-m}}} = \frac{4}{\left(D+|y|^2\right)^{3+\frac{1}{1-m}}} = 4V_*^{4-3m}$$

is integrable whenever $(d-6)m > (d-8)$. *Notice that when $m = m_* = (d-4)/(d-2)$, the weight is integrable.* We conclude by estimating $|A'| \leq k_0$ so that

$$I[g] = \int_{\mathbb{R}^d} |\nabla g|^2 V_* \,\mathrm{d}y \leq 2W_1^{3-2m}\mathcal{I}[w] + 4W_1^{2(1-m)}k_0 \int_{\mathbb{R}^d} g^4 V_*^{4-3m}\,\mathrm{d}y.$$

This concludes the proof. □

5.2 Evolution properties of the Fisher information

We now describe some further properties of the Fisher information $\mathcal{I}[w(s)]$ as a function of time, such as the fact that it is uniformly bounded for large s and it goes to zero as $s \to +\infty$. We prove a new differential inequality for the Fisher information. Indeed, by Proposition 2.6 of [7], it is easy to see that the Fisher information is finite almost everywhere and is the time derivative of the entropy almost everywhere so that:

$$\mathcal{F}[w(s_0)] - \mathcal{F}[w(s)] = \int_{s_0}^{s} \mathcal{I}[w(\xi)]\,\mathrm{d}\xi \tag{5.8}$$

taking the limits $s \to \infty$ and $s_0 \to 0$, recalling that $0 \leq \mathcal{F}[w(0)] < +\infty$, $0 \leq \mathcal{F}[w(s)] \to 0$ as $s \to +\infty$, we can conclude that $\mathcal{I}[w(s)]$ is integrable (and nonnegative) on $(0,+\infty)$.

Proposition 5.2 *In addition to the running assumptions, suppose that $v(0) - V_D \geq 0$. Then, the following differential inequality for the Fisher information holds true*

$$\frac{\mathrm{d}\mathcal{I}[w(s)]}{\mathrm{d}s} \;\leq\; \kappa_1\mathcal{I}[w(s)] \;-\; \kappa_2\mathcal{I}^2[w(s)] \tag{5.9}$$

the constant κ_1 depends on m, d, s_0, and the constant κ_2 depends on m, d, the relative mass $\int_{\mathbb{R}^d}(v_0 - V_D)\,\mathrm{d}x$, W_0 and W_1. Moreover, $\mathcal{I}[w(s)]$ goes to zero as $s \to \infty$.

Remark. The time derivative of the Fisher information is usually controlled by means of the Bakry-Emery method (cf. [3]) that allows to obtain spectral gap estimates. Such an estimate cannot hold when $m = m_*$ since there is no spectral gap. The above proposition can be viewed as a substitute for the Bakry-Emery method and gives a solution for asymptotic estimates in applications with no spectral gap.

Proof. The proof is divided in several steps. We use the notation $\Omega = \left(\frac{v^{m-1} - V_D^{m-1}}{m-1}\right)$ in this section for brevity. Note that for large s, Ω is uniformly bounded and $|\Omega| \leq V_{D_0}^{m-2}|v - V_D|$.

• Expression of the derivative. We first perform a formal time-derivative of $\mathcal{I}$, but in this case is convenient to write it in terms of v and V_D instead of w, where we recall that $w = v/V_D$

$$
\begin{aligned}
\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}s} &= \frac{\mathrm{d}}{\mathrm{d}s}\int_{\mathbb{R}^d} |\nabla\Omega|^2\, v\,\mathrm{d}y \\
&= 2\int_{\mathbb{R}^d} \nabla\Omega\cdot\nabla\left(v^{m-2}\frac{\mathrm{d}v}{\mathrm{d}s}\right) v\,\mathrm{d}y + \int_{\mathbb{R}^d} |\nabla\Omega|^2\,\frac{\mathrm{d}v}{\mathrm{d}s}\,\mathrm{d}y = (A)+(B)
\end{aligned}
\tag{5.10}
$$

Now we treat the two terms separately.

• Estimating the term (A): We have

$$
(A) = 2\int_{\mathbb{R}^d} v\nabla\Omega\cdot\nabla\left(v^{m-2}\frac{\mathrm{d}v}{\mathrm{d}s}\right)\mathrm{d}y = -2\int_{\mathbb{R}^d} \nabla\cdot[v\nabla\Omega]\,v^{m-2}\frac{\mathrm{d}v}{\mathrm{d}s}\,\mathrm{d}y. \tag{5.11}
$$

Using the equation, $v_s = \nabla\cdot\Omega$, we get

$$
(A) = -2\int_{\mathbb{R}^d} [\nabla\cdot(v\nabla\Omega)]^2\, v^{m-2}\,\mathrm{d}y = -2\int_{\mathbb{R}^d} [\nabla\cdot(v\nabla\Omega)]^2\,\frac{\Omega^2}{\Omega^2 v^{2-m}}\,\mathrm{d}y. \tag{5.12}
$$

Then we have

$$
\begin{aligned}
(A) &\le_{(i)} -2\frac{\left[\int_{\mathbb{R}^d} |\nabla\cdot(v\nabla\Omega)||\Omega|\,\mathrm{d}y\right]^2}{\int_{\mathbb{R}^d}\Omega^2 v^{2-m}\,\mathrm{d}y} \le_{(ii)} -2\frac{\left[-\int_{\mathbb{R}^d} v|\nabla\Omega|^2\,\mathrm{d}y\right]^2}{\int_{\mathbb{R}^d}\Omega^2 v^{2-m}\,\mathrm{d}y} \\
&\le_{(iii)} -2\frac{\mathcal{I}^2}{c_2\int_{\mathbb{R}^d}(v(0)-V_D)\,\mathrm{d}y} := -\kappa_2\mathcal{I}^2,
\end{aligned}
\tag{5.13}
$$

where in (i) we have used the Hölder inequality

$$
\int \frac{h_1^2}{h_2}\,\mathrm{d}\mu \ge \frac{\left[\int h_1\,\mathrm{d}\mu\right]^2}{\int h_2\,\mathrm{d}\mu},
$$

while in (ii) we use integration by parts, after noticing that $|a||b| \ge ab$. The point (iii) relies on fact that the difference between two Barenblatt solutions behaves like V_D^{2-m}, and on the fact that $V_{D_0} \le v(t) \le V_{D_1}$, so that

$$
\begin{aligned}
\int_{\mathbb{R}^d} \Omega^2 v^{2-m}\,\mathrm{d}y &= \int_{\mathbb{R}^d}\left(\frac{w^{m-1}-1}{m-1}\right)^2 V_D^{2(m-1)} v^{2-m}\,\mathrm{d}y \\
&= \int_{\mathbb{R}^d}\left(\frac{w^{m-1}-1}{m-1}\right)^2 V_{D_1}^{m}\,\mathrm{d}y \\
(a) &\le \max\{W_0^{m-2}, W_1^{m-2}\}\int_{\mathbb{R}^d} |w-1|^2 V_{D_1}^m\,\mathrm{d}y \\
&= \max\{W_0^{m-2}, W_1^{m-2}\}\int_{\mathbb{R}^d} |v-V_D|^2\frac{V_{D_1}^m}{V_D^2}\,\mathrm{d}y \\
(b) &\le c_0\max\{W_0^{m-2}, W_1^{m-2}\}\int_{\mathbb{R}^d} |v-V_D|^2 V_{D_1}^{m-2}\,\mathrm{d}y \\
(c) &\le c_0c_1\max\{W_0^{m-2}, W_1^{m-2}\}\int_{\mathbb{R}^d} |v-V_D| V_D^{2-m}V_D^{m-2}\,\mathrm{d}y \\
(d) &= c_2\int_{\mathbb{R}^d}|v-V_D|\,\mathrm{d}y = c_2\int_{\mathbb{R}^d}(v-V_D)\,\mathrm{d}y = c_2\int_{\mathbb{R}^d}(v(0)-V_D)\,\mathrm{d}y,
\end{aligned}
$$

where in (a) we have used (5.6), namely

$$W_1^{m-2}|w-1| \le \left|\frac{(w^{m-1}-1)}{(m-1)}\right| \le W_0^{m-2}|w-1|,$$

while in (b) we have used $V_D \ge c_0 V_{D_1}$ and in (c) we have used $|v - V_D| \le c_1 V_D^{2-m}$. In the last step (d) we have used hypothesis (H2') together with the fact that $v(0) - V_D \ge 0$ and conservation of relative mass, proved in Proposition 2.3 of [7].

• Estimating the term (B). We shall use the celebrated Bénilan-Crandall estimates [4], that for solutions to the un-rescaled FDE $\partial_t u = \Delta u^m/m$ read

$$\partial_t u(t,x) \le \frac{u(t,x)}{(1-m)t} \qquad \text{for any } t>0$$

if $m<1$, even for $m \le 0$. We perform the scaling to the Fokker-Plank equation, like in section 2, so that the Bénilan–Crandall estimates read

(5.14)

$$\begin{aligned}
\partial_s v(s,y) &\le \frac{2}{[d(1-m)-2](1-m)}\left[\frac{d}{d(1-m)-2} + \frac{1}{(1-m)\left(\mathrm{e}^{s(1-m)[d(1-m)-2]/2}-1\right)}\right] v(s,y) \\
&\le \frac{2}{[d(1-m)-2](1-m)}\left[\frac{d}{d(1-m)-2} + \frac{1}{(1-m)\left(\mathrm{e}^{s_0(1-m)[d(1-m)-2]/2}-1\right)}\right] v(s,y) \\
&= \kappa_1(m,d,s_0) v(s,y),
\end{aligned}$$

if $s \ge s_0 > 0$. We remark that $\kappa_1 \to +\infty$ when $s_0 \to 0$ but this will not be a problem. We finally estimate (B)

$$\begin{aligned}
(B) &= \int_{\mathbb{R}^d} |\nabla\Omega|^2\, \partial_s v \,\mathrm{d}y \le \mathcal{C}(m,d,s_0) \int_{\mathbb{R}^d} |\nabla\Omega|^2\, v\,\mathrm{d}y \\
&= \kappa_1(m,d,s_0)\, \mathcal{I}[w(s)].
\end{aligned} \tag{5.15}$$

This calculation is formal and has to be justified, but before we do that let us draw a first consequence.

• Integrating the Differential Inequality. We obtained a closed differential inequality for the Fisher information $\mathcal{I}[w(s)] = \mathcal{I}(s)$

$$\frac{\mathrm{d}\mathcal{I}(s)}{\mathrm{d}s} - \kappa_1 \mathcal{I}(s) + \kappa_2 \mathcal{I}^2(s) \le 0$$

which is of Bernoulli type and can be estimated explicitly. Indeed, the exact solution on $(s_1, s) \subseteq [0, +\infty)$ of the Bernoulli ordinary differential equation $Z'(s) - \kappa_1 Z(s) + \kappa_2 Z^2(s) = 0$ is given by

$$Z(s) = \frac{\mathrm{e}^{\kappa_1(s-s_1)}}{\left[Z_0^{-1} + \int_{s_1}^{s} \mathrm{e}^{\kappa_1(\xi-s_1)}\kappa_2\,\mathrm{d}\xi\right]} \le \frac{\mathrm{e}^{\kappa_1(s-s_1)}}{\kappa_2 \int_{s_1}^{s} \mathrm{e}^{\kappa_1(\xi-s_1)}\,\mathrm{d}\xi} \le c\frac{\kappa_1}{\kappa_2} \tag{5.16}$$

if $s \ge s_1 + 1 := s_0$, for a suitable $c>1$ which can be taken to be arbitrarily close to one by choosing s_1 large enough. By comparison it is clear that $\mathcal{I}(s) \le Z(s) \le c\kappa_1/\kappa_2$, provided $\mathcal{I}(s_0) \le Z(s_0)$. Therefore, for all $c>1$ and $0 < s_0 \le s$,

$$\mathcal{I}[w(s)] \le \frac{c\kappa_1}{\kappa_2}. \tag{5.17}$$

The constant κ_2 depends on m, d, the relative mass $\int_{\mathbb{R}^d}(v_0 - V_D)\,\mathrm{d}x$, W_0 and W_1; the constant κ_1 depends on m, d, s_0 and $\kappa_1 \to +\infty$ when $s_0 \to 0$.

