Global Positivity Estimates and Harnack Inequalities for the Fast Diffusion Equation

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Abstract

We investigate local and global properties of positive solutions to the fast diffusion equation $u_t = \Delta u^m$ in the range $(d-2)_+/d < m < 1$, corresponding to general nonnegative initial data. For the Cauchy problem posed in the whole Euclidean space $\mathbb{R}^d$ we prove sharp *Local Positivity Estimates (Weak Harnack Inequalities) and Elliptic Harnack inequalities*; we use them to derive sharp *Global Positivity Estimates and a Global Harnack Principle*. For the mixed initial and boundary value problem posed in a bounded domain of $\mathbb{R}^d$ with homogeneous Dirichlet condition, we prove Weak and Elliptic Harnack Inequalities.

Keywords. Nonlinear evolutions, Fast Diffusion, Harnack Inequalities, Positivity, Asymptotics.

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Introduction

It is well-known that the solutions of the Heat Equation $u_t = \Delta u$ posed in the whole space with nonnegative data at $t = 0$ become positive and smooth for all positive times and all points of space. The same positivity property is true in many other settings, e.g., for nonnegative solutions posed in a bounded space domain with zero boundary conditions. Such properties of positivity and smoothness are shared by the Fast Diffusion Equation

$$u_t = \Delta u^m, \quad 0 < m < 1,$$

but this happens under certain conditions on the exponent and data and with quite different quantitative estimates. The aim of this paper is to show precise estimates for the positivity of nonnegative solutions of the Fast Diffusion Equation in the 'good' exponent range $m_c < m < 1$, $m_c = (d-2)_+/d$. This exponent restriction is essential if we want to avoid the extinction phenomenon for the solutions of the Cauchy problem posed in the whole space $\mathbb{R}^d$, cf. [3, 6]. It also affects different a priori estimates that we use.

In a first step, we obtain sharp local bounds from below for the solutions at times $t > 0$ in terms of weighted mass estimates on the initial data. The estimates lose accuracy for small times $0 < t \le t_c$, i.e., in an initial interval which is needed for the diffusion process to transmit the information, but they are increasingly sharp as t grows. Such lower bound estimates were first used in elliptic equations, cf. [23, 24, 22, 12, 19], and were called *weak Harnack inequalities*. Our lower estimate takes the form

$$\inf_{x \in B_R(x_0)} u(t,x) \ge \overline{M}_R(x_0)\, H(t/t_c) > 0,$$

where $\overline{M}_R(x_0)$ is the average initial mass in the ball $B_R(x_0)$ and H and t_c are precisely defined in Theorem 1.1. This estimate is the equivalent for $m < 1$ of the famed Aronson-Caffarelli estimates for the Porous Medium Equation [1]. Note that in parabolic equations the bound on the local infimum of the solution must be taken at a later time than the L^p norm that controls it. For the linear parabolic case in a general setting see e.g. [21].

We next notice that, contrary to the PME, the FDE does not suffer from the problem of finite speed of propagation with its waiting times and free boundaries, and we are able to translate the local estimate into a global lower bound: for every time $t > 0$ we can insert a suitable Barenblatt solution below our solution,

$$u(t,x) \ge \mathcal{B}(t - \tau_1, x; M_c),$$

and the parameters τ_1, M_c defining that Barenblatt solution can be calculated in terms of the initial information, see Theorem 1.2.

As a consequence of the local lower bounds, combined with well-known upper bounds (smoothing effects), we derive Elliptic-like Harnack inequalities for continuous nonnegative solutions to the Fast Diffusion Equation in the same range $m_c < m < 1$. The result says that there exists a positive constant $\mathcal{H}$, depending only on m and d, R and a ratio of initial local and global masses such that: for any $t \ge t_c(R)$:

$$\sup_{x \in B_R} u(t,x) \le \mathcal{H} \inf_{x \in B_R} u(t,x).$$

Elliptic-like Harnack inequalities compare the infimum and the supremum of the solution at the same time level, roughly speaking with no need (of positivity) information on the values of the solution at previous times. For this reason, Elliptic-like inequalities can be viewed as an improvement of the Intrinsic Harnack estimates (see [13] or Section 2 for further details), because they show that at a fixed time, the parabolic solution possesses an elliptic behavior. This also suggests that somehow the parabolic problem inherits elliptic-regularity properties from the associated stationary elliptic problems. This fact is suggested also by the celebrated paper of Berryman and Holland [6] in which they show, in the bounded domain case, that the solution of the FDE converges up to scaling to the solution of an associated elliptic problem as time approaches the extinction time.

As a consequence of the global lower bounds, we derive a Global Harnack Principle for continuous nonnegative solutions to the Fast Diffusion Equation in the same range $m_c < m < 1$. The result is nothing but a lower and upper estimate in terms of suitable Barenblatt solutions. This result can be compared with the global principle introduced by DiBenedetto and Kwong, [14], in the context of bounded Euclidean domains. We work in the whole space; in comparison, the role played by the function distance to border is replaced in our result by the decay rate at infinity of the Barenblatt solutions. These play a fundamental role in the lower and upper estimates, as reflected in Theorem 1.5; it is to be compared to the role played by Gaussian kernel in the Heat Equation.

In the limit of the upper and lower estimates for large times we arrive at the asymptotic convergence in relative error introduced in [25]. We also show that the results do not extend to other exponent ranges: on the lower side, uniform local estimates of our kind are not true for $m \le m_c$, as a counterexample based on [8] shows. On the other hand, similar results hold for the Heat Equation, but they are not exactly as strong.

In Section 2 we consider the application of these techniques to the problem posed in a bounded domain with zero Dirichlet boundary conditions and the same restrictions on m. Here, the occurrence of extinction in a finite time $T > 0$ cannot be avoided and our positivity estimates are valid for intermediate times $t \in I = [t_c, T_c]$ with $0 < t_c < T_c < T$ (these times are explicitly computed in terms of the data). Therefore, we lose accuracy in the initial time as before, and we also lose the later times where the solution starts going down to zero because of the influence of the boundary conditions.

The consequences in terms of Harnack inequalities are therefore less important, and they have to be discussed in the context of the existing literature on Harnack inequalities for this problem. As far as we know, [13, 14] are the two more important papers on Harnack inequalities (and Harnack's principles as well) for this kind of problem. We will make a more detailed analysis of the issue and precedents in Section 2. The problem of stabilization as $t \to T$ will be studied in a separate paper, cf. [7].

In the sequel, the letters $a_i, b_i, C_i, K, k_i, \lambda_i, \mu$ are used to denote universal positive constants that depend only on m and d. The constant ϑ is fixed to the value $\vartheta = 1/(2-d(1-m)) > 0$.

1 Positivity and Harnack estimate for Fast Diffusion Equations on $\mathbb{R}^d$

In this section we prove Positivity Estimates (=weak Harnack estimates) for the Cauchy problem for the Fast Diffusion Equation posed in the whole Euclidean space $\mathbb{R}^d$:

$$\begin{cases} u_t = \Delta(u^m) & \text{in } Q = (0, +\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^d, \end{cases} \tag{1.1}$$

in the range $(d-2)^+/d = m_c < m < 1$. We then derive Elliptic Harnack inequalities. In the results, we fix a point $x_0 \in \mathbb{R}^d$ and consider different balls $B_R = B_R(x_0)$ with $R > 0$. We introduce the following measures of the local mass

$$M_R(x_0) = \int_{B_R} u_0(x)\,\mathrm{d}x, \quad \overline{M}_R(x_0) = M_R/R^d.$$

More precisely, we should write $M_R(u_0, x_0), \overline{M}_R(u_0, x_0)$, but we will even drop the variable x_0 when no confusion is feared.

1.1 Local Positivity estimate

This is the intrinsic positivity result that shows in a quantitative way that solutions are positive for all $(x,t) \in Q$. This type of result is also called weak Harnack inequality, and also half Harnack inequality or lower Harnack inequality, meaning that it is half of the full pointwise comparison that Harnack inequalities imply.

Theorem 1.1 *There exists a positive function $H(t)$ such that for any $t > 0$ and $R > 0$ the following bound holds true for all continuous nonnegative solutions u to* (1.1) *with $m_c < m < 1$:*

$$\inf_{x \in B_R(x_0)} u(t,x) \ge \overline{M}_R(x_0)\, H(t/t_c), \tag{1.2}$$

Function $H(\eta)$ is positive and takes the precise form

$$H(\eta) = \begin{cases} K\eta^{-d\vartheta} & \text{for } \eta \ge 1, \\ K\eta^{1/(1-m)} & \text{for } \ \eta \le 1. \end{cases} \tag{1.3}$$

The characteristic time is given by

$$t_c = C\, M_R^{1-m} R^{1/\vartheta}. \tag{1.4}$$

Constants $C, K > 0$ depend only on m and d.

Proof. Without loss of generality we assume that $x_0 = 0$. The proof is a combination of several steps. Different positive constants that depend on m and d are denoted by C_i. The value we get for the constants C and K in the above statement is given at the end of the proof.

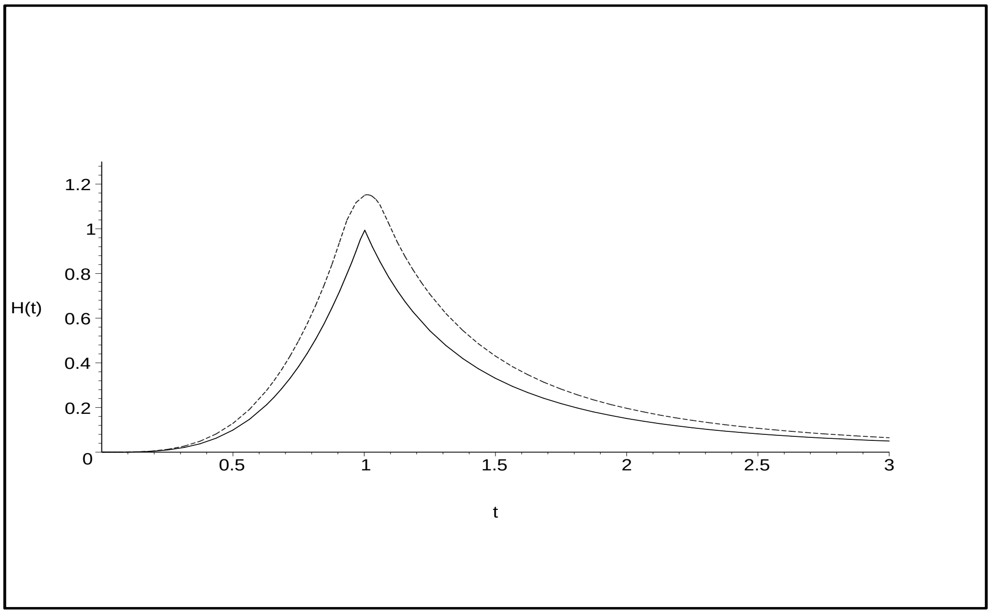

Figure 1: Approximative graphic of the functions $u(t,x)$ (dots) and $H(t)$ (line)

