ASYMPTOTICS OF THE POROUS MEDIA EQUATION VIA SOBOLEV INEQUALITIES

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Abstract. Let M be a compact Riemannian manifold without boundary. Consider the porous media equation $\dot{u} = \triangle(u^m)$, $u(0) = u_0 \in \mathrm{L}^q$, $\triangle$ being the Laplace–Beltrami operator. Then, if $q \geq 2 \vee (m-1)$, the associated evolution is $\mathrm{L}^q - \mathrm{L}^\infty$ regularizing at any time $t > 0$ and the bound $\|u(t)\|_\infty \leq C(u_0)/t^\beta$ (1) holds for $t < 1$ for suitable explicit $C(u_0), \gamma$. For large t it is shown that, for general initial data, $u(t)$ approaches its time–independent mean with quantitative bounds on the rate of convergence. Similar bounds are valid when the manifold is not compact, but $u(t)$ approaches $u \equiv 0$ with different asymptotics. The case of manifolds with boundary and homogeneous Dirichlet, or Neumann, boundary conditions, is treated as well. The proof stems from a new connection between logarithmic Sobolev inequalities and the contractivity properties of the nonlinear evolutions considered, and is therefore applicable to a more abstract setting.

1. Introduction and setup of the problem

One of the most important features of logarithmic Sobolev inequalities (LSI in the sequel) involving a Dirichlet form $\mathcal{E}$ is their connection with contractivity properties of the semigroup whose generator is the self–adjoint operator associated to $\mathcal{E}$. This is well–known since the pioneering work of L. Gross [22] and later developments, for an excellent discussion of which we refer to the book of E.B. Davies [15]. Informally speaking, the validity of a LSI implies hypercontractivity of the semigroup, while the validity of a certain family of LSI implies, under certain technical conditions, *ultracontractivity* of the semigroup, i.e. its operator boundedness from any L^p to L^∞.

The proof of such results relies heavily on the theory of symmetric Markov semigroups and in particular on a number of features which are typical of the *linear* setting, e.g. the spectral Theorem and duality and interpolation results.

The present paper is an investigation on the following question: do logarithmic Sobolev inequalities imply ultracontractive–like bounds for solutions to the *nonlinear* evolution equation known as the porous media equation? Our aim is to show that the answer is in most aspects affirmative. Before entering into details we remark that the first connection between logarithmic Sobolev inequalities and contractivity properties of nonlinear evolutions was discovered by E. Carlen and M. Loss in [9] for a class of equations including the Burgers and the 2–D Navier–Stokes equation. Later on in [12] it was shown that a new family of logarithmic Sobolev inequalities involving the p–energy functional can be applied to study the L^p–L^q smoothing and the asymptotics of the nonlinear parabolic equation driven by the p–Laplacian: this is not surprising in view of the fact that the p–energy functional is a *nonlinear Dirichlet form* in the sense of [13].

While a large amount of literature concerns the asymptotics of solutions of the porous media equation, a direct connection between such properties and the validity of LSI seems never to have been investigated. We notice however that in the recent papers [10], [17], [18], to which we refer for further references, approaches using relative entropy estimates and/or Gagliardo–Nirenberg–type inequalities with optimal constants are outlined in the special setting of the whole Euclidean space, sometimes with some extra restriction on dimension or on the value of m, and in most results requiring positivity of the initial datum. We should however comment that the knowledge of the best constant in the Gagliardo –Nirenberg inequalities is used in [18] to prove *finer* space time decay properties in terms of the Barenblatt solutions, an explicit space–time function which takes the role of a fundamental solution for the problem at hand and has nice space–time scaling properties.

It is our aim to show that LSI can be used to investigate the L^q –L^∞ *smoothing* of the porous media equation, and in particular its short and long time asymptotics, for initial data which are in most cases nor bounded nor positive, in the context of Riemannian manifolds which can have finite or infinite volume and may have or not a boundary, in which case homogeneous Dirichlet or Neumann boundary conditions are assumed. Since our approach is essentially functional analytic, the various cases can be dealt with almost at the same time. It does not require the knowledge of sharp constants in the Sobolev inequalitites to yield the conclusions, although of course such conclusions are the analogue in the present case of only the on–diagonal heat kernel bounds of the linear case. We stress however that the present setting is choosen as a *model* one, and that our results hold in a much more abstract setting which we briefly discuss in Remark 1.2.

We now describe in more detail the setting of our result and give some more detailed comparison with the existing literature.

Let (M, g) be a smooth, connected and compact Riemannian manifold without boundary, whose dimension is denoted by d and is assumed to be not smaller than three. We shall denote by ∇ the Riemannian gradient and by $\mathrm{d}x$ the Riemannian measure on (M, g). We consider the diffusion problem

$$\left\{\begin{array}{ll} \dot{u} = \triangle(u^m), & \text{on } (0,+\infty)\times M \\ u(0,\cdot) = u_0 \in L^q(M), & \text{on } M \end{array}\right. \tag{1.1}$$

where $q \geq 1$, $m > 1$, and we define $u^m = |u|^{m-1}u = |u|^m \mathrm{sgn}(u)$ as usual in the literature for the equation at hand. It is hopeless to give any complete account of the literature on this problem. It is therefore without any claim of completeness that we refer to [26] for a new Riemannian point of view in the study of the *Euclidean* porous media equation, to [3], [7], [32], [33], [34] for the setting of this problem in the whole $\mathbb{R}^d$, to [20], [27], [32] [35], for Euclidean bounded domains with homogeneous Dirichlet conditions, to [1] for the Neumann problem in Euclidean bounded domains. Such papers should also be meant as the source for a more complete bibliography.

By the term *solution* of (1.1) we shall mean its *weak* solution corresponding to an initial datum $u_0 \in L^1(M)$. This means that

$$\begin{aligned} u \in\ & \mathrm{C}^0([0,+\infty);\mathrm{L}^1(M)) \cap W^{1,1}_{loc}((0,+\infty);\mathrm{L}^1(M)) \\ u(0) = u_0 & \\ u(t)^m \in\ & \mathrm{W}^{1,2}(M) \text{ for a.e. } t>0 \end{aligned}$$

and that, for any $T > 0$ and any positive and bounded test function

$$\varphi \in \mathrm{C}^1\left([0,T]\times M\right), \quad \varphi(T,\cdot) = 0,$$

one has:

$$\begin{aligned} \int_M u_0(x)\varphi(0,x)\,\mathrm{d}x = &- \int_0^T \int_M u(t,x)\varphi'(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T \int_M \left(\nabla\left(u(t,x)^m\right)\right)\cdot\nabla\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t : \end{aligned}$$

see for example [8], pg 48–49. Other authors give slightly different notions of weak solution, for example see [21], [29], [32], [33], [34], [35]. Existence of such solutions has been established in [1] (see Thm. 0.2) for the Neumann problem in bounded Euclidean domains, and we just remark that their method works in our setting with the appropriate notational modifications. We can now state our first result, which can be considered as the main one in the present paper, together with Theorem 1.5. It corresponds to the fact, well–known in the linear case, that the validity of a suitable Sobolev inequality of the form (1.3) (or of the corresponding family of logarithic Sobolev inequalities, see the next Section for details on this) implies *short* times ultracontractive bounds for the evolution at hand (see [15]).

Theorem 1.1 (L^q–L^∞ bounds for short times)**.** *Let (M,g) be a smooth, connected and compact Riemannian manifold without boundary, of dimension $d \geq 3$. Consider a weak solution to the problem (1.1) with $m > 1$ and initial datum $u_0 \in L^q(M)$, $m_0 \leq q \leq +\infty$ with $m_0 = 2 \vee (m-1)$. Then the following ultracontractive bound holds true for all $t \in (0,1]$*

$$\|u(t)\|_\infty \leq C \frac{\|u_0\|_q^\gamma e^{E_0 \|u_0\|_{m_0}^{m-1} t}}{t^\alpha} \tag{1.2}$$

where $\alpha = \frac{1}{m-1}\left[1 - \left(\frac{q}{q+m-1}\right)^{d/2}\right]$, the constant γ depends only on q, m, d, while the constants C, E_0 depend only on $m, q, d, \mathrm{Vol}(M)$ and on the constant A appearing in the Sobolev inequality

$$\|u - \overline{u}\|_{2d/(d-2)} \leq A \|\nabla u\|_2 \,. \tag{1.3}$$

One should notice that $\alpha \to d/(2q)$ as $m \downarrow 1$ as expected. An inspection of the proof will show also that $\gamma = (q/(q+m-1))^{d/2}$, so that $\gamma \to 1$ in such limit.

Remark 1.2. It will be apparent from the proof that the validity of such result depends only on the validity of suitable Sobolev inequality and from the fact that ∇ is a TM–valued *derivation.* The setting could therefore be generalized as in [14] to cover the case in which the operator $\triangle$ is replaced by the self–adjoint operator associated to the Dirichlet form

$$\mathcal{E}(u) = \|\partial u\|^2_{L^2(E,\mu)},$$

where E is a C^*–monomodule over $C_0(TM)$ and ∂ is a closed *derivation* from $L^2(X, \mathrm{d}x)$ to $L^2(E, \mathrm{d}\mu)$, μ being a finite Radon measure on M. A Sobolev inequality of the type (1.3), for a suitable d (not necessarily coinciding with the dimension of M nor integer) has to be required, and all the constants appearing in the conclusions will depend on such d. Although we could address our discussion from the beginning in such general setting, we have preferred to make the paper more readable and to concentrate on the usual porous media equation, especially because of its physical importance. As a special, particularly simple situation which could be dealt with we mention however at least the case in which $\triangle$ is replaced by a *sublaplacian* associated to a collection of vector fields on M, provided a Sobolev inequality hold, as e.g. in the Hörmander case.

