DIRECT AND REVERSE GAGLIARDO–NIRENBERG INEQUALITIES FROM LOGARITHMIC SOBOLEV INEQUALITIES

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Abstract. We investigate the connection between the validity of certain logarithmic Sobolev inequality and the validity of suitable generalizations of Gagliardo–Nirenberg inequalities. A similar connection holds between reverse logarithmic Sobolev inequalities and a new class of reverse Gagliardo–Nirenberg inequalities, valid for a suitable class of functions.

0. Introduction

The main concern of this paper is to investigate the connections between *logarithmic Sobolev inequalities* (LSI in the sequel) and generalizations of Gagliardo–Nirenberg inequalities (GNI in the sequel). The typical LSI inequality we shall be concerned with in the first part of the paper will be of the form

$$\int_X \log\left[\frac{|u(x)|}{\|u\|_p}\right]\frac{|u(x)|^p}{\|u\|_p^p}\mathrm{d}\mu(x) \le c_1 \log\left[c_2\frac{\|\nabla u\|_p}{\|u\|_p}\right] \quad \forall f\in C_c^\infty(X), \tag{0.1}$$

μ being a positive Radon measure on a Riemannian manifold X and ∇ the Riemannian gradient, c_1, c_2 being suitable positive constants and $\|\cdot\|_p$ denoting the L^p norm. The manifold setting is chosen for the sake of notational simplicity only and could be generalized in many respects: for example the role of the operator ∇ could be taken by a (vector valued) *derivation* (see e.g. [13] for details), but certain discrete settings could be discussed as well (see [1]).

Such inequalities have a long history since the pioneering work of Gross [17], who proved the equivalence between the validity of a weaker form of such inequalities in the case $p=2$ and hypercontractivity of the linear heat semigroup. It was proved later that (0.1) is indeed equivalent to ultracontractivity of the heat semigroup (see also [2]). This can be seen for example by noticing that, by applying the numerical inequality $\log x \le \varepsilon x + \log(1/\varepsilon)\ \forall x, \varepsilon > 0$, to the r.h.s. of (0.1), one proves a family of LSI of the form considered in [14]. The proof of ultracontractivity then follows by methods which are by now standard.

More recently, it has been shown in [12], [7], [8] that the validity of such a LSI for $p \ge 2$ implies ultracontractive–like bounds of the form

$$\|u(t)\|_\infty \le \frac{C}{t^\alpha}\|u(0)\|_2^\beta$$

for the solutions $u(t)$ to suitable classed of *nonlinear* evolution equations including the porous media equation and the heat equation driven by the p–Laplacian (see also [9] and [15] for a generalization to a doubly nonlinear evolution equation).

Since on the other hand it is known that, in the linear case, ultracontractivity for the heat semigroup is equivalent to the validity of the usual Sobolev inequality $\|u\|_{2d/(d-2)} \le C\|\nabla u\|_2$ (and to the validity of Nash inequalities as well), it is not surprising that the validity of (0.1) is connected to the validity of Sobolev inequalities involving the p-energy functional $\|\nabla u\|_p$, or to inequalities of Nash type involving such functional.

This is indeed a consequence of the results of [4], [5] [6], in which it is proved that logarithmic Sobolev inequalities imply Nash–type inequalities (which are a special case of GNI), and of [1] (see also [18]), in which it is shown that the validity of any single GNI implies the validity of a whole class of them: in Section 2 we discuss this point with a few more details.

Our aim here is to investigate further this connection. We shall proceed in this direction by first showing that the entropic functional

$$J(p,u)=\int_X \log\left[\frac{|u(x)|}{\|u\|_p}\right]\frac{|u(x)|^p}{\|u\|_p^p}\mathrm{d}\mu(x)$$

can be used to bound both from below and from above the variation of the convex function $p\mapsto \log\|u\|_p^p$. This is the content of our first result, Theorem 1.1. This will allow us first to prove the following inequality, which we call 4–norms inequality. It is a generalization of the GNI and reads:

$$\|u\|_q^{\frac{1}{d}}\|u\|_p^{\left(\frac{1}{s}-\frac{1}{q}\right)}\le C\|\nabla u\|_p^{\left(\frac{1}{s}-\frac{1}{q}\right)}\|u\|_s^{\frac{1}{d}}, \tag{0.2}$$

where $0<s\le q\le p$ and $d\ge 1$, d being a parameter having the role of a dimension. GNI inequalities can then be proved with the help of the results of [1].

We next prove *reverse* analogues of the above 4-norms inequalities, as a consequence of reverse LSI which we prove in Section 3. In fact we first discuss the consequences of the validity of a *reverse* LSI in the form given by [16], [19], [20] adapted to the real case. We shall mostly specialize to the Euclidean case, the underlying measure being chosen either as the Gaussian measure or as the Lebesgue one. This is because these are among the main cases in which we are able to prove the validity of a reverse LSI for suitable classes of function. We hope that such inequalities can be used in order to study the possible validity of reverse hypercontractivity for suitable classes of data. We shall then use Theorem 1.1 to prove new *reverse* analogue of inequality (0.2), from which reverse GNI (and in particular reverse Sobolev inequalities) will then follow, again for a suitable class of functions.