• JUSTIFICATION OF THE CALCULATION. The differentiation of $\mathcal{I}$ performed above contains calculations that are not justified in principle since they involve differentiations and integrations by parts in integrals over the whole space that are not justified a priori. Therefore, we introduce a cutoff function $\zeta_n(y)$ for the integrand of $\mathcal{I}$ and define

$$\mathcal{I}_n = \int_{\mathbb{R}^d} |\nabla\Omega|^2\, v\zeta_n^2\,\mathrm{d}y.$$

We assume that ζ_n has value 1 if $|y| \le n$, value 0 of $|y| \ge 2n$, and $|\nabla\zeta_n| \le 1/n$, $|\Delta\zeta_n| \le 1/n^2$. Then we have

$$\frac{\mathrm{d}\mathcal{I}_n}{\mathrm{d}s} = (A_n) + (B_n) \tag{5.18}$$

and the two terms are as before but for the cutoff factor. There is no problem with (B_n). But (A_n) produces extra terms that we must control. Indeed,

$$\begin{aligned}(A_n) &= 2\int_{\mathbb{R}^d} v\nabla\Omega\cdot\nabla\left(v^{m-2}\frac{\mathrm{d}v}{\mathrm{d}s}\right)\zeta_n^2\,\mathrm{d}y\\ &= -2\int_{\mathbb{R}^d}\nabla\cdot(v\nabla\Omega)\,v^{m-2}\frac{\mathrm{d}v}{\mathrm{d}s}\zeta_n^2\,\mathrm{d}y - 2\int_{\mathbb{R}^d} v^{m-1}\frac{\mathrm{d}v}{\mathrm{d}s}\left(\nabla\Omega\cdot\nabla\zeta_n^2\right)\,\mathrm{d}y.\end{aligned}$$

When we replace dv/ds by its value according to the equation, the first of the two terms of the last expression becomes

$$(A_{n1}) := -2\int_{\mathbb{R}^d}|\nabla\cdot(v\nabla\Omega)|^2\, v^{m-2}\zeta_n^2\,\mathrm{d}y\left(= -2\int_{\mathbb{R}^d}|v_s|^2 v^{m-2}\zeta_n^2\,\mathrm{d}y\right). \tag{5.19}$$

which has a convenient negative sign. We now perform integration by parts on this term, an operation that is now perfectly justified, and we get much as before:

$$(A_{n1}) = -2\int_{\mathbb{R}^d}[\nabla\cdot(v\nabla\Omega)]^2\,\frac{\Omega^2}{\Omega^2 v^{2-m}}\zeta_n^2\,\mathrm{d}y \le -2\frac{\left[\int_{\mathbb{R}^d}|\nabla\cdot(v\nabla\Omega)|\,|\Omega|\zeta_n^2\,\mathrm{d}y\right]^2}{\int_{\mathbb{R}^d}\Omega^2 v^{2-m}\zeta_n^2\,\mathrm{d}y}$$

The numerator of the last term is larger than $|\int_{\mathbb{R}^d}\nabla\cdot(v\nabla\Omega)\,\Omega\zeta_n^2\,\mathrm{d}y|$, hence

$$(A_{n1}) \le -\kappa_2\left|\int_{\mathbb{R}^d}(\nabla\cdot(v\nabla\Omega))\,\Omega\zeta_n^2\,\mathrm{d}y\right|^2.$$

with the notation that we have used above. Let us calculate the integral: after integrating by parts, it gives a term as before plus a term where ζ_n^2 is differentiated, as follows:

$$\int_{\mathbb{R}^d} v\,|\nabla\Omega|^2\,\zeta_n^2\,\mathrm{d}y + 2\int_{\mathbb{R}^d} v\Omega\nabla\Omega\cdot\zeta_n\nabla\zeta_n\,\mathrm{d}y = (X_n') + (X_n'')$$

The first term is $(X_n') = \mathcal{I}_n$, as before, while the new term, (X_n''), can be tackled as follows. We separate by Hölder a factor like $\mathcal{I}_n^{1/2}$ (but we only need to integrate in the annulus $R_n = \{n \le |y| \le 2n\}$ so it goes to zero as $n \to \infty$) and we still have another factor:

$$\int_{\mathbb{R}^d} v\,|\Omega|^2\,|\nabla\zeta_n|^2\,\mathrm{d}y \le C\int_{R_n} v|V_{D_0}^{m-1} - V_{D_1}^{m-1}|^2|\nabla\zeta_n|^2\,\mathrm{d}y \le C\int_{R_n}\frac{v}{n^2}dy$$

and this tends to zero as $n \to \infty$ for $m > m_*$. For $m \le m_*$ we calculate differently,

$$\int_{R_n} v|V_{D_0}^{m-1} - V_{D_1}^{m-1}|^2 |\nabla\zeta_n|^2 \,\mathrm{d}y \le C \int_{R_n} \frac{V_D^{m-1}}{n^2} |v - V_D| dy \le C \int_{R_n} |v - V_D| dy,$$

that goes to zero as $n \to \infty$, but in a uniform way we only know that is bounded a priori. In any case, raising to the square we get an estimate of the form

$$(A_{n1}) \le -\kappa_2' \mathcal{I}_n^2 + \kappa_2'' \mathcal{I}, \tag{5.20}$$

with constants uniform in n.

• We now consider the other new term

$$(A_{n2}) = -4 \int_{\mathbb{R}^d} v^{m-1} \zeta_n \frac{\mathrm{d}v}{\mathrm{d}s} \left(\nabla\Omega \cdot \nabla\zeta_n\right) \,\mathrm{d}y.$$

Use Hölder and the numerical inequality $2ab \le \varepsilon a^2 + b^2/\varepsilon$, to separate a term like (A_{n1}) in formula (5.19), with a factor ε which is convenient to be absorbed by the term already existing that has a negative sign in front, as we have remarked. We are then left with another factor of the form

$$(A_{n22}) \le C \int_{\mathbb{R}^d} v^m \left|\nabla\Omega\right|^2 |\nabla\zeta_n|^2 \,\mathrm{d}y \sim C \int_{n \le |y| \le 2n} \frac{v^m}{n^2} |\nabla(v^{m-1} - V^{m-1})|^2 \,\mathrm{d}y.$$

Since $v^{m-1}(s,y) \sim |y|^2$ as $|y| \to \infty$, and we get an equivalent expression

$$(A_{n22}) \le C \int_{n \le |y| \le 2n} \left|\nabla\Omega\right|^2 v \,\mathrm{d}y \le C\mathcal{I}_{2n}.$$

If we had $C\mathcal{I}_n$ instead of $C\mathcal{I}_{2n}$ we would have ended. The estimate for (B_n) has no problems.

• To solve the difficulty we take a little detour. We integrate the inequality obtained so far for $\mathrm{d}\mathcal{I}_n/\,\mathrm{d}s$ to get the integrated inequality:

$$\mathcal{I}_n(s_2) - \mathcal{I}_n(s_1) \le k \int_{s_1}^{s_2} \mathcal{I}_n \,\mathrm{d}s + C \int_{s_1}^{s_2} \mathcal{I}_{2n} \,\mathrm{d}s \tag{5.21}$$

But the right-hand side is bounded above by the integral

$$(C + k) \int_{s_1}^{s_2} \mathcal{I} \,\mathrm{d}s \tag{5.22}$$

and this is known to be bounded by the relative entropy. Moreover, since the integral $\int_{s_1}^{\infty} \mathcal{I} \,\mathrm{d}s$ is finite, for every $\varepsilon > 0$ there exists a s_ε such that

$$\int_{s_\varepsilon}^{\infty} \mathcal{I} \,\mathrm{d}s \le \varepsilon. \tag{5.23}$$

We conclude that $\mathcal{I}_n(s_2) - \mathcal{I}_n(s_1) \le \varepsilon$ when $s_\varepsilon \le s_1 \le s_2$. Combining this half continuity with the integrability of $\mathcal{I}_n(s) \le \mathcal{I}(s)$ given by (5.23), we obtain by an easy calculus lemma that

$$\mathcal{I}_n(s) \le C_1\varepsilon \tag{5.24}$$

for all $s \geq 2s_\varepsilon$, with C_1 uniform in n. We conclude that $\mathcal{I}(s) = \lim_n \mathcal{I}_n(s)$ is bounded for all large times and goes to zero as $s \to \infty$.

• Coming back to the differential inequality satisfied by $\mathcal{I}_n$ we have proved that $\mathcal{I}_n' \leq c_1 \mathcal{I} - c_2 \mathcal{I}_n^2$. Integrating this differential inequality in time between s_1 and s_2 with $s_1 < s_2$ sufficiently large we get $I_n(s_2) - I_n(s_1) \leq c_1 \int_{s_1}^{s_2} I(s)\,\mathrm{d}s - c_2 \int_{s_1}^{s_2} I_n(s)^2\,\mathrm{d}s$ so that, passing to the limit as $n \to +\infty$ and using both monotone convergence and the boundedness of I as a function of time we get $I(s_2) - I(s_1) \leq c_1 \int_{s_1}^{s_2} I(s)\,\mathrm{d}s - c_2 \int_{s_1}^{s_2} I(s)^2\,\mathrm{d}s$, which is an equivalent form of our statement. □

5.3 Comparing linear and nonlinear entropies

The quantitative comparison of linear and nonlinear entropies concludes the preliminary results needed for the nonlinear entropy method. Under Assumptions (H1")-(H2"), the relative entropy is well defined.

Lemma 5.3 (An equivalence result) *Let $m < 1$. If w satisfies* (H1")-(H2")*, then*

$$\frac{F[w]}{2W_1^{2-m}} \leq \mathcal{F}[w] \leq \frac{F[w]}{2W_0^{2-m}}. \tag{5.25}$$

We recall that $F[w] = \int_{\mathbb{R}^d} |w-1|^2 V_{D_*}^m\,\mathrm{d}x$.

The short proof of this result has been given first in [7] but we repeat it here for reader's convenience.

Proof. For $a > 0$, let $\phi_a(w) := \frac{1}{1-m}\left[(w-1) - (w^m-1)/m\right] - a\,(w-1)^2$. We compute $\phi_a'(w) = \frac{1}{1-m}\left[1 - w^{m-1}\right] - 2a\,(w-1)$ and $\phi_a''(w) = w^{m-2} - 2a$, and note that $\phi_a(1) = \phi_a'(1) = 0$. With $a = W_1^{m-2}/2$, ϕ_a'' is positive on (W_0, W_1), which proves the lower bound after multiplying by V_D^m and integrating over $\mathbb{R}^d$. With $a = W_0^{m-2}/2$, ϕ_a'' is negative on (W_0, W_1) which proves the upper bound. □

Equivalently, we may write

$$\frac{F[w]}{2W_1^{2-m}} \leq \frac{F[w]}{2\sup\limits_{\mathbb{R}^d}|w|^{2-m}} \leq \mathcal{F}[w] \leq \frac{F[w]}{2\inf\limits_{\mathbb{R}^d}|w|^{2-m}} \leq \frac{F[w]}{2W_0^{2-m}}. \tag{5.26}$$

5.4 The entropy bounds a suitable L^p-norm

Lemma 5.4 *Let $m < 1$. If w satisfies* (H1")-(H2")*, then*

$$\|w-1\|_{\mathrm{L}^{2+\frac{m}{1-m}}(\mathbb{R}^d)}^{2+\frac{m}{1-m}} \leq \overline{D}_m F[w], \tag{5.27}$$

where $\overline{D}_m$ is given at the end of the proof.