Once that the short time estimate has been given, the long–time behaviour in the L^∞ norm is relatively simple to obtain. While in the linear case the compactness of the state space and the spectral Theorem force exponential time decay of solutions, in the present case two different possibilities can occur, as pointed out in the Euclidean case in [1]. In fact we have first the following

Corollary 1.3. *Assume that $\overline{u} = 0$, where*

$$\overline{u} = \frac{1}{\mathrm{Vol}\, M} \int_M u \, \mathrm{d}x.$$

Then for any $q \geq 2$, $t > 2$ we have:

$$\|u(t)\|_\infty \leq \frac{C}{\left(B(t-1) + \|u_0\|_q^{-(m-1)}\right)^{\frac{\gamma}{m-1}}} \tag{1.4}$$

In particular, for any $\varepsilon \in (0,1)$ and $t > 2$ one has, setting $\gamma = \left(\frac{q}{q+m-1}\right)^{d/2}$:

$$\|u(t)\|_\infty \le \frac{C\|u_0\|_q^{\varepsilon\gamma}}{(B(t-1))^{\gamma\frac{1-\varepsilon}{m-1}}} \tag{1.5}$$

and in addition, for any $t > 0$ and $r \in [2, +\infty)$ there exists $B_r > 0$ such that the following absolute bound holds:

$$\|u(t)\|_r \le \frac{1}{(B_r t)^{\frac{1}{m-1}}}. \tag{1.6}$$

Classes of solutions with L^∞ initial data and for which the stated time decay is sharp have been given, for the Euclidean Neumann problem, in [1]. In such paper a bound of a similar nature is in fact proved for bounded initial data in the Euclidean Neumann setting.

It has been pointed out in [1], at least for *bounded* initial data, that the situation is entirely different for *positive* (or negative) solutions. In fact, the next corollary shows that there is L^∞ exponential decay for solutions corresponding to such class of data, with rate depending on the datum itself, a fact first noticed in [1] for the Neumann problem and L^∞ data.

Corollary 1.4. *Assume that the initial datum u_0 has non–zero mean. Then there exists $K = K(u) > 0$ such that the bound*

$$\|u(t) - \overline{u_0}\|_\infty \le K\mathrm{e}^{-\sigma t}\|u_0 - \overline{u_0}\|_2^\gamma \tag{1.7}$$

holds for any $t \ge 1$, with $\sigma = m\gamma\lambda_1(|\overline{u}_0|/2)^{m-1}$ and λ_1 the constant appearing in the Poincaré inequality

$$\|\nabla u\|_2^2 \ge \lambda_1\|u - \overline{u}\|_2^2.$$

The next result deals with manifold of infinite volume, for which the Sobolev inequality

$$\|u\|_{\frac{2d}{d-2}} \le A^* \|\nabla u\|_2 \,. \tag{1.8}$$

is *assumed* to hold. This is perhaps the results which bears the closest similarity with the linear case. The validity of (1.8) is a strong assumption on the manifold and is well–known to be equivalent to a number of other geometrical or analytical conditions, some of which will be briefly recalled in the next Section. Existence of solutions to this problem can be proved as in [8].

Theorem 1.5. *Let (M, g) be a smooth connected Riemannian manifold without boundary, of infinite volume and of dimension $d \ge 3$, such that the $\mathbb{R}^d$–type Sobolev inequality (1.8) hold. Consider a weak solution to the problem (1.1) with $m > 1$ and initial datum $u_0 \in L^q(M)$, $2 \le q \le +\infty$. Then the following ultracontractive bound holds true for all $t > 0$*

$$\|u(t)\|_\infty \le C\,\frac{\|u_0\|_q^\gamma}{t^\alpha} \tag{1.9}$$

where

$$\begin{aligned} \alpha &= \frac{d}{2q + d(m-1)} \\ \gamma &= \frac{2q}{2q + d(m-1)} \end{aligned} \tag{1.10}$$

where C depends only on $m, q, d, \mathrm{Vol}(M)$ and on the constant A appearing in the Sobolev inequality (1.3).

Remark 1.6. For the Euclidean setting see [3], [21], Thm 1.1, [25], Thm 4, [31], [32], [33], [34]), Thm 3.1 and 3.2. The asymptotics of the porous media equation in $\mathbb{R}^d$ are usually expressed in terms of a comparison with the so–called Barenblatt solutions. No such solution is available in our context. Moreover the dependence on the initial data is usually not explicitely investigated, and in most such results some further restriction on initial data (like positivity and/or boundedness) are required.

We now turn to the case of compact incomplete manifolds with smooth boundary. First we require homogeneous Dirichlet boundary conditions, i.e. formally $u \equiv 0$ on ∂M, which should be meant by the requirement that the above definition of weak solution is modified so that $W_0^{1,2}$ takes the place of $W^{1,2}$ there. The conclusions are identical to the ones given in the previous Theorem for small times, since the Sobolev inequality one starts from is formally identical, but for large times the asymptotics are the same as in the case of compact manifolds without boundary. Existence of solution in this setting can be proved as it has been done in [27] in the case of bounded Euclidean domains with smooth boundary and homogeneous Dirichlet boundary conditions.

Theorem 1.7. *Let (M, g) be a compact smooth connected Riemannian manifold of dimension $d \geq 3$, with smooth boundary ∂M. Consider a weak solution to the problem (1.1) corresponding to homogeneous Dirichlet boundary conditions, with $m > 1$ and initial datum $u_0 \in L^q(M)$, $2 \leq q \leq +\infty$. Then (1.9) holds, for all $t > 0$, with the same exponents given in Theorem 1.5. Moreover for $t > 1$ one also has bounds of the type given Theorem 1.1, namely (1.4), (1.5), (1.6), with the same exponents appearing there, but with $u(t)$ approaching $u \equiv 0$.*

For the Euclidean setting see Aronson–Peletier [4], Evans [20], Pazy [27], Vazquez [32], [35]. We in particular comment that the only paper in which an ultracontractive–like bound of the kind proved here is present is [27], in the setting of bounded Euclidean domains, but such results are not sharp for large times as shown in [4].

The situation for homogeneous Neumann boundary conditions, i.e. formally with $\nabla u \equiv 0$ on ∂M (and in this case the definition of weak solution is identical to the previous one), is completely identical to the one given in Theorem 1.1. The existence of solutions in this case has again been established in [1].

Theorem 1.8. *Let (M, g) be a compact smooth connected Riemannian manifold of dimension $d \geq 3$, with smooth boundary ∂M. Consider a weak solution to the problem (1.1) and corresponding to homogeneous Neumann boundary conditions, with $m > 1$ and initial datum $u_0 \in L^q(M)$, $2 \leq q \leq +\infty$. Then all the conclusions of Theorem 1.1 and of Corollaries 1.3, 1.4 hold.*

This result is consistent with the known long–time asymptotics of the porous media equation in bounded *Euclidean domains* with Neumann boundary conditions given in [1], Thm.3.3 and Thm. 3.4 for *bounded* data.

Remark 1.9. There is a version of our results for incomplete non compact manifolds with infinite volume and smooth boundary, provided the appropriate Sobolev inequality holds.

1.1. **The approach using Nash inequalities.** It is well–known that, in the linear case, ultracontractive bounds can be proved by using directly Sobolev inequalities or, even more directly, using Nash inequalities (see e.g. [15], pg. 79). In fact, in this latter approach, one usually first proves an $\mathrm{L}^1 - \mathrm{L}^2$ bound and then proceeds by duality and interpolation. Although there is, to our knowledge, no suitable duality Theorem at one's disposal in the present context, it is worth noticing that this approach can be pushed forward, in some situation, to prove $\mathrm{L}^2 - \mathrm{L}^q$ bounds for $q \neq \infty$. We now sketch the argument which shows this. Let us then choose M non–compact, with no boundary and such that the Sobolev inequality (1.8) holds. It is then well–known that a suitable family of Gagliardo–Nirenberg inequalities holds. More precisely, one has

$$\|f\|_r \le C\|\nabla f\|_2^{\vartheta}\|f\|_s^{1-\vartheta} \tag{1.11}$$

where $s>0$, $\vartheta\in(0,1)$ and, setting $2^*=2d/(d-2)$, one has $1/r=\vartheta/2^*+(1-\vartheta)/s$. This follows directly starting from the Sobolev inequality but has been proved in great generality in [6].

Take then any $p\in[2,+\infty)$ and a solution of the equation at hand, and compute formally (for suitable positive constants, always indicated hereafter by C)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p &= -C\|\nabla|u(t)|^{(p+m-1)/2}\|_2^2 \le -C\frac{\||u(t)|^{(p+m-1)/2}\|_r^{2/\vartheta}}{\||u(t)|^{(p+m-1)/2}\|_s^{2(1-\vartheta)/s\vartheta}}\\ &= -C\frac{\|u(t)\|_{r(p+m-1)/2}^{(p+m-1)\vartheta}}{\|u(t)\|_{s(p+m-1)/2}^{(1-\vartheta)(p+m-1)\vartheta}}.\end{aligned}$$

Choose now the parameters in (1.11) so that

$$s=\frac{4}{p+m-1},\ \vartheta=\frac{(p+m-1)(p-2)d}{p[(p+m-1)d-2(d-2)]},\ r=\frac{2p}{p+m-1},$$

this being compatible with the constraints on the parameters in (1.11). Since the L^1–norm of the solution decreases along the evolution we arrive at the inequality

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_p^p &\le -C\frac{\|u(t)\|_p^{(p+m-1)/\vartheta}}{\|u(0)\|_2^{(p+m-1)(1-\vartheta)/\vartheta}}\\ &= -C\frac{\|u(t)\|_p^{p(1+\varepsilon)}}{\|u(0)\|_2^{2[2p+d(m-1)]/[d(p-2)]}}\end{aligned}$$

By setting $a(t):=\|u(t)\|_p^p$ one then has proved a differential inequality of the form $\dot a\le -Ka^{1+\varepsilon}$ with $\varepsilon>0$ given by with

$$\varepsilon=\frac{4+d(m-1)}{d(p-2)}$$

so that one readily gets, noticing in addition that $a(0)>0$:

$$\|u(t)\|_p \le C\frac{\|u(0)\|_2^{\gamma}}{t^{\alpha}} \tag{1.12}$$

with

$$\gamma=\frac{2[2p+d(m-1)]}{p[4+d(m-1)]},\ \alpha=\frac{d(p-2)}{p[4+d(m-1)]}.$$

This is a *supercontractive–type* bound, and similar L^q–L^p bounds can be proved similarly. However the present method of proof seems not adequate to reach the limiting case $p=\infty$. In fact it is a tedious but straightforward task to verify that the proportionality constant in (1.12) tends to $+\infty$ in such a limit. This is the reason which motivated our use of LSI in this paper.

