Ultracontractive bounds for nonlinear evolution equations governed by the subcritical p-Laplacian

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Abstract. We consider the equation $\dot{u} = \triangle_p(u)$ with $2 \le p < d$ on a compact Riemannian manifold. We prove that any solution $u(t)$ approaches its (time independent) mean $\overline{u}$ with quantitative bounds on the rate of convergence $\|u(t) - \overline{u}\|_\infty \le C\|u_0 - \overline{u}\|_r^\gamma / t^\beta$ for any $q \in [2, +\infty]$ and $t > 0$. The proof is based upon the connection between logarithmic Sobolev inequalities and decay propertiesof nonlinear semigroups.

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1. Introduction

Let (M, g) be a smooth, connected and compact Riemannian manifold without boundary, whose dimension is denoted by d with $d \ge 3$. Let ∇ be the Riemannian gradient and $\mathrm{d}x$ the Riemannian measure and consider, for $2 \le p < d$ (the *subcritical case*), the following functional:

$$\mathcal{E}_p(u) = \int_M |\nabla u|^p \, \mathrm{d}x \tag{1.1}$$

for any $u \in \mathrm{L}^2(M)$, where we adopt the convention that $\mathcal{E}_p(u)$ equals $+\infty$ if the distributional gradient of u does not belong to $\mathrm{L}^p(M)$. It is well–known that $\mathcal{E}_p$ is a convex, lower semicontinuous functional. The subgradient of the functional $\mathcal{E}_p/p$, denoted by $\triangle_p$, generates a (nonlinear) strongly continuous nonexpansive semigroup $\{T_t : t \ge 0\}$ on $\mathrm{L}^2(M)$. On smooth functions, the operator $\triangle_p$ reads

$$\triangle_p u = \mathrm{div}\left(|\nabla u|^{p-2} \nabla u\right),$$

$|\cdot| = |\cdot|_x$ indicating the norm in the tangent space at x. We refer to [16] as a complete general reference for parabolic equations driven by operators of p–Laplace type in the Euclidean setting, there can be found also existence results (for weak solutions) in Euclidean setting as well as other properties of the solution.

It should be remarked here that the function $u(t,x) := (T_t u)(x)$ is also a weak solution in the sense of [16] of the equation $\dot{u} = \triangle_p u$, so we will speak equivalently of weak solution or time evolution associated to the semigroup at hand. To be more precise, by *weak solution* to equation

$$\begin{cases} \dot{u} = \triangle_p(u), \text{ on } (0,+\infty)\times M \\ u(0,\cdot) = u_0 \in \mathrm{L}^2(M) \end{cases} \tag{1.2}$$

corresponding to the initial datum $u_0 \in \mathrm{L}^2(M)$ we mean that

$$u \in \mathrm{L}^p((0,T);W^{1,p}(M)) \cap C([0,T];\mathrm{L}^2(M))$$

for any $T>0$ and that, for any positive and bounded test function

$$\varphi \in W^{1,2}(0,T;\mathrm{L}^2(M)) \cap \mathrm{L}^p((0,T);W^{1,p}(M)), \quad \varphi(T) = 0,$$

one has:

$$\int_M u_0(x)\varphi(0,x)\,\mathrm{d}x = -\int_0^T\int_M u(t)\varphi'(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_M |\nabla u(t,x)|^{p-2}\,\nabla u(t,x)\cdot\nabla\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

Let us denote by $\overline{u}$ the mean of an integrable function u:

$$\overline{u} = \frac{1}{\mathrm{vol}(M)}\int_M u\;\mathrm{d}x.$$

Let finally $u(t) := T_t u$ be the time evolution associated to the semigroup at hand and to the initial datum $u(0) = u \in \mathrm{L}^1(M)$ (or the weak solution to problem (1.2), as well). Then $\overline{u(t)}$ does not depend upon time, so that it equals $\overline{u}$: we prove this fact by means of abstract semigroup theory in Lemma 3.1.