Proof. We first state some inequalities between Barenblatt solutions with different constants. Consider

$$\frac{\partial V_D}{\partial D} = -\frac{1}{1-m}\left[D + |y|^2\right]^{-\frac{2-m}{1-m}} = -\frac{1}{1-m} V_D^{2-m} \leq 0\,.$$

Hence, for any $0 < D_1 < D_0$

$$\frac{D_0 - D_1}{1-m} V_{D_0}^{2-m} \leq |V_{D_1} - V_{D_0}| \leq \frac{D_0 - D_1}{1-m} V_{D_1}^{2-m}$$

Moreover, it is easy to see that if $0 < D_1 \le D_0$

$$V_{D_0}^{1-m}(y) = \frac{1}{D_0 + |y|^2} \le \frac{1}{D_1 + |y|^2} = V_{D_1}^{1-m}(y) \le \left(1 + \frac{D_0}{D_1}\right) \frac{1}{D_0 + |y|^2} = \left(1 + \frac{D_0}{D_1}\right) V_{D_0}^{1-m}(y).$$

The above inequalities prove that $|w-1|^{m/(1-m)}$ is bounded by a multiple of V_*^m. Indeed, by hypothesis (H1') we have that $0 < D_0 < D_* < D_1$, so that $V_{D_0} - V_* \le v(s) - V_* \le V_{D_1} - V_*$, and since $w = v/V_*$

$$|w - 1| = \left|\frac{v(s) - V_*}{V_*}\right| \le \frac{|V_{D_1} - V_{D_0}|}{V_*} \le \frac{D_0 - D_1}{1 - m} \frac{V_{D_1}^{2-m}}{V_*}$$

Thus,

$$\begin{aligned} \|w - 1\|_{\mathrm{L}^{2+\frac{m}{1-m}}(\mathbb{R}^d)}^{2+\frac{m}{1-m}} &= \int_{\mathbb{R}^d} |w - 1|^2 |w - 1|^{\frac{m}{1-m}} \,\mathrm{d}y \\ &\le \left(\frac{D_0 - D_1}{1 - m}\right)^{\frac{m}{1-m}} \left(1 + \frac{D_*}{D_1}\right)^{\frac{m(2-m)}{(1-m)^2}} \int_{\mathbb{R}^d} |w - 1|^2 V_*^m \,\mathrm{d}y := \overline{D}_m F[w]. \quad \square \end{aligned}$$

Remarks. (i) The estimate proves that $w - 1 \in \mathrm{L}^{2+\frac{m}{1-m}}(\mathbb{R}^d)$, whenever the initial entropy is finite, since we know, joining inequalities (5.25) and (5.27):

$$\|w(s) - 1\|_{\mathrm{L}^{2+\frac{m}{1-m}}(\mathbb{R}^d)}^{2+\frac{m}{1-m}} \le F[w(s)] \le 2\overline{D}_m W_1^{2-m} \mathcal{F}[w(s)] \le 2\overline{D}_m W_1^{2-m} \mathcal{F}[w_0], \tag{5.28}$$

since the nonlinear entropy is decreasing in time. Moreover, we have also proved that

$$\|w(s) - 1\|_{\mathrm{L}^{2+\frac{m}{1-m}}(\mathbb{R}^d)}^{2+\frac{m}{1-m}} \le 2\overline{D}_m W_1^{2-m} \mathcal{F}[w(s)] \tag{5.29}$$

and we shall show below that entropy goes to zero as $s \to +\infty$.

(ii) As an easy consequence, letting $w - 1 = (v - V_*)/V_*$ and using the fact that $V_* \le C$, we obtain

$$\|v - V_*\|_{\mathrm{L}^{2+\frac{m}{1-m}}(\mathbb{R}^d)}^{2+\frac{m}{1-m}} \le \overline{D}_m F[w]\,. \tag{5.30}$$

(iii) For $m = m_*$ we have $2 + \frac{m}{1-m} = d/2$.

5.5 Comparing linear and nonlinear Fisher information

With the above remarks we can improve on Lemma 5.1 that compares the linear and nonlinear Fisher information:

Proposition 5.5 *Under the same assumptions of Lemma* 5.1, *we have*

$$I[g] = \int_{\mathbb{R}^d} |\nabla g|^2 \, V_* \,\mathrm{d}y \le k_1 \mathcal{I}[w] + k_3 \mathcal{F}^{1+\sigma}[w] \tag{5.31}$$

for any $m < 1$, where $g = (w - 1)V_*^{m-1}$, $\sigma = 2/[d + 2 + m/(1 - m)] > 0$, $k_1 = 2W_1^{3-2m}$, and $k_3 > 0$ is given at the end of the proof.

Proof. We estimate the second term of the inequality of Lemma 5.1 in the following way

$$\begin{aligned}\int_{\mathbb{R}^d} g^4 V_*^{4-3m}\,\mathrm{d}y &= \int_{\mathbb{R}^d} \left(|w-1|V^{m-1}\right)^4 V_*^{4-3m}\,\mathrm{d}y = \int_{\mathbb{R}^d} \left(|w-1|\right)^4 V_*^{m}\,\mathrm{d}y\\ &\le \|w-1\|_\infty^2 \int_{\mathbb{R}^d} \left(|w-1|\right)^2 V_*^{m}\,\mathrm{d}y = \|w-1\|_\infty^2\, F[w]\end{aligned}$$

Now we recall the interpolation inequality (7.7) with $j=0$

$$\|f\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \;\le\; \mathcal{C}_d\, \|f\|_{C^1(\mathbb{R}^d)}^{\frac{d}{d+p}}\, \|f\|_p^{\frac{p}{d+p}} \tag{5.32}$$

then we apply it to $f=w-1$ and we let $p=2+m/(1-m)$. We get:

$$\begin{aligned}\|w-1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} &\le \mathcal{C}_{p,d}\, \|w-1\|_{C^1(\mathbb{R}^d)}^{\frac{d}{d+p}} \left(\|w(s)-1\|_{\mathrm{L}^{2+\frac{m}{1-m}}(\mathbb{R}^d)}^{2+\frac{m}{1-m}}\right)^{\frac{1}{d+p}}\\ &\le \mathcal{C}_{p,d} M_1 \left(2\overline{D}_m W_1^{2-m}\mathcal{F}[w]\right)^{\frac{1}{d+2+\frac{m}{1-m}}} := k_3 \mathcal{F}[w]^{\sigma/2}\end{aligned} \tag{5.33}$$

where $\sigma = 2/[d+2+m/(1-m)]>0$ for any $m<1$ and we used inequality (5.28) and the fact that $\|w-1\|_{C^1(\mathbb{R}^d)}^{\frac{d}{d+p}} \le M_1$ by Theorem 7.2. Thus we have proved that

$$\int_{\mathbb{R}^d} g^4 V_*^{4-3m}\,\mathrm{d}y \le \|w-1\|_\infty^2\, F[w] \le k_3 \mathcal{F}[w]^{1+\sigma}.$$

The expression of k_3 is then

$$k_3 = \mathcal{C}_{p,d} M_1 \left(2\overline{D}_m W_1^{2-m}\right)^{\frac{1}{d+2+\frac{m}{1-m}}}$$

where $\|w-1\|_{C^1(\mathbb{R}^d)}^{\frac{d}{d+p}} \le M_1$, $\overline{D}_m = \frac{D_0-D_*}{1-m}\frac{D_*-D_1}{1-m}\left(1+\frac{D_*}{D_1}\right)^{\frac{2-m}{1-m}}$, and $\mathcal{C}_{p,d}$ is the constant of the interpolation inequality 7.7 with $j=0$ and $p=2+m/(1-m)$. $\square$

Remarks. (i) The above proposition holds for any $m<1$ and allow to conclude that $I(s)\to 0$ as $s\to+\infty$, since we already know that both $\mathcal{I}(s)$ (cf. Proposition 5.2) and $\mathcal{F}(s)$ tend to zero as $t\to+\infty$.

(ii) When $m=m_*$, we obtain that $\sigma = 4/(3d)$. But in this critical case we shall need another finer comparison between the linear and the nonlinear Fisher information that hold only when $m=m_*$, namely we would like to have that there exists $s_0>0$ and a constant $k_4>0$, such that

$$I[g(s)] \le k_4 \mathcal{I}[w(s)]. \tag{5.34}$$

for any $s\ge s_0$, where $g=(w-1)V_*^{m_*-1}$. Unfortunately the above inequality is not guaranteed for all times $s\ge s_0$. In the next section we will prove a weaker version of this statement, sufficient to our scopes, namely we will show that the above estimate (5.34) holds on a family of intervals $[s_{1,k}, s_{2,k}]$ that is sufficiently dense as $s\to\infty$. The technical details will be postponed to Appendix A4.

6 Proofs of the main results in the critical case

In this section we shall always take $m=m_*$, and we shall show that the nonlinear flow converges with the same rate as the linear case, cf. Section 4.4. We shall use the relationship between the entropy functional $\mathcal{F}$ and the Fisher information $\mathcal{I}$, namely $\mathrm{d}\mathcal{F}/\,\mathrm{d}s = -\mathcal{I}$. In view of the absence of any spectral gap (or Hardy-Poincaré inequality) inequality, valid instead in the case $m \neq m_*$, we have to proceed differently. The Gagliardo-Nirenberg inequality, that in the linear case give the correct decay of the linearized entropy in the $\mathrm{L}^2(V_*^{2-m}\,\mathrm{d}x)$-norm, turn out to work as well in the nonlinear case as the previous proposition started to show.

Proof of Theorem 3.1. Notice first that $\|g\|_\infty$ is finite and bounded as a function of time. In fact, by hypothesis (H1') we know that

$$
\begin{aligned}
|g(s,y)| = |w(s,y)-1|V_{D_*}^{m-1} &= \left|\frac{v - V_{D_*}}{V_{D_*}}\right| V_{D_*}^{m-1} \le c_0 |V_{D_1}(y) - V_{D_0}(y)| V_{D_*}^{m-2} \\
&\le c_1 V_{D_*}^{2-m} V_{D_*}^{m-2} = c_1
\end{aligned}
$$

for all $y \in \mathbb{R}^d$ and all $s > 0$, where c_i are a positive constant depending only on m, D_0, D_1, D_*. The inequality $|V_{D_1} - V_{D_0}| \le c V_{D_*}^{2-m}$ can be proved easily using the explicit expression of the pseudo–Barenblatt solutions (see the proof of Lemma 5.4).

Next we prove that I is bounded as a function of time. Indeed by Lemma 5.1 we observe that

$$
I[g] \le k_1 \mathcal{I}[w] + k_2 \int_{\mathbb{R}^d} g^4 V_{D_*}^{4-3m}\,\mathrm{d}y \le k_1 \mathcal{I}[w] + k_2 k_3 \|g\|_\infty^4 \tag{6.1}
$$

where we have noticed that, for $m = m_*$, $V_{D_*}^{4-3m} = \left(1+|x|^2\right)^{-(d+4)/2}$ is integrable. It has been proved in Proposition 5.2 that $\mathcal{I}$ is also bounded.

By conservation of relative mass, cf. Proposition 2.3 of [7], we know that $\|g(s)\|_1 = \|g(0)\|_1$, where we have used the fact that $v_0 - V_{D_*}$ above is taken also nonnegative with $\int (v_0 - V_{D_*})\,\mathrm{d}y = M > 0$ in this part of the proof. This implies that the ratio $I/M = I_{m_*}[g(s)]/\|g(s)\|_1^2$ is bounded as a function of time.

We shall use now the Gagliardo-Nirenberg inequalities of Proposition 4.10 taking $v = g(t)$, putting $F = \|g\|^2_{L^2(V^{2-m}\,\mathrm{d}x)}$, I the linear Dirichlet form, and $M = \|g\|_{L^1(V^{2-m}\,\mathrm{d}x)}$:

$$
F^3 \le K_1 I M^4. \tag{6.2}
$$

The validity of such inequalities depends on the boundedness of the ratio $I_{m_*}[g(s)]/\|g(s)\|^2_{L^1(V^{2-m}\,\mathrm{d}x)}$, which is ensured along the evolution, as above mentioned.