The paper is organized as follows. Section 2 contains a short proof of the relevant LSI starting from the appropriate Sobolev inequalities, and some properties of the Young functional which are of particular relevance in the case of manifolds of infinite volume. Section 3 contains the proofs of the main results for manifolds of finite volume, while the case of manifolds of infinite volume is dealt with in section 4. The Appendix contains some elementary computations used in Section 3.

It is a pleasure to thank the referee for his careful reading of the manuscript and especially for noticing an error in the first version of this paper. We also thank N. Alikakos for a helpful discussion.

2. Some inequalities

Our goal here is first to recall a well known argument which shows how to deduce a family of logarithmic Sobolev inequalities from ordinary Sobolev inequalities (see [15]). It will give somewhat crude bounds on the so–called local norm function in some cases, but will work for what follows. A more pleasant form of LSI involving $u-\overline{u}$ instead of u could be proved as well, but it would not be adequate for the sequel. Later on we prove some bounds for Young functionals which will be crucial in dealing with the case of manifolds with infinite volume.

2.1. **Compact manifolds without boundary.** In this section (M,g) will denote a smooth connected compact manifold without boundary and dimension $d\geq 3$. Then the Poincarè–Sobolev inequality:

$$\|u-\overline{u}\|_{2^*}\leq A\,\|\nabla u\|_2\,. \tag{2.1}$$

holds for every $u\in W^{1,2}(M)$, with $2^*=\frac{2d}{d-2}$. See [5] or [23] for a proof.

Proposition 2.1. *There exists $c_1,c_2>0$ such that the logarithmic Sobolev inequality:*

$$\int_M |f|^2\log\left(\frac{|f|}{\|f\|_2}\right)^2\,\mathrm{d}x\leq\frac{d}{2}\left[c_1\varepsilon\left|\overline{f}\right|^2+c_2\varepsilon\|\nabla f\|_2^2-\|f\|_2^2\log\varepsilon\right] \tag{2.2}$$

holds true for any $\varepsilon>0$, for all $f\in W^{1,2}(M)$. The constants c_1 and c_2 depend only on d, on $\mathrm{Vol}(M)$ *and on the constant appearing in the Sobolev inequality (1.3).*

Proof. It suffices to consider the case in which f is nonnegative and it is such that $\|f\|_2=1$, so that $\mathrm{d}\mu(x)=f(x)^2\,\mathrm{d}x$ is a probability measure. Then Jensen's inequality implies (see [15], Th. 2.4.4):

$$\int_M f^2\log(f)\,\mathrm{d}x\leq\frac{d}{4}\log\|f\|_{2^*}^2\leq\frac{d}{4}\left(\log(\varepsilon)+\varepsilon\|f\|_{2^*}^2\right)$$

since $\log(t)\leq\varepsilon t-\log(\varepsilon)$ for all $t,\varepsilon>0$. Notice also that

$$\|f\|_{2^*}-\|\overline{f}\|_{2^*}\leq\|f-\overline{f}\|_{2^*}\leq A\|\nabla f\|_2$$

so that $\|f\|_{2^*}\leq A\|\nabla f\|_2+\|\overline{f}\|_{2^*}$. Since of course $\|\overline{f}\|_{2^*}=\left|\overline{f}\right|\mathrm{Vol}(M)^{1/2^*}$ we then have, by the Sobolev inequality:

$$\|f\|_{2^*}^2\leq 2A^2\|\nabla f\|_2^2+2\left|\overline{f}\right|^2\mathrm{Vol}(M)^{2/2^*}. \tag{2.3}$$

Finally we get

$$\begin{aligned}\int_M f^2\log(f)\,\mathrm{d}x&\leq\frac{d}{4}\left[-\log(\varepsilon)+\varepsilon\|f\|_{2^*}^2\right]\\&\leq\frac{d}{4}\left[-\log(\varepsilon)+\varepsilon\left(2A^2\|\nabla f\|_2^2+2\left|\overline{f}\right|^2\mathrm{Vol}(M)^{2/2^*}\right)\right]\end{aligned}$$

And thus the statement follows. □

Notice that the Sobolev inequality (2.1) does not hold for non compact, complete manifolds of finite volume (see [23], pg.56), nor does the $\mathbb{R}^d$–type inequality:

$$\|u\|_{2^*}\leq A^*\,\|\nabla u\|_2\,.$$

(see again [23]).

2.2. **Non compact manifolds without boundary.** We now assume (M, g) to be a smooth connected non compact manifold without boundary, and of infinite volume. It will also be assumed that the $\mathbb{R}^d$–type inequality:

$$\|u\|_{2^*} \le A^* \|\nabla u\|_2 . \tag{2.4}$$

holds true. This kind of inequality does not hold in general and it is actually equivalent to several other geometric or analytic conditions. To give a sample of such conditions we remind that (2.4) is actually equivalent to the Faber–Krahn inequality:

$$\begin{aligned}&\text{There exist } \Lambda > 0 \text{ such that for any } \Omega \subset\subset M\\ &\lambda_1^M(\Omega) \ge \Lambda \mathrm{Vol}(M)^{-2/d}\end{aligned}$$

where $\lambda_1^M(\Omega)$ is the ground state eigenvalue of the Laplace–Beltrami operator with Dirichlet boundary conditions on $\partial\Omega$. See also [5], [11] for other conditions. It is also well–known that the stronger Sobolev inequality:

$$\|u\|_{\frac{d}{d-1}} \le A_1 \|\nabla u\|_1$$

is equivalent to isoperimetric inequalities, or to diagonal estimates for heat kernels, or to suitable volume growth estimates. For other equivalent conditions valid in the more general subelliptic setting see also [36].

The proof of the following Proposition is identical to the previous one, so it is omitted.

Proposition 2.2. *There exists $c > 0$ such that the logarithmic Sobolev inequality:*

$$\int_M |f|^2 \log\left(\frac{|f|}{\|f\|_2}\right)^2 \mathrm{d}x \le \frac{d}{2}\left[\|f\|_2^2 \log\left(\frac{1}{\varepsilon}\right) + c\varepsilon \|\nabla f\|_2^2\right] \tag{2.5}$$

holds true for any $\varepsilon > 0$, for all $f \in W^{1,2}(M)$, provided M is a manifold for which (2.4) holds. The constant c depends only on the constant appearing in the Sobolev inequality (2.4).

2.3. **Incomplete manifolds.** In this section (M, g) will denote a smooth, connected and compact Riemannian manifold with smooth boundary ∂M. Then the Sobolev inequality:

$$\|u\|_{2^*} \le \overline{A}\, \|\nabla u\|_2 . \tag{2.6}$$

holds true for every $u \in W_0^{1,2}(M)$, with $2^* = \frac{2d}{d-2}$. If we are interested in solutions to the porous media equations corresponding to Dirichlet boundary conditions then we need a LSI for functions in $W_0^{1,2}(M)$, given as follows:

Proposition 2.3. *The logarithmic Sobolev inequality (2.5) holds true for any $\varepsilon > 0$, for all $f \in W_0^{1,2}(M)$.*

If, instead, one considers the solutions corresponding to Neumann boundary conditions the Sobolev inequality to start with is (2.1) and the conclusion is the following:

Proposition 2.4. *The logarithmic Sobolev inequality (2.2) holds true for any $\varepsilon > 0$, for all $f \in W^{1,2}(M)$.*

2.4. **The Young functional.** Let us introduce the following Young functional, defined on the space $[1, +\infty) \times X$, where $X = \bigcap_{p=1}^{\infty} \mathrm{L}^p(M)$, by

$$J(r, u) = \int_M \log\left(\frac{|u(x)|}{\|u\|_r}\right) \frac{|u(x)|^r}{\|u\|_r^r}\, \mathrm{d}x$$

for $(r, u) \in [1, +\infty) \times X$. We also introduce the functional on $[1, +\infty) \times X$:

$$N(r, u) = \log \|u\|_r^r$$

It is well–known that for fixed $u \in X$, N is a convex function of the variable r and thus twice differentiable for a.e. $r \geq 1$, with $N^{'}$ non decreasing and $N^{''}$ positive for a.e. r. Now we state some useful properties of the Young functional which depends on the above facts and on the following well–known Lemma (see [28]):

Lemma 2.5. *Let $u : M \to \mathbb{R}$ be such that $\|u\|_{p,\mu} = \left(\int_M |u(x)|^p \, d\mu(x)\right)^{1/p}$ is finite for some $p > 0$, where μ is a probability measure. Then*

$$\lim_{p \downarrow 0} \|u\|_{p,\mu} = \exp\left(\int_M \log|u(x)| d\mu(x)\right)$$

We shall collect below some properties of J and N needed in the next Section.

Proposition 2.6. *The Young functional satisfies the following properties:*
(a) for a.a. $r \geq 1$

$$J(r,u) = \frac{d}{dr}N(r,u) - \frac{1}{r}N(r,u);$$

(b) for any positive $u \in X$, J satisfies the identity

$$J\left(r, u^{\gamma}\right) = \gamma J(\gamma r, u), \ \forall \gamma > 0 \ s.t. \ \gamma r \geq 1;$$

(c) J satisfies the bound

$$J(r,u) \geq \frac{J(1,u)}{r}, \ \forall r \geq 1; \tag{2.7}$$

(d) for any positive $u \in X$, J satisfies the bound

$$J\left(r, u^{s+h}\right) \geq J\left(r, u^{s}\right) \tag{2.8}$$

for all $r \geq 1$, $h \geq 0$, $s > 0$ such that $rs \geq 1$.