Theorem 1.1. *Let (M,g) be a smooth, connected and compact Riemannian manifold without boundary and with dimension $d>2$. Consider, for any $t>0$, the solution $u(t)$ to the problem (1.2) with $u(0)\in L^q(M)$ with $q\ge 1$. Then the following ultracontractive bound holds true for all $t\in(0,1]$:*

$$\|u(t)-\overline{u}\|_\infty \le \frac{C(p,q,d,\overline{A},\mathrm{Vol}(M))}{t^\beta}\,\|u(0)-\overline{u}\|_q^\gamma \tag{1.3}$$

with:

$$\beta = \frac{d}{pq+d(p-2)}, \quad \gamma = \frac{pq}{pq+d(p-2)} \tag{1.4}$$

where $\overline{A}$ are the constants appearing in the Sobolev inequality

$$\|u-\overline{u}\|_{pd/(d-p)} \le \overline{A}\,\|\nabla u\|_p$$

If $t > 1$ one has instead, for all data belonging to $L^2(M)$:

$$\|u(t) - \overline{u}\|_\infty \leq \frac{C(p, 2, d, \overline{A}, \mathrm{Vol}(M))}{\left(Bt + \|u(0) - \overline{u}\|_2^{2-p}\right)^{\frac{2p}{(2p+d(p-2))(p-2)}}} \tag{1.5}$$

and in particular, for any $\varepsilon \in [0, 1]$:

$$\|u(t) - \overline{u}\|_\infty \leq \frac{C(p, 2, d, \overline{A}, \mathrm{Vol}(M))\|u(0) - \overline{u}\|_2^{\frac{2p(1-\varepsilon)}{(2p+d(p-2))}}}{(Bt)^{\frac{2p\varepsilon}{(2p+d(p-2))(p-2)}}} \tag{1.6}$$

where

$$B = \frac{(p-2)}{\overline{A}^p \mathrm{Vol}(M)^{p\frac{2p+d(p-2)}{2(p-d)}}}.$$

The proof will show that identical conclusions hold for the solutions to the equation $\dot{u} = \triangle_p u$ in bounded Euclidean domains, or in compact manifolds with smooth boundary, with homogeneous Neumann boundary conditions.

Corollary 1.2 (absolute bound)**.** *For all $t > 2$, all $\varepsilon \in (0, 1)$ and all initial data u_0 in $\mathrm{L}^1(M)$ there exists $c_\varepsilon > 0$ such that*

$$\|u(t) - \overline{u}\|_\infty \leq c_\varepsilon t^{-(1-\varepsilon)/(p-2)}. \tag{1.7}$$

independently of the initial datum u_0. Moreover, if the initial datum belongs to $L^r(M)$ with $\|u(0)\|_r < 1$ then

$$\|u(t) - \overline{u}\|_\infty \leq c_\varepsilon \|u(0) - \overline{u}\|_r^\varepsilon t^{-(1-\varepsilon)/(p-2)}$$

for all $t \geq 2\|u(0) - \overline{u}\|_r^{2-p}$.

The proof of this corollary is identical to the proof of the corollary (1.2) of [5] since the proof presented there is independent on the range of p.

A few comments on the *sharpness of the bound* are now given:

- (compact manifold or Neumann cases). It is known from the results of [1] that a lower bound of the form

$$\|u(t) - \overline{u}\|_2 \geq \frac{C}{t^{1/(p-2)}}$$

holds for any L^2 data and all t sufficiently large. A similar bound for the L^∞ norm thus holds as well. Hence the bounds in Corollary 1.2 are close to the optimal ones for large time. For small times a comparison with the Barenblatt solutions ([16]) shows that the power of time is the correct one for data belonging to L^1, while for data in L^q with $q > 1$ the L^∞ our result is better in the sense that norm diverges at a *slower* rate depending on q, a property which is familiar in the theory of linear ultracontractive semigroups but which seems to have not been explicitly stated so far in the nonlinear context.

- (Dirichlet case). A similar result can be shown on compact manifolds with smooth boundary, homogeneous Dirichlet boundary conditions being assumed.

The main difference is in the fact that the solutions approach zero when t tends to infinity. The proof stems from the appropriate Sobolev inequality for functions in $W_0^{1,p}(M)$ and is easier than in the previous case. For short time remarks similar to the Neumann case hold. By using the optimal logarithmic Sobolev inequality of [14] for the p–energy functional, bounds which are sharp also for general L^1 data and small times can be proved easily by the present methods in the Euclidean case.