We now prove an entropy - entropy production inequality. We obtain a differential inequality for the entropy $\mathcal{F}$, by comparing it with the Fisher information $\mathcal{I}$ via Gagliardo-Nirenberg inequalities,

$$
\begin{aligned}
\mathcal{F}^3[w] &\le^{(a)} \left[\frac{1}{2} W_0^{m-2} \int_{\mathbb{R}^d} |w-1|^2 V_{D_*}^m\,\mathrm{d}y\right]^3 = \left[\frac{1}{2} W_0^{m-2}\right]^3 F^3 \\
&\le^{(b)} \left[\frac{1}{2} W_0^{m-2}\right]^3 K_1 I M^4 = K_2 I M^4
\end{aligned} \tag{6.3}
$$

where (a) follows from (5.25) of Lemma 5.3, while in (b) we used the Gagliardo-Nirenberg inequality (6.2) above.

(i) In order to continue the argument, we assume for the moment that the initial datum satisfies $v_0 \ge V_{D_*}$ and is radially symmetric so that $g_0 = (w_0 - 1)V_{D_*}^{m-1}$ is *nonnegative*. This extra assumption will be removed afterwards. Under it we will prove in Appendix A4 that there is an infinite sequence of intervals of times $[s_{1,k}\,, s_{2,k}] \subset [2k, 2k+2]$ (hence, $s_{2,k} \le s_{1,k+1}$) such that

$$I[g(s)] \le k_4\, \mathcal{I}[g(s)] \qquad \text{for all } s \in \bigcup_{k\in\mathbb{N}} [s_{1,k}\,, s_{2,k}] \tag{6.4}$$

for a constant k_4 that does not change along the evolution. We shall prove moreover that the length of each of such intervals is at least $1/2$ for all $k \ge k_0$, which in particular implies that

$$\sum_{k=k_0}^{n} (s_{2,k} - s_{1,k}) \ge \sum_{k=k_0}^{n} \frac{1}{2} = \frac{n-k_0}{2} \ge c\, s_{2,n} \tag{6.5}$$

whenever $n \ge n_0$ is large, for a suitable $c > 0$. Then, recalling that $\mathcal{I} = -\,\mathrm{d}\mathcal{F}/\,\mathrm{d}s$, and using (6.3) we conclude that

$$\mathcal{F}^3 \le K_2\, k_4\, M^4\, I \le k_5 \mathcal{I} = -k_5 \frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t}\,,$$

and an integration oven the interval $[s_{1,k}, s_{2,k}]$ gives

$$\sum_{k=1}^{n} \left(\frac{1}{\mathcal{F}(s_{2,k})^2} - \frac{1}{\mathcal{F}(s_{1,k})^2} \right) \ge \frac{1}{k_5} \sum_{k=1}^{n} (s_{2,k} - s_{1,k}) \ge \frac{c}{k_5} s_{2,n}\,.$$

This implies

$$\frac{1}{\mathcal{F}(s_{2,n})^2} - \frac{1}{\mathcal{F}(s_{1,1})^2} \ge \frac{c}{k_5} s_{2,n},$$

since the intermediate terms are such that

$$-\frac{1}{\mathcal{F}(s_{1,k})^2} + \frac{1}{\mathcal{F}(s_{2,k-1})^2} \le 0$$

because $\mathcal{F}(s)$ is non-increasing and $s_{2,k-1} \le s_{1,k}$. The monotonicity of the function $\mathcal{F}(s)$ allows then to conclude that for all $s \in [s_{2,k}, s_{2,k+1}]$

$$\frac{1}{\mathcal{F}(s)^2} \ge \frac{1}{\mathcal{F}(s_{2,k})^2} \ge \frac{1}{\mathcal{F}(s_{1,1})^2} + \frac{c}{k_5}\, s_{2,k}, \tag{6.6}$$

Using the fact that $s \le s_{2,k+1} \le s_{2,k} + 4$ we get

$$\mathcal{F}(s) \le \frac{1}{\left[\mathcal{F}(s_{1,1})^{-2} + c\, k_5^{-1}\, s\right]^{\frac{1}{2}}} \le \frac{1}{(c_0 + c_1 s)^{\frac{1}{2}}} \tag{6.7}$$

for large times s and some positive constants c_0, c_1. We have thus proved that the nonlinear entropy decays with the same rate as the linear one, when the initial relative mass is nonzero.

(ii) Proof without extra restrictions. The arguments used above and in Appendix A4 are valid changing h into $-h$ and g into $-g$ under the same a priori bounds. Hence, the conclusion is valid for negative and radial initial difference $v_0 - V_{D_*} \le 0$.

To deal with the general case where $v_0 - V_{D_*}$ is not radial or does not have a sign, we use the maximum principle, after writing $|v_0(x) - V_{D_*}(x)| \le f(|x|)$. By comparison we have $v_1 \le v \le v_2$, where v_1 and v_2 are the solutions corresponding to initial data $V_{D_*} - f$ and $V_{D_*} + f$ resp. For the corresponding $w = v/V_{D_*}$, $h = w - 1$ and $g = h(D_{D_*} + y^2)$ a similar comparison holds. Thus, $w_1 \le w \le w_2$, where $w_1 \le 1$ and $w_2 \ge 1$ are the solutions with radial initial data $1 \pm (f/V_{D_*})$, hence functions of $r = |y|$ and s. Same idea applies to g. Take now into account the form of the entropy

$$\mathcal{F}[w] := \frac{1}{1-m}\int_{\mathbb{R}^d} \Psi(w) V_{D_*}^m \,\mathrm{d}y, \qquad \text{with} \quad \Psi(w) = (w-1) - \frac{1}{m}(w^m - 1). \tag{6.8}$$

We note that $\Psi(w)$ is convex and has a zero minimum at $w = 1$. Since we have just proved that the decay result holds for both g_1 and g_2, the statement also holds for g, even if we do not assume that $v_0 - V_{D_*}$ is nonnegative or radial. □

Proof of Corollary 3.2. We recall the following facts proved in [7], Lemma 6.2 under the running assumptions, (H1) and (H2). First we have that for any $\vartheta \in [0, \frac{2-m}{1-m}]$, there exists positive constants K_ϑ, K_2 such that

$$\left\| |x|^\vartheta (v - V_{D_*}) \right\|_2 \le K_\vartheta \left(\mathcal{F}[w]\right)^{1/2} \ .$$

Moreover

$$\|v - V_{D_*}\|_2 \le K_2 \left(\mathcal{F}[w]\right)^{1/2} \ .$$

We now recall the result of Lemma 3.6 of [7]

$$\|v(s) - V_{D_*}\|_{C^\alpha(\mathbb{R}^d)} \le \mathcal{H} \|v(s) - V_{D_*}\|_\infty \quad \forall\, t \ge t_0 \ . \tag{6.9}$$

for a suitable $\alpha \in (0,1)$, and we combine it with the interpolation inequality (7.6), with $\lambda = -\alpha\, d < 0 = \mu < 1/2 = \nu$, $C = \mathcal{C}_{-\alpha d,\,0,\,1/2}$

$$\|v(s) - V_{D_*}\|_\infty \le C \|v(s) - V_{D_*}\|_{C^\alpha}^{\vartheta} \|v(s) - V_{D_*}\|_2^{1-\vartheta} \le C\, \mathcal{H}^\vartheta \|v(s) - V_{D_*}\|_\infty^{\vartheta} \|v(s) - V_{D_*}\|_2^{1-\vartheta}$$

where $\vartheta = 1/(2 + \alpha\, d)$. This implies

$$\|v(s) - V_{D_*}\|_\infty \le C^{1/(1-\vartheta)}\, \mathcal{H}^{\vartheta/(1-\vartheta)} \|v(s) - V_{D_*}\|_2 \le \mathcal{K}_\vartheta \left(\mathcal{F}[w]\right)^{1/2} \quad \forall\, t \ge t_0 \ .$$

From Hölder's inequality,

$$\|v(s) - V_{D_*}\|_q \le \|v(s) - V_{D_*}\|_\infty^{(q-2)/q} \|v(s) - V_{D_*}\|_2^{2/q} \le \mathcal{K}_q \left(\mathcal{F}[w]\right)^{1/2}$$

for all $q \in [2, \infty]$, we deduce that $\|v(s) - V_{D_*}\|_q$ decays with the same rate as $\left(\mathcal{F}[w]\right)^{1/2}$.

If $q \in (1,2)$, we know from Lemma 6.2. of [7] that there exists a positive constant $K(q)$ such that

$$\|v - V_{D_*}\|_q \le K(q) \left(\mathcal{F}[w]\right)^{1/2} ,$$

This and the known decay of $\mathcal{F}$ proves (ii).

To prove (iii), use first (7.7) with the choice $p = \infty$, i.e.

$$\|f\|_{C^j(\mathbb{R}^d)} \le \mathcal{C}_{j,d} \|f\|_{C^{j+1}(\mathbb{R}^d)}^{\frac{j}{(j+1)}} \|f\|_\infty^{\frac{1}{j+1}} \tag{6.10}$$

for any $j \in \mathbb{N}$, and the decay of the L^∞ norm, namely $\|v - V_{D_*}\|_\infty \le K s^{-1/4}$ to get

$$\|v(s) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \le H_j s^{-\frac{1}{4(j+1)}} \quad \forall\, s \ge s_0 \ ,$$

where in fact H_j depends on s itself and tends to zero as $s \to +\infty$, so that the bound can be improved. Indeed we iterate the procedure putting such bound for the C^{j+1} norm into (6.10) to get a new bound for the C^j norm. In fact what we get after h steps is, for any fixed $s \geq s_0$, $j \in \mathbb{N}$:

$$\|v(s) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \leq \frac{\mathcal{C}_{j,d}^{\sum_0^{h-1}\left(\frac{j}{j+1}\right)^h} k^{\frac{1}{j}\sum_1^{h+1}\left(\frac{j}{j+1}\right)^h} H_j^{\left(\frac{j}{j+1}\right)^h}}{s^{k_h}}$$

where the value of k_h will be determined later. Notice in first place that the numerator of the above expression remain finite as $h \to \infty$, for any fixed $s \geq s_0$, $j \in \mathbb{N}$. As for k_h, by construction it satisfies the recursion relation

$$k_0 = \frac{1}{4(j+1)}, \quad k_{h+1} = \frac{j}{j+1} k_h + \frac{1}{4(j+1)}.$$

subtracting $1/4$ to both sides of the latter equation gives

$$k_{h+1} - \frac{1}{4} = \frac{j}{j+1}\left(k_h - \frac{1}{4}\right)$$

which immediately gives $k_h = \frac{1}{4} - \left(\frac{j}{j+1}\right)^h \frac{j}{4(j+1)}$, thus proving that $k_h \to 1/4$ as $h \to +\infty$. □

Proof of Corollary 3.4. We have proved that $\mathcal{F}[w(s)] \leq c_0 s^{-1/2}$ and by Lemma 5.3 we also know that $F[w] \leq c_1 \mathcal{F}[w]$. By Lemma 5.4 with $m = m_*$, we have

$$\|w(s) - 1\|_{d/2}^{d/2} = \|w(s) - 1\|^{2+\frac{m_*}{1-m_*}}_{\mathrm{L}^{2+\frac{m_*}{1-m_*}}(\mathbb{R}^d)} \leq \overline{D}_{m_*} F[w(s)] \leq c_2 \mathcal{F}[w(s)] \leq c_3 s^{-1/2} \tag{6.11}$$

Moreover, (5.33) yields

$$\|w(s) - 1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq c_4 \mathcal{F}[w(s)]^{\frac{1}{d+2+\frac{m_*}{1-m_*}}} = c_4 \mathcal{F}[w(s)]^{2/(3d)} \leq c_5 s^{-1/(3d)} \tag{6.12}$$

Interpolating between these bounds shows that, for $q \in [d/2, +\infty]$:

$$\|w(s) - 1\|_q \leq \|w - 1\|_\infty^{[q-(d/2)]/q} \|w(s) - 1\|_{d/2}^{d/(2q)} \leq c_6 s^{-\frac{1}{3}\left(\frac{1}{d} + \frac{1}{q}\right)}.$$