This paper is organized as follows. In the first Section we prove the main property of the entropy functional which will be used in the sequel, including the aforementioned Theorem 1.1. In Section 2 we state and prove the fact that the validity of a suitable LSI implies the validity of inequality (0.2), and then some remarks on the connections with the validity of GNI. Section 3 is devoted to the proof of reverse LSI, Sobolev and Gagliardo–Nirenberg inequalities.

1. Basic Entropy Inequalities

In this section we prove the basic inequalities concerning the functional $J(p,u)$, defined w.r.t. a general positive Radon measure μ, which will be the starting point in proving both direct and reverse Gagliardo-Nirenberg inequalities.

Theorem 1.1. *The following inequality:*

$$\|u\|_p\,\mathrm{e}^{\frac{q-p}{qp}J(1,u^p)}\le\|u\|_q\le\|u\|_p\,\mathrm{e}^{\frac{q-p}{qp}J(1,u^q)} \tag{1.1}$$

holds true for any $0<p\le q$, *any* $u\in \mathrm{L}^p(X,\mu)\cap\mathrm{L}^q(X,\mu)$.

Proof. It is well-known that the functional

$$N(r,u) = \log \|u\|_r^r = \log \int_X |u(x)|^r \mathrm{d}\mu(x)$$

defined over $(0,+\infty) \times \bigcap_{p>0} \mathrm{L}^p(X,\mu)$ is convex w.r.t. $r>0$, moreover its first derivative:

$$\frac{\mathrm{d}}{\mathrm{d}r} N(r,u) = J(r,u) + \frac{1}{r} N(r,u)$$

is non–decreasing w.r.t $r>0$. For more details one can refer to section (2.4) of [8].

By the convexity of N one has, for $0<p\le q$:

$$N'(p) \le \frac{N(q,u)-N(p,u)}{q-p} \le N'(q)$$

which becomes:

$$(q-p)\left[J(p,u) + \log \|u\|_p\right] \le \log \frac{\|u\|_q^q}{\|u\|_p^p} \le (q-p)\left[J(q,u) + \log \|u\|_q\right]$$

or equivalently

$$\mathrm{e}^{(q-p)J(p,u)} \|u\|_p^{q-p} \le \frac{\|u\|_q^q}{\|u\|_p^p} \le \mathrm{e}^{(q-p)J(q,u)} \|u\|_q^{q-p}.$$

The latter inequalities are then clearly equivalent to the assertion. □

We now collect here some useful properties of the entropy functional which will be of some help later on.

Proposition 1.2. *The functional J satisfies the following properties:*

$$J\left(r,u^{\gamma}\right) = \gamma J(\gamma r, u) \tag{1.2}$$

for all $\gamma, r>0$;

$$J\left(r, u^{s+h}\right) \ge J\left(r,u^s\right) \tag{1.3}$$

for all $r,s>0$, $h\ge 0$.

Proof. The first statement is an immediate consequence of the definition of J. As for the second one we first prove that the map $\beta \mapsto J(1,u^\beta)$ is nondecreasing. In fact, it is well–known that the map

$$\alpha \mapsto \log \|u\|_{1/\alpha}.$$

is convex (see e.g. [1]). By taking derivatives one then has that the map:

$$\alpha \mapsto -\frac{1}{\alpha} J\left(\frac{1}{\alpha}, u\right)$$

is nondecreasing. Thus, the map $\beta \mapsto \beta J(\beta,u)$ is increasing as well so that the same property is satisfied, by the previous result, by the functional $\beta \mapsto J(1,u^\beta)$.

Finally

$$sJ(s,u^{r+h}) = J(1,u^{s(r+h)}) \ge J(1,u^{sr}) = sJ(s,u^r).$$

□

2. 4–norms inequalitites via entropy and LSI

In this section we draw the main consequences of the lower bound appearing in equation (1.1) of Theorem 1.1, by making use of the inequalities of Proposition 1.2.

We shall first prove the following Proposition.

Proposition 2.1. *For any $0<p<\rho$ the validity of p-LSI implies the validity of ρ-LSI.*

Proof. We compute

$$\begin{aligned}\rho J(\rho,u) &= p\frac{\rho}{p}J\left(p\frac{\rho}{p},u\right)=pJ\left(p,u^{\frac{\rho}{p}}\right)\le\frac{d}{p}\log\left[\mathcal{L}_p\frac{\left\|\nabla\left(|u|^{\frac{\rho}{p}}\right)\right\|_p^p}{\left\||u|^{\frac{\rho}{p}}\right\|_p^p}\right]\\
&=\frac{d}{p}\log\left[\mathcal{L}_p\left(\frac{\rho}{p}\right)^p\frac{\int_X|u|^{p\left(\frac{\rho}{p}-1\right)}|\nabla u|^p\,\mathrm{d}\mu}{\|u\|_\rho^\rho}\right]\\
&\le\frac{d}{p}\log\left[\mathcal{L}_p\left(\frac{\rho}{p}\right)^p\frac{\||u|^{\rho-p}\|_{\sigma'}\,\||\nabla u|^p\|_\sigma}{\|u\|_\rho^\rho}\right]\\
&=\frac{d}{p}\log\left[\mathcal{L}_p\left(\frac{\rho}{p}\right)^p\frac{\|u\|_\rho^{\rho-p}\,\|\nabla u\|_\rho^p}{\|u\|_\rho^\rho}\right]=d\log\left[\mathcal{L}_p^{\frac{1}{p}}\frac{\rho}{p}\frac{\|\nabla u\|_\rho}{\|u\|_\rho}\right]\end{aligned}$$

where we also used Hölder inequality with the choice of the two conjugate exponents $\sigma=\frac{\rho}{p}>1$ and $\sigma'=\frac{\rho}{\rho-p}>1$. This concludes the proof. $\square$