A comparison with some previous results is now given. While a discussion of similar problems *in the whole* $\mathbb{R}^n$ has been given long ago in [19] by entirely different methods, and recently improved in [12], nothing seem to have appeared, apart of some estimates of a somewhat similar nature given in [16] (in any case the Neumann case and the compact manifold case are not discussed there) concerning asymptotics of evolution equations driven by the p–Laplacian in bounded domains before the recent work [8]. In this paper a similar discussion is given for the *Euclidean* p–Laplacian with Dirichlet boundary conditions on a bounded Euclidean domain: the solution approaches zero, instead of $\overline{u}$, in the course of time. In [10] a generalization of such results to a much larger class of operators is given, but Dirichlet boundary conditions are still assumed. The Dirichlet boundary conditions determine the form of the Sobolev inequalities on which our work relies and thus the situation is different from the very beginning. We shall also comment later on the case of Dirichlet boundary conditions is much easier and can be dealt with in the present case as well, and that the case of Neumann boundary conditions displays exactly the same properties discussed in Theorem (1.1). In the case of the present type of evolutions it seems that even the fact that $u(t)$ approaches $\overline{u}$ in the course of time is new. Similar results have been proved in [5] in the case $p > d$.

2. Entropy and Logarithmic Sobolev Inequalities

In this section we will prove a family of logarithmic Sobolev inequalities, which will be an essential tool in the rest of the paper. They involve the entropy or Young functional below:

$$J(r,u) = \int_M \log\left(\frac{|u|}{\|u\|_r}\right)\frac{|u|^r}{\|u\|_r^r}\,\mathrm{d}x \tag{2.1}$$

well defined for any $r \geq 1$ and $u \in X = \bigcap_{p=1}^{+\infty} \mathrm{L}^p(M)$.

Proposition 2.1. *The logartithmic Sobolev inequality*

$$pJ(p,u) \leq \frac{d}{p}\left[\varepsilon A \frac{\|\nabla f\|_p^p}{\|f\|_p^p} + \varepsilon \mathrm{Vol}(M)^{p/p^*}\frac{|\overline{f}|^p}{\|f\|_p^p} - \log\varepsilon\right] \tag{2.2}$$

holds true for any $\varepsilon > 0$, *for all* $f \in W^{1,p}(M)$, $1 \leq p < d$, $d \geq 2$. *Here*

$$A = 2^p \overline{A}$$

and $\overline{A}$ is the constant appearing in the classical Sobolev inequality:

$$\|u-\overline{u}\|_{p^*} \le \overline{A}\,\|\nabla u\|_p\,, \qquad p^* = \frac{p\ d}{d-p} \tag{2.3}$$

Proof. First we notice that

$$\begin{aligned}\|u\|_{p^*} - |\overline{u}|\mathrm{Vol}(M)^{1/p^*} &= \|u\|_{p^*} - \|\overline{u}\|_{p^*} \\ &\le \|u-\overline{u}\|_{p*} \le \overline{A}\|\nabla u\|_p.\end{aligned}$$

Thus

$$\begin{aligned}\|u\|_{p^*}^p &\le \left(\overline{A}\|\nabla u\|_p + |\overline{u}|\mathrm{Vol}(M)^{1/p^*}\right)^p \\ &\le 2^{p-1}\left(\overline{A}\|\nabla u\|_p^p + |\overline{u}|^p\mathrm{Vol}(M)^{p/p^*}\right)\end{aligned} \tag{2.4}$$

where we have used the numerical Young inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$.
Now we prove the LSI (2.2):