• *Reduction.* By comparison we may assume $\mathrm{supp}(u_0) \subset B_R(0)$. Indeed, a general $u_0 \geq 0$ is greater than $u_0\eta$, η being a suitable cutoff function compactly supported in B_R and less than one. If v is the solution of the fast diffusion equation with initial data $u_0\eta$ (existence and uniqueness are well known in this case), we obtain:

$$\int_{B_R} u(0,x)\,\mathrm{d}x \geq \int_{B_R} u_0(x)\eta(x)\,\mathrm{d}x = M_R$$

and if the statement holds true for v, then

$$\inf_{x\in B_R} u(t,x) \geq \inf_{x\in B_R} v(t,x) \geq H(t/t_c)\overline{M}_R.$$

• *A priori estimates.* The second step is based on the well-known a priori upper estimates (see e.g. [17], Theorem 2.2 or [28]) rewritten in an equivalent form:

$$u(t,x) \leq C_1\|u_0\|_1^{2\vartheta} t^{-d\vartheta}. \tag{1.5}$$

We remark that $\|u_0\|_1 = M_R$ since u_0 is nonnegative and supported in B_R, so that we get

$$u(t,x) \leq C_1 M_R^{2\vartheta} t^{-d\vartheta}$$

for any $x \in \mathbb{R}^d$, while $\vartheta = 1/(2+d(m-1))$.
Let $b = 2 - 1/d$, an integration over $B_{2^b R}$ gives then:

$$\int_{B_{2^b R}} u(t,x)\,\mathrm{d}x \leq C_2\, M_R^{2\vartheta} R^d\, t^{-d\vartheta} \tag{1.6}$$

where $C_2 = C_1 2^{bd}\omega_d$.

• *Integral estimate.* The third step uses Herrero-Pierre's estimate (cf. Lemma 3.1 of [17]), a property that can be labeled as weak conservation of mass and reads: for any $R, r > 0$ and $s, t \geq 0$ one has

$$\int_{B_{2R}} u(s,x)\,\mathrm{d}x \leq C_3 \left[\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x + \frac{|s-t|^{1/(1-m)}}{r^{(2-d(1-m))/(1-m)}}\right].$$

We let $s = 0$ and rewrite it in a form more useful for our purposes:

$$\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x \geq \frac{M_R}{C_3} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2-d(1-m)}{1-m}}}. \tag{1.7}$$

We recall that $M_{2R} = M_R$ since u_0 is nonnegative and supported in B_R.

• *Aleksandrov Principle.* The fourth step consists in using the well-known Reflection Principle in a slightly different form. This principle reads:

$$\int_{B_{2R+r}\setminus B_{2^b R}} u(t,x)\,\mathrm{d}x \leq A_d\, r^d u(t,0) \tag{1.8}$$

where A_d and $b = 2 - 1/d$ are chosen as in (3.4) in Appendix, and one has to remember of the condition $r \geq (2^{(d-1)/d} - 1)2R$. We refer to Proposition (3.1) and formula (3.4) in the Appendix for more details.

• We now put together all the previous calculations:

$$\begin{aligned}\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x &= \int_{B_{2R}} u(t,x)\,\mathrm{d}x + \int_{B_{2R+r}\setminus B_{2^b R}} u(t,x)\,\mathrm{d}x \\ &\leq C_2 \frac{M_R^{2\vartheta} R^d}{t^{d\vartheta}} + A_d\, r^d u(t,0)\end{aligned}$$

this follows by (1.6) and (1.8). Next, we use (1.7) to obtain:

$$\frac{M_R}{C_3} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2-d(1-m)}{1-m}}} \leq \int_{B_{2R+r}} u(t,x)\,\mathrm{d}x \leq C_2 \frac{M_R^{2\vartheta} R^d}{t^{d\vartheta}} + A_d\, r^d u(t,0)$$

And finally we obtain:

$$\begin{aligned} u(t,0) &\geq \frac{1}{A_d}\left[\left(\frac{M_R}{C_3} - C_2 \frac{M_R^{2\vartheta} R^d}{t^{d\vartheta}}\right)\frac{1}{r^d} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2}{1-m}}}\right] \\ &= \frac{1}{A_d}\left[\frac{B(t)}{r^d} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2}{1-m}}}\right]\end{aligned} \tag{1.9}$$

• We now obtain the claimed estimate for $t > t_c^*$. To this end, we check when $B(t)$ is positive:

$$B(t) = \frac{M_R}{C_3} - C_2 \frac{M_R^{2\vartheta} R^d}{t^{d\vartheta}} > 0 \Longleftrightarrow t > (C_3 C_2)^{1/(d\vartheta)} M_R^{1-m} R^{1/\vartheta} = t_c^* \tag{1.10}$$

Now, assuming $t \ge t_c = 2t_c^* > t_c^*$ we optimize the function

$$f(r) = \frac{1}{A_d}\left[\frac{B(t)}{r^d} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2}{1-m}}}\right]$$

with respect to $r(t) = r > 0$ and we obtain that it attains its maximum in $r = r_{max}(t)$. Then one has to check that $r_{max}(t) > \left(2^{(d-1)/d} - 1\right) 2R$, to this end one has to optimize the function $r_{max}(t)$ with respect to $t \in (t_c, +\infty)$, then find the minimum attained at $t = t_{min}$ and, after straightforward calculations, one gets that the condition $r_{max}(t_{min}) > \left(2^{(d-1)/d} - 1\right) 2R$ is nothing more than a lower bound on the constants C_2 and C_3, but since they are constants appearing in upper bound estimates, they can be chosen arbitrarily large. A detailed proof of this fact is given in the domain case, using a different parametrization of the time interval, since there the explicit value of $r_{max}(t_{min})$ is needed also for other purposes and gives rise to conditions on the radius R. Here, it is sufficient to choose C_2 and C_3 sufficiently large.

After a few straightforward computations, we show that the maximum value is attained for all $t > t_c$ as follows:

$$f(r_{max}) = A_d \frac{[d(1-m)]^{2\vartheta-1}}{2^{2\vartheta}\vartheta}\left[\frac{1}{C_3} - C_2 \frac{M_R^{2\vartheta-1} R^d}{t^{d\vartheta}}\right]^{2\vartheta} \frac{M_R^{2\vartheta}}{t^{d\vartheta}} > 0$$

We get in this way the estimate:

$$u(t,0) \ge A_d \frac{[d(1-m)]^{2\vartheta-1}}{2^{2\vartheta}\vartheta}\left[\frac{1}{C_3} - C_2 \frac{M_R^{2\vartheta-1} R^d}{t^{d\vartheta}}\right]^{2\vartheta} \frac{M_R^{2\vartheta}}{t^{d\vartheta}} = K_1\, H_1(t) \frac{M_R^{2\vartheta}}{t^{d\vartheta}}.$$

A straightforward calculation shows that the function

$$H_1(t) = \left[\frac{1}{C_3} - C_2 \frac{M_R^{2\vartheta-1} R^d}{t^{d\vartheta}}\right]^{2\vartheta}$$

is non-decreasing in time, thus if $t \ge t_c$:

$$H_1(t) \ge H_1(t_c) = \left(\frac{1}{2C_3}\right)^{2\vartheta}$$

and finally we obtain:

$$u(t,0) \ge K_1\, H_1(t) \frac{M_R^{2\vartheta}}{t^{d\vartheta}} \ge K_1\, H_1(t_c) \frac{M_R^{2\vartheta}}{t^{d\vartheta}} = \frac{K_1}{(2C_3)^{2\vartheta}} \frac{M_R^{2\vartheta}}{t^{d\vartheta}}$$

So we have proved that

$$u(t,0) \ge \frac{K_1}{(2C_3)^{2\vartheta}} \frac{M_R^{2\vartheta}}{t^{d\vartheta}} \tag{1.11}$$

for $t > t_c = 2\,(C_3 C_2)^{1/(d\vartheta)}\, M_R^{1-m} R^{1/\vartheta} = C M_R^{1-m} R^{1/\vartheta}$.

• *From the center to the infimum.*

Now we want to obtain a positivity estimate for the infimum of the solution u in the ball $B_R = B_R(0)$. Suppose that the infimum is attained in some point $x_m \in \overline{B_R}$, so that $\inf_{x\in B_R} u(t,x) = u(t,x_m)$, then one can apply (1.11) to this point and obtain:

$$u(t,x_m) \ge \frac{K_1}{(2C_3)^{2\vartheta}} \frac{M_{2R}^{2\vartheta}(x_m)}{t^{d\vartheta}} \tag{1.12}$$

for $t > t_c(x_m) = 2\,(C_3C_2)^{1/(d\vartheta)}\, M_R^{1-m}(x_m) R^{1/\vartheta}$. Since the point $x_m \in \overline{B_R(0)}$ then it is clear that $B_R(0) \subset B_{2R}(x_m) \subset B_{4R}(x_0)$, and this leads to the inequality:

$$M_{2R}(x_m) \ge M_R(0) \quad \text{and} \quad M_{2R}(x_m) \le M_{4R}(0)$$

since $M_\varrho(y) = \int_{B_\varrho(y)} u_0(x)\,\mathrm{d}x$ and $u_0 \ge 0$. Thus, we have found that:

$$\begin{aligned}\inf_{x\in B_R(0)} u(t,x) = u(t,x_m) &\ge \frac{K_1}{(2C_3)^{2\vartheta}} \frac{M_{2R}^{2\vartheta}(x_m)}{t^{d\vartheta}} \\ &\ge \frac{K_1}{(2C_3)^{2\vartheta}} \frac{M_{2R}^{2\vartheta}(0)}{t^{d\vartheta}} = \frac{K_1}{(2C_3)^{2\vartheta}} \frac{M_{R}^{2\vartheta}(0)}{t^{d\vartheta}}\end{aligned} \tag{1.13}$$

for $t > t_c(0) = CM_{4R}^{1-m}(0)R^{1/\vartheta} = CM_R^{1-m}(0)R^{1/\vartheta}$, noticing that $M_{4R}(0) = M_{2R}(0) = M_R(0)$, since $\mathrm{supp}(u_0) \subset B_R(0)$. Finally we obtain the claimed estimate for $t \ge t_c(0)$

$$\inf_{x\in B_R(0)} u(t,x) \ge \frac{K_1}{(2C_3)^{2\vartheta}} \frac{M_R^{2\vartheta}(0)}{t^{d\vartheta}} = \frac{K_1}{(2C_3)^{2\vartheta}} \frac{t_c^{d\vartheta}}{t^{d\vartheta}} \frac{M_R^{2\vartheta}(0)}{t_c^{d\vartheta}} \tag{1.14}$$

which is exactly (1.2).