Proof. Property (a) follows by the previous Lemma (2.5) and Property (b) is a direct application of the definition of J. As for (c) we know by (a) that, for fixed $u \in X$, J is a.e. differentiable w.r.t. r. One also has:

$$\frac{\mathrm{d}}{\mathrm{d}r}J(r,u) + \frac{1}{r}J(r,u) = N^{''}(r,u) \geq 0.$$

which gives (c) by integration. To prove (d) we first use (b) and (c):

$$J\left(1, u^{r+h}\right) - J\left(1, u^{r}\right) = \frac{r+h}{r} J\left(\frac{r+h}{r}, u^{r}\right) - J\left(1, u^{r}\right) \geq \frac{r+h}{r}\frac{r}{r+h} J\left(1, u^{r}\right) - J\left(1, u^{r}\right) = 0$$

for positive h. Statement (2.8) finally follows again by using (b). □

Remark 2.7. Let us define

$$\widetilde{J}(r,u) = rJ(r,u) = \int_M \log\left(\frac{|u(x)|^r}{\|u\|_r^r}\right) \frac{|u|^r}{\|u\|_r^r} \, \mathrm{d}x.$$

Part (b) and (d) of the above Proposition then imply

$$\widetilde{J}(r+h,u) \geq \widetilde{J}(r,u)$$

for any $r \geq 1$,$h \geq 0$, and fixed $u \in X$.

3. Manifolds with finite volume

We assume now that M has finite volume and that the initial datum $u_0 \in \mathrm{L}^\infty(M)$. The latter assumption will be removed later on. We will need the following well–known properties of the evolution at hand:

- the mean value of the initial datum is conserved by the evolution, $\overline{u(t)} = \overline{u(0)}$ for all t;
- the evolution considered is a semigroup which leaves each L^p invariant for all $p \in [1, +\infty]$ and moreover the L^p norm decreases along the evolution, $\|u(t)\|_p \le \|u(0)\|_p$ for all p and t;
- the evolution is non–expansive on L^1, $\|u(t) - v(t)\|_1 \le \|u(0) - v(0)\|_1$.

See e.g. [8], [20], [24], [29], [32], [33], [34], [35], whose arguments are valid in the present context as well. Hereafter we shall assume, with no loss of generality, that $\mathrm{Vol}(M) = 1$

Lemma 3.1. *Let u be a weak solution to the problem (1.1) corresponding to an initial datum $u_0 \in L^\infty(M)$, and let $m > 0$, $r \ge 2$, $t \ge 0$. Then*

$$\varphi_r(s) = \int_M |u(s,x)|^r \,\mathrm{d}x = \|u(s)\|_r^r$$

is differentiable in the time variable $s \ge 0$ for a.e. $r \ge 1$ and

$$\dot\varphi_r(s) = \frac{-4r(r-1)m}{\varrho^2} \left\| \nabla \left(|u(s)|^{\varrho/2} \right) \right\|_2^2$$

where $\varrho = r + m - 1$

Proof. We proceed formally by computing

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\varphi_r(s) &= \int_M \frac{\mathrm{d}}{\mathrm{d}s} |u(s,x)|^r \,\mathrm{d}x = \\
&= r \int_M \operatorname{sgn} u(s,x) |u(s,x)|^{r-1} \dot u(s,x) \,\mathrm{d}x = \\
&= -r(r-1)m \int_M |u(s,x)|^{r+m-3} |\nabla u(s,x)|^2 \,\mathrm{d}x = \\
&= \frac{-4r(r-1)m}{(r+m-1)^2} \int_M \left| \nabla \left(|u(s,x)|^{\frac{r+m-1}{2}} \right) \right|^2 \,\mathrm{d}x.
\end{aligned}$$

This can be justified as in [12], Lemma 3.1, with minor modifications on the proof given there for the evolution equation driven by the p–Laplacian. See also [1], pg. 764–766 , [7], pg. 174–175, [27], pg. 257–258 for an alternative approach. □

An immediate consequence of the above result is the following.

Lemma 3.2. *With the same assumptions of Lemma 3.1, let $r : [0, t) \to [q, +\infty]$, $q \ge 2$, be a C^1 nondecreasing function such that $r(0) = q$ and $r(s) \to +\infty$ as $s \uparrow t$, and let $\varrho(s) = r(s) + m - 1$. Then*

$$\begin{aligned}
\frac{d}{ds} \log \|u(s)\|_{r(s)}^{r(s)} = &\frac{-4r(r-1)m}{\varrho(s)^2} \left\| \nabla \left(|u|^{\varrho(s)/2} \right) \right\|_2^2 + \\
&+ \dot r(s) \int_M \log\left(|u(s,x)| \right) |u(s,x)|^{r(s)} \,\mathrm{d}x
\end{aligned}$$

Proof. Let $\varphi(r,s) = \|u(s)\|_r^r$. Then

$$\frac{\mathrm{d}}{\mathrm{d}s}\varphi(r(s),s)) = \varphi_s(r,s)|_{r=r(s)} + \dot r(s)\varphi_r(r,s)|_{r=r(s)}$$

and the thesis follows from the above Lemma. □

Lemma 3.3. *With the same assumptions of Lemma 3.1, then*

$$\frac{d}{ds}\log\|u(s)\|_{r(s)} = \frac{\dot r(s)}{r(s)} J\left(r(s),u(s)\right) - \frac{4m(r(s)-1)}{\varrho(s)^2}\frac{\left\|\nabla\left(|u|^{\varrho(s)/2}\right)\right\|_2^2}{\|u(s)\|_{r(s)}^{r(s)}}$$

where J is the Young functional.

Proof.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} &= -\frac{\dot r(s)}{r(s)}\log\|u(s)\|_{r(s)} + \frac{1}{r(s)\|u(s)\|_{r(s)}^{r(s)}}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)}^{r(s)} \\
&= -\frac{\dot r(s)}{r(s)}\log\|u(s)\|_{r(s)} - \frac{4m(r(s)-1)}{\varrho(s)^2}\frac{\left\|\nabla\left(|u|^{\varrho(s)/2}\right)\right\|_2^2}{\|u(s)\|_{r(s)}^{r(s)}} \\
&+ \frac{\dot r(s)}{r(s)}\frac{1}{\|u(s)\|_{r(s)}^{r(s)}}\int_M \log\left(|u(s,x)|\right)|u(s,x)|^{r(s)}\,\mathrm{d}x \\
&= \frac{\dot r(s)}{r(s)}\int_M \log\left(\frac{|u(s,x)|}{\|u(s)\|_{r(s)}}\right)\frac{|u|^{r(s)}}{\|u\|_{r(s)}^{r(s)}}\,\mathrm{d}x - \frac{4m(r(s)-1)}{\varrho(s)^2}\frac{\left\|\nabla\left(|u(s)|^{\varrho(s)/2}\right)\right\|_2^2}{\|u(s)\|_{r(s)}^{r(s)}}
\end{aligned}$$

Using the definition of J we get the assertion. □

Lemma 3.4. *With the same assumptions of Lemma 3.1, then*

$$\begin{aligned}
\frac{d}{ds}\log\|u(s)\|_{r(s)} \le{}& \frac{\dot r(s)}{r(s)}J\left(r(s),u(s)\right) - \frac{4m}{c_2\varepsilon}\frac{r(s)-1}{\varrho(s)^2}\frac{\|u(s)\|_{\varrho(s)}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}\left[\frac{2}{d}J\left(1,|u(s)|^{\varrho(s)}\right)+\log(\varepsilon)\right] \\
&+ \frac{4mc_1}{c_2}\frac{r(s)-1}{\varrho(s)^2}\frac{\|u(s)\|_{\varrho(s)/2}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}
\end{aligned} \tag{3.1}$$

where c_1, c_2 are the constants appearing in the logarithmic Sobolev inequality (2.2)

Proof. First we rewrite the logarithmic Sobolev inequality as follows

$$\left\|\nabla\left(|u(t)|^{\frac{\varrho(s)}{2}}\right)\right\|_2^2 \ge \frac{2\left\||u(t)|^{\frac{\varrho(s)}{2}}\right\|_2^2}{c_2 d\varepsilon}\left[J\left(1,|u(t)|^{\varrho(s)}\right)+\frac{d}{2}\log(\varepsilon)\right] - \frac{c_1\left|\overline{|u(t)|^{\frac{\varrho(s)}{2}}}\right|^2}{c_2}$$

since $\left\||u(t,x)|^{\varrho(s)/2}\right\|_2^2 = \|u(t,x)\|_{\varrho(s)}^{\varrho(s)}$ and $\left|\overline{|u(t)|^{\frac{\varrho(s)}{2}}}\right|^2 = \|u(s)\|_{\varrho(s)/2}^{\varrho(s)}$.
Now apply this inequality to the one of the previous Lemma to get the thesis. □

Lemma 3.5. *Under the running assumptions, one has, setting $m_0 = 1 \vee (m-1)$:*

$$\begin{aligned}\frac{d}{ds}\log\|u(s)\|_{r(s)} \le \frac{\dot r(s)}{r(s)}\frac{d(m-1)}{2\varrho(s)}\log\|u(s)\|_{r(s)} + \frac{\dot r(s)}{r(s)}\frac{d}{2\varrho(s)}\log\left(\frac{8mr(s)(r(s)-1)}{c_2 d\dot r(s)\varrho(s)}\right)\\ + \frac{4mc_1\|u_0\|_{m_0}^{m-1}}{c_2(q+m-1)}.\end{aligned}$$

Proof. We shall use the interpolation inequality

$$\|u\|_{\varrho/2} \le \|u\|_{m-1}^{(m-1)/\varrho}\|u\|_r^{r/\varrho},$$

valid whenever $r \ge m-1$. Therefore $\|u\|_{\varrho/2}^{\varrho}/\|u\|_r^r \le \|u\|_{m-1}^{m-1} \le \|u\|_{m_0}^{m-1}$.