$$\begin{aligned}pJ(p,u) &= \int_M \log\left(\frac{|u(x)|^p}{\|u\|_p^p}\right)\frac{|u(x)|^p}{\|u\|_p^p}\,\mathrm{d}x = \frac{d-p}{p}\int_M \log\left(\frac{|u(x)|^{\frac{p^2}{d-p}}}{\|u\|_p^{\frac{p^2}{d-p}}}\right)\frac{|u(x)|^p}{\|u\|_p^p}\,\mathrm{d}x \\ &\le \frac{d-p}{p}\log\left(\int_M \frac{|u(x)|^{\frac{p^2}{d-p}+p}}{\|u\|_p^{\frac{p^2}{d-p}+p}}\,\mathrm{d}x\right) \\ &= \frac{d-p}{p}\log\left(\frac{\|u\|_{\frac{pd}{d-p}}^{\frac{pd}{d-p}}}{\|u\|_p^{\frac{pd}{d-p}}}\right) = \frac{d}{p}\log\frac{\|u\|_{p^*}^p}{\|u\|_p^p} \\ &\le \frac{d}{p}\log\left(\frac{2^{p-1}\overline{A}\|\nabla u\|_p^p + 2^{p-1}\mathrm{Vol}(M)^{p/p^*}|\overline{u}|^p}{\|u\|_p^p}\right) \\ &\le \frac{d}{p}\varepsilon 2^{p-1}\overline{A}\frac{\|\nabla u\|_p^p}{\|u\|_p^p} + \frac{d}{p}\varepsilon 2^{p-1}\mathrm{Vol}(M)^{p/p^*}\frac{|\overline{u}|^p}{\|u\|_p^p} - \log\varepsilon.\end{aligned}$$

Indeed, we first used Jensen inequality for the probability measure $\frac{|u(x)|^p}{\|u\|_p^p}\,\mathrm{d}x$, then the inequality (2.4) and finally the numerical inequality $\log(t) \le \varepsilon t - \log\varepsilon$, which holds for any $\varepsilon, t > 0$.
The proof is thus complete. □

3. Preliminary Results

We first recall two facts proved in [5] for the case $p > d$ remarking that their proof do not depend upon the range of p.

Lemma 3.1. *The semigroup $\{T_t\}_{t\ge 0}$ associated with the functional $\mathcal{E}_p$ satisfies the properties:*

- *$\overline{T_t u} = \overline{u}$ for any $u \in L^1(M)$ and any $t \geq 0$;*
- *$T_t u = T_t(u - \overline{u}) + \overline{u}$ for all $u \in L^1(M)$.*

In view of the above lemma it is clear that it suffices to prove theorem (1.1) for data with zero mean.

Lemma 3.2. *Let u be a weak solution to the equation (1.2) corresponding to an essentially bounded initial datum $u_0 \in \mathrm{L}^\infty(M)$ with zero mean. Let also $r : [0,t) \to [2,+\infty)$ be a monotonically non-decreasing C^1 function. Then*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} &= \frac{\dot r(s)}{r(s)} J\left(r(s), u(s)\right) \\ &\quad - \left(\frac{p}{r(s)+p-2}\right)^p \frac{(r(s)-1)}{\|u(s)\|_{r(s)}^{r(s)}} \left\|\nabla\left(|u(s)|^{\frac{r(s)+p-2}{p}}\right)\right\|_p^p\end{aligned} \tag{3.1}$$

Lemma 3.3. *Let u be a weak solution to the equation (1.2) corresponding to an essentially bounded initial datum $u_0 \in \mathrm{L}^\infty(M)$ with zero mean.*
Let also $r : [0,t) \to [2,+\infty)$ be a monotonically non-decreasing C^1 function. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} &\leq -\frac{\dot r(s)}{r(s)}\frac{d(p-2)}{pr(s)+d(p-2)}\log\|u(s)\|_{r(s)} + \\ &- \frac{\dot r(s)}{r(s)}\frac{d}{pr(s)+d(p-2)}\log\left(\frac{p^{p+2}}{dA}\frac{r(s)^3(r(s)-1)}{\dot r(s)(r(s)+p-2)^p(pr(s)+d(p-2))}\right) \\ &+ K\|u(0)\|_2^{p-2}\end{aligned} \tag{3.2}$$

where $K = \mathrm{Vol}(M)^{(3/2)}p/A$.

Proof. We can rewrite the LSI (2.2) in the following form:

$$\|\nabla f\|_p^p \geq \frac{p\|f\|_p^p}{\varepsilon A d}\left[J(1, f^p) + \frac{d}{p}\log(\varepsilon)\right] - \frac{|\overline{f}|^p}{A}.$$