• The last step consists in obtaining a lower estimate when $0 \le t \le t_c$.
To this end we consider the fundamental estimate of Bénilan-Crandall [4]:

$$u_t(t,x) \le \frac{u(t,x)}{(1-m)t}.$$

This easily implies that the function:

$$u(t,x)t^{-1/(1-m)}$$

is non-increasing in time, thus for any $t \in (0,t_c)$ we have that

$$u(t,x) \ge u(t_c,x) \frac{t^{1/(1-m)}}{t_c^{1/(1-m)}}$$

in order to obtain inequality (1.2) for $0 < t < t_c$ is now sufficient to apply the inequality valid for $t > t_c$ to the r.h.s. in the above inequality. The proof of formula (1.2) is complete in all cases. Constant C has the value $C = 2\,(C_3C_2)^{1/(d\vartheta)}$, while K is given by

$$K = 2^{-(d+4)\vartheta+1} \frac{[d(1-m)]^{2\vartheta-1}}{A_d\, C_2\, C_3^{2\vartheta+1}} \tag{1.15}$$

□

Remarks. **(1) Scaling.** We could have simplified the rather cumbersome writing of the formulas by a convenient use of rescaling. Thus, the renormalized function

$$\widehat{u}(x,t) = \frac{1}{\overline{M}_R} u(Rx, Tt) \tag{1.16}$$

is again a solution of the equation precisely if $T = t_c$, but it has initial mass $\widehat{M} = 1$ in the ball of radius $\widehat{R} = 1$. In this way we can dispense with chasing M's and R's in the proof. We have not followed this idea since we fear that, but for the real expert, such kind of calculation is less transparent. But the reader will notice that the result of Theorem 1.1 has been deliberately written in this a-dimensional form.

(2) Characteristic time. Notice that t_c is an increasing function of M_R and R. This is in contrast with the porous medium case $m > 1$ where t_c decreases with M_R. This difference explains some of the different consequences of the lower estimate, which in this case does not restrict the allowed growth of the initial data as $|x| \to \infty$ in the existence theory, as it does for $m > 1$ (cf. [1]).

(3) Minimax problem. Suppose that we want to obtain the best of the lower bounds when t varies. This happens for $t/t_c \approx 1$ and the value is

$$u(t_c, 0) \geq C_3\, M_R R^{-d},$$

which is just proportional to the average. At this time also the maximum is controlled by the average (see the upper estimate).

(4) The proof we present of the weak Harnack inequality follows the general outline of the proof done for the case $m > 1$ by Chasseigne and one of the authors in [11].

(5) We find in the literature on Harnack inequalities expressions of the form

$$\Phi_p(u, r) = \int_{B_r(0)} |u|^p \, dx.$$

in that notation, our $M_R(u_0)$ equals $\Phi_1(u_0, R)$.

(6) The behaviour of H is optimal in the limits $t \gg 1$ and $t \approx 0$ as the Barenblatt solutions show. If we perform the explicit computation for the Barenblatt solution in the worst case where the mass is placed on the border of the ball B_{R_0}, it gives (see (1.18) below)

$$\mathcal{B}(0,t) = \frac{M_R^{2\vartheta}\, t^{1/(1-m)}}{(b_1 t^{2\vartheta} + b_2 t_c^{2\vartheta})^{1/(1-m)}}. \tag{1.17}$$

1.2 Global Positivity estimate

The consideration of the Barenblatt solutions as example leads us to examine what is the form of the positivity estimate when we move far away from a ball in space. Indeed, we can get a global estimate by carefully inserting a Barenblatt solution with small mass below our solution. Let us recall that the Barenblatt solution of mass M is given by the formula

$$\mathcal{B}(t,x;M) = \frac{t^{1/(1-m)}}{\left[\dfrac{b_1 t^{2\vartheta}}{M^{2\vartheta(1-m)}} + b_2|x|^2\right]^{1/(1-m)}}. \tag{1.18}$$

and also that

$$t_c = C\, M_R^{(1-m)} R^{1/\vartheta}.$$

The following Theorem can be viewed as a Weak Global Harnack Principle, since it leads to the Global Harnack Principle, which will be derived in the next subsection. Notice that the parameters of the Barenblatt subsolution have a different form in the two cases $t \ge t_c$ and $0 < t < t_c$

Theorem 1.2 (I) *There exist $\tau_1 \in (0, t_c)$ and $M_c > 0$ such that for all $x \in \mathbb{R}^d$ and $t \ge t_c$*

$$u(t,x) \ge \mathcal{B}(t-\tau_1, x; M_c). \tag{1.19}$$

where we can take $\tau_1 = \lambda\, t_c$ and $M_c = k\, M_R$ for some universal constants $\lambda, k > 0$ which depend only on m and d.

(II) *For any $0 < \varepsilon < t_c$ we have the global lower bound valid for $t \ge \varepsilon$*

$$u(t,x) \ge \mathcal{B}(t-\tau(\varepsilon), x; M(\varepsilon)), \tag{1.20}$$

with $\tau(\varepsilon) = \lambda\,\varepsilon$ and

$$M(\varepsilon) = (\varepsilon/t_c)^{1/(1-m)} M_c = k_1 \left(\varepsilon/R^{1/\vartheta}\right)^{1/(1-m)}. \tag{1.21}$$

Proof. The main result is the first, the point of stating (II) is to have an estimate for small times (with a smaller time shift), at the price of having a subsolution with smaller mass. Let us point out that the last constant $k_1 = k\, C^{-1/(1-m)}$.

We divide the proof in a number of steps; the proof of (I) consists of steps (i)–(iii).

(i) Let us first argue for $x \in B_R(0)$ at time $t = t_c$. As a consequence of our local estimate (1.1) at $t = t_c$, one gets:

$$u(t_c, x) \ge K \frac{M_R}{R^d}$$

for all $|x| \le R$. Hence, (1.19) is implied in this region by the inequality

$$K\frac{M_R}{R^d} \ge \mathcal{B}(t_c - \tau_1, x; M_c) = \frac{(t_c-\tau_1)^{1/(1-m)}}{\left[\dfrac{b_1(t_c-\tau_1)^{2\vartheta}}{M_c^{2\vartheta(1-m)}} + b_2|x|^2\right]^{1/(1-m)}} \tag{1.22}$$

Now we choose $\tau_1 = \lambda t_c$ with a certain $\lambda \in (0,1)$. We put $\mu = 1-\lambda \in (0,1)$ so that $t_c - \tau_1 = \mu\, t_c$. With this choice, (1.22) is equivalent to

$$\frac{b_1(\mu\, t_c)^{2\vartheta}}{M_c^{2\vartheta(1-m)}} + b_2|x|^2 \ge \frac{R^{d(1-m)}\mu\, t_c}{M_R^{1-m} K^{1-m}}$$

putting $x = 0$ and using the value of t_c, it is implied by the condition:

$$M_c = k\, M_R, \quad k \le b_1^{1/(2\vartheta(1-m))} K^{1/2\vartheta}\, (\mu\, C)^{d/2}\,. \tag{1.23}$$

(ii) We now extend the comparison to the region $|x| \ge R$, again at time $t = t_c$. We take as domain of comparison the exterior space-time domain

$$S = (\tau_1, t_c) \times \{x \in \mathbb{R}^d : |x| > R\}.$$

Both functions in estimate (1.19) are solutions of the same equation, hence we need only compare them on the parabolic boundary. Comparison at the initial time $t = \tau_1$ is clear since $B(t_c - \tau_1, x; M_c)$ vanishes. The comparison on the lateral boundary where $|x| = R$ and $\tau_1 \le t \le t_c$ amounts to

$$K \frac{M_R}{R^d} \left(\frac{t}{t_c}\right)^{1/(1-m)} \ge \frac{(t-\tau_1)^{1/(1-m)}}{\left[\dfrac{b_1(t-\tau_1)^{2\vartheta}}{M_c^{2\vartheta(1-m)}} + b_2 R^2\right]^{1/(1-m)}}\,. \tag{1.24}$$

Raising to the power $(1-m)$ and using the value of t_c, we get

$$\frac{K^{1-m} t}{R^2 C} \ge \frac{t-\tau_1}{\dfrac{b_1(t-\tau_1)^{2\vartheta}}{M_c^{2\vartheta(1-m)}} + b_2 R^2}\,,$$

or

$$K^{1-m} \frac{b_1(t-\tau_1)^{2\vartheta}}{M_c^{2\vartheta(1-m)}} + K^{1-m} b_2 R^2 \ge (1 - \frac{\tau_1}{t}) R^2 C\,. \tag{1.25}$$

If we have fixed τ_1 as before and we define $M_c = k\, M_R$ with $k = k(m,d)$ small enough, this inequality is true for $\tau_1 \le t \le t_c$.

(iii) Using now the Maximum Principle in S, the proof of (1.19) is thus complete for $t = t_c$ in the exterior region. Since the comparison holds in the interior region by step (i), we get a global estimate at $t = t_c$.

(iv) We now prove part (II) of the Theorem. We only need to prove it at $t = \varepsilon$. We recall that λ and M_c are as defined in part (I). We know that

$$t_c - \tau_1 = \mu\, t_c, \text{with } \mu \in (0,1)$$

Using the Bénilan-Crandall estimate, we have for $0 < t < t_c$:

$$u(t,x) \ge u(t_c, x) \frac{t^{1/(1-m)}}{t_c^{1/(1-m)}}\,,$$

together with the above estimate (1.19), we can see that:

$$
\begin{aligned}
u(t,x) &\geq u(t_c,x)\frac{t^{1/(1-m)}}{t_c^{1/(1-m)}} \geq \frac{t^{1/(1-m)}}{t_c^{1/(1-m)}}\mathcal{B}(t_c-\tau_1,x;M_c)\\
&= \frac{t^{\frac{1}{1-m}}}{t_c^{\frac{1}{1-m}}}\frac{(\mu\, t_c)^{\frac{1}{1-m}}}{\left[\frac{b_1(\mu\, t_c)^{2\vartheta}}{M_c^{2\vartheta(1-m)}}+b_2|x|^2\right]^{\frac{1}{1-m}}} = \frac{(\mu\, t)^{1/(1-m)}}{\left[\frac{b_1(\mu\, t)^{2\vartheta}}{M_c^{2\vartheta(1-m)}t^{2\vartheta}t_c^{-2\vartheta}}+b_2|x|^2\right]^{\frac{1}{1-m}}}\\
&= \mathcal{B}\left(\mu\, t, x\,;\frac{M_c t^{1/(1-m)}}{t_c^{1/(1-m)}}\right) = \mathcal{B}(t-\tau,x\,;M_c(t))
\end{aligned}
$$

once one let $t-\tau=\mu\, t$ and M_c as above. The proof of (1.20) is thus complete. □

A consequence of this result is the following lower asymptotic behaviour that is peculiar of the FDE evolution.