Theorem 2.2. *(4–Norms Inequality)*
Suppose that the following LSI holds true for some $p,d>0$:

$$pJ(p,u)\le d\log\left[\mathcal{L}_p^{1/p}\frac{\|\nabla u\|_p}{\|u\|_p}\right] \tag{2.1}$$

Then the following inequality holds true:

$$\|u\|_q^{\frac{1}{d}}\|u\|_p^{\left(\frac{1}{s}-\frac{1}{q}\right)}\le\mathcal{L}_p^{\frac{1}{p}\left(\frac{1}{s}-\frac{1}{q}\right)}\|\nabla u\|_p^{\left(\frac{1}{s}-\frac{1}{q}\right)}\|u\|_s^{\frac{1}{d}},\quad \textit{whenever}\quad 0<s\le q\le p. \tag{2.2}$$

Moreover if $\rho\ge p$ then (2.1) implies the following:

$$\|u\|_q^{\frac{1}{d}}\|u\|_\rho^{\left(\frac{1}{s}-\frac{1}{q}\right)}\le\left(\mathcal{L}_p^{\frac{1}{p}}\frac{\rho}{p}\right)^{\left(\frac{1}{s}-\frac{1}{q}\right)}\|\nabla u\|_\rho^{\left(\frac{1}{s}-\frac{1}{q}\right)}\|u\|_s^{\frac{1}{d}},\ \textit{whenever}\quad 0<s\le q\le p\le\rho. \tag{2.3}$$

Proof. We will prove the claimed inequality by combining the r.h.s. of the entropy inequality (1.1) and the p-LSI rewritten in the form:

$$\mathrm{e}^{pJ(p,u)}\le\mathcal{L}_p^{d/p}\frac{\|\nabla u\|_p^d}{\|u\|_p^d}. \tag{2.4}$$

To this end we need the monotonicity property for the entropy functional (1.3) that we recall here:

$$qJ(q,u)=J\left(1,u^q\right)\le pJ(p,u)=J\left(1,u^p\right)\ \text{for any } p\ge q>0.$$

Using this together with r.h.s. of the entropy inequality (1.1) one obtains

$$\frac{\|u\|_q}{\|u\|_s} \le \mathrm{e}^{\frac{q-s}{sq} J(1,u^q)} \le \mathrm{e}^{\frac{q-s}{sq} J(1,u^p)}$$

for any $p \ge q \ge s > 0$. Now we will combine this last inequality with the LSI (2.4) in order to obtain:

$$\frac{\|u\|_q}{\|u\|_s} \le \mathrm{e}^{\frac{q-s}{sq} J(1,u^p)} \le \exp\left[\frac{q-s}{sq}\log\left(\mathcal{L}_p^{d/p}\frac{\|\nabla u\|_p^d}{\|u\|_p^d}\right)\right]$$

or equivalently:

$$\frac{\|u\|_q}{\|u\|_s} \le \mathcal{L}_p^{\frac{d}{p}\left(\frac{1}{s}-\frac{1}{q}\right)}\frac{\|\nabla u\|_p^{d\left(\frac{1}{s}-\frac{1}{q}\right)}}{\|u\|_p^{d\left(\frac{1}{s}-\frac{1}{q}\right)}} \tag{2.5}$$

for any $p \ge q \ge s > 0$. This is clearly equivalent to the assertion.
The last part follows from the first and from Proposition 2.1 □

Given the above result, the GNI are a consequence of the results of [1]. Although the following results are known since the paper [1], for completeness and for the reader's convenience we recall concisely how to proceed in this direction from our starting point.

• p-Nash Inequalities

Let $p, d > 0$ be fixed. The first consequence of inequality (2.2) (just by letting $q = p$) is the following family of p-Nash inequalities:

$$\|u\|_p^{1+\frac{ps}{d(p-s)}} \le \mathcal{L}_p^{\frac{1}{p}}\|\nabla u\|_p\|u\|_s^{\frac{ps}{d(p-s)}}, \qquad \text{whenever } 0 < s < p \tag{2.6}$$

Similarly inequality (2.3) implies a family of ρ-Nash inequalities with $\rho \ge p$ and with proportionality constant $\mathcal{L}_p^{1/p}\frac{\varrho}{p}$. We first stress that the above result holds for any $p > 0$. We also comment that our denomination *p-Nash inequalities* is due to their similarity with the celebrated Nash inequality:

$$\|u\|_2^{1+2/d} \le C\|\nabla u\|_2\|u\|_1^{2/d}.$$

The above remark does not distinguish in a relevant way the fact that the parameter p is larger or smaller than the parameter d. The following remarks deal with some more detailed consequences of the above results, which take into account such differences.