To improve such bound we insert it in the interpolation inequality (7.7) and use Theorem 7.2 as well to get

$$\|w(s) - 1\|_{C^j(\mathbb{R}^d)} \leq \frac{C}{s^{\frac{q}{3}\left(\frac{1}{d} + \frac{1}{q}\right)\frac{k-j}{d+qk}}}$$

for any $q \in [d/2, +\infty]$, $k > j \in \mathbb{N}$. As a function of q the exponent of s is nonincreasing, so we choose $q = d/2$ to get

$$\|w(s) - 1\|_{C^j(\mathbb{R}^d)} \leq \frac{C}{s^{\frac{k-j}{d(2+k)}}}.$$

To optimize in k we should take $k = \infty$, which is not allowed, so that for any fixed $\varepsilon > 0$ we take k large enough so that

$$\|w(s) - 1\|_{C^j(\mathbb{R}^d)} \leq \frac{C}{s^{\frac{1-\varepsilon}{d}}}$$

as claimed. Putting this bound back into (7.7) with $j = 0$ and using what is known so far for the decay of the L^p norm we get a decay of the form $\|w - 1\|_\infty \leq C s^{-\alpha}$, α being given (with an inessential renaming of the free parameter ε) by

$$\alpha(p, k) = \left(\frac{1-\varepsilon}{d}\right)\frac{d}{d+pk} + \frac{1}{3}\left(\frac{1}{d} + \frac{1}{p}\right)\frac{pk}{d+pk}.$$

Maximizing α w.r.t. to p when ε is small enough yields again $p = d/2$, so that after some calculation we get the exponent $\alpha(d/2, k) = \frac{1}{d} - \frac{2\varepsilon}{d(2+k)}$. This proves the claim for the L^∞ norm and hence also for all L^q norms with $q \in (d/2, +\infty)$ by interpolation. $\square$

7 Proofs for fast diffusion with $m \neq m_*$ revisited

The previous method allows for shorter proofs of the convergence when $m \neq m_*$, and at the same time some minor improvements of paper [7]. We recall that, in the case $m \neq m_*$, the spectral gap inequality

$$F[g(s)] \leq \lambda_{m,d}^{-1}\, I[g(s)] \tag{7.1}$$

holds true, and the best constant is known for $m < m_*$, since [6, 7]. We also recall the result of Proposition 5.5

$$I[g] = \int_{\mathbb{R}^d} |\nabla g|^2\, V_{D_*}\, \mathrm{d}y \leq k_1 \mathcal{I}[w] + k_3 \mathcal{F}^{1+\sigma}[w]$$

where, in particular, $k_1 = 2W_1^{3-2m}$. From these bounds we get

$$\begin{aligned} \mathcal{F}(w) &\leq^{(a)} \frac{1}{2} W_0^{m-2} \int_{\mathbb{R}^d} |w-1|^2 V_{D_*}^m \,\mathrm{d}y = \frac{1}{2} W_0^{m-2} F \\ &\leq^{(b)} \frac{1}{2} W_0^{m-2} \lambda_{m,d}^{-1} I[g] \\ &\leq^{(c)} \frac{1}{2} W_0^{m-2} \lambda_{m,d}^{-1} \left[k_1 \mathcal{I}[w] + k_3 \mathcal{F}^{1+\sigma}[w] \right] \end{aligned} \tag{7.2}$$

where in (a) we compared the linear and nonlinear entropies via inequality (5.25), in (b) the above spectral gap inequality, and in (c) the above mentioned Proposition 5.5. We may rewrite the latter formula as a differential inequality:

$$\mathcal{F}' + W_0^{2-m} W_1^{2m-3} \lambda_{m,d} \mathcal{F} - \frac{W_1^{2m-3}}{2} k_3 \mathcal{F}^{1+\sigma} \leq 0$$

so that by comparison with its explicit solution, we get

$$\mathcal{F}[w(s)] \leq \frac{\mathrm{e}^{-k_4 (s-s_0)}}{\left[\mathcal{F}[w(s_0)]^{-\sigma} + \frac{k_3 W_1^{2m-3}}{2k_4} \left(\mathrm{e}^{-\sigma k_4 s} - \mathrm{e}^{-\sigma k_4 s_0} \right) \right]^{\frac{1}{\sigma}}} \leq k_5 \mathrm{e}^{-k_4 s}$$

provided $\mathcal{F}[w(s_0)]$ is small enough, a property which holds for s_0 large enough. This gives an exponential decay of the entropy, with a rate $k_4 = W_0^{2-m} W_1^{2m-3} \lambda_{m,d}$. Note that $\lambda_{m,d}$ is the optimal constant in the Hardy- Poincaré inequality (7.1), known for $m < m_*$ since [7]. Then we can proceed as in [7] to show that the optimal rate is given by $\lambda_{m,d}$. Indeed, one can substitute W_0 and W_1 with $\inf_{\mathbb{R}^d} |w|$ and $\sup_{\mathbb{R}^d} |w|$ respectively, allowing them to depend on time. Then prove that they both tend to 1 when $s \to +\infty$, so that $k_4 \to \lambda_{m,d}$; this can be done in view of the uniform convergence of the relative error $\|w(t) - 1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \to 0$ as $s \to +\infty$ together with a Gronwall-type argument. For more details we refer to Section 6.3 of [7].

Remarks

(i) When $m = m_*$ the above steps do not hold since we do not have a spectral gap for the linearized generator. This is one of the reasons which forced us to use Gagliardo-Nirenberg inequalities which, instead, compare the Fisher information with a power of the entropy.

(ii) This method simplifies and complements some proofs of [7] when $m \neq m_*$, but also gives a more detailed proof of the case $m \leq 0$ that was only briefly treated in [7]. We finally emphasize the analysis of the present paper covers the case $m = 0$, that is logarithmic diffusion, even in dimension $d = 4$ since in that case $m_* = 0$ so that no spectral gap holds.

(iii) The interpolations made in the proof of Theorem 3.2 are valid also in the case $m \neq m_*$ and allow to improve the convergence rate of the derivatives, proving that the rate is always given by $\lambda_{m,d}$. We state here this improved version of the main asymptotic Theorem of [7]

Theorem 7.1 (Convergence with rate, $m \neq m_*$) *Under the assumptions of Theorem 2.1, if $m \neq m_*$, there exists $t_0 \geq 0$ such that the following properties hold:*

(i) *For any $q \in [q_*, \infty]$, there exists a positive constant C_q such that*

$$\|v(s) - V_{D_*}\|_q \leq C_q \, \mathrm{e}^{-\lambda_{m,d}\, s} \quad \forall\, s \geq s_0 \,.$$

(ii) *For any $\vartheta \in [0, (2-m)/(1-m)]$, there exists a positive constant K_ϑ such that*

$$\big\|\, |x|^\vartheta (v(s) - V_{D_*}) \big\|_2 \leq K_\vartheta \, \mathrm{e}^{-\lambda_{m,d}\, s} \quad \forall\, s \geq t_0 \,.$$

(iii) *For any $j \in \mathbb{N}$, there exists a positive constant H_j such that*

$$\|v(s) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \leq H_j \, \mathrm{e}^{-\lambda_{m,d}\, s} \quad \forall\, s \geq s_0 \,.$$

The constants C_q, K_ϑ and H_j depend on s_0, m, d, v_0, D_0, D_1, and q, ϑ and j; s_0 also depends on D_0 and D_1. It is remarkable that the decay rate of the nonlinear problem is given exactly by $\lambda_{m,d}$. Rescaling back to the original equation, we obtain results in terms of intermediate asymptotics, cf. Corollary 3.3 or Corollary 1.3 of [7].

Appendices

A1. Calculation of curvatures. Proof of Lemma 4.2

We can use well-known formulas for the Ricci tensor as a function of the metric data:

$$R_{ij} = g^{km} R_{ikjm}, \quad R_{ikjm} = \frac{1}{2}\left(\partial^2_{kj} g_{im} + \partial^2_{im} g_{kj} - \partial^2_{km} g_{ij} - \partial^2_{ij} g_{km}\right) + g_{np}(\Gamma^n_{kj}\Gamma^p_{im} + \Gamma^n_{km}\Gamma^p_{ij}).$$

but in the case of conformal transformation there is a worked out relation between the Ricci tensors of two metrics $\mathbf{g}$ and $\widetilde{\mathbf{g}}$ in terms of the conformal factor relating them. Precisely, if $\widetilde{\mathbf{g}} = (1/\varphi^2)\mathbf{g}$, where φ is a scalar, the formula reads as follows [5]:

$$\widetilde{R} - R = \frac{1}{\varphi^2}\left[(d-2)\varphi\nabla^2\varphi + (\varphi\Delta_{\mathbf{g}}\varphi - (d-1)\mathbf{g}(\nabla\varphi, \nabla\varphi)\cdot \mathbf{g}\right].$$

where $R = (R_{ij})$ is the Ricci tensor of $\mathbf{g}$, $\widetilde{R} = \widetilde{R}_{ij}$ is the Ricci tensor of $\widetilde{\mathbf{g}}$, ∇ denotes the gradient, ∇^2 the Hessian and $\Delta_{\mathbf{g}}$ the Laplace-Beltrami operator with respect to $\mathbf{g}$. Specializing the formula to the case $\mathbf{g} = \delta_{ij}$, so that $R_{ij} = 0$, we get in coordinates

$$\widetilde{R}_{ij} = \frac{1}{\varphi^2}\left[(d-2)\varphi\partial^2_{ij}\varphi + (\varphi\Delta\varphi - (d-1)|\nabla\varphi|^2)\delta_{ij}\right].$$

Put now $\widetilde{g}_{ij} = (1+|x|^2)^{-1}\delta_{ij}$ so that $\varphi = (1+|x|^2)^{1/2}$. Then $\partial_i\varphi = x_i(1+|x|^2)^{-1/2}$,

$$\partial^2_{ij}\varphi = -x_ix_j(1+|x|^2)^{-3/2} + (1+|x|^2)^{-1/2}\delta_{ij}, \quad \Delta\varphi = -|x|^2(1+|x|^2)^{-3/2} + d(1+|x|^2)^{-1/2}.$$

Applying the last formula we get after some calculations

$$\widetilde{R}_{ij} = -\frac{(d-2)x_ix_j}{(1+|x|^2)^2} + \left[\frac{(d-2)|x|^2 + 2(d-1)}{(1+|x|^2)^2}\right]\delta_{ij}.$$

There is a clear form of these expressions when we take the particular point $\widehat{x} = (X, 0, \cdot, 0)$ which implies no loss of geometrical generality since the metric is conformal and radial, hence invariant under rotations in the space. We get

$$\widetilde{R}_{11}(\widehat{x}) = \frac{2(d-1)}{(1+X^2)^2}; \qquad \widetilde{R}_{ii}(\widehat{x}) = \frac{(d-2)X^2 + 2(d-1)}{(1+X^2)^2} \quad \forall i = 2, \cdots, d, \tag{7.3}$$

and $\widetilde{R}_{ij}(\widehat{x}) = 0$ for all $i \neq j$. Both eigenvalues tend to zero as $|x| \to +\infty$ with different rates. It immediately follows that the symmetric tensor Ric is positive; indeed, given $\xi \in \mathbb{R}^d$, we have

$$\widetilde{R}_{ij}(\widehat{x})\xi_i\xi_j \geq \frac{2(d-1)}{(1+X^2)^2}|\xi|^2 > 0,$$

and the same is true for all $x \in \mathbb{R}^d$ by invariance under rotations. If one wants to visualize the behaviour of this manifold, it is convenient to look at Ricci curvatures given by

$$\widetilde{r}_1 = \frac{\widetilde{R}(e_1, e_1)}{\widetilde{g}(e_1, e_1)} = \frac{2(d-1)}{(1+X^2)}, \qquad \widetilde{r}_i = \frac{\widetilde{R}(e_i, e_i)}{\widetilde{g}(e_i, e_i)} = \frac{2(d-1) + (d-2)X^2}{(1+X^2)}, \tag{7.4}$$

Note that the transversal curvatures tend to $(d-2)$ as $|x| \to \infty$ while the curvature in the radial direction behaves like $O(|x|^{-2})$. This clearly shows the difference in the behaviour of the curvatures in radial and transversal directions which is typical of a cigar manifold.