Then we apply it to the function $f = |u(s,x)|^{(r(s)+p-2)/p}$ and obtain:

$$\begin{aligned}\left\|\nabla|u(s)|^{(r(s)+p-2)/p}\right\|_p^p &\geq \frac{p\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\varepsilon A d}\left[J(1, u(s)^{r(s)+p-2}) + \frac{d}{p}\log(\varepsilon)\right] \\ &\quad - \frac{\left|\overline{|u(s)|^{(r(s)+p-2)/p)}}\right|^p}{A}\end{aligned} \tag{3.3}$$

since $\||u(s)|^{(r(s)+p-2)/p}\|_p^p = \|u(s)\|_{r(s)+p-2}^{r(s)+p-2}$. Then we apply this to the inequality (3.1) of previous lemma and we obtain:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} &= \frac{\dot r(s)}{r(s)} J\left(r(s), u(s)\right) \\ &\quad - \left(\frac{p}{r(s)+p-2}\right)^p \frac{r(s)-1}{\|u(s)\|_{r(s)}^{r(s)}} \left\|\nabla\left(|u(s)|^{\frac{r(s)+p-2}{p}}\right)\right\|_p^p\end{aligned} \tag{3.4}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} \le \frac{\dot r(s)}{r(s)^2}J\left(1,u(s)^{r(s)}\right) - \frac{p^{p+1}(r(s)-1)}{\varepsilon Ad(r(s)+p-2)^p}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\times$$
$$\times\left[J\left(1,u(s)^{r(s)+p-2}\right) - \frac{d}{p}\log\varepsilon\right] + R(p,r(s),u(s),A,d)$$

since $J\left(1,u(s)^{r(s)}\right) = r(s)J\left(r(s),u(s)\right)$, where

$$R = \frac{p^p(r(s)-1)}{A(r(s)+p-2)^p}\frac{\left|\overline{|u(s)|^{(r(s)+p-2)/p}}\right|^p}{\|u(s)\|_{r(s)}^{r(s)}}.$$

Now choose

$$\varepsilon = \varepsilon(s) = \frac{r(s)^3}{\dot r(s)}\frac{p^{p+2}(r(s)-1)}{Ad(r(s)+p-2)^p(p(r(s)+d(p-2)))}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}$$
$$= \varepsilon_1\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}$$

and obtain from (3.4):

(3.5)
$$\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} \le \frac{\dot r(s)}{r(s)^2}\left[J\left(1,u(s)^{r(s)}\right) - \frac{pr(s)}{pr(s)+d(p-2)}J\left(1,u(s)^{r(s)+p-2}\right)\right.$$
$$\left.-\frac{pdr(s)}{p(pr(s)+d(p-2))}\log\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\right]$$
$$-\frac{\dot r(s)}{r(s)^2}\frac{pdr(s)}{p(pr(s)+d(p-2))}\log\varepsilon_1 + R$$
$$\le \frac{\dot r(s)}{r(s)^2}\left[J\left(1,u(s)^{r(s)}\right) - \frac{pr(s)}{pr(s)+d(p-2)}J\left(1,u(s)^{r(s)+p-2}\right)\right.$$
$$\left.-\frac{(p-2)pdr(s)}{p(pr(s)+d(p-2))}J\left(1,u(s)^{r(s)}\right) - \frac{(p-2)pdr(s)}{p(pr(s)+d(p-2))}\log\|u(s)\|_{r(s)}^{r(s)}\right]$$
$$-\frac{\dot r(s)}{r(s)^2}\frac{pdr(s)}{p(pr(s)+d(p-2))}\log\varepsilon_1 + R$$
$$= \frac{\dot r(s)}{r(s)^2}\frac{pr(s)}{(pr(s)+d(p-2))}\left[J\left(1,u(s)^{r(s)}\right) - J\left(1,u(s)^{r(s)+p-2}\right)\right]$$
$$-\frac{\dot r(s)}{r(s)^2}\frac{(p-2)pdr(s)}{p(pr(s)+d(p-2))}\log\|u(s)\|_{r(s)}$$

$$-\frac{\dot r(s)}{r(s)^2}\frac{pdr(s)}{p(pr(s)+d(p-2))}\log\varepsilon_1+R$$
$$\le -\frac{\dot r(s)}{r(s)}\frac{(p-2)d}{pr(s)+d(p-2)}\log\|u(s)\|_{r(s)}-\frac{\dot r(s)}{r(s)}\frac{d}{pr(s)+d(p-2)}\log\varepsilon_1+R$$