Corollary 1.3 *We have*

$$
\liminf_{|x|\to\infty} u(t,x)\,|x|^{2/(1-m)} \geq c(m,d)\, t^{1/(1-m)}. \tag{1.26}
$$

The constant $c(m,d)=(2m/\vartheta(1-m))^{1/(1-m)}$ *of the Barenblatt solution is sharp.*

This result has been proved by Herrero and Pierre (see Thm. 2.4 of [17]) by similar methods. Here, it easily follows from the estimates of Theorem 1.2 which provides an exact lower bound for all times, not only for large times.

Remarks. (1) In order to complement the previous lower estimates, let us review what is known about estimates from above. These depend on the behaviour of the initial data as $|x|\to\infty$. Recall only that constant data produce the constant solution, that does not decay. Under the decay assumption on the initial datum $u_0\in \mathrm{L}^1_{loc}(\mathbb{R}^d)$:

$$
\int_{|y-x|\leq|x|/2}|u_0(y)|\mathrm{d}y = O\left(|x|^{d-\frac{2}{1-m}}\right) \text{ as } |x|\to\infty, \tag{1.27}
$$

it has been proved by entirely different methods in [25] that

$$
\lim_{|x|\to\infty} u(t,x)\,|x|^{2/(1-m)} \leq c(m,d)\,(t+S)^{1/(1-m)}.
$$

where $S>0$ depends on the constant in the bound (1.27) as $|x|\to\infty$. The time shift S is needed in the asymptotic behaviour of u as $|x|\to\infty$. Actually, when the initial datum has an exact decay at infinity, $u_0\sim a|x|^{-2/(1-m)}$ we have more:

$$
\lim_{|x|\to\infty} u(t,x)\,|x|^{2/(1-m)} = C\ (t+S)^{1/(1-m)}\,.
$$

with $C=2m/\vartheta(1-m)$ and $S=a^{1-m}/C$, and this cannot be improved as the delayed Barenblatt solutions show. Moreover, there exists a t_0 such that u^{1-m} is convex as a function of x for $t>t_0$, cf. [18].

(2) In comparison with the upper bounds, we have shown that global lower estimates need a time shift τ (in the other direction, explicitly calculated), but in the limit we can put $\tau = 0$, as one can see above. Moreover, the behaviour at infinity is independent of the mass (a fact that is false for the heat equation), hence all Barenblatt solutions with different free constant b_1 behave in the same way in the limit as $|x| \to \infty$, cf. [25].

(3) We can also get better results if we consider radially-symmetric initial data (always in our range of parameters $m_c < m < 1$), cf. [10].

(4) The last remark concerns the mass. The asymptotic behaviour is independent of the mass, thus we can let the mass grow until we reach the total mass, that can be infinite since we only assumed that the initial datum is locally integrable. In case the global mass of the initial datum is finite we can prove local Elliptic Harnack inequality, and a Global Harnack Principle (provided the initial data behaves "well" at infinity), as we will see in the next section.

1.3 Harnack Inequality for FDE on $\mathbb{R}^d$

We now show that the positivity result implies a full local Harnack inequality and a global Harnack principle on the whole Euclidean space.

In this section we will consider $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, $u_0 \ge 0$ and we let

$$M_\infty = \int_{\mathbb{R}^d} u_0(x)\,\mathrm{d}x, \qquad M_R = \int_{B_R} u_0(x)\,\mathrm{d}x \tag{1.28}$$

for some $R > 0$, $x_0 \in \mathbb{R}^d$.

Theorem 1.4 (Elliptic Harnack Inequality)
Let $u(t,x)$ satisfy the same hypothesis as Theorem 1.1. If moreover $u_0 \in L^1(\mathbb{R}^d)$, there exists a positive constant $\mathcal{H}$, depending only on m and d on the ratio M_R/M_∞, such that for any $t \ge t_c(M_R, R)$:

$$\sup_{x\in B_R} u(t,x) \le \mathcal{H} \inf_{x\in B_R} u(t,x). \tag{1.29}$$

If moreover u_0 is supported in B_R, then the constant $\mathcal{H}$ is universal and depends only on m and d.

Proof. First we remark that the exact expression for t_c is given in Theorem1.1. The well known a priori estimates used above, see (1.5), can be rewritten in an equivalent form:

$$\sup_{x\in B_R} u(t,x) \le C_1 M_\infty^{2\vartheta} t^{-d\vartheta} = C_1 \left[\frac{M_\infty}{M_R}\right]^{2\vartheta} M_R^{2\vartheta} t^{-d\vartheta}.$$

Now using (1.2) in a slightly different form (see (1.14)) when $t > t_c$

$$\inf_{x\in B_R} u(t,x) \ge K\, M_R^{2\vartheta} t^{-d\vartheta} \ge K\, C_1^{-1} \left[\frac{M_R}{M_\infty}\right]^{2\vartheta} \sup_{x\in B_R} u(t,x)$$

that is (1.29) with $\mathcal{H} = K^{-1}\, C_1\, [M_\infty/M_R]^{2\vartheta}$. This concludes the proof. $\square$

Under a further control on the initial data, we can transform the local Harnack Principle into a global version. We recall that b_i, λ_1, k_1, and C_i are constants that depend only on m and d, while rest of the parameters depend also on the data as expressed.

Theorem 1.5 (Global Harnack Principle)
Let $u_0 \in L^1(\mathbb{R}^d)$, $u_0 \ge 0$ and

$$u_0(x)\,|x|^{2/(1-m)} \le A. \tag{1.30}$$

for $|x| \ge R_0$. Then, for any time $\varepsilon > 0$ there exist constants τ_1, τ_2, M_1 and M_2, such that for any $(t,x) \in (\varepsilon, \infty) \times \mathbb{R}^d$ we have the following upper and lower bounds:

$$\mathcal{B}(t-\tau_1, x; M_1) \le u(t,x) \le \mathcal{B}(t+\tau_2, x; M_2) \tag{1.31}$$

where $\tau_1 = \lambda_1 \varepsilon$, $\tau_2 = \tau(\varepsilon, A, t_s)$, $M_1 = M(\varepsilon)$ as given in Theorem 1.2 and $M_2 = k_2(\varepsilon, A, \tau_2)\, M_\infty$, while

$$t_c = C\, M_R^{1-m}\, R^{1/\vartheta}, \qquad t_s = C_5\, M_\infty^{1-m} R_0^{1/\vartheta}.$$

Proof. In view of Theorem 1.2, we only have to prove the upper bound. Just recall that in order to adapt the notation we set

$$M_1 = \begin{cases} k_1 (\varepsilon\, t_c)^{1/(1-m)} M_R = k_1 \left(\lambda_1 \varepsilon R^{-1/\vartheta}\right)^{1/(1-m)}, & \text{if } \varepsilon \in (0, t_c) \\ k_1\, M_R\,, & \text{if } \varepsilon > t_c \end{cases}$$

Let us fix $\varepsilon > 0$. We have to find suitable M_2 and τ_2 such that

$$u(t,x) \le \mathcal{B}(t+\tau_2, x; M_2)$$

for any $(t,x) \in (\varepsilon, \infty) \times \mathbb{R}^d$. Using the Comparison Principle, we only need to prove that estimate for $t = \varepsilon$. It will be done in three steps: first, we show that given $\varepsilon, R_1 > 0$, we can find M_2 and τ_2 such that

$$u(\varepsilon, x) \le \mathcal{B}(\varepsilon + \tau_2, x; M_2) \tag{1.32}$$

for any $|x| \le R_1$ by using the uniform boundedness of the solutions due to the smoothing effect; then, we estimate the solution at $t = \varepsilon$ by using a suitable barrier which is valid for $|x| \ge R_1$; finally, we calculate the parameters M_2 and τ_2 such that the corresponding Barenblatt lies on top of the barriers at $t = \varepsilon$ in the whole space. Once this plan is clear, the computations are long and tedious, but the result easy to foresee.

- *Upper Estimates in a Ball*

First we show that one can choose M_2 and τ_2 such that (1.32) holds for any $|x| \le R_1$. In view of the well known $\mathrm{L}^1 - \mathrm{L}^\infty$ estimates for the solutions of the FDE, $u(t,x) \le C_1\, M_\infty^{2\vartheta}\, t^{-d\vartheta}$, we can impose the condition

$$C_1 \frac{M_\infty^{2\vartheta}}{\varepsilon^{d\vartheta}} \le \mathcal{B}(\varepsilon + \tau_2, x; M_2) \tag{1.33}$$

Here, $M_\infty = \|u_0\|_1$ is the total mass and C_1 is the best constant in the smoothing effect, cf. [28]. By the explicit form of the Barenblatt solution $\mathcal{B}$, we are reduced to prove that

$$C_1^{1-m}\frac{M_\infty^{2\vartheta(1-m)}}{\varepsilon^{d\vartheta(1-m)}} \le \frac{\varepsilon+\tau_2}{b_1(\varepsilon+\tau_2)^{2\vartheta}M_2^{-2(1-m)\vartheta}+b_2x^2}$$

for $|x|^2 \le R_1$, that can be written as

$$\frac{b_1(\varepsilon+\tau_2)^{2\vartheta}}{M_2^{2(1-m)\vartheta}} + b_2R_1^2 \le C_1^{m-1}\frac{(\varepsilon+\tau_2)\,\varepsilon^{d\vartheta(1-m)}}{M_\infty^{2\vartheta(1-m)}}\,.$$

This is implied by the following two conditions

$$M_2 \ge \left(2\,b_1C_1^{1-m}\right)^{1/(2(1-m)\vartheta)}\left(\frac{\varepsilon+\tau_2}{\varepsilon}\right)^{d/2}M_\infty \tag{1.34}$$

$$R_1^2 \le \frac{C_1^{m-1}}{2\,b_2}\frac{(\varepsilon+\tau_2)\,\varepsilon^{d\vartheta(1-m)}}{M_\infty^{2\vartheta(1-m)}}. \tag{1.35}$$

- *Upper Barrier outside a ball*

We want to estimate the behaviour of the solution outside a ball, namely when $|x|$ larger than a certain R_1, always at time $t=\varepsilon$. To this end we are going to consider the singular variations of the Barenblatt solution. Suppose first that estimate (1.30) holds in the whole space, i.e., with $R_0=0$. Then, if we choose

$$S \ge b_2\,A^{1/(1-m)}, \tag{1.36}$$

then, it is easy to see that $u_0(x) \le \mathcal{U}(x,0)$, where $\mathcal{U}(t,x;S)$ is the singular solution obtained as a limit of the when $M\to\infty$, namely:

$$\mathcal{U}(t,x;S) = \left(\frac{t+S}{b_2|x|^2}\right)^{1/(1-m)}$$

Where $S>0$ is not fixed a priori, it will be fixed by the asymptotic information on the initial datum. It is known that $\mathcal{U}$ is a supersolution of the equation defined in the spatial region $|x|>0$. Since $\mathcal{U}$ takes the value $\mathcal{U}(t,0)=+\infty$ for all $t>0$, we conclude from the Maximum Principle that under this condition on S, $u(t,x)\le\mathcal{U}(t,x)$ in the $D=\{(x,t):|x|>0\}$, hence in Q. In this way, we have obtained an upper barrier away from $x=0$ that decays in the correct form at infinity.