Finally, the value of the scalar curvature follows from the formula $R = g^{ij}R_{ij}$. Since we are in a conformal situation, it can be deduced in a direct way from the Yamabe formula [39]

$$\widetilde{R} = -\frac{4(d-1)}{d-2}\frac{\Delta w}{w^{(d+2)/(d-2)}}, \quad \text{with } w = g^{(d-2)/4},$$

where $\mathbf{g}$ is the conformal factor, here $(1+|x|^2)^{-1}$, cf. the formulas e. g. in [37], pages 211-212. In order to obtain the results stated in Lemma 4.2 we only need to eliminate the tildes from $\widetilde{R}_{ij}$ and $\widetilde{R}$. $\square$

A2. Explicit representation of the cigar

We give here a simple parametric representation for the cigar-like manifold $(M, \mathbf{g})$. We want to represent it as a hypersurface in $\mathbb{R}^{d+1}$. The radial symmetry of the metric suggests to represent such imbedded manifold as $z = f(|y|)$ with variables $(y, z) \in \mathbb{R}^{d+1}$, where $y \in \mathbb{R}^d$ and $z \in \mathbb{R}$, having a unique chart $x \in \mathbb{R}^d \mapsto (y, z)$ given by the formulas

$$r = |y| = \Phi(\varrho) \qquad \text{and} \qquad z = \Psi(\varrho),$$

where $\varrho = |x| \geq 0$. We fix $\Phi(0) = \Psi(0) = 0$. We want the Euclidean metric in $\mathbb{R}^{d+1}$ to induce the metric on the hypersurface. We know that the infinitesimal length element in the radial direction satisfies

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \mathrm{d}z^2 = \left[\Psi'^2(\varrho) + \Phi'^2(\varrho)\right] \mathrm{d}\varrho^2 = \frac{\mathrm{d}\varrho^2}{1+\varrho^2}$$

which implies the relation $\Psi'^2(\varrho) + \Phi'^2(\varrho) = 1/(1+\varrho^2)$. On the other hand, the length calculation for the transversal part gives

$$\left[\frac{\Phi(\varrho)}{\varrho}\right]^2 = \frac{1}{1+\varrho^2}.$$

Solving the above two equations gives

$$\Phi(\varrho) = \frac{\varrho}{(1+\varrho^2)^{1/2}}, \qquad \Psi'(\varrho) = \frac{\varrho(2+\varrho^2)^{1/2}}{(1+\varrho^2)^{3/2}}.$$

We see that $r = \Phi(\varrho)$ goes from 0 to 1 as $0 < \varrho < \infty$. Analyzing the behaviour of $\Psi(\varrho)$, one concludes that

$$\Psi(\varrho) \approx \varrho^2 \quad \text{when } \varrho \approx 0, \qquad \Psi(\varrho) \approx \log \varrho \quad \text{when } \varrho \gg 1.$$

This is the representation of a cigar. The point out that the transversal radius at infinity is constant; actually, $\Phi(\varrho) \to 1$ as $\varrho \to +\infty$.

A3. Some Technicalities

We recall here some technical facts that we used in the proofs. First we recall Theorem 2.4 of [7],

Theorem 7.2 (Uniform C^k regularity) *Let $m < 1$ and $w \in \mathrm{L}^\infty_{\mathrm{loc}}((0,T)\times\mathbb{R}^d)$ be a solution of* (2.9)*. Then for any $k \in \mathbb{N}$, for any $s_0 \in (0,T)$,*

$$\sup_{s \geq s_0} \|w(s)\|_{C^k(\mathbb{R}^d)} < +\infty\,. \tag{7.5}$$

We also needed an interpolation Lemma due to Gagliardo [24], cf. also Nirenberg, [30, p. 126].

Lemma 7.3 *Let λ, μ and ν be such that $-\infty < \lambda \leq \mu \leq \nu < \infty$. Then there exists a positive constant $\mathcal{C}_{\lambda,\mu,\nu}$ independent of f such that*

$$\|f\|_{1/\mu}^{\nu-\lambda} \leq \mathcal{C}_{\lambda,\mu,\nu} \|f\|_{1/\lambda}^{\nu-\mu}\, \|f\|_{1/\nu}^{\mu-\lambda} \quad \forall\, f \in \mathcal{C}(\mathbb{R}^d)\,, \tag{7.6}$$

where $\|\cdot\|_{1/\sigma}$ stands for the following quantities: (i) If $\sigma > 0$, then $\|f\|_{1/\sigma} = \left(\int_{\mathbb{R}^d} |f|^{1/\sigma}\,\mathrm{d}x\right)^\sigma$. (ii) If $\sigma < 0$, let k be the integer part of $(-\sigma d)$ and $\alpha = |\sigma| d - k$ be the fractional (positive) part of σ.

Using the standard multi-index notation, where $|\eta| = \eta_1 + \ldots + \eta_d$ *is the length of the multi-index* $\eta = (\eta_1, \ldots \eta_d) \in \mathbb{Z}^d$*, we define*

$$\|f\|_{1/\sigma} = \begin{cases} \max\limits_{|\eta|=k} \left|\partial^\eta f\right|_\alpha = \max\limits_{|\eta|=k} \sup\limits_{x,y\in\mathbb{R}^d} \dfrac{\left|\partial^\eta f(x) - \partial^\eta f(y)\right|}{|x-y|^\alpha} = |f\|_{C^\alpha(\mathbb{R}^d)} & \textit{if } \alpha > 0\ , \\ \max\limits_{|\eta|=k} \sup\limits_{z\in\mathbb{R}^d} \left|\partial^\eta f(z)\right| := \|f\|_{C^k(\mathbb{R}^d)} & \textit{if } \alpha = 0\ . \end{cases}$$

As a special case, we observe that $\|f\|_{-d/j} = \|f\|_{C^j(\mathbb{R}^d)}$.

(iii) By convention, we note $\|f\|_{1/0} = \sup_{z\in\mathbb{R}^d} |f(z)| = \|f\|_{C^0(\mathbb{R}^d)} = \|f\|_\infty$.

Remark. The following special case of the above interpolation inequality (7.6) has been used in the paper: let $k > j \in \mathbb{N}$ and $\lambda = -k/d \le \mu = -j/d \le \nu = 1/p$. Inequality (7.6) becomes

$$\|f\|_{C^j(\mathbb{R}^d)} \ \le\ \mathcal{C}_{j,k,p}\, \|f\|_{C^k(\mathbb{R}^d)}^{\frac{d+jp}{d+kp}}\, \|f\|_p^{\frac{p(k-j)}{d+kp}} \tag{7.7}$$

for any $k > j \in \mathbb{N}$ and $p > 0$.

A4. Complete proof of the estimates of Section 6

In the proof of Theorem 3.1 in Section 6 we have assumed that for every solution under the stated conditions there is an infinite sequence of intervals of *good times* $[s_{1,k}\,, s_{2,k}] \subset [2k, 2k+2]$ with $s_{2,k} < s_{1,k+1}$ for all k, such that

$$I[g(s)] \le k_4\, \mathcal{I}[g(s)] \qquad \textit{for all}\ \ s \in \bigcup_{k\in\mathbb{N}} [s_{1,k}\,, s_{2,k}]. \tag{7.8}$$

Recall that in view of hypothesis (H2) and the discussion made in Section 6, we may assume that g is radially symmetric and positive. We will also prove that the length, $l_k = s_{2,k} - s_{1,k}$ of the intervals in our construction is at least $1/2$ for all $k \ge k_0$.

The proof of these facts is long, and would have broken the flow of the proof of Theorem 3.1: this is the reason why we put it here. The main point in getting (7.8) consists in obtaining stronger estimates of the remainder term in the inequality of Lemma 5.1 than the ones obtained in Proposition 5.5. We restate here Lemma 5.1 for convenience of the reader:

Let $0 < W_0 \le w \le W_1 < +\infty$*, be a measurable function on* $\mathbb{R}^d$*, with* $W_0 < 1$ *and* $W_1 > 1$*, and assume that* $\mathcal{I}(w) < +\infty$*. Then for any* $m < 1$ *the following inequality holds true*

$$I[w(s)] \le k_1 \mathcal{I}[w(s)] + R[w(s)], \qquad \textit{with}\ \ R[w(s)] = k_2 \int_{\mathbb{R}^d} g(s,y)^4 V_*(y)^{4-3m}\, \mathrm{d}y\,, \tag{7.9}$$

where $g = (w-1)V_*^{m-1}$*;* k_1 *and* k_2 *are positive constants.*

We need to control the remainder term $R[w(s)]$ to proceed with the asymptotic estimate. Note that V_*^{4-3m} is integrable for $m = m^*$: in fact, for such a value of m we have

$$R[w(s)] = k_2 \int_{\mathbb{R}^d} \frac{g(s,y)^4}{(1+y^2)^{(d+4)/2}}\, \mathrm{d}y \tag{7.10}$$

Put now $N(s) = N[g(s)] = \|g(\cdot, s)\|_\infty$, the supremum of g for fixed time $s > 0$. We know that $N(s)$ is uniformly bounded in time. Then we have

$$R[w(s)] \le k_3\,[N(s)]^4 \tag{7.11}$$

We want to estimate the decay of $N(s)$ in time in terms of the linearized Fisher information $I[w]$. Suppose for a moment that we can prove that the remainder term is small relatively to $I[w]$, more precisely that $R[w(s)] \le \frac{1}{2} I[w(s)]$ for all large s. In that case we conclude that $I[w(s)] \le 2k_1 \mathcal{I}[w(s)]$, and the desired estimate (7.8) easily follows. Hence, we need to prove that

$$N(s)^4 \le \frac{I[g(s)]}{K} \tag{7.12}$$

with $K > k_3$, say $K = 2k_3$. This is a most convenient estimate on the values of g.

Unfortunately, even under such assumption it is not clear that the last inequality holds at all times, or even at all large times. Therefore, we shall be cautious and call *good times* those times at which (7.12) holds with $K \ge 2k_3$. The frequency and density of the intervals of such times is important, as the end of proof of Theorem 3.1 shows.

We will now proceed with the proof of the existence of the time intervals stated at the beginning of this section. They will consist only of so-called good times. The proof is split into two parts, namely

(i) *Controlling the remainder term away from the origin.* This is the part where we use the fact the $|v_0 - V_*| \le \tilde f$ for a *radially symmetric* $\tilde f$;

(ii) *Transforming the outer control into a control on a small ball,* namely we will control $\sup_{r \ge R} g(r, s)$ for small $R > 0$. Due to the peculiarities of the parabolic Harnack inequality, we shall prove that such a control only takes place for a large set of so-called good times, this sufficing for our goals.