We used first the fact that

$$\log\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\ge(p-2)\left[J\left(r(s),u(s)\right)+\log\|u(s)\|_{r(s)}\right] \tag{3.6}$$

which follows from two basic facts. First, the function $N(r,u)=\log\|u\|_r^r$ is convex with respect to the variable $r\ge 1$, so its derivative is an increasing function of $r\ge 1$. Moreover $N^{'}(r,u)=J(r,u)+\log\|u\|_r$, so the following inequality:

$$N(r+p-2,u)-N(r,u)\ge N^{'}(r,u)(p-2)=\left[J(r,u)+\log\|u\|_r\right](p-2)$$

holds if $p\ge 2$ and leads to (3.6).
The last estimate is obtained by the following monotonicity property of the Young functional

$$J\left(1,u^r\right)-J\left(1,u^{r+p-2}\right)\le 0,\quad\text{if } p\ge 2$$

the proof of the fact that $J\left(1,u^r\right)$ is a non-decreasing function of $r\ge 1$ is a consequence of the convexity (w.r.t. the variable r) of the function:

$$\phi(r,u)=\log\|u\|_{1/r}.$$

We refer to [3] for a proof of such fact, but comment that it is equivalent to the well known interpolation inequality:

$$\|u\|_{1/r}\le\|u\|_{1/p}^{\theta}\|u\|_{1/q}^{1-\theta}$$

valid when $\frac{1}{r}=\frac{\theta}{p}+\frac{1-\theta}{q}$. Now deriving ϕ respect to r gives us:

$$\frac{\mathrm{d}}{\mathrm{d}r}\phi(r,u)=-\frac{1}{r}J\left(\frac{1}{r},u\right)$$

thus, as derivative of a convex functions, $-\frac{1}{r}J\left(\frac{1}{r},u\right)$ is non-decreasing.
Our next goal will be to give an estimate on the term R. To this end we use an Hölder and an interpolation inequality to yield

$$\|u\|_{(r+p-2)/p}\le\mathrm{Vol}(M)^{1/(r+p-2)}\|u\|_{(r+p-2)/(p-1)}$$
$$\le\mathrm{Vol}(M)^{1/(r+p-2)}\|u\|_1^{(p-2)/(r+p-2)}\|u\|_r^{r/(r+p-2)}.$$

Therefore

$$\frac{\left|\overline{|u(s)|^{(r(s)+p-2)/p}}\right|^p}{\|u(s)\|_{r(s)}^{r(s)}}=\frac{\|u(s)\|_{(r(s)+p-2)/p}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}$$
$$\le\mathrm{Vol}(M)\frac{\|u(s)\|_{(r(s)+p-2)/(p-1)}^{(r(s)+p-2)}}{\|u(s)\|_{r(s)}^{r(s)}}\le\mathrm{Vol}(M)\|u(s)\|_1^{p-2}.$$

The statement finally follows by the bounds

$$\frac{p^p(r(s)-1)}{A(r(s)+p-2)^p} \le \frac{p^p}{A(q+p-2)^{p-1}} \le \frac{p}{A}$$

valid because $r(s) \ge q \ge 1$ and $p \ge 2$ by assumption, together with the Hölder inequality and the L^2 contraction property of the evolution at hand:

$$\|u(s)\|_1 \le \mathrm{Vol}(M)^{(1/2)}\|u(s)\|_2 \le \mathrm{Vol}(M)^{(1/2)}\|u(0)\|_2 = \mathrm{Vol}(M)^{(1/2)}\|u_0\|_2$$

which is well known to hold for any $s > 0$. □

Lemma 3.4. *Let u be a weak solution to the equation (1.2) corresponding to an essentially bounded initial datum $u_0 \in L^\infty(M)$ with zero mean.*
Let also $r : [0,t) \to [2,+\infty)$ be a monotonically non-decreasing C^1 function. Then the following differential inequality holds true for any $s \ge 0$:

$$\frac{\mathrm{d}}{\mathrm{d}s}y(s) + p(s)y(s) + q(s) \le 0 \tag{3.7}$$

With

(3.8)

$$\begin{aligned} y(s) &= \log \|u(s)\|_{r(s)} \\ p(s) &= \frac{\dot r(s)}{r(s)}\frac{d(p-2)}{pr(s)+d(p-2)} \\ q(s) &= \frac{\dot r(s)}{r(s)}\frac{d}{pr(s)+d(p-2)}\log\left(\frac{p^{p+2}r(s)^3(r(s)-1)}{Ad\dot r(s)(pr(s)+d(p-2))(r(s)+p-2)^p}\right) \\ &- K\|u_0\|_2^{p-2} \end{aligned}$$