By the above mentioned contraction property $\|u(t)\|_r \le \|u(s)\|_r$ if $t \ge s$ and $r \in [1,+\infty]$ we then get that

$$\frac{\|u(s)\|_{\varrho(s)/2}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}} \le \|u_0\|_{m_0}^{m-1}.$$

We can rewrite inequality (3.1) of the previous Lemma as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} \le & \frac{\dot r(s)}{r(s)}J\left(r(s),u(s)\right) - \frac{8m(r(s)-1)}{\varepsilon d c_2\varrho(s)}\frac{\|u(s)\|_{\varrho(s)}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}\left[J\left(\varrho(s),u(s)\right) + \frac{d}{2\varrho(s)}\log(\varepsilon)\right]\\ & + \frac{4mc_1}{c_2}\frac{r(s)-1}{\varrho(s)^2}\|u_0\|_{m_0}^{m-1}\end{aligned}$$

since $J\left(1,u^\alpha\right) = \alpha J(\alpha,u)$. Now choose

$$\varepsilon = \frac{r(s)(r(s)-1)}{\dot r(s)\varrho(s)}\frac{8m}{dc_2}\frac{\|u(s)\|_{\varrho(s)}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}$$

so the previous inequality becomes:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} \le & \frac{\dot r(s)}{r(s)}\left[J\left(r(s),u(s)\right) - J\left(\varrho(s),u(s)\right) - \frac{d}{2\varrho(s)}\log\frac{\|u(s)\|_{\varrho(s)}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}\right]\\ & - \frac{\dot r(s)}{r(s)}\frac{d}{2\varrho(s)}\log\left(\frac{8mr(s)(r(s)-1)}{c_2 d\dot r(s)\varrho(s)}\right) + \frac{4mc_1}{c_2}\frac{r(s)-1}{\varrho(s)^2}\|u_0\|_{m_0}^{m-1}\\ \le & \frac{\dot r(s)}{r(s)}\left[\log\frac{\|u(s)\|_{\varrho(s)}}{\|u(s)\|_{r(s)}} - \frac{d}{2\varrho(s)}\log\frac{\|u(s)\|_{\varrho(s)}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}\right]\\ & - \frac{\dot r(s)}{r(s)}\frac{d}{2\varrho(s)}\log\left(\frac{8mr(s)(r(s)-1)}{c_2 d\dot r(s)\varrho(s)}\right) + \frac{4mc_1}{c_2(q+m-1)}\|u_0\|_{m_0}^{m-1} =\\ \le & \frac{\dot r(s)}{r(s)}\left[\left(1-\frac{d}{2}\right)\log\|u(s)\|_{\varrho(s)} + \left(\frac{dr(s)}{2\varrho(s)}-1\right)\log\|u(s)\|_{r(s)}\right] +\\ & - \frac{\dot r(s)}{r(s)}\frac{d}{2\varrho(s)}\log\left(\frac{8mr(s)(r(s)-1)}{c_2 d\dot r(s)\varrho(s)}\right) + \frac{4mc_1}{c_2(q+m-1)}\|u_0\|_{m_0}^{m-1}\end{aligned}$$

since

$$J(r(s),u) - J(\varrho(s)) - \log\frac{\|u(s)\|_{\varrho(s)}}{\|u(s)\|_{r(s)}} = N^{'}(r,u) - N^{'}(\varrho,u) \le 0$$

because $\varrho(s) = r(s) + m - 1 \geq r(s)$ since $m > 1$ and N' is non decreasing with respect to r. We have also used the fact that $(r(s)-1)/\varrho(s)^2 \leq 1/(q+m-1)$.
Notice now that Hölder inequality implies

$$\|u(s)\|_{\varrho(s)} \geq \|u(s)\|_{r(s)}$$

since $\mathrm{Vol}(M) = 1$, and

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s} \log \|u(s)\|_{r(s)} \leq& \frac{\dot{r}(s)}{r(s)} \left[\left(1 - \frac{d}{2}\right) \log \|u(s)\|_{r(s)} + \left(\frac{dr(s)}{2\varrho(s)} - 1\right) \log \|u(s)\|_{r(s)} \right] \\ &- \frac{\dot{r}(s)}{r(s)} \frac{d}{2\varrho(s)} \log \left(\frac{8mr(s)(r(s)-1)}{c_2 d\dot{r}(s)\varrho(s)} \right) + \frac{4mc_1}{c_2(q+m-1)} \|u_0\|_{m_0}^{m-1} \\ =& -\frac{\dot{r}(s)}{r(s)} \frac{d}{2} \left(1 - \frac{r(s)}{\varrho(s)}\right) \log \|u(s)\|_{r(s)} - \frac{\dot{r}(s)}{r(s)} \frac{d}{2\varrho(s)} \log \left(\frac{8mr(s)(r(s)-1)}{c_2 d\dot{r}(s)\varrho(s)} \right) \\ &+ \frac{4mc_1}{c_2(q+m-1)} \|u_0\|_{m_0}^{m-1}\end{aligned}$$

This concludes the proof. □

The previous Lemma gives a closed differential inequality for $\log \|u(s)\|_{r(s)}$ which we rephrase hereafter.

Proposition 3.6. *Let u be a weak solution to (1.1) corresponding to an essentially bounded initial datum $u_0 \in L^\infty(M)$, with $\|u_0\|_\infty \leq 1$. Let $r : [0, t) \to [q, +\infty)$ be a C^1 non decreasing function such that $r(0) = q$ and $r(s) \to +\infty$ as $s \uparrow t$ and let $\varrho(s) = r(s) + m - 1$. If we let*

$$\begin{aligned} y(s) =& \log \|u(s)\|_{r(s)} \\ p(s) =& \frac{\dot{r}(s)}{r(s)} \frac{d(m-1)}{2\varrho(s)} \\ q(s) =& \frac{\dot{r}(s)}{r(s)} \frac{d}{2\varrho(s)} \log \left(\frac{8mr(s)(r(s)-1)}{c_2 d\dot{r}(s)\varrho(s)} \right) + \frac{4mc_1}{c_2(q+m-1)} \|u_0\|_{m_0}^{m-1}. \end{aligned}$$

Then the following differential inequality holds true $\forall s \geq 0$:

$$\frac{\mathrm{d}y\,(s)}{\mathrm{d}s} + p\,(s)\,y\,(s) + q\,(s) \leq 0$$

so that $y\,(s) \leq y_L\,(s)$, provided $y\,(0) \leq y_L\,(0)$, where

$$y_L\,(s) = \exp\left(-\int_0^s p\,(\lambda)\,d\lambda \right) \left[y_L\,(0) - \int_0^s q\,(\lambda) \exp\left(\int_0^\lambda p\,(\eta)\,d\eta \right) d\lambda \right]$$

is a solution of the ordinary differential equation

$$\frac{\mathrm{d}y\,(s)}{\mathrm{d}s} + p\,(s)\,y\,(s) + q\,(s) = 0$$

Lemma 3.7. *Let us fix $t > 0$. Then the solution y_L to the ordinary differential equation of the previous Proposition, with the choices*

$$r\,(s) = \frac{qt}{t-s}, \quad y_L(0) = y(0) = \log \|u_0\|_q$$

satisfies :

$$\omega\,(t) = \lim_{s \uparrow t^-} e^{y_L(s)} \leq C_0 \frac{e^{E_0 \|u_0\|_{m_0}^{m-1} t} \|u_0\|_q^\gamma}{t^\alpha}$$

where C_0, E_0 depend on $m, q, d\, M$ and on the Sobolev constant appearing in (1.3), and moreover

$$\alpha = \frac{1}{m-1}\left[1-\left(\frac{q}{q+m-1}\right)^{d/2}\right], \qquad \gamma = \left(\frac{q}{q+m-1}\right)^{d/2}.$$

Proof. With the running choice of $r(s)$ we want to calculate

$$y_L\left(s\right) = \exp\left(-\int_0^s p\left(\lambda\right) d\lambda\right)\left[y_L\left(0\right) - \int_0^s q\left(\lambda\right)\exp\left(\int_0^\lambda p\left(\eta\right) d\eta\right) d\lambda\right]$$

Thus we have:

$$P(s) = \int_0^s p(s)\mathrm{d}s = \int_0^s \frac{d(m-1)}{2r(\lambda)(r(\lambda)+m-1)}\dot{r}(\lambda)\mathrm{d}\lambda = \frac{d}{2}\left[\log\left(\frac{r(s)}{\varrho(s)}\right) + \log\left(\frac{q+m-1}{q}\right)\right],$$

$$\mathrm{e}^{-P(t)} = \lim_{s\uparrow t}\mathrm{e}^{-P(s)} = \left(\frac{q}{q+m-1}\right)^{d/2}$$

Now let

$$Q(t) = \lim_{s\uparrow t}\int_0^s q(\lambda)\mathrm{e}^{P(\lambda)}\mathrm{d}\lambda = \sum_{i=1}^{3} Q_i(t)$$

where Q_1 is the contribution to the integral which corresponds to the term $q_1(s) = \frac{d\dot{r}(s)}{2\varrho(s)r(s)}\log\left(\frac{8m}{c_2 d}\right)$, $Q_2(s)$ is the contribution corresponding to the term $q_2(s) = \frac{d\dot{r}(s)}{2\varrho(s)r(s)}\log\left(\frac{r(s)(r(s)-1)}{\dot{r}(s)\varrho(s)}\right)$, $Q_3(s)$ is the contribution corresponding to the term $q_3(s) = \frac{4mc_1}{c_2(q+m-1)}\|u_0\|_{m_0}^{m-1}$. We then compute:

$$\begin{aligned} Q_1(t) =& \lim_{s\uparrow t} Q_1(s) = \lim_{s\uparrow t}\frac{d}{2}\left(\frac{q+m-1}{q}\right)^{d/2}\log\left(\frac{8m}{c_2 d}\right)\int_0^s \frac{1}{\varrho(\lambda)}\left(\frac{r(\lambda)}{\varrho(\lambda)}\right)^{d/2}\frac{\dot{r}(\lambda)}{r(\lambda)}\mathrm{d}\lambda = \\ &= I_1(m,q,d)\left(\frac{q+m-1}{q}\right)^{d/2}\frac{d}{2}\log\left(\frac{8m}{c_2 d}\right) \end{aligned}$$

for a suitable $I_1(m,q,d)$.