Part (i). The control of a radial g far away from the origin

In the calculation that follows we drop the s-dependence for convenience since time does not enter in the argument. Let $g(r)$ be a *nonnegative* continuous function such that $g(r) \to 0$ as $r \to \infty$ and let

$$\mathcal{M}_R = \int_R^\infty \frac{g(r)}{(1+r^2)^{d/2}} r^{d-1}\,\mathrm{d}r < \infty\,, \qquad \text{and} \qquad I_R = \int_R^\infty \frac{|g'(r)|^2}{(1+r^2)^{(d-2)/2}} r^{d-1}\,\mathrm{d}r < \infty\,.$$

We put the powers in a way such that it is clear that for $r > 1$ we are dealing with the radial case and we are merely asking the mass and the linearized Fisher information to be finite. Now pick $\alpha > 0$, $R_1 > R > 1$ and calculate

$$\begin{aligned} g^{1+\alpha}(R) - g^{1+\alpha}(R_1) &= -\alpha \int_R^{R_1} g'(r) g^\alpha(r)\,\mathrm{d}r \le \alpha \int_R^{R_1} |g'(r)| g(r)^\alpha\,\mathrm{d}r \le \alpha \int_R^{R_1} |g'(r)| r^{\frac12} \frac{g(r)^\alpha}{r^{\frac12}}\,\mathrm{d}r \\ &\le \alpha \left[\int_R^{R_1} |g'(r)|^2 r\,\mathrm{d}r\right]^{\frac12} \left[\int_R^{R_1} \frac{g(r)^{2\alpha}}{r}\,\mathrm{d}r\right]^{\frac12} \end{aligned}$$

Now, if we assume $\alpha \ge 1/2$, the last integral can be bounded as follows:

$$\int_R^{R_1} \frac{|g(r)|^{2\alpha}}{r}\,\mathrm{d}r \le \sup_{r \ge R} |g(r)|^{2\alpha - 1} \int_R^\infty \frac{|g(r)|}{r}\,\mathrm{d}r,$$

so that for $\alpha \ge 1/2$ we have obtained

$$g^{1+\alpha}(R) - g^{1+\alpha}(R_1) \le \alpha \left[\sup_{r \ge 1} |g(r)|^{2\alpha-1} \int_1^\infty \frac{|g(r)|}{r}\,\mathrm{d}r\right]^{\frac12} \left[\int_1^\infty |g'(r)|^2 r\,\mathrm{d}r\right]^{\frac12}\,.$$

Letting $R_1 \to \infty$, and assuming that $g(R) \to 0$ as $R \to \infty$, we get

$$g^{1+\alpha}(R) \leq \alpha \left[\sup_{r\geq 1} |g(r)|^{2\alpha-1} \int_1^\infty \frac{|g(r)|}{r} \, \mathrm{d}r\right]^{\frac{1}{2}} \left[\int_1^\infty |g'(r)|^2 r \, \mathrm{d}r\right]^{\frac{1}{2}}.$$

Taking the supremum over $R \geq 1$ on the l.h.s. and simplifying we get:

$$\left[\sup_{R\geq 1} g(R)\right]^4 \leq \alpha^{\frac{8}{3}} \left[\int_1^\infty \frac{|g(r)|}{r} \, \mathrm{d}r\right]^{\frac{4}{3}} \left[\int_1^\infty |g'(r)|^2 r \, \mathrm{d}r\right]^{\frac{4}{3}} \leq c\,\mathcal{M}_1^{\frac{4}{3}}\, I_1^{\frac{4}{3}} \leq \tilde{c} I^{\frac{4}{3}}.$$

This is a very good estimate because it says that the supremum of g^4 outside the unit ball is not only proportional to I as expected in so-called better times, but even more: it is proportional to a higher power of I. Now, recall that $I[w(s)] \to 0$ as $s \to \infty$. If the same could be done near $r = 0$ the proof that every large s is a good time would be complete.

The previous calculation can be done in the complement of the ball of radius R as small as we like and then $g(R)$ will depend also on an inverse power of R, because of the presence of the factors $1 + r^2$ in the denominators of the last quantities. We now get in the last line for $0 < R < 1$

$$\left[\sup_{r\geq R} |g(r)|\right]^4 \leq \alpha^{\frac{8}{3}} \left[\int_R^\infty \frac{|g(r)|}{r} \, \mathrm{d}r\right]^{\frac{4}{3}} \left[\int_R^\infty |g'(r)|^2 r \, \mathrm{d}r\right]^{\frac{4}{3}} \leq \frac{C}{R^{8(d-1)/3}} \mathcal{M}_R^{\frac{4}{3}}\, I_R^{\frac{4}{3}}.$$

The estimate blows up at $R = 0$. Therefore, we cannot let $R \to 0$ to get an estimate for $N(s)$ for any $x \in \mathbb{R}^d$.

Justifying that g goes to zero at infinity. To conclude part (i) of the proof, it remains to prove that $g(R) \to 0$ as $R \to \infty$. Choose R_n such that

$$\int_{R_n}^\infty |g'|^2 r \, \mathrm{d}r < \frac{1}{4n^2} \qquad \text{and} \qquad \int_{R_n}^\infty \frac{|g|^2}{r} \, \mathrm{d}r < \frac{1}{4n^2}$$

and define $\widetilde{R}_n = \min\left\{r \geq R_n \,; g^2(r) \leq \frac{1}{2n^2}\right\}$. Indeed the set in the r.h.s. is not empty since g/r is integrable at infinity: this is not compatible with g being everywhere larger than a positive constant for all $r \geq R_n$. Notice that $\widetilde{R}_n \geq R_n$ and $g^2(\widetilde{R}_n) \leq 1/(2n^2)$. Hence, for all $R \geq R_n$:

$$g^2(R) = g^2(\widetilde{R}_n) + 2\int_{\widetilde{R}_n}^R g\, g' \, \mathrm{d}r \leq \frac{1}{2n^2} + 2\left|\int_{\widetilde{R}_n}^R g|g'| \, \mathrm{d}r\right| \leq \frac{1}{2n^2} + 2\left[\int_{R_n}^\infty |g'|^2 \, \mathrm{d}r\right]^{\frac{1}{2}} \left[\int_{R_n}^\infty g^2 \, \mathrm{d}r\right]^{\frac{1}{2}} \leq \frac{1}{n^2}$$

Therefore, $0 \leq g(R) \leq 1/n$ for all $R \geq R_n$. The proof of part (i) is now complete.

Part (ii). Transforming the outer control into a control on a small ball.

In part (i) we have estimated the supremum of a radial g^4 outside a ball of radius $R > 0$ in terms of $I[g(s)]^{4/3}$, so that the problem is to estimate the supremum inside a ball as well, hopefully in terms of $I[g(s)]^{1+\alpha}$, at least in the form $\varepsilon I[g(s)]$. We are unable to prove that for all (sufficiently large) times. To circumvent such a difficulty we have to make use of a rather complicated argument that takes into account the possibility that such estimate does not hold because of possible bad behaviour of g at points near the origin. We begin by carefully labeling the times. We say that a time $s \in [0, \infty)$ belongs to the class of *good times* $\mathcal{G}_K$, if

$$N(s)^4 = \sup_y (g(s,y))^4 < \frac{I[g(s)]}{K}$$

We are not claiming that some half-line $[T,\infty) \subset \mathcal{G}_{2k_3}$, which would finish the proof in the simple way. Finally, we say that a time is *very good*, $s \in \mathcal{V}_C$, if

$$N(s) \le C\, I[g(s)]^{4/3} \tag{7.13}$$

for some $C > 0$, in the spirit of the radial estimate away from the origin. Note that since $I(s) \to 0$ we have the inclusion of very good times with constant C into the good times with any constant $K > 0$ if s is large enough.

Harnack inequality. The study of points near the space origin is based on classical regularity theory for linear or quasilinear parabolic equations in divergence form. We are going to use the version of the celebrated paper by Aronson-Serrin [1] . We consider the equation satisfied by the error function

$$h(s,y) = w(s,y) - 1 = \frac{v(s,y)}{V_{D_*}(y)} - 1. \tag{7.14}$$

It can be written in the standard form

$$\partial_s h = \nabla \cdot \mathbf{A}(y,h,\nabla h) + B(y,h,\nabla h). \tag{7.15}$$

In fact, starting with the equation satisfied by w, we have
(7.16)

$$\partial_s h = \partial_s w = \frac{1}{V_*}\nabla \cdot \left[w V_* \nabla\left(\frac{w^{m-1}-1}{m-1}V_*^{m-1}\right)\right] = \frac{1}{V_*}\nabla \cdot \left[(h+1) V_* \nabla\left(\frac{(h+1)^{m-1}-1}{m-1}V_*^{m-1}\right)\right]$$

so that we can identify

$$\mathbf{A} = (h+1)\nabla\left(\frac{(h+1)^{m-1}-1}{m-1}V_*^{m-1}\right), \quad B = (h+1)\frac{\nabla V_*}{V_*}\cdot \nabla\left[\frac{(h+1)^{m-1}-1}{m-1}V_*^{m-1}\right]$$

and we have to check that the structure conditions are satisfied by $\mathbf{A}$ and B in a compact ball with constants that do not depend on s. In fact, the structure conditions are satisfied in the homogeneous form of [1], which means that, in the notation of that paper, the terms $f, g, h = 0$ in the structure condition for $\mathbf{A}, B$ vanish. Checking this is a straightforward calculation involving also the known bounds on w namely $0 < W_0 \le w = h+1 \le W_1$. We note in passing that since we already know that $w \to 1$ uniformly in $\mathbb{R}^d$ as $s \to \infty$, the lower and upper bounds W_0, W_1 can be taken closer and closer to 1 if we restrict the time to $s \ge s_0$ and s_0 is large enough.

In any case, we conclude that the Harnack inequality has the standard form, as stated below. This also implies a similar Harnack inequality for g if we work on a bounded space domain, say, in $B_1(0)$. We state it next. Take $T > 0$ large and consider the parabolic cylinders

$$Q = [T-2, T] \times B_1(0)\,, \qquad Q_{1/2} = [T-1/2, T] \times B_1(0)\,, \qquad \widetilde{Q} = [T-2, T-1] \times B_1(0)\,.$$

The parabolic Harnack inequality on the disjoint cylinders $Q_{1/2}$ and $\widetilde{Q}$ is then

$$\inf_{Q_{1/2}} g(s,y) \ge c \sup_{\widetilde{Q}} g(s,y) = c\,\widetilde{N} \tag{7.17}$$

for some positive constant $c < 1$ depending only on structural constants. Note that since g is continuous on Q all the above suprema are attained at some points.

Evolution of the maximum of h. The equation for h can be written as

$$
\begin{aligned}
\partial_s h = {} & \nabla\cdot\left[(h+1)\nabla\left(\frac{(h+1)^{m-1}-1}{m-1}V_*^{m-1}\right)\right] + V_*^{m-2}(h+1)^{m-1}\nabla V_*\cdot\nabla h \\
& + V_*^{m-3}|\nabla V_*|^2(h+1)[(h+1)^{m-1}-1] = \\
& = \nabla h\cdot\nabla\left(\frac{(h+1)^{m-1}-1}{m-1}V_*^{m-1}\right) + (h+1)\nabla\cdot[(h+1)^{m-2}V_*^{m-1}\nabla h] \\
& + mV_*^{m-2}(h+1)^{m-1}\nabla V_*\cdot\nabla h \\
& + (h+1)[(h+1)^{m-1}-1]\nabla\cdot(V_*^{m-2}\nabla V_*) + V_*^{m-3}|\nabla V_*|^2(h+1)[(h+1)^{m-1}-1]
\end{aligned}
$$

In particular, using the fact h is small we get

$$\partial_s h = \text{second and first order terms in } h + C(r,h)\,h \tag{7.18}$$

where $C(r,h)\le \overline{k}$ for a suitable $\overline{k}$ independent of r and depending only on the known a priori bounds for h. Then, as a consequence of the Maximum Principle, cf. e.g. [1], the maximum $N(s)$ obeys the growth rate

$$N'(s)\le \overline{k}N(s). \tag{7.19}$$

In fact the function $H(s,y)=N(s_0)e^{\overline{k}(s-s_0)}$ is an explicit supersolution in the whole space for $s\ge s_0$.

The structure of good times. Alternative. Now we are going to prove that

Lemma 7.4 *For every time interval $(T-2,T)$ with T large enough there is at least a subinterval of length 1/2 consisting of very good times.*

Proof. We can use the cylinders $Q_{1/2}$ and $\widetilde{Q}$ as in the previous paragraph on the Harnack inequality. The idea is to consider separately the two possibilities: (a) either the maximum in x of h at every time of the lower cylinder is taken outside the ball $B_2(0)$, or (b) the maximum at one of such times, say s_0, is taken inside.