• Gagliardo-Nirenberg Inequalities

In the previous remark we proved that the validity of a p-LSI implies the validity of a 4-Norms inequality such as (2.2) and then the validity of a family of p-Nash inequalities, which are a special case of GNI:

$$\|u\|_r \le \mathcal{C}_p^{\frac{\vartheta}{p}}\ \|\nabla u\|_p^{\vartheta}\ \|u\|_s^{1-\vartheta} \tag{2.7}$$

for any $p, r, s, d > 0$ and $\vartheta \in [0, 1]$ such that

$$\frac{1}{r} = \vartheta\left(\frac{1}{p} - \frac{1}{d}\right) + (1-\vartheta)\frac{1}{s} \tag{2.8}$$

where $\mathcal{C}_p \propto \mathcal{L}_p$, $\mathcal{L}_p$ being the constant in the p-LSI.
We will recall hereafter that this fact actually guarantees the validity of all the GNI above, once the relative position of p and d is fixed: see also [1], Th. 10.2. To this end we will need some results of [1].

$\star$ The Subcritical case: $0 < p < d$

By Theorem 3.1 of [1] it is known that a single GNI of the form (2.7) implies the validity of the other GNI inequalities corresponding to $0 < p < d$ fixed, while $\vartheta \in [0, 1]$ and $r, s > 0$ are related as in (2.8). Then a p-LSI of the form (2.1) implies the whole family of GNI (2.7) mentioned above, via the validity of a p-Nash inequality. This family also contains as a special case the classical p-Sobolev inequality:

$$\|u\|_{\frac{pd}{d-p}} \le \mathcal{C}_p \|\nabla u\|_p$$

$\star$ The Critical case: $p = d$

By Theorem 3.3 of [1] it is known that a single GNI of the form (2.7) implies the validity of the other GNI corresponding to $p = d > 0$, $0 < s < r < +\infty$, $\vartheta = 1 - \frac{s}{r}$.

With the help of Theorem 3.2.6 of [18], we can also show that the above mentioned family of GNI implies some versions of Moser-Trudinger inequalities. See [1], Theorem 3.4 for details.

$\star$ The Supercritical case: $p > d$

By Theorem 3.2 of [1] it is known that a single GNI of the form (2.7) implies the validity of the other GNI inequalities corresponding to $p > d > 0$ fixed, while $0 < s < r \le +\infty$, $\vartheta \in [0, 1]$ are related as in (2.8). In particular by letting $r \to +\infty$ we get

$$\|u\|_\infty \le \mathcal{C}_p^{\frac{\vartheta}{p}} \; \|\nabla u\|_p^\vartheta \; \|u\|_s^{1-\vartheta} \tag{2.9}$$

for all $0 < s < +\infty$. This last family contains a version of the well–known Morrey inequality.

$\bullet$ Other Gagliardo-Nirenberg Inequalities

In this last remark we focus our attention on the main consequences of the second 4-Norms inequality (2.3). We proved in a previous remark, using Theorem 2.2, that the validity of a suitable p-LSI implies the validity of a 4-Norms inequality such as (2.3) and then the validity of a ρ-Nash inequality, provided $\rho \ge p$. This fact leads to prove that a p-LSI implies a larger family of GNI:

$$\|u\|_r \le \mathcal{G}_\rho^\vartheta \; \|\nabla u\|_\rho^\vartheta \; \|u\|_s^{1-\vartheta} \tag{2.10}$$

hold as well, whenever $0 < p \le \rho$, $\vartheta \in [0, 1]$ with $\mathcal{G}_q \propto \mathcal{L}_p^{\frac{1}{p}} \frac{q}{p}$ and $\frac{1}{r} = \vartheta\left(\frac{1}{\rho} - \frac{1}{d}\right) + (1 - \vartheta)\frac{1}{s}$.

Thus we can extend the above remarks simply by replacing p with ϱ, and $\mathcal{L}_p^{1/p}$ with $\mathcal{L}_p^{1/p}(\rho/p)$. Informally speaking we recalled that, fixed $p > 0$, a single p-LSI implies a family of ρ-GNI of the type (2.10), with $\rho \ge p$ (this being the content of [1], Section 8) and hence all the ρ–versions of Sobolev, Moser-Trudinger and Morrey inequalities.

3. Reverse Inequalities

In this section we start by proving a new family of reverse logarithmic Sobolev inequalities in a general setting. This reverse LSI will give as a direct consequence a reverse Sobolev inequality, while put together with a reverse 4-norms inequality will give a family of reverse Gagliardo-Nirenberg inequalities as well.

As far as we know, reverse LSI first appeared in the works of S.B. Sontz [19], [20] in the setup of Segal–Bargmann spaces. After these pioneering works, a paper [16] of F. Galaz-Fontes, L.Gross and S.B. Sontz gave a generalization of such reverse LSI over complex manifolds and investigated the connection between this reverse LSI and reverse hypercontractivity.