$$\begin{aligned} Q_2(t) =& \lim_{s\uparrow t} Q_2(s) = \lim_{s\uparrow t}\frac{d}{2}\left(\frac{q+m-1}{q}\right)^{d/2}\int_0^s \frac{1}{\varrho(\lambda)}\times \\ &\times\left(\frac{r(\lambda)}{\varrho(\lambda)}\right)^{d/2}\frac{\dot{r}(\lambda)}{r(\lambda)}\left[\log\left(\frac{1}{qt}\right) + \log\left(\frac{r(\lambda)^2(r(\lambda)-1)}{\varrho(\lambda)}\right)\right]\mathrm{d}\lambda = \\ &= -\, I_2^{'}(m,q,d)\frac{d}{2}\left(\frac{q+m-1}{q}\right)^{d/2}\log(qt) + I_2^{''}(m,q,d)\frac{d}{2}\left(\frac{q+m-1}{q}\right)^{d/2} \end{aligned}$$

for suitable $I_2^{'}(m,q,d)$, $I_2^{''}(m,q,d)$. To give a bound for $Q_3(t)$ first notice that $\left(\frac{r(\lambda)}{\varrho(\lambda)}\right)^{d/2} \le 1$ so that

$$\begin{aligned} Q_3(t) = \lim_{s\uparrow t} Q_3(s) =& -\lim_{s\uparrow t}\frac{4mc_1}{c_2(q+m-1)}\left(\frac{q+m-1}{q}\right)^{d/2}\|u_0\|_{m_0}^{m-1}\int_0^s\left(\frac{r(\lambda)}{\varrho(\lambda)}\right)^{d/2}\mathrm{d}\lambda \ge \\ &\ge -\frac{4mc_1}{c_2(q+m-1)}\left(\frac{q+m-1}{q}\right)^{d/2}\|u_0\|_{m_0}^{m-1}t \end{aligned}$$

This easily implies the stated bounds. □

Proof of Theorem 1.1. We let $y(s), y_L(s), \omega(s), C_0, E_0$ have the meaning of the previous Lemma, and notice that the aforementioned contraction property in L^p for the evolution implies that

$$\|u(t)\|_{r(s)} \le \|u(s)\|_{r(s)} = \exp\left(\log \|u(s)\|_{r(s)}\right) = \mathrm{e}^{y(s)} \le \mathrm{e}^{y_L(s)}.$$

Now recalling that $r(s) \to +\infty$ as $s \uparrow t$, we deduce

$$\|u(t)\|_\infty = \lim_{s\uparrow t} \|u(t)\|_{r(s)} \le \lim_{s\uparrow t} \|u(s)\|_{r(s)} = \lim_{s\uparrow t} \mathrm{e}^{y(s)} \le \lim_{s\uparrow t} \mathrm{e}^{y_L(s)} = \omega(t).$$

we have thus proved, for any positive t:

$$\|u(t)\|_\infty \le \omega(t) \le C_0 \frac{e^{E_0\|u_0\|_{m_0}^{m-1} t}}{t^\alpha} \|u_0\|_q^\gamma. \tag{3.2}$$

We now remove the requirement that the initial datum u_0 belong to $\mathrm{L}^\infty(M)$. To this end, given $u_0 \in \mathrm{L}^q(M)$, with $q \ge m_0$, take a sequence $\{v_n\} \subset \mathrm{L}^\infty(M)$, converging in L^q (and hence in L^{m-1} too) to u_0. Let $v_n(t)$ be the solution to the evolution equation at hand, corresponding to the data $v_n \in \mathrm{L}^\infty$. It follows by the small time estimate obtained above that, for $t \in (0,1]$:

$$\|v_n(t)\|_\infty \le C(t) e^{E_0\|v_n\|_{m_0}^{m-1}} \|v_n\|_q^\gamma$$

where $C(t) = C_0/t^\alpha$. This proves that the sequence $v_n(t)$ is bounded in $\mathrm{L}^\infty(M)$ for any fixed $t \in (0,1]$. Possibly by passing to a subsequence, we can assume that such sequence converges, in the weak* topology of $\mathrm{L}^\infty(M)$, to a function $U(t) \in \mathrm{L}^\infty(M)$, which thus satisfies, by the weak* lower semicontinuity of the $\mathrm{L}^\infty(M)$ norm, the bound:

$$\|U(t)\|_\infty \le C(t)\|u_0\|_q^\gamma e^{E_0\|u_0\|_{m_0}^{m-1}}$$

Now we want to identify the weak* limit $U(t)$, with the solution $u(t)$ corresponding to the datum $u_0 \in \mathrm{L}^q(M)$. To this end we use the above mentioned L^1 contraction property:

$$\|v_n(t) - u(t)\|_1 \le \|v_n - u_0\|_1$$

to conclude that $v_n(t) \to u(t)$ in L^1 since u_n converges to u_0 in $\mathrm{L}^q(M)$, and thus in $\mathrm{L}^1(M)$, so that $U(t) = u(t)$ for $t \in (0,1]$.

Proof of Corollary 1.3. We first prove an L^r–L^r time decay for functions with zero mean value and assuming that the data are essentially bounded (this can be removed later as done above). Suppose $r \ge 2$ and let $\varrho = r + m - 1$ as above. Lemma 3.1 then shows that

$$\frac{\mathrm{d}}{dt}\|u(t)\|_r^r = \frac{-4r(r-1)m}{\varrho^2} \left\|\nabla\left(|u(t)|^{\varrho/2}\right)\right\|_2^2.$$

Lemma 3.2 of [1], valid since u has zero mean, and Hölder inequality then imply, B denoting a positive constant which depends on r, m, d:

$$\frac{\mathrm{d}}{dt}\|u(t)\|_r^r \le -B\|u(t)\|_\varrho^\varrho \le -B\|u(t)\|_r^{r[\varrho/r]}$$

since $\varrho d/(d-2) > r$. We have obtained a closed differential inequality for $\varphi(t) = \|u(t)\|_r^r$, namely:

$$\dot\varphi(t) \le -B\varphi(t)^{\varrho/r}$$

for a suitable C. This yields the bound:

$$\|u(t)\|_r \le \frac{1}{\left(Bt + \|u_0\|_r^{-(m-1)}\right)^{1/(m-1)}} \tag{3.3}$$

which gives immediately (1.6). We now show that the evolution is L^2–L^a regularizing for the class of data at hand and any $a \in [2, +\infty)$, this being of course relevant only when $m > 3$. Indeed, again first taking bounded data and then approximating, and using the interpolation inequality:

$$\|u\|_r \le \|u\|_2^{1-\lambda} \|u\|_\varrho^\lambda, \qquad \lambda = \frac{(r+m-1)(r-2)}{r(r+m-3)}$$

we find, using the previous calculation and the fact that $\|u(t)\|_2$ decreases:

$$\begin{aligned}\frac{\mathrm{d}}{dt}\|u(t)\|_r^r &\le -B\|u(t)\|_\varrho^\varrho \\ &\le -B\frac{(\|u(t)\|_r^r)^{(r+m-3)/(r-2)}}{\|u_0\|_2^{(m-1)/(r-2)}}.\end{aligned}$$

Integration of the above inequality yields

$$\|u(t)\|_r \le K\frac{\|u_0\|_2^{(r-2)/[r(m-1)]}}{t^{(r-2)/[r(m-1)]}}$$

thus showing the claimed regularization. Finally the claim:

$$\|u(t)\|_\infty \le C\mathrm{e}^{E_0\|u(t-1)\|_{m_0}^{m-1}}\|u(t-1)\|_q^\gamma \le \frac{C}{(B(t-1)+\|u_0\|_q^{-(m-1)})^{\gamma/(m-1)}}$$

follows by using first the L^q–L^∞ smoothing property for small times and the expression of γ given in Lemma 3.7, the semigroup property next together with the above L^1–L^{m_0} regularizing property and with the absolute bound for $\|u(t-1)\|_{m_0}$. The other assertions follows from the above together with the numerical inequality

$$a + b \ge a^\varepsilon b^{1-\varepsilon}$$

valid for any $a, b \ge 0$ and for any $\varepsilon \in (0, 1)$.

Proof of Corollary 1.4. As for solutions with non–zero mean we shall first prove that they converge as $t \to \infty$ to $\overline{u_0}$ in the L^∞ norm. In fact assume first that $u_0 \in L^\infty$ and that the mean value of the datum is strictly positive. Then convergence in all L^p norms with $p \neq \infty$ can be shown exactly as in [1], Th. 1.4.

To prove that convergence also takes place strongly in L^∞ we first notice now that, adapting the arguments of [19], it follows that the solutions of the equation considered are spatially Hölder continuous at each time $t > 0$, the proportionality constants and the Hölder exponents not depending on $t \ge 1$. In fact one needs the validity of condition (1.7) of that paper, which does indeed hold in the present setting. In fact one has to control, uniformly in $t > 1$, the quantities $\|u(t)\|_2$ and $\|u\|_{L^2(M\times(1,t))}$: the first one decreases in time, while the other one is uniformly bounded in time as a consequence, e.g., of [8], proof of Prop. 1 and Lemma 3 (see also [30]). If one assumes that convergence does *not* take place strongly in L^∞ then there is a sequence $t_n \to +\infty$ such that $|u(x) - \overline{u_0}| \ge c$ for a suitable fixed c, all n sufficiently large and all x in a set K_n of non–zero measure, depending on n. L^p convergence shows that the measure of K_n must tend to zero as $n \to +\infty$. This fact and the Hölder continuity of each function $u(t)$ easily yields a contradiction. Thus convergence also takes place strongly in L^∞.

We use this fact to notice that the strategy of proof of Theorem 1.1 can be adapted to obtain bounds on the quantity $\|u(s) - \overline{u_0}\|_{r(s)}$. Since the above result implies that, for sufficiently large time,

$|u(s)| \geq |u(s) - \overline{u_0}|$ one first has for any $r \geq 2$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(s) - \overline{u}_0\|_r^r &= -r(r-1)m \int_M |u(s,x) - \overline{u_0}|^{r-2}|u(s,x)|^{m-1}|\nabla u(s,x)|^2\,\mathrm{d}x \\ &\leq -r(r-1)m \int_M |u(s,x) - \overline{u_0}|^{r+m-3}|\nabla u(s,x)|^2\,\mathrm{d}x \\ &= \frac{-4r(r-1)m}{(r+m-1)^2} \int_M \left|\nabla \left(|u(s,x) - \overline{u_0}|^{\frac{r+m-1}{2}}\right)\right|^2 \mathrm{d}x\end{aligned} \tag{3.4}$$

so that in particular the quantity $\|u(s) - \overline{u}_0\|_r$, for any $r \geq 2$, decreases in time. This inequality for $\|u(s) - \overline{u}_0\|_r^r$ is exactly the same given in Lemma 3.1 for $\|u(s)\|_r^r$. One can proceed exactly as in the sequel of Section 3 to prove an estimate of the form

$$\|u(t) - \overline{u_0}\|_\infty \leq \frac{C}{(t-S)^\alpha}\|u(S) - \overline{u_0}\|_2^\gamma e^{E_0\|u(S)-\overline{u_0}\|_{m_0}^{m-1}(t-S)}.$$

valid for $t > S \geq T(u_0)$ sufficiently large and for a suitable function A. In fact, although it is obviously necessary in the proof of the above results to require that the initial datum belongs to L^{m_0}, one may anyway choose $r(s) = qt/(t-s)$ with no restriction on q, and in fact we choose here $q = 2$. The values of γ and α are then those given in Lemma 3.7 with the choice $q = 2$. Although in the statement of Theorem 1.1 it was required that $t < 1$, the conclusion there has in fact been proved for any time, although it was not interesting for t large.