In case $R_0>0$, we have to use a further modification of the Barenblatt solution where the free constant b_1 becomes negative, and we write

$$\mathcal{U}(t,x;B_1,S) = \left(\frac{t+S}{b_2|x|^2 - B_1\,(t+S)^{2\vartheta}}\right)^{1/(1-m)}$$

This function has a singularity on the surface $|x| = R_U(t)$ where the denominator vanishes and is a solution of the equation for $|x|>R_U(t)$. In order to compare $u(t,x)$ and $\mathcal{U}(t,x;B_1,S)$

in that exterior region, we only need to control that the inequality holds on the parabolic boundary. We settle the inequality at $t = 0$ by putting $R_U(0) = R_0$, i.e.,

$$B_1\, S^{2\vartheta} = b_2 R_0^2, \tag{1.37}$$

and $S \geq b_2\, A^{1/(1-m)}$ as before. The comparison on the curved lateral boundary offers no difficulty since $\mathcal{U} = +\infty$ there. We conclude that

$$u(x,t) \leq \mathcal{U}(t,x;B_1,S)$$

for all $t > 0$ and $|x| > R_U(t)$. The free constants B_1, τ_2, M_2 and S are subject to some further relations in the next step. We will use $\mathcal{U}(\varepsilon,x;B_1,S)$ as an upper barrier for $u(\varepsilon,t)$ in the exterior domain $|x| \geq R_U(\varepsilon)$.

• *Upper Estimates in the whole space*

Since we have two different upper barriers at $t = \varepsilon$, we only have to choose a Barenblatt with parameters M_2 and τ_2 that stays on top of the lower of the barriers at every point. We first determine the point R_1 where the barriers meet at time ε. We get

$$b_2\, R_1^2 = \frac{(\varepsilon+S)\varepsilon^{d\vartheta(1-m)}}{C_1^{1-m} M_\infty^{2\vartheta(1-m)}} + B_1\,(\varepsilon+S)^{2\vartheta}. \tag{1.38}$$

This is the value of R_1 that we have to use in the first step, and the calculation done in the first step takes care of the interior region. For $|x| > R_1$, we have $u(\varepsilon,x) \leq \mathcal{U}(\varepsilon+S,x)$, and we still have to impose the condition

$$\mathcal{U}(\varepsilon+S,x) \leq \mathcal{B}(\varepsilon+\tau_2,x;M_2).$$

This is true if

$$\frac{b_1(\varepsilon+\tau_2)^{2\vartheta}}{M_2^{2\vartheta(1-m)}} + \frac{b_2 R_0^2(\varepsilon+S)^{d\vartheta(1-m)}(\varepsilon+\tau_2)}{S^{2\vartheta}} \leq b_2 R_1^2 \frac{\tau_2 - S}{\varepsilon+S}.$$

which can be further calculated using the value of R_1 as

$$\frac{b_1(\varepsilon+\tau_2)^{2\vartheta}}{M_2^{2\vartheta(1-m)}} + \frac{b_2 R_0^2(\varepsilon+S)^{2\vartheta}}{S^{2\vartheta}} \leq (\tau_2 - S)\frac{(\varepsilon+\tau_2)^{2\vartheta-1}}{C_1^{1-m} M_\infty^{2\vartheta(1-m)}}. \tag{1.39}$$

We still have to check the compatibility of conditions (1.34), (1.35), (1.36), and (1.39) to finish the computation of the upper Barenblatt for $t \geq \varepsilon$. We proceed as follows:

(i) We fix $S = b_2\, A^{1/(1-m)}$.

(ii) Using (1.38) to define R_1, condition (1.35) is equivalent to

$$\tau_2 \geq \varepsilon + 2S + K\varepsilon\left(\frac{1}{S}+\frac{1}{\varepsilon}\right)^{2\vartheta},$$

where $K = 2b_2 C_1^{1-m} M_\infty^{2\vartheta(1-m)} R_0^2$, which has dimensions of a power of a second characteristic time, $K = t_s^{2\vartheta}$.

(iii) Estimate (1.39) is implied by the two conditions

$$\frac{b_1(\varepsilon+\tau_2)^{2\vartheta}}{M_2^{2\vartheta(1-m)}} \le \frac{1}{2}(\tau_2 - S)\frac{(\varepsilon+\tau_2)^{2\vartheta-1}}{C_1^{1-m}M_\infty^{2\vartheta(1-m)}}$$

$$\frac{b_2 R_0^2(\varepsilon+S)^{2\vartheta}}{S^{2\vartheta}} \le \frac{1}{2}(\tau_2 - S)\frac{(\varepsilon+\tau_2)^{2\vartheta-1}}{C_1^{1-m}M_\infty^{2\vartheta(1-m)}}$$

The second gives

$$(\tau_2 - S)(\varepsilon+\tau_2)^{2\vartheta-1} \ge t_s^{2\vartheta}\left(\frac{\varepsilon+S}{S}\right)^{2\vartheta},$$

while the first gives

$$\frac{\tau_2 - S}{\tau_2+\varepsilon} \ge 2b_1 C_1^{1-m}(M_\infty/M_2)^{2\vartheta(1-m)},$$

(iv) We have to add (1.34) which is very similar:

$$\left(\frac{\tau_2+\varepsilon}{\varepsilon}\right)^{2\vartheta-1} \ge 2b_1 C_1^{1-m}(M_\infty/M_2)^{2\vartheta(1-m)},$$

This allows to find $\tau_2 = f(\varepsilon, S, t_s)$ and then M_2/M_∞ as a function of τ_2, ε, S. □

1.4 Asymptotic behaviour. Relative Error Estimates

The second author has proved in [25] the so-called Relative Error Estimates (REE) for the FDE in the same range of parameters, namely

$$\lim_{t\to\infty}\left\|\frac{u(t,\cdot)-\mathcal{B}(t,\cdot\,;M)}{\mathcal{B}(t,\cdot\,;M)}\right\|_\infty = 0\,,$$

where $\mathcal{B}$ is the Barenblatt solution with the same mass (the result is independent of a possible shift in time or space). This is related to our Theorem 1.5 as follows: for every $\varepsilon > 0$ we can find a Barenblatt solution with mass $M_1(\varepsilon) < M_\infty$ and another one with mass $M_2(\varepsilon) > M_\infty$ that serve as lower bound, resp. upper bound for the solution for all times $t \ge \varepsilon$. It is clear from the maximum principle that $M_1(\varepsilon)$ increases with time while $M_2(\varepsilon)$ decreases. The asymptotic result says that

$$\lim_{\varepsilon\to\infty} M_1(\varepsilon) = \lim_{\varepsilon\to\infty} M_2(\varepsilon) = M_\infty.$$

Theorem 1.5 adds to this asymptotic statement a more precise quantitative information that is valid not only for large times, but also for arbitrary small times. The solution thus inherits positivity and boundedness properties directly from the Barenblatt solutions that serve as upper and lower bounds from the very beginning.

Usually, it is said that the Barenblatt solution of the nonlinear equations is a 'poor cousin' of the fundamental solution of the Heat Equation since there is no representation formula as in the linear case. The above results show that in the good fast diffusion range $m_c < m < 1$

it is a stronger model in some respects. Thus, a consequence of this powerful Global Harnack Principle, obviously valid for the Barenblatt solutions, is that the behaviour at infinity (i.e. for $|x| \to \infty$ and/or $t \to \infty$) of the Barenblatt solution is always the same, independent of the mass. This uniformity property is not shared by the Heat Equation nor by the Porous Medium Equation and shows how much more the fast diffusion process regularizes data.

1.5 Different behaviour in the cases $m \notin (m_c, 1)$

In the above considerations, it is essential that the range of parameters is $m_c < m < 1$, since when $m \notin (m_c, 1)$ different phenomena hold. We refer to [28] for a detailed and exhaustive exposition and as a source for more complete bibliography. Let us discuss here the question of possible uniform lower bounds.

Concerning the basic problem of optimal space for existence, H. Brezis and A. Friedman proved in [8] that there can be no solution of the equation if $m \le m_c$ when the initial data is a Dirac mass, so that we lose our main model. But at least solutions exist[1] for all initial data $u_0 \in L^1_{loc}(\mathbb{R}^d)$, when $0 < m < m_c$, and moreover they are global in time, $u \in C([0,\infty) : L^1_{loc}(\mathbb{R}^d))$. Moreover, as Brezis and Friedman proved, the limit of any reasonable approximation is $u(x,t) = M\,\delta(x)$, so that no diffusion takes place at all. This can be viewed by a quite simple example, which is a 'lite-version' of the result of Brezis and Friedman:

Let $\varphi \ge 0$ be a bounded and continuous function with $M = \int_{\mathbb{R}^d} \varphi(x)\,\mathrm{d}x > 0$, and consider it as the initial data for a solution of the FDE with $0 < m < m_c$. Let $T > 0$ be the extinction time of that solution. Put $\vartheta = 2 - d(1-m) < 0$, and use the scaling transformation

$$u_k(x,t) = k^n u(kx, k^{-|\vartheta|}t)$$

to construct solutions u_k with the same initial mass and with extinction times $T_k = k^{|\vartheta|}T \to \infty$. It is easy to show that for every $t > 0$ we have $u(x,t) \to M\,\delta(x)$ as $k \to \infty$ (in the weak sense).

Here is a related result

Consider the Barenblatt solutions $\mathcal{B}_m$ with $m > m_c$ and let $m \to m_c$. We have that

$$\lim_{m \to m_c} \mathcal{B}_m(x,t;M) = M\,\delta(x)$$

in $Q = \mathbb{R}^d \times (0,\infty)$. $\square$

These facts show that the Dirac mass is not diffused by the FDE with critical or subcritical exponent, so that a Dirac delta at $x = 0$ that does not change in time.

As a consequence of this example, controlling the initial mass of a solution in a given ball $B_R(0)$ does not allow us to get any kind of locally uniform lower estimate (take the approximations to a Dirac delta placed at $x_0 \in B_R(0)$ and estimate the value of $u(0,t)$). It follows that

Proposition 1.6 *Locally uniform positivity estimates, and a posteriori any kind of Harnack inequalities, are false for general initial data.*

[1] Such an existence result is not guaranteed when moreover $m \le 0$.

This quite simple example shows that the range of parameters we consider in this paper is optimal from below, if we want the initial datum u_0 to be as general as possible.