Although reverse LSI are in a sense typical of the complex setting, we shall show that they have some real analogue. One should comment that a reverse LSI of a different type appears in [11]. We notice here once for all that, hereafter, we deal with spaces of *real valued* functions.

We start by proving the main theorems of this section, concerning reverse inequalities with respect to a positive measure, absolutely continuous w.r.t. a reference measure, indicated by $\mathrm{d}x$:

$$\mathrm{d}\mu(x) = m(x)\,\mathrm{d}x,$$

m being a function for which $\triangle m$ makes sense as a locally integrable function (hereafter, $\triangle$ denotes the Laplace–Beltrami operator). In this section we shall indicate explicitly in the L^q norms the measure with respect to which they are taken (namely, e.g., $\|\cdot\|_{q,\mu}$), since we shall make more than one possible choice of measure hereafter.

We denote by $\mathcal{C}$ the class of functions v for which the following integration by parts formula:

$$\int_X v(x)\triangle(m(x))\,\mathrm{d}x = \int_X \triangle(v(x))m(x)\,\mathrm{d}x. \tag{3.1}$$

We notice that, if $f \in H^1(X,\mu)$ (i.e. $f, \nabla f \in \mathrm{L}^2(X,\mu)$) and $f^2 \in \mathcal{C}$ then the Leibniz rule implies that $\int_X f\triangle f \mathrm{d}\mu$ is finite.

Theorem 3.1. *(Reverse Logarithmic Sobolev Inequality)*
Let f be a measurable, real valued function such that:
(i) $f \in H^1(X,\mu)$, $f^2 \in \mathcal{C}$;
(ii) f satisfies the inequality:

$$\int_X f(x)(\triangle f)(x)\mathrm{d}\mu(x) \geq 0 \tag{3.2}$$

(iii) There exists $c > 0$ such that

$$B(c,m) = \int_X \mathrm{e}^{\frac{(\triangle m)(x)}{cm(x)}}\,\mathrm{d}\mu(x) < +\infty. \tag{3.3}$$

Then the following reverse LSI holds true:

$$2\|\nabla f\|_{2,\mu}^2 \leq c\int_X f^2 \log\left(\frac{f^2}{\|f\|_{2,\mu}^2}\right)\mathrm{d}\mu + c\log(B(c,m))\|f\|_{2,\mu}^2. \tag{3.4}$$

Proof. For the proof we adapt to our setting a method of L. Gross and S.B. Sontz [20]. We first recall Young's inequality:

$$st \le s\log(s) - s + \mathrm{e}^t$$

which holds true for any $s > 0$ and $t \in \mathbb{R}$. The choice $s = cf(x)^2 > 0,\ t = \frac{(\triangle m)(x)}{c\ m(x)}$ together with an integration over (X, μ), will lead to

$$\int_X cf(x)^2 \frac{(\triangle m)(x)}{c\ m(x)} \mathrm{d}\mu(x) \le \int_X cf(x)^2 \log\left(cf(x)^2\right) \mathrm{d}\mu(x) - \int_X cf(x)^2 \mathrm{d}\mu(x) + \int_X \mathrm{e}^{\frac{(\triangle m)(x)}{c\ m(x)}} \mathrm{d}\mu(x).$$

Beside noticing that $\log(cf(x)^2) = \log(c) + \log(f(x)^2)$, $\mathrm{d}\mu(x) = m(x)\,\mathrm{d}x$ and $B(c,m) = \int_X \mathrm{e}^{\frac{(\triangle m)(x)}{cm(x)}} \mathrm{d}\mu(x) < +\infty$ by hypothesis, we obtain

$$\begin{aligned} &\int_X f^2(x)(\triangle m)(x)\,\mathrm{d}x \\ &\le c\int_X f(x)^2 \log\left(f(x)^2\right) \mathrm{d}\mu(x) + (c\log(c) - c)\int_X f(x)^2 \mathrm{d}\mu(x) + B(c,m). \end{aligned} \tag{3.5}$$

An integration by parts, allowed by our assumptions, will then give us

$$\begin{aligned} \int_X f(x)^2 (\triangle m)(x))\,\mathrm{d}x &= \int_X (\triangle f^2)(x) m(x)\,\mathrm{d}x \\ &= 2\int_X |\nabla f(x)|^2\, m(x)\,\mathrm{d}x + 2\int_X f(x)(\triangle f)(x) m(x)\,\mathrm{d}x \\ &\ge 2\int_X |\nabla f(x)|^2\, m(x)\,\mathrm{d}x. \end{aligned}$$

and the latter inequality follows from our assumption $\int_X f(x)(\triangle f)(x)m(x)\,\mathrm{d}x \ge 0$.