Another consequence of inequality (3.4) is the fact that, applying again Lemma 3.2 of [1], valid since $u(s) - \overline{u_0}$ has zero mean, one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(s) - \overline{u}_0\|_r^r \leq C\|u(s) - \overline{u}_0\|_\varrho^\varrho$$

which can be dealt with as in the previous Corollary, showing that

$$\|u(t) - \overline{u_0}\|_r \leq \frac{1}{\left(Bt + \|u_0 - \overline{u_0}\|_r^{-(m-1)}\right)^{1/(m-1)}}$$

and the absolute bound $\|u(t) - \overline{u_0}\|_r \leq B/t^{1/(m-1)}$ as well, both for $t > T$.

Again for $t \geq T(u_0)$, one can assume that $u(x,t) \geq \overline{u}_0/2$. Therefore:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t) - \overline{u}_0\|_2^2 &\leq -2m\left[\frac{\overline{u}_0}{2}\right]^{m-1} \int_M |\nabla u(t,x)|^2\,\mathrm{d}x \\ &\leq -2m\lambda_1\left[\frac{\overline{u}_0}{2}\right]^{m-1} \int_M |u(t,x) - \overline{u}_0|^2\,\mathrm{d}x\end{aligned}$$

by the Poincarè inequality. This gives us a closed differential inequality for the function

$$f(t) = \|u(t) - u_0\|_2^2,$$

namely $\dot{f}(t) \leq -2\sigma f(t)$, for any $t \geq T(u_0)$. Thus we get:

$$\|u(t) - \overline{u_0}\|_2 \leq \mathrm{e}^{2\sigma(T-t)}\|u(T) - \overline{u_0}\|_2 \tag{3.5}$$

By the semigroup property we then get for $t \geq 2T$:

$$\begin{aligned}\|u(t)-\overline{u_0}\|_\infty &\leq \frac{C}{t^\alpha}\|u(t/2)-\overline{u_0}\|_2^\gamma e^{E_0 t\|u(t/2)-\overline{u_0}\|_{m_0}^{m-1}/2} \\ &\leq \frac{C}{t^\alpha}\|u(t/2)-\overline{u_0}\|_2^\gamma \\ &\leq \frac{Ce^{2\sigma(T-(t/2))}}{t^\alpha}\|u(T)-\overline{u_0}\|_2^\gamma \\ &\leq Ce^{-\sigma t}\|u_0-\overline{u_0}\|_2^\gamma\end{aligned}$$

where we used the absolute bound in the second step.

The assumption $u_0 \in \mathrm{L}^\infty$ can be removed using the L^2–L^∞ regularizing property of the evolution, so that this concludes the proof in the case $\overline{u_0} > 0$. Finally, if $\overline{u_0} < 0$ we notice that if $u(t,x)$ is the solution corresponding to the initial datum $u_0(x)$ then $-u(t,x)$ is the solution corresponding to the initial datum $-u_0(x)$.

Remark 3.8. One should also comment that in [1] it is shown how to prove the stated absolute bound also in the L^∞ norm if the data are L^∞ as well.

Remark 3.9. The calculations for incomplete manifolds with Neumann boundary conditions are identical to the previous ones since the Sobolev inequality one starts with is the same and since the solutions are Hölder continuous up to the boundary due to the homogeneous Neumann condition. We have therefore proved Theorem 1.8 as well.

4. Manifolds with infinite volume

We prove now Theorem 1.5, using the appropriate LSI given in Section 2. Their form allows us to treat both the small time and the large time case at the same time, and a key ingredient will be the monotonicity property (2.8) of the Young functional. The first Lemmata of Section 3 do not depend on the assumption $\mathrm{Vol}\, M < +\infty$, so we start with an analogue of Lemma 3.4:

Lemma 4.1. *Let $u \in X = \bigcap_{p\geq 1} L^p(M)$ be a weak solution to (1.1) corresponding to an essentially bounded initial datum u_0. Let $m > 1$ (actually $m > 0$ suffices here), and let $r : [0,t) \to [q,+\infty]$, $q \geq 2$, be a C^1 non decreasing function, such that $r(0) = q$ and $r(s) \to +\infty$ as $s \uparrow t$ and let $\varrho(s) = r(s)+m-1$, then $\forall \varepsilon > 0$:*

$$\frac{d}{ds}\log\|u(s)\|_{r(s)} \leq \frac{\dot r(s)}{r(s)} J\left(r(s),u(s)\right) - \frac{4m}{\varepsilon c}\frac{r(s)-1}{\varrho(s)^2}\frac{\|u(s)\|_{\varrho(s)}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}\left[\frac{2}{d}J(1,|u(s)|^{\varrho(s)}) + \log\varepsilon\right]. \tag{4.1}$$

The proof is similar to the one given in the previous Section, provided one uses the appropriate Sobolev inequality.

Lemma 4.2. *Under the assumption of Lemma 4.1 one has:*

$$\begin{aligned}\frac{d}{ds}\log\|u(s)\|_{r(s)} \leq &-\frac{\dot r(s)}{r(s)}\frac{d(m-1)}{2r(s)+d(m-1)}\log\|u(s)\|_{r(s)} + \\ &-\frac{\dot r(s)}{r(s)}\frac{d}{2r(s)+d(m-1)}\left[\log\left(\frac{r(s)(r(s)-1)(2r(s)+d(m-1))}{\dot r(s)\varrho(s)^2}\right) + \log\left(\frac{4m}{cd}\right)\right]\end{aligned} \tag{4.2}$$

Proof. Choose $\varepsilon > 0$ in the inequality (4.1) of previous Lemma as follows:

$$\varepsilon = \frac{r(s)^2}{\dot r(s)}\frac{2r(s)+d(m-1)}{2r(s)}\frac{r(s)-1}{\varrho(s)^2}\frac{8m}{dc}\frac{\|u(s)\|_{\varrho(s)}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}$$

And thus obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} \le{}& \frac{\dot r(s)}{r(s)^2}\left[J\left(1,u(s)^{r(s)}\right)-\frac{2r(s)}{2r(s)+d(m-1)}J\left(1,u(s)^{\varrho(s)}\right)\right]\\ &-\frac{\dot r(s)}{r(s)^2}\frac{d}{2}\frac{2r(s)}{2r(s)+d(m-1)}\log\frac{\|u(s)\|_{\varrho(s)}^{\varrho(s)}}{\|u(s)\|_{r(s)}^{r(s)}}\\ &-\frac{\dot r(s)}{r(s)^2}\frac{d}{2}\frac{2r(s)}{2r(s)+d(m-1)}\log\left(\frac{r(s)^2}{\dot r(s)}\frac{2r(s)+d(m-1)}{2r(s)}\frac{r(s)-1}{\varrho(s)^2}\frac{8m}{cd}\right)\end{aligned}$$

Now we prove the inequality:

$$\log\frac{\|u\|_\varrho^\varrho}{\|u\|_r^r}=\log\frac{\|u\|_{r+m-1}^{r+m-1}}{\|u\|_r^r}\ge(m-1)\left(J(r,u)+\log\|u\|_r\right). \tag{4.3}$$

This in fact a consequence of the the fact that, with the notations of Proposition 2.6, N is a convex function whose derivative $N'(r)$ coincides with $J(r,u)+N(r)/r$ so that, since $m>1$:

$$N(r+m-1)\ge N(r)+(m-1)\left[J(r,u)+\frac{N(r)}{r}\right],$$

which is in fact (4.3). Hence:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)}\le\frac{\dot r(s)}{r(s)^2}\frac{d(m-1)}{2}\log\|u(s)\|_{r(s)}\\ &-\frac{\dot r(s)}{r(s)^2}\frac{d}{2}\frac{2r(s)}{2r(s)+d(m-1)}\left(J\left(1,u^{r(s)}\right)-J\left(1,u^{\varrho(s)}\right)\right)-\frac{\dot r(s)}{r(s)}\frac{d}{2r(s)+d(m-1)}\\ &\times\left[\log\left(\frac{r(s)(r(s)-1)(2r(s)+d(m-1))}{\dot r(s)\varrho(s)^2}\right)+\log\left(\frac{4m}{cd}\right)\right]\end{aligned}$$

Now we use the monotonicity property of J

$$J\left(1,u^{r(s)}\right)-J\left(1,u^{\varrho(s)}\right)\le 0$$

to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)}\le{}&\frac{\dot r(s)}{r(s)^2}\frac{d(m-1)}{2}\log\|u(s)\|_{r(s)}-\frac{\dot r(s)}{r(s)}\frac{d}{2r(s)+d(m-1)}\\ \times&\left[\log\left(\frac{r(s)(r(s)-1)(2r(s)+d(m-1))}{\dot r(s)\varrho(s)^2}\right)+\log\left(\frac{4m}{cd}\right)\right]\end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.5. We have proved that

$$\frac{\mathrm{d}y\,(s)}{\mathrm{d}s}+p\,(s)\,y\,(s)+q\,(s)\le 0$$

which holds true $\forall s\ge 0$, provided

$$\begin{aligned}y(s)={}&\log\|u(s)\|_{r(s)}\\ p(s)={}&\frac{\dot r(s)}{r(s)}\frac{d(m-1)}{2r(s)+d(m-1)}\\ q(s)={}&\frac{\dot r(s)}{r(s)}\frac{d}{2r(s)+d(m-1)}\left[\log\left(\frac{r(s)(r(s)-1)(2r(s)+d(m-1))}{\dot r(s)\varrho(s)^2}\right)+\log\left(\frac{4m}{cd}\right)\right]\end{aligned}$$