In case *(b)*, T must be a good time if it is large enough as we show next. We take $s_0\in(T-2,T-1)$ as above and let N_1 be the corresponding maximum in the ball. Of course $N_1\le\widetilde{N}$. We now apply the growth rate of previous paragraph to obtain

$$N(s_2)\le C\,N_1\le C\widetilde{N}, \qquad C=e^{2\overline{k}}. \tag{7.20}$$

for every s_2 in the upper cylinder: $T-(1/2)\le s_2\le T$. On the other hand, for every $y\in B_1(0)$ we have for such s_2 the lower estimate $h(s_2,y)\ge c\widetilde{N}$. We conclude that the maximum and the minimum at all those times are related by a constant. This is also true for the function g up to a small change in the constant, hence

$$g(s_2,y'_M)\le C_1 g(s_2,y) \tag{7.21}$$

where now y'_M is the point of maximum of g in the ball $B_1(0)$. Now, we know that on the boundary $|y|=1$ there is a good estimate for g, more precisely, for such y of unit norm $g(s_2,y)$ satisfies the estimates that defines the *very good time*, and it does with a fixed constant C. We conclude that

$g(s_2, y)$, $|y| \geq 1$, also satisfies such an estimate with a possible worse constant $C' = C_1 C$. Since the estimate was true for $|y| \geq 1$ we are done.

Therefore, whenever T is not a good time in $\mathcal{G}_K$ with $K = 2k_3$, the first part of the alternative, *(a)*, must be true. But in that case for every time in the lower cylinder we know that the maximum of h is taken in the exterior of the ball B_1. If we look for the expression of $g = h\,(1 + |y|^2)$ this also means that the maximum of $g(s, \cdot)$ at times in $(T-2, T-1)$ is taken at an exterior point (maybe different). So all these times are very good. Recalling what was said before, they are good times in $\mathcal{G}_K$ if T is large enough. □

Choice of intervals of good times. We can apply the previous results letting $T = 2k + 2$, for $k \geq k_0$ and k_0 sufficiently large. The above lemma implies that there exists a subinterval $[s_{1,k}\,, s_{2,k}] \subset [2k, 2k+2]$ of length at least $1/2$ made of times in $\mathcal{G}_K$ with $K = 2k_3$.

8 Concluding remarks and open problems

8.1. The special situation has been studied for the critical exponent $m_* = (d-4)/(d-2)$ in dimensions $d \geq 3$ where our considerations make sense. Algebraic extensions have been shown to be fruitful or intriguing in some dynamical studies. Here, for $d = 2$ we formally have $m_* = \pm\infty$, which is an extreme situation for porous medium that has appeared in the literature (for instance, in connection with the mesa problem), [22, 9, 37], while for $d = 1$ we formally get $m_* = 3$, a value inside the porous medium range where nothing special has been shown to happen.

8.2. We pose the following questions:

• Are the rates obtained in this paper optimal for a certain class of data, as the linearized analysis suggests?

• Can we prove convergence, maybe with worse rates or without rates, for more general initial data? we recall that for $m > m_c$ all nonnegative initial data in $L^1(\mathbb{R}^d)$ are attracted towards a Barenblatt solution, with no rate in that generality.

• Find an explicit optimal dependence of the constant in the asymptotic formula with respect to the data.

• Assuming that we get an optimal rate of convergence, can we find a profile for the next level of approximation?

8.3. One may wonder if the techniques used in [6, 7] for the case $m \neq m_*$, which use Hardy-Poincaré inequalities, work also in the case m_*. We have partially given a negative answer to this question in Corollary 4.9, in which we have shown that no inequality of Hardy type can hold for the linearized Fisher information $I[w]$. However, one may wonder if modified versions of the Hardy-Poincaré inequalities, with logarithmic terms added in the spirit of the classical Hardy inequality in $\mathbb{R}^2$, allow to solve the problem. Thus, there is a family of valid Hardy inequalities (see below) in which the Dirichlet form involved has a logarithmic correction, but we are not able to prove the asymptotic results by means of such inequalities. It is then a further open problem to see whether this path may lead to the goal or not.

Proposition 8.1 *Let $d \geq 3$. We have*

$$\int_{\mathbb{R}^d} g^2 \, \mathrm{d}\mu_\alpha \leq \mathcal{H}_{\alpha,d} \int_{\mathbb{R}^d} |\nabla g|^2 \, \mathrm{d}\nu_\alpha \tag{8.1}$$

for any $g \in \mathcal{D}(\mathbb{R}^d)$ and for any $0 < \alpha \le \frac{d}{2} - 1$, where

$$
\begin{aligned}
d\mu_\alpha(y) &= \left(1+|y|^2\right)^{-\frac{d}{2}} \left[1+\log(1+|y|^2)\right]^{\alpha-1}\, dy, \\
d\nu_\alpha(y) &= \left(1+|y|^2\right)^{1-\frac{d}{2}} \left[1+\log(1+|y|^2)\right]^{\alpha+1}\, dy
\end{aligned} \tag{8.2}
$$

and

$$
\mathcal{H}_{\alpha,d} = \frac{2(d-2)}{\alpha\,(d-2-2\alpha)\,\min\left\{2\alpha\,,(d-2-2\alpha)\right\}}. \tag{8.3}
$$

Acknowledgements. G.G. acknowledges an ESF grant of the research group GLOBAL (Global and Geometric Aspects of Partial Differential equations), which allowed him to visit M.B. and J.L.V. at the Universidad Autónoma de Madrid, and the Departamento de Matemáticas of such University, where part of this work has been done. M.B. and J.L.V. were partially supported by Spanish Project MTM2005-08760-C02-01. M.B. and J.L.V. wish to thank the Dipartimento di Matematica of Politecnico di Torino where part of the work has been done. We thank J. Dolbeault for a careful reading of this article which resulted in a serious improvement.

References

[1] D. G. Aronson, J. Serrin, *Local Behavior of Solutions of Quasilinear Parabolic Equations,* Arch. Rat. Mech. Anal. **25** (1967), 81–122.

[2] D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, *Sobolev inequalities in disguise,* Indiana Univ. Math. J. **44** (1995), 1033–1074.

[3] D. Bakry and M. Émery, *Diffusions hypercontractives*, in Séminaire de probabilités, XIX, 1983/84, vol. 1123 of Lecture Notes in Math., Springer, Berlin, 1985, 177–206.

[4] P. Bénilan, M. G. Crandall, *Regularizing effects of homogeneous evolution equations.* Contributions to Analysis and Geometry (suppl. to Amer. Jour. Math.), Johns Hopkins Univ. Press, Baltimore, Md., 1981. Pp. 23-39.

[5] A. L. Besse, *"Einstein manifolds,"* Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10. Springer-Verlag, Berlin, 1987.

[6] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, J.-L. Vázquez, *Hardy-Poincaré inequalities and applications to nonlinear diffusions*, C. R. Math. Acad. Sci. Paris, **344** (2007), 431–436.

[7] A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo, J.L. Vázquez, *Asymptotics of the fast diffusion equation via entropy estimates,* Arch. Rat. Mech. Anal. **191** (2009), 347–385.

[8] M. Bonforte and J. L. Vázquez, *Global positivity estimates and Harnack inequalities for the fast diffusion equation*, J. Funct. Anal., 240 (2006), pp. 399–428.

[9] L.A. Caffarelli, A. Friedman, *Asymptotic behaviour of solutions of $u_t = \Delta u^m$ as $m \to \infty$*, Indiana Univ. Math. J. **36** (4) (1987), 711–718.

[10] J. A. Carrillo, A. Jüngel, P. A. Markowich, G. Toscani, A. Unterreiter, *Entropy dissipation methods for degenerate parabolic problems and generalized Sobolev inequalities*, Monatsh. Math., **133** (2001), 1–82.

[11] J. A. Carrillo, G. Toscani, *Asymptotic L^1-decay of solutions of the porous medium equation to self-similarity*, Indiana Univ. Math. J., **49** (2000), 113–142.

[12] J. A. Carrillo, J. L. Vázquez, *Fine asymptotics for fast diffusion equations*, Comm. Partial Differential Equations, **28** (2003), 1023–1056.

[13] E. Chasseigne, J. L. Vázquez, *Theory of extended solutions for fast-diffusion equations in optimal classes of data. Radiation from singularities,* Arch. Ration. Mech. Anal. **164** (2002), 133–187.

[14] B. Chow, D. Knopf, *"The Ricci flow: an introduction,"* Mathematical Surveys and Monographs, **110**. American Mathematical Society, Providence, RI, 2004.

[15] B. Chow, P. Lu, L. Ni, *Hamilton's Ricci flow.* Graduate Studies in Mathematics, 77. American Mathematical Society, Providence, RI; Science Press, New York, 2006.

[16] P. Daskalopoulos, R. Hamilton, *Geometric estimates for the logarithmic fast diffusion equation*, Comm. Anal. Geom. **12** (2004), no. 1-2, 143–164.

[17] P. Daskalopoulos, N. Sesum, *On the extinction profile of solutions to fast-diffusion*, J. reine angew. Math. **622** (2008), 95–120.

[18] E. B. Davies, *"Heat Kernels and Spectral Theory",* Cambridge University Press, 1989.

[19] M. Del Pino, J. Dolbeault, *Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. (9) **81** (2002), 847–875.

[20] M. del Pino, M. Saez. *On the Extinction Profile for Solutions of* $u_t = \Delta u^{(N-2)/(N+2)}$, Indiana Univ. Math. Journal **50**, 2 (2001), 612–628.

[21] J. Denzler, R. J. McCann, *Fast diffusion to self-similarity: complete spectrum, long-time asymptotics, and numerology*, Arch. Ration. Mech. Anal., **175** (2005), 301–342.

[22] C. M. Elliott, M. A. Herrero, J. R. King, J. R. Ockendon, *The mesa problem: diffusion patterns for* $u_t = \nabla \cdot (u^m \nabla u)$ *as* $m \to +\infty$, IMA J. Appl. Math. **37** *(1986), no. 2, 147–154.*

[23] M. Fukushima, *Dirichlet Forms and Markov Processes,* North Holland, 1980.

[24] E. Gagliardo, *Proprietà di alcune classi di funzioni in piú variabili, (Italian)* Ricerche Mat. **7** (1958) 102–137.

[25] R. S. Hamilton. *The Ricci flow on surfaces*, Contemporary Math. **71** (1988), 237–262.

[26] M. A. Herrero, M. Pierre, *The Cauchy problem for* $u_t = \Delta u^m$ *when* $0 < m < 1$, Trans. Amer. Math. Soc., **291** (1985), 145–158.

[27] J. Jost, *"Riemannian Geometry and Geometric Analysis",* Springer, 1998.

[28] C. Lederman, P. A. Markowich, *On fast-diffusion equations with infinite equilibrium entropy and finite equilibrium mass*, Comm. Partial Differential Equations **28** (2003), 301–332.

[29] R. Müller, *Differential Harnack inequalities and the Ricci flow.* EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2006.

[30] L. Nirenberg, *On elliptic partial differential equations,* Ann. Scuola Norm. Sup. Pisa **13** (1959), 115–162.

[31] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. Partial Differential Equations, **26** (2001), 101–174.

[32] P. Li, S.-T. Yau, *On the parabolic kernel of the Schrödinger operator,* Acta Math. **156** (1986), 153–201.

[33] P. Lu, L. Ni, J. L. Vázquez, C. Villani, Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds, J. Math. Pures Appl. (9) **91** (2009), no. 1, 1–19.

[34] L. Ni, *The entropy formula for the linear heat equation*, *J. Geom. Anal.* **14** (1) (2004), 85–96. And *Adenda to "The entropy formula for the linear heat equation"*, **14** (2) (2004), 339–334.

[35] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, ArXiv:math.DG/0211159 (2002)

[36] J. L. Vázquez, *Asymptotic behaviour for the porous medium equation posed in the whole space. Dedicated to Philippe Bénilan*, J. Evol. Equ. **3** (2003), 67–118.

[37] J. L. Vázquez, *"Smoothing and decay estimates for nonlinear diffusion equations"*, vol. 33 of Oxford Lecture Notes in Math. and its Applications, Oxford Univ. Press, 2006.

[38] J. L. Vázquez, *"The Porous Medium Equation.Mathematical Theory"*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

[39] H. Yamabe. *On a deformation of Riemannian structures on compact manifolds,* Osaka Math. J. **12** *(1960), 21–37.*