Now letting $\lambda > 0$ and replacing f by λf, in the inequality (3.5) will give:

$$\begin{aligned} 2\lambda^2 \int_X |\nabla f(x)|^2\,\mathrm{d}\mu(x) &\le c\lambda^2 \int_X f(x)^2 \log\left(\lambda^2 f(x)^2\right) \mathrm{d}\mu(x) \\ &\quad + (c\log(c) - c)\lambda^2 \int_X f(x)^2 \mathrm{d}\mu(x) + B(c,m). \end{aligned}$$

Now divide both members by λ^2 and obtain:

$$2\,\|\nabla f\|_{2,\mu}^2 \le c\int_X f(x)^2 \log\left(f(x)^2\right) \mathrm{d}\mu(x) + \left[c\log(\lambda^2) + c\log(c) - c\right] \|f\|_{2,\mu}^2 + \frac{B(c,m)}{\lambda^2}. \tag{3.6}$$

Optimizing with respect to λ^2 will give us the following value:

$$\lambda^2 = \frac{B(c,m)}{c\|f\|_{2,\mu}^2}$$

Substituting this value in (3.6) leads to:

$$2\|\nabla f\|_{2,\mu}^2 \le c\int_X f^2 \log(f^2)\mathrm{d}\mu + \left[c\log\left(\frac{B(m,c)}{c\|f\|_{2,\mu}^2}\right) + c\log(c) - c\right] \|f\|_{2,\mu}^2 + c\|f\|_{2,\mu}^2,$$

which is the claim. □

In what follows it will be useful to rewrite inequality (3.4) in the following form:

$$2\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} \le c\int_X \log\left(\frac{f^2}{\|f\|_{2,\mu}^2}\right)\frac{f^2}{\|f\|_{2,\mu}^2}\mathrm{d}\mu + c\log(B(c,m)) = cJ_\mu\left(1,f^2\right) + K \tag{3.7}$$

As a direct consequence of this theorem we obtain the following

Theorem 3.2. *(Reverse Sobolev inequality)*
Let $f \in \mathrm{L}^2(X,\mu)$ be a function such that a reverse LSI of the form (3.7) holds for some positive constants c and K. Then for any $\varepsilon > 0$ there exist a $M'_\varepsilon > 0$ such that

$$M'_\varepsilon \exp\left(\frac{2\varepsilon}{c(2+\varepsilon)}\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2}\right) \le \frac{\|f\|_{2+\varepsilon,\mu}^2}{\|f\|_{2,\mu}^2}.$$

In particular, there exists $M_\varepsilon > 0$ such that the reverse Sobolev inequality:

$$M_\varepsilon\|\nabla f\|_{2,\mu} \le \|f\|_{2+\varepsilon,\mu} \tag{3.8}$$

holds.

Proof. First we rewrite the reverse LSI (3.7)

$$\begin{aligned}\frac{1}{c}\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - \frac{K}{2c} \le & J_\mu(2,f) = \frac{1}{\varepsilon}\int_X \log\left(\frac{|f|^\varepsilon}{\|f\|_{2,\mu}^\varepsilon}\right)\frac{|f|^2}{\|f\|_{2,\mu}^2}\mathrm{d}\mu \\ & \le \log\int_X \frac{|f|^{2+\varepsilon}}{\|f\|_{2,\mu}^{2+\varepsilon}}\mathrm{d}\mu = \frac{2+\varepsilon}{2}\log\frac{\|f\|_{2+\varepsilon,\mu}^2}{\|f\|_{2,\mu}^2},\end{aligned}$$

where in the first line we used the property $J_\mu\left(1,f^2\right) = 2J_\mu(2,f)$ while in the second line we the Jensen inequality w.r.t. the probability measure $\frac{|f|^2}{\|f\|_{2,\mu}^2}\mathrm{d}\mu$. Now we use the numerical inequality $\log(x) \le x$ to obtain:

$$\log\left(\frac{2\varepsilon}{c(2+\varepsilon)}\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2}\right) - \frac{\varepsilon K}{c(2+\varepsilon)} \le \frac{2\varepsilon}{c(2+\varepsilon)}\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - \frac{\varepsilon K}{c(2+\varepsilon)} \le \log\frac{\|f\|_{2+\varepsilon,\mu}^2}{\|f\|_{2,\mu}^2}.$$

Exponentiating the three terms give us:

$$\exp\left(-\frac{\varepsilon K}{c(2+\varepsilon)}\right)\frac{2\varepsilon}{c(2+\varepsilon)}\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} \le \exp\left(\frac{2\varepsilon}{c(2+\varepsilon)}\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - \frac{\varepsilon K}{c(2+\varepsilon)}\right) \le \frac{\|f\|_{2+\varepsilon,\mu}^2}{\|f\|_{2,\mu}^2}. \tag{3.9}$$

□

As far as we know, reverse Sobolev inequalities appeared first in the work of S.B. Sontz [19], [20] in the context of the Segal–Bargmann spaces. Their connection with reverse hypercontractivity has been discussed in [16].

Theorem 3.3. *(Reverse 4–Norms Inequality)*
Let $f \in \mathrm{L}^2(X,\mu) \cap \mathrm{L}^q(X,\mu)$, with $q > 2$, be a function such that a reverse LSI of the form (3.7) holds true for some constants $c > 0$ and $K > 0$. Then the following inequality:

$$\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} \le \mathrm{e}^{K/2} \left[\frac{\|f\|_{q,\mu}}{\|f\|_{p,\mu}} \right]^{\frac{cqp}{2(q-p)}} \tag{3.10}$$

holds true for any $2 \le p \le q$ and any $c > 2$.