Thus $y(s) \le y_L(s)$, provided $y(0) \le y_L(0)$, where again

$$y_L(s) = \exp\left(-\int_0^s p(\lambda)\,d\lambda\right)\left[y_L(0) - \int_0^s q(\lambda)\exp\left(\int_0^\lambda p(\eta)\,d\eta\right)d\lambda\right]$$

is a solution of the ordinary differential equation:

$$\frac{\mathrm{d}y(s)}{\mathrm{d}s} + p(s)\,y(s) + q(s) = 0$$

so that

$$y(t) = \lim_{s\uparrow t} y(s) \le y_L(t) = \lim_{s\uparrow t} y_L(s) = \lim_{s\uparrow t} \mathrm{e}^{-P(s)}\left[y_L(0) - Q(s)\right].$$

Choose now $r(s) = qt/(t-s)$ and compute, as in Lemma 3.7:

$$\begin{aligned}
\mathrm{e}^{P(s)} &= \frac{2r(s)+d(m-1)}{r(s)}\frac{q}{2q+d(m-1)}\\
\mathrm{e}^{-P(t)} &= \lim_{s\uparrow t}\mathrm{e}^{-P(s)} = \frac{2q}{2q+d(m-1)}\\
Q(t) &= \lim_{s\uparrow t}\int_0^s q(\lambda)\mathrm{e}^{P(\lambda)}\mathrm{d}\lambda =\\
&= \lim_{s\uparrow t}\left[\frac{d(2q+d(m-1))}{2q}\log\left(\frac{4m}{dc}\right)\left(\frac{1}{2q+d(m-1)} - \frac{1}{2r(s)+d(m-1)}\right)\right.\\
&+ \int_{r(0)}^{r(s)}\frac{d(2q+d(m-1))}{q(2\lambda+d(m-1))^2}\log\left(\frac{2\lambda+d(m-1)}{(\lambda+m-1)^2}\lambda(\lambda-1)\right)\mathrm{d}\lambda + \frac{d}{2q}\log(t) + \frac{d}{2q}\log(q)\\
&\left. -2d(2q+d(m-1))\int_{(t-s)/t}^1\frac{\log(\eta)}{(2q+d(m-1)\eta)^2}\mathrm{d}\eta\right]
\end{aligned}$$

Recalling that $r(s) \to +\infty$ as $s \uparrow t$ we get:

$$y_L(t) = \frac{2q}{2q+d(m-1)}y_L(0) - \frac{d}{2q+d(m-1)}\log(t) + R$$

where R is a numerical constant depending no d, q, m, c. As in the previous Section we have:

$$\begin{aligned}
\log\|u(t)\|_\infty &= \lim_{s\uparrow t}\log\|u(t)\|_{r(s)} \le \lim_{s\uparrow t}\log\|u(s)\|_{r(s)} =\\
&= \lim_{s\uparrow t} y(s) \le \lim_{s\uparrow t} y_L(s) = y_L(t)
\end{aligned}$$

so that letting $y_L(0) = \log\|u(0)\|_q = y(0)$ one obtains:

$$\|u(t)\|_\infty \le \mathrm{e}^{y_L(t)} = \frac{\mathrm{e}^R}{t^\alpha}\|u(0)\|_q^\gamma$$

Provided α and γ are as in the statement. The assumption $u_0 \in \mathrm{L}^\infty$ can be removed exactly as in Section 3. □

Remark 4.3. The method of proof given above allows to prove also the statement of Theorem 1.7 for short times since the Sobolev inequality one starts with is the same. For the long time asymptotics one can however proceed exactly as in the proof of Theorem 1.1 when dealing with the L^r–L^r decay. The same arguments given there then allow to conclude the proof of Theorem 1.7.

Remark 4.4. It is possible to adapt several of the above arguments to prove similar results for the problem

$$\begin{cases} \dot{u} = \triangle\phi(u) \\ u(0,\cdot) = u_0 \in L^q(M) \end{cases} \tag{4.4}$$

provided

$$\begin{aligned} \phi(0) =& 0 \\ \phi'(s) \geq& P_\phi |s|^{m-1}, \quad \text{with } P_\phi > 0,\ m > 1. \end{aligned}$$

As general references for such equation in the Euclidean setting we mention [7], [20], [24], [27], [31].

References

1. N.D. Alikakos, R. Rostamian, *Large time behaviour of solutions of Neumann boundary value problem for the porous medium equation*, Indiana Univ. J. Math. **30** (1981), 749–785
2. D.G. Aronson, *Regularity of flows in porous medium: a survey*, in: Nonlinear diffusion equations and their equilibrium states (Berkeley, CA, 1986), Math. Sci. Res. Inst. Publ. **12** (1988), Springer Verlag
3. D.G. Aronson, P. Benilan, *Régularité des solutions de l' équation des milieux poreux dans* $\mathbb{R}^N$, C. R. Acad. Sc. Paris, A. **288**, (1979), 103–105.
4. D.G. Aronson, L.A. Peletier, *Large time behaviour of solutions of the porous medium equation in bounded domains*, J. Diff. Equations **39** (1981), 378–412.
5. T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Monographs in Mathematics, Springer–Verlag (1998).
6. D. Bakry, T. Coulhon, M. Ledoux, L. Saloff-Coste, *Sobolev inequalities in disguise*, Indiana Univ. Math. J. **44** (1995), 1033–1074.
7. P. Benilan, M.G. Crandall, *The continuous dependence on* φ *of solutions of* $u_t - \triangle\varphi(u) = 0$, Indiana Univ. J.Math. **30**, (1981), 161–177.
8. P. Benilan,*A strong regularity* L^p *for solution of the porous media equation*, Contrib. to Nonlin. PDE, Res. Notes in Math. **89**, (1983), 39–58,Pitmann, London.
9. E. Carlen, M. Loss, *Optimal smoothing and decay estimates for viscously damped conservation lows, with applications to the 2–D Navier Stokes equation*, Duke Math. J. **8**1 (1995), 135–157.
10. J.A. Carrillo, G. Toscani, *Asymptotic* L^1*–decay of solutions of the porous medium equation to self–similarity*, Indiana Univ. J. Math. **49** (2000), 113–142.
11. G. Carron, *Inégalités de Faber–Krahn et inclusion de Sobolev–Orlicz*, Potential Anal. **7**, (1997), 555–575.
12. F. Cipriani, G. Grillo, *Uniform bounds for solutions to quasilinear parabolic equations*, J. Diff. Equations. **177** (2001), 209–234.
13. F. Cipriani, G. Grillo, *Nonlinear markovian semigroups and nonlinear Dirichlet forms*, J. reine angew. Math. **5**62 (2003), 201–235.
14. F. Cipriani, G. Grillo, L^p*–*L^∞ *Hölder continuity for quasilinear parabolic equations associated to Sobolev derivations*, J. Math. Anal. Appl. **270** (2002), 267–290.
15. E.B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, 1989.
16. E.B. Davies, B. Simon, *Ultracontractivity and the heat kernel for Schrödinger operators and Dirichlet Laplacians*, J. Funct. Anal. **5**9 (1984), 335–395.
17. M. del Pino, J. Dolbeault, *Generalized Sobolev inequalities and asymptotic behaviour in fast diffusion and porous medium problem*, Preprint Ceremade 9905 (1999), available at http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/GLSFD6.ps.
18. M. del Pino, J. Dolbeault, *Best constant for Gagliardo–Nirenberg inequalities and applications to nonlinear diffusions*, J. Math. Pures Appl. **8**1 (2002), 847–875.
19. E. DiBenedetto, A. Friedman, *Hölder estimates for nonlinear degenerate parabolic systems*, J. Reine Angew. Math. **3**57 (1985), 1–22.
20. L.C. Evans, *Applications of nonlinear semigroup theory to certain partial differential equations*, in: Nonlinear Evolution Equations, M.G. Crandall Ed. Academic Press, New York (1978), 163–188.
21. A. Friedman, S. Kamin, *The asymptotic behaviour of gas in an* n*–dimensional porous medium*, Trans. AMS **262**, n.2, (1980), 551–563.
22. L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math **97** (1976), 1061–1083.
23. E. Hebey, *Sobolev Spaces on Riemannian Manifolds*, LNM **1635** Springer–Verlag (1996).

24. N. Igbida, *Large time behaviour of the solutions to some degenerate parabolic equations*, Commun. P.D.E. **26** (2001), 1385–1408.
25. S. Kamin, J.L. Vázquez, *Asymptotic behaviour of solutions of the porous medium equation with changing sign*, SIAM J. Math. Anal. **22** (1991), 34–45.
26. F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Commun. P.D.E. **26** (2001), 101–174.
27. A.Pazy, *The Lyapunov method for semigroups of nonlinear contractions in Banach spaces*, J. d'Analyse Math. **40** (1981), 239–262.
28. W. Rudin, *Real and Complex Analysis*, McGraw–Hill (1974).
29. R.E. Showalter, *Monotone Operators in Banach Spaces and Nonlinear Partial Differential Equations*, AMS Mathematical Surveys and Monographs, **49**, AMS (1996).
30. J.L. Vázquez, *An introduction to the mathematical theory of the porous medium equation*, in: Shape optimization and free boundaries, NATO ASI Ser., Ser. C **3**80, 347-389 (1992).
31. J.L. Vázquez, *Symétrization pour* $u_t = \triangle\varphi(u)$ *et applications*, C. R. Acad. Sc. Paris, A. **295**, (1982), 71–74.
32. J.L. Vázquez, *An introduction to the mathematical theory of the porous medium equation*, Adv. Sci. Ser. C. Math. Phys. Sci. **380** Kluver Acad. Publ (1992), 347–389.
33. J.L. Vázquez, *Asymptotic behaviour for the porous medium equation in the whole space*, Preprint (2002).
34. J.L. Vázquez, *Asymptotic behaviour for the porous medium equation in the whole space*, Notas del curso de doctorado (1997).
35. J.L. Vázquez, *Asymptotic behaviour for the porous medium equation in a Bounded Domain. The Dirichlet problem.*, Notas del Curso de Doctorado (1997).
36. N. Varopoulos, L. Saloff–Coste, T. Coulhon: *Analysis and Geometry on Groups*, Cambridge University Press, 1992.

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