In particular, choosing $r(s) = \frac{qt}{t-s}$, one gets the bound:

$$y(t) = \lim_{s\to t^-} y(s) \le \lim_{s\to t^-} y_L(s) = y_L(t)$$

with

$$\begin{aligned} y_L(t) = &\frac{pq}{pq+d(p-2)}y_L(0) \\ &- \frac{d}{pq+d(p-2)}\log(t) + c_2(p,q,d,\mathrm{Vol}(M))\|u_0\|_2^{p-2}t + c_1(p,q,d) \end{aligned} \tag{3.9}$$

Proof. The fact that $y(s)$ satisfies the differential inequality (3.7) follows immediately by the inequality (3.2) of lemma (3.3), by our choice of $p(s)$ and $q(s)$. Therefore $y(s) \le y_L(s)$ for any $s \ge 0$ provided $y(0) \le y_L(0)$ where $y_L(s)$ is a solution to:

$$\frac{\mathrm{d}}{\mathrm{d}s}y_L(s) + p(s)y_L(s) + q(s) = 0$$

i.e.

$$y_L(s) = \mathrm{e}^{-P(s)}\left[y_L(0) - \int_0^s q(\lambda)\mathrm{e}^{P(\lambda)}\mathrm{d}\lambda\right] = \mathrm{e}^{-P(s)}\left[y_L(0) - Q(s)\right]$$

where

$$P(s) = \int_0^s p(\lambda)\mathrm{d}\lambda, \quad Q(s) = \int_0^s q(\lambda)\mathrm{e}^{P(\lambda)}\mathrm{d}\lambda.$$

Choosing $r(s)$ as in the statement one gets, after straightforward calculations and beside noticing that $r(0) = q$ and $r(s) \to +\infty$ as $s \to t^-$:

$$\mathrm{e}^{-P(t)} = \lim_{s\to t^-} \mathrm{e}^{-P(s)} = \frac{pq}{pq + d(p-2)}$$

and

$$\begin{aligned} Q(t) &= \lim_{s\to t^-} \mathrm{Q(s)} \\ &= \frac{d}{pq + d(p-2)} \frac{pq + d(p-2)}{pq} \log\left(\frac{p^{p+2}qt}{Ad}\right) \\ &+ c_0(p,q,d) + c_2(p,q,d,\mathrm{Vol}(M))\|u_0\|_2^{p-2}t \end{aligned}$$

for suitable numerical constants $c_0(p,q,d)$ and $c_2(p,q,d,\mathrm{Vol}(M))$. □

End of proof of Theorem 1.1.
The following contractivity property holds true for all $0 \le s \le t$:

$$\|u\,(t)\|_r \le \|u\,(s)\|_r$$

Therefore by the previous results one has, for all such s and t :

$$\|u\,(t)\|_{r(s)} \le \|u\,(s)\|_{r(s)} = \exp\left(\log\|u\,(s)\|_{r(s)}\right) = e^{y(s)} \le e^{y_L(s)}$$

whence, letting $s \to t^-$, and recalling that $r\,(s) \to +\infty$ as $s \to t^-$, we deduce:

$$\begin{aligned} \|u\,(t)\|_\infty &= \lim_{s\to t^-} \|u\,(t)\|_{r(s)} \le \lim_{s\to t^-} \|u\,(s)\|_{r(s)} \\ &= \lim_{s\to t^-} e^{y(s)} \le \lim_{s\to t^-} e^{y_L(s)} = e^{y_L(t)}. \end{aligned}$$

By the explicit form for $e^{y_L(t)}$ we can now prove the bound (1.3) for small times: it is sufficient to let $y(t) = \log\|u(t)\|_\infty$, $y(0) = y_L(0) = \log\|u(0)\|_q = \log\|u_0\|_q$. So we obtain:

$$\|u(t)\|_\infty \le \mathrm{e}^{c_1(p,q,d)+c_2(p,q,d,\mathrm{Vol}(M))\|u_0\|_2^{p-2}t} \frac{\|u_0\|_q^{\frac{pq}{pq+d(p-2)}}}{t^{\frac{d}{pq+d(p-2)}}}$$