Proof. We combine the first part of entropy inequality (1.1), with $2 \le p \le q$:

$$\|f\|_{p,\mu}\, \mathrm{e}^{\frac{q-p}{qp} J_\mu(1,f^p)} \le \|f\|_{q,\mu}$$

rewritten in the form

$$c J_\mu(1, f^p) \le c \frac{qp}{q-p} \log \left(\frac{\|f\|_{q,\mu}}{\|f\|_{p,\mu}} \right)$$

with the reverse LSI (3.4) rewritten in the form:

$$2 \frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - K \le c J_\mu\left(1, f^2\right)$$

We notice that we can glue these inequalities using the monotonicity of the Young functional $J_\mu\left(1, u^2\right) \le J_\mu(1, u^p)$ for any $2 \le p$. Hence we obtain

$$2 \frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - K \le c J_\mu\left(1, u^2\right) \le c J_\mu(1, u^p) \le c \frac{qp}{q-p} \log \left(\frac{\|f\|_{q,\mu}}{\|f\|_{p,\mu}} \right)$$

exponentiating and using the numerical inequality $x \le \mathrm{e}^x$ will finally give

$$\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} \mathrm{e}^{-\frac{K}{2}} \le \exp\left(\frac{\|\nabla f\|_{2,\mu}^2}{\|f\|_{2,\mu}^2} - \frac{K}{2} \right) \le \left[\frac{\|f\|_{q,\mu}}{\|f\|_{p,\mu}} \right]^{\frac{cqp}{2(q-p)}}.$$

□

Corollary 3.4. *(Reverse Gagliardo-Nirenberg inequalities)*
Let $f \in \mathrm{L}^2(X,\mu) \cap \mathrm{L}^q(X,\mu)$, with $q > 2$, be a function such that a reverse LSI of the form (3.7) holds true for some constants $c > 0$ and $K > 0$. Then the following family of reverse GNI

$$\|\nabla f\|_{2,\mu}^{\vartheta} \|f\|_{2,\mu}^{1-\vartheta} \le \mathrm{e}^{\frac{K\vartheta}{4}} \|f\|_{q,\mu} \tag{3.11}$$

holds true for any $q > 2$, where $\vartheta = \frac{2(q-2)}{cq}$ and $K > 0$ is the constant (3.3) in the reverse LSI (3.7).

Proof. Just let $p = 2$ in the 4-norms inequality (3.10). □

Remark 3.5. Notice that condition (3.2) becomes an identity for harmonic functions in the space (X,μ), i.e. those functions f which satisfy $\triangle f = 0$ μ-almost everywhere in X. Condition (3.2) is also fulfilled when the integral appearing there is finite and when moreover either f is a non-negative subharmonic function of (i.e. when $f \ge 0$ and $\triangle f \ge 0$ a.e.) or f is a non-positive superharmonic (i.e. when $f \le 0$ and $\triangle f \le 0$ a.e.) function of X. It is also satisfied, in the Euclidean case, by positive convex functions or by negative concave functions of $\mathrm{L}^2(X,\mu)$, if the corresponding integral appearing in (3.2) exists.

3.1. **The Gaussian setup.** In this section first we draw the main consequences of the above results in the Gaussian setup i.e. when $(X,\mu) = (\mathbb{R}^d,\gamma)$ where γ is the Gaussian measure

$$\mathrm{d}\gamma(x) = (2\pi)^{-d/2}\mathrm{e}^{-|x|^2/2}\,\mathrm{d}x.$$

We then prove some families of reverse Sobolev, 4-Norms and Gagliardo-Nirenberg inequalities. The validity of the last inequalities depends on the reverse LSI (3.4) and hence depends on the condition (3.2) which reads, in the present context:

$$\int_{\mathbb{R}^d} |f(x)|^2 \left(|x|^2 - d\right) \mathrm{d}\gamma(x) \geq 2 \int_{\mathbb{R}^d} |\nabla f(x)|^2 \,\mathrm{d}\gamma(x) \tag{3.12}$$

This inequality becomes an identity when we deal with harmonic functions. In the present setting this inequality plays the role of the identities of V. Bargmann (see [3], pg. 210) and of E. A. Carlen (see [10]), which hold in the Segal–Bargmann space. Inequality (3.12) holds for a class of functions that includes harmonic functions: such class will play the role, in our context, played by Segal–Bargmann functions in the complex case.