Let us now comment that the results discussed above have been motivated by similar properties of the heat equation flow. It has to be noted that there are slight differences in favor of the fast diffusion case. Indeed, if one considers as initial datum $u_0 = \delta_y$, then it is easy to see that the shifted fundamental solution

$$E_y(t,x) = (4\pi\, t)^{-d/2} \mathrm{e}^{-|x-y|^2/t}$$

does not satisfy the condition

$$c_1\, E_0(t,x) \le E_y(t,x) \le c_2 E_0(t,x)$$

for some universal constants $c_i > 0$, which is however satisfied by the Barenblatt solutions if $m_c < m < 1$.

2 Positivity and Harnack estimates for Fast Diffusion Equations on a domain

In this section we will prove local Positivity Estimates (Weak Harnack) and Elliptic Harnack inequalities for the Fast Diffusion Equation in the range $(d-2)^+/d = m_c < m < 1$ in an Euclidean domain $\Omega \subset \mathbb{R}^d$.

$$\begin{cases} u_t = \Delta(u^m) & \text{in } Q = (0,+\infty)\times\Omega \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u(t,x) = 0 & \text{for } t>0 \text{ and } x\in\partial\Omega \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^d$ is an open connected domain with sufficiently smooth boundary. Since we are interested in lower estimates, by comparison we may assume that Ω is bounded without loss of generality.

As a precedent, E. DiBenedetto and Y. C. Kwong prove an Intrinsic Harnack inequality (see [13], Thm. 2.1):

There exist constants $0 < \delta < 1$ and $C > 1$ depending on d and m such that for every point $P_0 = (t_0, x_0) \in Q_T$, $Q_T = (0,T)\times\Omega$, we have

$$\inf_{x\in B_R} u(t_0+\theta, x) \ge C\, u(t_0, x_0) \tag{2.2}$$

provided $u(t_0, x_0)$ is strictly positive and

$$(t_0-\tau, t_0+\tau)\times B_R(x_0) \subset Q_T, \quad \tau = u(t_0,x_0)^{1-m} R^2.$$

The constant $\theta = \delta\,\tau$ depends on the positive value of u at P_0. It is a local property and thus it holds both for the case of the whole space and for the domain case.

Our main result takes the form of a precise lower estimate for the values in question, and will thus ensure that such intrinsic Harnack inequality will hold for all positive times not too close to the extinction time.

We also prove an Elliptic Harnack inequality for intermediate times, i.e. for $t \in I = [t_c, T_c]$ with $0 < t_c < T_c < T$, where t_c and T_c are computed in terms of the initial datum, which follows from our sharp result on positivity. There is a difference between the above estimate and our elliptic Harnack inequality: we calculate explicitly all the constants. As before, we can say that our results somehow "support" the results of [14], in the sense that we ensure positivity in a quantitative way, and thus a posteriori their result holds true for times not too close to the extinction time.

We can use Theorem 2.1 to give a quantitative improvement to the global Harnack principle of E. DiBenedetto, Y. C. Kwong and V. Vespri, [14].

2.1 Weak Harnack Inequality

This is the intrinsic positivity result that shows in a quantitative way that solutions are positive for all $(x,t) \in Q$. In the result we fix a point $x_0 \in \Omega$ and consider different balls $B_R = B_R(x_0)$ with $R > 0$, included in Ω.

Theorem 2.1 *Let u be a continuous nonnegative solution to (2.1), with $m_c < m < 1$. There exists times $0 < t_c^* < T_c^* \le T^*$, where T^* is the finite extinction time, and a positive function $H(t)$ such that for any $t \in (0, T_c^*)$ and $R > 0$ such that*

$$R \le \Lambda \, dist\,(x_0, \partial\Omega) \tag{2.3}$$

the following bound holds true:

$$\inf_{x \in B_R} u(t,x) \ge \overline{M}_R \, H(t/t_c^*), \tag{2.4}$$

where $\overline{M}_R = M_R/R^d$, $M_R = \int_{B_R} u_0(x)\,\mathrm{d}x$. Function $H(t)$ is positive and takes the precise form

$$H(\eta) = \begin{cases} K\eta^{-d\vartheta} & \text{for } 1 \le \eta \le T_c^*/t_c^*, \\ K\eta^{1/(1-m)} & \text{for } \ \eta \le 1 \end{cases} \tag{2.5}$$

The times $0 < t_c^* \le T_c^* \le T^*$ are given by

$$\begin{aligned} t_c^* &= \tau_c (2R)^{1/d\vartheta} M_R^{1-m}, \\ T_c^* &= \tau_c' \, [\, dist(x_0, \partial\Omega) - 2R \,]\, M_R^{1-m}. \end{aligned} \tag{2.6}$$

Constants C, K, τ_c, τ_c', $\Lambda > 0$ depend only on d and m.

Proof. The proof is a combination of several steps. Without loss of generality we assume that $x_0 = 0$. Different positive constants that depend on m and d are denoted by C_i.The precise values we get for C, K, τ_c, τ_c' and Λ are given at the end of the proof.

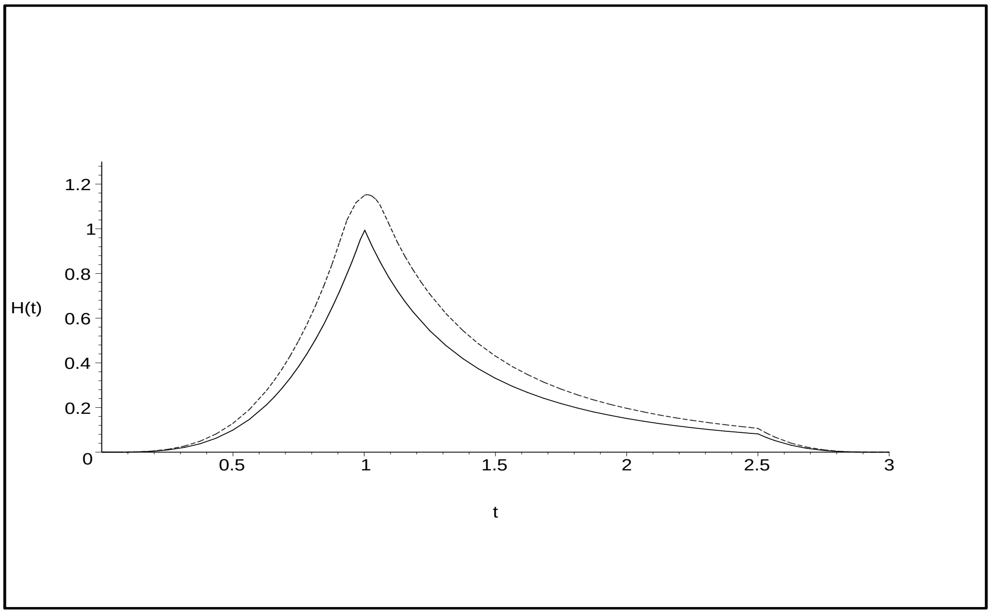

Figure 2: Approximative graphic of the functions $u(t,x)$ (dots) and $H(t)$ (line)

• *Reduction.* By comparison we may assume $\mathrm{supp}(u_0) \subset B_{R_0}(0)$, since the same argument made in the proof of Theorem 1.1 works also in this case.

• *Lower bounds on the extinction time.* In order to get a lower bound for the extinction time in terms of local mass information, we use property which can be labeled as weak conservation of mass, and has been proved in lemma (3.1) of [17]. It reads: for any $R, r > 0$ and $s, t \geq 0$ one has

$$\int_{B_{2R}} u(s,x)\,\mathrm{d}x \leq C_3 \left[\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x + \frac{|s-t|^{1/(1-m)}}{r^{(2-d(1-m))/(1-m)}} \right]. \tag{2.7}$$

Now letting $t = T^*$, so that $u(T^*, x) = 0$, and $s = 0$ so that $\int_{B_{2R}} u(0,x)\,\mathrm{d}x = M_R$, we get

$$T^* \geq \frac{M_R^{1-m}\, r^{1/\vartheta}}{C_3^{1-m}} \geq \frac{M_R^{1-m}\, [dist\,(0, \partial\Omega) - 2R]^{1/\vartheta}}{C_3^{1-m}} \tag{2.8}$$

since $r \in (0, dist\,(0, \partial\Omega) - 2R)$.

• *A priori estimates.* The second step again is similar to the analogous step in proof of Theorem 1.1, so we will omit details. We rewrite the well known a priori estimates (see e.g. [20], Proposition 6.5, or [28]), after an integration over $B_{2^b R}$, in the form

$$\int_{B_{2^b R}} u(t,x)\,\mathrm{d}x \leq C_2\, M_R^{2\vartheta} R^d\, t^{-d\vartheta} \tag{2.9}$$

since u_0 is nonnegative and supported in B_R. Here $C_2 = C_1 2^{bd} \omega_d$.

• *Integral estimate.* Again in this step we are going to use the estimate (2.7). We let $s = 0$ and we rewrite it in a form more useful to our purposes (remember that $M_{2R} = M_R$ since u_0 is supported in B_R):

$$\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x \ge \frac{M_R}{C_3} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{1}{\theta(1-m)}}}. \tag{2.10}$$

we now remark that r and R are such that $B_{2R+r} \subset \Omega$.

• *Aleksandrov Principle.* The fourth step consists in using the well-known Reflection Principle in a slightly different form (see proposition (3.1) and formula (3.4) in the Appendix for more details). This principle reads:

$$\int_{B_{2R+r}\setminus B_{2^b R}} u(t,x)\,\mathrm{d}x \le A_d\, r^d u(t,0) \tag{2.11}$$

where A_d and $b = 2 - 1/d$ are chosen as in (3.4) in Appendix, and one has to remember of the condition $r \ge (2^{(d-1)/d} - 1)2R$.