To conclude the proof for small times, we prove an L^2–L^2 time decay estimate for arbitrary time. We compute, for initial data with zero mean

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 &= -2\|\nabla u\|_p^p \\ &\le -2\overline{A}^{-p}\|u(t)\|_{p^*}^p \\ &\le -2\overline{A}^{-p}\mathrm{Vol}(M)^{-p\frac{2p+d(p-2)}{p-d}}\|u(t)\|_2^p \end{aligned}$$

where we have used the Sobolev inequality in the first step and the constant $\overline{A}$ appearing in (2.3). Thus, setting $f(t) = \|u(t)\|_2^2$ we have proved that

$$\dot{f}(t) \le -2\overline{A}^{-p}\mathrm{Vol}(M)^{-p\frac{2p+d(p-2)}{p-d}} f(t)^{p/2}.$$

This yields the bound, valid for all positive t:

$$\|u(t)\|_2 \le \frac{1}{\left(Bt + \|u(0)\|_2^{2-p}\right)^{1/(p-2)}}$$

where we have set

$$B = \frac{(p-2)}{\overline{A}^p \mathrm{Vol}(M)^{p\frac{2p+d(p-2)}{p-d}}}.$$

This last estimate also gives the so called absolute bound:

$$\|u(t)\|_2 \le \frac{1}{(Bt)^{1/(p-2)}}$$

The absolute bound, together with the L^q–L^∞ smoothing property above and with the semigroup property yields the bound:

$$\begin{aligned}
\|u(t)\|_\infty &\le \mathrm{e}^{c_1(p,q,d)+c_2(p,q,d,\mathrm{Vol}(M))\|u(t/2)\|_2^{p-2}t/2} \frac{\|u(t/2)\|_q^{\frac{2p}{2p+d(p-2)}}}{(t/2)^{\frac{d}{pq+d(p-2)}}} \\
&\le \mathrm{e}^{c_1(p,q,d)+c_2(p,q,d,\mathrm{Vol}(M))B} \frac{\|u(t/2)\|_q^{\frac{2p}{2p+d(p-2)}}}{(t/2)^{\frac{d}{pq+d(p-2)}}} \\
&\le C(p,q,d,\overline{A},\mathrm{Vol}(M)) \frac{\|u(0)\|_q^{\frac{2p}{2p+d(p-2)}}}{t^{\frac{d}{pq+d(p-2)}}}
\end{aligned}$$

in the last step we used the L^q contraction property, which is well known to hold for any $q \ge 1$ and $t \ge 0$ and we obtained the desired bound for small times, at least for essentially bounded initial data.
To deal with the case of general L^q–data, it suffices to refer to the discussion given in [8], which does not depend either upon the value of p or on the Euclidean setting. This concludes the proof for small times.

To deal with the case of large times, we use again the above L^2–L^∞ decay, the L^2–L^2 time decay, together with the above absolute bound and the semigroup property to yield, for all positive t:

$$\begin{aligned}
\|u(t)\|_\infty &\le \mathrm{e}^{c_1(p,q,d)+c_2(p,q,d,\mathrm{Vol}(M))\|u(t/2)\|_q^{p-2}t/2}\|u(t/2)\|_2^{\frac{2p}{2p+d(p-2)}} \\
&\le \frac{\mathrm{e}^{c_1(p,q,d)+c_2(p,q,d,\mathrm{Vol}(M))2/B}}{\left(B(t/2) + \|u(0)\|_2^{2-p}\right)^{\frac{2p}{(2p+d(p-2))(p-2)}}}
\end{aligned}$$

The latter statement is obtained from the numerical inequality

$$a + b \ge a^\varepsilon b^{1-\varepsilon}$$

valid for all positive a, b and all $\varepsilon \in (0,1)$. Putting $a = Bt$ and $b = \|u(0)\|_2^{2-p}$ we thus get, for all $t > 1$

$$\|u(t)\|_\infty \le \frac{C(p,2,d,\overline{A},\mathrm{Vol}(M))\|u(0)\|_2^{\frac{2p(1-\varepsilon)}{(2p+d(p-2))}}}{(Bt)^{\frac{2p\varepsilon}{(2p+d(p-2))(p-2)}}}$$

□

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