Theorem 3.6. *(Reverse Inequalities, Gaussian case)*
Let f be a smooth, compactly supported function such that:
(A) f satisfies the inequality:

$$\int_{\mathbb{R}^d} f(x)(\triangle f)(x)\mathrm{d}\gamma(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(x)(\triangle f)(x)\mathrm{e}^{-|x|^2/2}\,\mathrm{d}x \geq 0 \tag{3.13}$$

Then for any $c > 2$ there exists a positive constant $B(c)$ such that the following reverse LSI holds true:

$$2\|\nabla f\|_{2,\gamma}^2 \leq c \int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\|f\|_{2,\gamma}^2}\right) \mathrm{d}\gamma + B(c,d)\|f\|_{2,\gamma}^2 \tag{3.14}$$

where

$$B(c,d) = d\left(-1 + \frac{1}{2}\log\left(\frac{2c}{c-2}\right)\right) \tag{3.15}$$

Moreover the following inequalities hold:

(a) Reverse Sobolev inequality*: for any $\varepsilon > 0$ and $c > 2$ there exist a constant $G_{\varepsilon,c} > 0$ such that the following inequality:*

$$G_{\varepsilon,c}\|\nabla f\|_{2,\gamma} \leq \|f\|_{2+\varepsilon,\gamma} \tag{3.16}$$

where $G_{\varepsilon,c} = \frac{2\varepsilon}{c(2+\varepsilon)}\mathrm{e}^{-\frac{\varepsilon B(c,d)}{c(2+\varepsilon)}}$ and $B(c,d)$ is given by (3.15).

(b) Reverse GNI inequalities*: for any $c > 2$ there exists a positive constant $N(c,d,p,q)$ such that:*

$$\|\nabla f\|_{r,\gamma}^{\vartheta}\|f\|_{p,\gamma}^{1-\vartheta} \leq N\|f\|_{q,\gamma} \tag{3.17}$$

holds true for any $0<r\le 2$ *and* $2\le p<q$*, where* $\vartheta=\frac{4(q-p)}{cqp}$*,* $N=\mathrm{e}^{\frac{B(c,d)(q-p)}{cqp}}$ *and* $B(c)$ *is given by (3.15).*

Proof. First we prove the reverse LSI (3.14). This is a consequence of Theorem 3.1 together with some calculations. In fact, the assumption (i) of such Theorem is satisfied for the present class of functions. Moreover the Gaussian density $\gamma(x)=(2\pi)^{-d/2}\mathrm{e}^{-|x|^2/2}$ on $\mathbb{R}^d$ satisfies the following identity:

$$(\triangle\gamma)(x)=(2\pi)^{-d/2}\left(|x|^2-d\right)\mathrm{e}^{-|x|^2/2}$$

for any $x\in\mathbb{R}^d$. Then we compute the constant $B(c,\gamma)$:

$$B(c,\gamma)=\int_{\mathbb{R}^d}\mathrm{e}^{\frac{(\triangle\gamma)(x)}{c\ \gamma(x)}}\,\mathrm{d}\gamma(x)=\int_{\mathbb{R}^d}\mathrm{e}^{\frac{|x|^2-d}{c}}\,\mathrm{d}\gamma(x)=\mathrm{e}^{-d/c}\left(\frac{2c}{c-2}\right)^{d/2}=B(c,d).$$

This proves the claimed reverse LSI (3.14).

Reverse Sobolev inequalities (3.16) are just a direct consequence of the reverse LSI (3.14), exactly as it has been done in the general case.

Reverse GNI (3.17) are consequence of the 4-Norms inequality (3.10), which hold in the present case as well, together with Hölder inequality, in fact $\|f\|_{r,\gamma}\le\|f\|_{s,\gamma}$, whenever $0<r<s$ since the Gaussian measure on $\mathbb{R}^d$ is a probability measure. Moreover this implies

$$\frac{\|\nabla f\|_{r,\gamma}}{\|f\|_{s,\gamma}}\le\frac{\|\nabla f\|_{2,\gamma}}{\|f\|_{2,\gamma}},\qquad\text{whenever}\begin{cases}0<r\le 2\\ s\ge 2\end{cases}$$

which combined with the reverse 4-norms inequality (3.10) gives us

$$\frac{\|\nabla f\|_{r,\gamma}^2}{\|f\|_{s,\gamma}^2}\le\mathrm{e}^{\frac{B(c,d)}{2}}\left[\frac{\|f\|_{q,\gamma}}{\|f\|_{p,\gamma}}\right]^{\frac{cqp}{2(q-p)}},\qquad\text{whenever}\begin{cases}0<r\le 2\\ s\ge 2\\ 2\le p<q.\end{cases}$$

Finally let $s=p$ and obtain:

$$\|\nabla f\|_{r,\gamma}^{\frac{4(q-p)}{cqp}}\|f\|_{p,\gamma}^{1-\frac{4(q-p)}{cqp}}\le\mathrm{e}^{\frac{B(c,d)(q-p)}{cqp}}\|f\|_q$$

with $0<r\le 2$, $2\le p<q$. Letting $\vartheta=\frac{4(q-p)}{cqp}$ and $N=\mathrm{e}^{\frac{B(c,d)(q-p)}{cqp}}$ gives the claimed reverse GNI (3.17). This concludes the proof. □

We remark that the above class of reverse Gagliardo–Nirenberg inequalities contains as a special case, a reverse Moser inequality, by letting $r=p=2$ and $q>2$:

$$\|\nabla f\|_{2,\gamma}^{\vartheta}\|f\|_{2,\gamma}^{1-\vartheta}\le N\|f\|_{q,\gamma},\qquad\vartheta=\frac{2(q-2)}{cq},\ c>2.$$

The stated reverse of Moser inequality is obtained letting $q=2\left(1+\frac{1}{d}\right)>2$.

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