• We now put together all the previous calculations:

$$\begin{aligned}\int_{B_{2R+r}} u(t,x)\,\mathrm{d}x &= \int_{B_{2^b R}} u(t,x)\,\mathrm{d}x + \int_{B_{2R+r}\setminus B_{2^b R}} u(t,x)\,\mathrm{d}x \\ &\le C_2\, M_R^{2\vartheta} R^d\, t^{-d\vartheta} + A_d r^d u(t,0)\end{aligned}$$

this follows by (1.6) and (2.11). Now we are going to use the (1.7) to obtain:

$$\frac{M_R}{C_3} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{1}{\theta(1-m)}}} \le \int_{B_{2R+r}} u(t,x)\,\mathrm{d}x \le \frac{C_2\, M_R^{2\vartheta} R^d}{t^{d\vartheta}} + A_d r^d u(t,0)$$

And finally we obtain:

$$u(t,0) \ge \frac{1}{A_d}\left[\left(\frac{M_R}{C_3} - \frac{C_2\, M_R^{2\vartheta} R^d}{t^{d\vartheta}}\right)\frac{1}{r^d} - \frac{t^{\frac{1}{1-m}}}{r^{\frac{2}{1-m}}}\right] = \frac{1}{A_d}\left[\frac{A(t)}{r^d} - \frac{B(t)}{r^{2/(1-m)}}\right] \tag{2.12}$$

• Now we would like to obtain the claimed estimate for $t > t_c^*$. To this end we seek whether $A(t)$ is positive:

$$A(t) = \frac{M_R}{C_3} - C_2 \frac{M_R^{2\vartheta} R^d}{t^{d\vartheta}} > 0 \Longleftrightarrow t > (C_3 C_2)^{1/(d\vartheta)}\, M_R^{1-m} R^{1/\vartheta} = t_c^* \tag{2.13}$$

Now we have to check if $t_c^* \le T^*$. By (2.8) one knows that a sufficient condition is that $t_c^* \le T_c^* = C_3^{m-1} M_R^{1-m}\left[dist\,(0,\partial\Omega) - 2R\right]^{1/\vartheta} \le T^*$, that is:

$$R \le \frac{dist\,(0,\partial\Omega)}{2 + C_3^{1-m+1/d\vartheta} C_2^{1/d\vartheta}} \tag{2.14}$$

Now, assuming $t \in (t_c^*, T_c^*)$ temporarily fixed, we optimize the function

$$f(r) = \frac{1}{A_d}\left[\frac{A(t)}{r^d} - \frac{B(t)}{r^{2/(1-m)}}\right]$$

with respect to $r = r(t) \in (0, dist(0, \partial\Omega) - 2R)$ and we obtain that it attains its maximum in $r = r_{max}(t)$:

$$r_{max}(t) = \left[\frac{2}{d(1-m)}\right]^{\vartheta(1-m)} t^{\vartheta} \left[\frac{M_R}{C_3} - \frac{C_2 M_R^{2\vartheta} R^d}{t^{d\vartheta}}\right]^{-\vartheta(1-m)} \tag{2.15}$$

At this point is necessary to check the conditions

$$(2^{(d-1)/d} - 1)2R < r_{max}(t) < dist(0, \partial\Omega) - 2R$$

To this end is useful to get a simpler parametrization of the time interval (t_c^*, T_c^*), indeed

$$t_\alpha = \alpha t_c^* = \alpha \left(C_3 C_2\right)^{1/(d\vartheta)} M_R^{1-m} R^{1/\vartheta}$$

maps the time interval (t_c^*, T_c^*) into $(1, \alpha_c)$, where

$$\alpha_c = \frac{T_c^*}{t_c^*} = C_3^{1-m+1/d\vartheta} C_2^{1/d\vartheta} \left(\frac{dist(0, \partial\Omega)}{R} - 2\right)$$

And

$$r_{max}(t_\alpha) = \left(\frac{2}{d(1-m)}\right)^{\vartheta(1-m)} C_3^{1-m+1/d\vartheta} C_2^{1/d\vartheta} \frac{\alpha^{\vartheta}}{\left(1 - \alpha^{-d\vartheta}\right)^{\vartheta(1-m)}} R$$

Optimizing now this function w.r.t. $\alpha \in (1, \alpha_c)$ will lead to the value

$$\alpha_{min} = 1 + \vartheta d(1-m)$$

and in order to guarantee the fact that $\alpha_{min} \le \alpha_c$ we impose the condition

$$R \le \frac{dist\,(0, \partial\Omega)}{2 + \left((1 + \vartheta d(1-m)) C_3^{1-m+1/d\vartheta} C_2^{1/d\vartheta}\right)^{\vartheta}}$$

Moreover, it is tedious but straightforward to verify that:

$$(2^{(d-1)/d} - 1)2R < r_{max}(t_{\alpha_c}) \le dist\,(x_0, \partial\Omega) - 2R$$

the first inequality becomes nothing else but a lower bound on the constants C_2 and C_3, but since they are constants used in upper estimates, they can be chosen arbitrarily large. The second inequality is guaranteed by the hypothesis $R \le \Lambda\, dist(0, \partial\Omega)$. Now going back to the standard time parametrization we proved that:

$$f(r_{max}(t)) = A_d \frac{[d(1-m)]^{2\vartheta - 1}}{2^{2\vartheta}\vartheta} \left[\frac{1}{C_3} - C_2 \frac{M_R^{2\vartheta-1} R^d}{t^{d\vartheta}}\right]^{2\vartheta} \frac{M_R^{2\vartheta}}{t^{d\vartheta}} > 0$$

for all $t \in (t_{\alpha_{min}}, T_c^*) \subset (t_c^*, T^*)$. We thus found the estimate:

$$u(t, 0) \ge A_d \frac{[d(1-m)]^{2\vartheta-1}}{2^{2\vartheta}\vartheta} \left[\frac{1}{C_3} - C_2 \frac{M_R^{2\vartheta-1} R^d}{t^{d\vartheta}}\right]^{2\vartheta} \frac{M_R^{2\vartheta}}{t^{d\vartheta}} = K_1\, A(t) \frac{M_R^{2\vartheta}}{t^{d\vartheta}}$$

a straightforward calculation shows that the function

$$A(t) = \left[\frac{1}{C_3} - C_2 \frac{M_R^{2\vartheta-1} R^d}{t^{d\vartheta}}\right]^{2\vartheta}$$

is non-decreasing in time, thus if $t \geq t_{\alpha_{min}}$:

$$A(t) \geq A(t_{\alpha_{min}}) = \left(\frac{1-(1+\vartheta d(1-m))^{-d\vartheta}}{2C_3}\right)^{2\vartheta}$$

and finally we obtain:

$$u(t,0) \geq K_1\, A(t)\frac{M_R^{2\vartheta}}{t^{d\vartheta}} \geq K_1\, A(t_{\alpha_{min}})\frac{M_R^{2\vartheta}}{t^{d\vartheta}}$$

So we proved that

$$u(t,0) \geq K\frac{M_R^{2\vartheta}}{t^{d\vartheta}} \tag{2.16}$$

for $t \in (t_{\alpha_{min}}, T_c^*)$, with

$$K = \frac{A_d}{(2C_3)^{2\vartheta}} \frac{[d(1-m)]^{2\vartheta-1}}{2^{2\vartheta}\vartheta} \left[1-(1+\vartheta d(1-m))^{-d\vartheta}\right]^{2\vartheta}.$$

• *From the center to the infimum.*
Now we want to obtain positivity estimate for the infimum of the solution u in the ball $B_R = B_R(0)$. Suppose that the infimum is attained in some point $x_m \in \overline{B_R}$, so that $\inf_{x\in B_R} u(t,x) = u(t,x_m)$, then one can apply (2.16) to this point and obtain:

$$u(t,x_m) \geq K\frac{M_{2R}^{2\vartheta}(x_m)}{t^{d\vartheta}} \tag{2.17}$$

for $t_{\alpha_{min}}(x_m) < t < T_c^*(x_m) < T^*$. Since the point $x_m \in \overline{B_R(0)}$ then it is clear that $B_R(0) \subset B_{2R}(x_m) \subset B_{4R}(0)$ and this leads to the equality:

$$M_{2R}(x_m) = M_R(0) = M_{4R}(0)$$

since $M_\varrho(y) = \int_{B_\varrho(y)} u_0(x)\,\mathrm{d}x$, $\mathrm{supp}(u_0) \subset B_R(0)$ and $u_0 \geq 0$.
This equalities will imply then that the times:

$$\begin{aligned} t_{\alpha_{min}}(x_m) &= (1+\vartheta d(1-m))(C_3C_2)^{1/d\vartheta}(2R)^{1/\vartheta} M_{2R}(x_m) \\ &= (1+\vartheta d(1-m))(C_3C_2)^{1/d\vartheta}(2R)^{1/\vartheta} M_R(0) = t^*_{min}(0) \geq t_{\alpha_{min}}(0) \end{aligned}$$

and

$$\begin{aligned} T_c^*(x_m) &= C_3^{m-1}\left[dist\,(0,\partial\Omega) - 4R\right]^{1/\vartheta} M_{2R}^{1-m}(x_m) \\ &= C_3^{m-1}\left[dist\,(0,\partial\Omega) - 4R\right]^{1/\vartheta} M_R^{1-m}(0) \leq T_c^*(0) \end{aligned}$$

Thus, we have found that:

$$\begin{aligned}\inf_{x\in B_R(0)} u(t,x) = u(t,x_m) \geq K\frac{M_R^{2\vartheta}(x_m)}{t^{d\vartheta}} &= K\frac{M_R^{2\vartheta}(0)}{t^{d\vartheta}}\\ &= K\frac{t_{min}^{*\,d\vartheta}(0)}{t^{d\vartheta}}\frac{M_R^{2\vartheta}(0)}{t_{min}^{*\,d\vartheta}(0)}\end{aligned} \tag{2.18}$$

for $t_c^* = t_{min}^*(0) < t < T_c^*(0) < T^*$ which is exactly (2.4).

• The last step consists in obtaining a lower estimate when $0 \leq t \leq t_c^*$.
The same argument used in the proof of Theorem 1.1, based on Bénilan-Crandall estimate (cf. [4]) will thus give

$$u(t,x) \geq \frac{t^{1/(1-m)}}{t_c^{*\,1/(1-m)}} u(t_c^*,x)$$

in order to obtain inequality (2.4) for $0 < t < t_c^*$ is now sufficient to apply the inequality valid for $t > t_c^*$ to the r.h.s. in the above inequality.

• The values of the constants K and C are given by:

$$\begin{aligned}
K &= \frac{A_d}{(2C_3)^{2\vartheta}}\frac{[d(1-m)]^{2\vartheta-1}}{2^{2\vartheta}\vartheta}\frac{\left[1-(1+\vartheta d(1-m))^{-d\vartheta}\right]^{2\vartheta}}{2^d C_3 C_2(1+\vartheta d(1-m))}.\\
C &= C_3^{1-m+1/d\vartheta}C_2^{1/d\vartheta}\\
\tau_c &= (1+\vartheta d(1-m))(C_3C_2)^{1/d\vartheta}\\
\tau_c' &= 1/C_3^{1-m}\\
\Lambda &= \min\left(\frac{1}{(2+C)}, \frac{1}{2+((1+\vartheta d(1-m))C)^{\vartheta}}\right)
\end{aligned}$$

The proof is complete. □

2.2 Elliptic Harnack Inequality

In this section we want to obtain local Elliptic Harnack inequalities for intermediate times, in analogy to what has been done in the whole space. As already mentioned above, this result somehow "supports" quantitatively the results of [13, 14]. We can conclude that for small times ($0 < t < t_c$) a weaker Intrinsic Harnack inequality is valid (see [13] or equivalently (2.2)), for intermediate times ($t_c < t < T_c$) there holds an Elliptic Harnack Inequality (see below). We point out that for times close to the extinction time an Elliptic Harnack inequality is still valid, as is proved by the authors via accurate asymptotic estimates in a forthcoming paper [7].

Theorem 2.2 *Let $u(t,x)$ satisfy the same hypothesis as theorem (2.1). If moreover $u_0 \in L^1(\Omega)$, then there exists a positive constant $\mathcal{H}$, depending only on m, d and on the ratio:*

$$\frac{M_\Omega}{M_{R_0}} = \frac{\int_\Omega u_0(x)\,\mathrm{d}x}{\int_{B_{R_0}} u_0(x)\,\mathrm{d}x},$$

such that for any $t_c^* < t < T_c^* < T^*$:

$$\sup_{x \in B_{R_0}} u(t,x) \le \mathcal{H} \inf_{x \in B_{R_0}} u(t,x).$$

If moreover u_0 is supported in B_{R_0}, then the constant $\mathcal{H}$ is universal and depends only on m, d.

Proof. The proof is formally the same as in Theorem 1.4 , since the upper bounds are the same, (once one replace M_∞ with M_Ω) and use (2.4) when $t_c^* < t < T_c^*$. □

2.3 Global Harnack Principle

Passing now from the local to the global point of view, we should mention that the Global Harnack Principle in the case of bounded domains, has been proved in [14].

E. DiBenedetto, Y. C. Kwong and V. Vespri investigate some regularity properties of the FDE problem posed on bounded domains. They prove a global Harnack principle (Theorem 1.1, [14]):

For any $\varepsilon \in (0,T)$ there exist constants c, C depending only upon d, m, $\|u_0\|_{1+m}$, $\mathrm{diam}(\Omega)$, $\partial\Omega$ and ε, such that for all $(t,x) \in (0,T) \times \Omega$, $t > \varepsilon$

$$c\,\mathrm{dist}(x,\partial\Omega)^{1/m}(T-t)^{1-m} \le u(t,x) \le C\,\mathrm{dist}(x,\partial\Omega)^{1/m}(T-t)^{1-m} \tag{2.19}$$

This global Harnack principle will give further regularity of the solutions (namely space analyticity and time Holder continuity), and holds on bounded domains depending on some further global regularity of the initial datum. As a consequence of this global Harnack principle, they also prove a rather peculiar property of such solutions, namely:

$$u(t_0,x_0) \ge \gamma_0 \sup_{|x-x_0|<R} u(t_0,x)$$

valid for a $R > 0$ so small that the box

$$(t_0 - \tau, t_0 + \tau) \times B_R(x_0) \subset Q_T, \quad \tau = u(t_0,x_0)^{1-m} R^2,$$

but again the box depends on the positivity value of u in the point (t_0, x_0).

The difference between the $\mathbb{R}^d$ case and the bounded domain case is that in the case of whole space $\mathbb{R}^d$ the general solution $u(x,t)$ is estimated from above and from below in terms of the Barenblatt solution, while in the case of a bounded domain it is bounded between $d(x)^{1/m}(T-t)^{1/(1-m)}$, which is essentially the solution obtained by separation of variables.

We should conclude by saying that the global version of the Elliptic Harnack inequality is the Global Harnack Principle, that is nothing more than an accurate lower and upper bound with the same "comparison function", both in the case of the whole space and in the case of bounded domain.

As far as we know, it is an interesting open problem to find such global principle in unbounded domains.

3 Appendix

Here we prove the Reflection Principle of Aleksandrov in a slightly different form, more useful to our purposes. Other forms of the same principle, in different settings can be found, for example in [15]), Proposition 2.24 (pg. 51) or in [2], Lemma 2.2.

We also notice that it is sufficient to consider the Dirichlet problem on a suitable ball in order to achieve the stated positivity results, namely consider:

$$\begin{cases} u_t = \Delta(u^m) & \text{in } (0,T^*) \times B_{4R}(0) \\ u(0,x) = u_0(x) & \text{in } B_{4R}(0) \\ u(t,x) = 0 & \text{for } 0 < t < T^* \text{ and } x \in \partial B_{4R}(0) \end{cases} \tag{3.1}$$

with $\mathrm{supp}(u_0) \subset B_R(0) \subset B_{4R}(0) \subset \Omega$, where $T^* > 0$ is the finite extinction time. Let u_B denote the solution to the above problem (3.1), while let u_Ω denote the solution to the problem (2.1). It is clear then that u_B is a subsolution to the problem (2.1) so that $u_B \le u_\Omega$ and thus local positivity result for u_B will imply local positivity result for u_Ω. Note however that since the solutions have extinction in finite time and u_B disappears before u_Ω, we are renouncing to obtain estimates near the extinction time of u_Ω.

Proposition 3.1 (Local Aleksandrov's Reflection Principle)
Let $B_{\lambda R}(x_0) \subset \mathbb{R}^d$ be an open ball with center in $x_0 \in \mathbb{R}^d$ of radius λR with $R > 0$ and $\lambda > 2$. Let u be a solution to problem

$$\begin{cases} u_t = \Delta(u^m) & \text{in } (0,+\infty) \times B_{\lambda R}(x_0) \\ u(0,x) = u_0(x) & \text{in } B_{\lambda R}(x_0) \\ u(t,x) = 0 & \text{for } t > 0 \text{ and } x \in \partial B_{\lambda R}(x_0) \end{cases} \tag{3.2}$$

with $\mathrm{supp}(u_0) \subset B_R(x_0)$*. Then, for any $t > 0$ one has:*

$$u(t,x_0) \ge u(t,x_2)$$

for any $t > 0$ and for any $x_2 \in D_{\lambda,R}(x_0) = B_{\lambda R}(x_0) \setminus B_{2R}(x_0)$. Hence,

$$u(t,x_0) \ge |D_{\lambda,R}(x_0)|^{-1} \int_{D_{\lambda,R}(x_0)} u(t,x)\,\mathrm{d}x = \fint_{D_{\lambda,R}(x_0)} u(t,x)\,\mathrm{d}x \tag{3.3}$$

Remark. Formula (3.3) can be viewed as a *local mean value inequality*, it has been derived here from the Aleksandrov principle, but it is interesting by itself and moreover is independent of the range of m: one can apply the same argument to any $m > 0$. Formula (3.3) states indeed that the mean value of the solution of evolution equations of diffusive type, over an annulus is less than the value in the center of that ball where mass was concentrated at the beginning. This property is crucial in the proof of the positivity estimates and, a posteriori, of the Harnack inequality. It will be very useful to obtain it also for other kinds of diffusion equation, since one could prove Harnack inequalities at least for the variable coefficient case, or for the FDE or PME on a Riemannian manifold, provided some other a priori estimate holds, but such estimates are more common in literature.

However, we will use the mean value inequality (3.3) in a slightly different form:

$$\int_{B_{R+r}(x_0)\setminus B_R(x_0)} u(t,x)\,\mathrm{d}x \le A_d r^d u(t,x_0) \tag{3.4}$$

with $r \ge \mu R$, $\mu > 1$, and a suitable positive constant $A_{d,\mu}$. This inequality can easily be obtained from (3.3), noticing that for $r \ge \mu R$ one has

$$(R+r)^d \le c_1\left(R^d + r^d\right)$$

for a constant c_1 that depends on d and $\mu > 1$. Then, we get $(R+r)^d - R^d \le (c_1 - 1)\,R^d + r^d \le c_2\, r^d$, so that

$$|B_{2R+r}(x_0) \setminus B_{2^bR}(x_0)| = \omega_d\left[(R+r)^d - R^d\right] \le A_d\, r^d$$

with $A_d = \omega_d c_2$ where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

Proof. Now we are going to prove the Local Aleksandrov's Reflection Principle. This proof borrows some ideas from the proof of the Aleksandrov's Reflection Principle found in [15].

We can assume without loss of generality that $x_0 = 0$ and we will write B_R instead of $B_R(0)$. The support of the initial datum thus is $\mathrm{supp}(u_0) \subset B_R \subset B_{\lambda R}$. To this end consider the sets

$$\Omega_+ = B_{\lambda R} \cap H_+, \quad \Omega_- = B_{\lambda R} \cap H_-,$$

where H is the hyperplane tangent to the sphere of radius $a \ge R > 0$. By a change of variables we can assume for sake of simplicity, that the equation of the such hyperplane is $H = \{x \in \mathbb{R}^d \mid x_1 = a\}$ and it splits the whole space into two parts $H_+ = \{x \in \mathbb{R}^d \mid x_1 > a\}$ and $H_- = \{x \in \mathbb{R}^d \mid x_1 < a\}$. Associated to this one also has the reflection $\pi(z) = \pi(z_1, z_2, \ldots, z_n) = (2a - z_1, z_2, \ldots, z_n)$.
Moreover, it is easy to see that $\pi(\Omega_+) \subset \Omega_-$, since $a > 0$, that $x_0 \in \Omega_*$ if $a < \lambda R/2$.
Now consider two solutions to the problem on $\Omega_* = \pi(\Omega_+)$: $u_1(t,x)$ is the restriction to the set Ω_* of the solution $u(t,x)$ to the Dirichlet problem in the whole $\Omega = B_{\lambda R}(0)$, while $u_2(t,x) = u(t,\pi(x))$ in the "reflected solution", i.e., the reflection by π of u restricted to Ω_+. This is function is still a solution of the FDE.
Now we compare both functions in the parabolic domain $Q_1 = \Omega_* \times (0,T)$. It is clear that both are solutions in that domain. As for the initial data, we have $u_1(0,x) \ge u_2(0,x)$ in $\pi(\Omega_+)$ since $u_2(0,x) = 0$ while $u_1(0,x) = u_0(x) \ge 0$. As for the boundary conditions, we have $u_1(0,x) = u_2(0,x)$ on the piece of the boundary $H \cap \Omega^*$. On the rest of the boundary, the part that has been reflected from $\partial\Omega_+$ we have $u_1(t,x) \ge 0 = u_2(t,x)$. This implies that for any $t > 0$

$$u(t,x_1) \ge u(t,x_2)$$

provided $x_1 \in \Omega_*$, $x_2 \in \Omega_+$ and $x_1 = \pi(x_2)$. Now letting a moving in the range $(R, \lambda R/2)$ will complete the proof of the first statement. We have that $u(t,0) \ge u(t,x_2)$ for any x_2 in the ray $2R < |x_2| < \lambda R$ and this implies that $u(t,0) \ge u(t,x_2)$ for any $x_2 \in D_{\lambda,R}$ and lead to

$$\begin{aligned}\int_{D_{\lambda,R}} u(t,x)\,\mathrm{d}x &= \int_R^{\lambda R}\int_{\partial B_\varrho} u(t,\varrho\,\sigma)\mathrm{d}\sigma\mathrm{d}\varrho \le \int_R^{\lambda R}\int_{\partial B_\varrho} u(t,0)\mathrm{d}\sigma\mathrm{d}\varrho \\ &= \int_{D_{\lambda,R}} u(t,0)\,\mathrm{d}x = |D_{\lambda,R}|\,u(t,0)\end{aligned}$$

which proves the last statement. The proof is complete □

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