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ULTRACONTRACTIVITY AND CONVERGENCE TO EQUILIBRIUM FOR SUPERCRITICAL QUASILINEAR PARABOLIC EQUATIONS ON RIEMANNIAN MANIFOLDS

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Abstract. Let (M,g) be a compact Riemannian manifold without boundary and dimension $d \geq 3$. Let $u(t)$ be a solution to the problem $\dot{u} = \triangle_p u$, $u(0) = u_0$, $\triangle_p$ being the Riemannian p–Laplacian with $p > d$. Let also $\overline{u}$ be the (time–independent) mean of $u(t)$. We will prove ultracontractive estimates of the type $\|u(t) - \overline{u}\|_\infty \leq C\|u(0) - \overline{u}\|_q^\gamma / t^\beta$. The constant C depends only on p and q, on geometric quantities of M and on the dimension of the manifold, while the exponents β and γ depend only on p and q and differ according to the regimes $t \to 0$ and $t \to +\infty$. Similar bounds hold when $\triangle_p$ is replaced by the subelliptic p–Laplacian associated to a collection of Hörmander vector fields. We also prove the L^q–L^∞ Hölder continuity of the solutions, and apply similar methods to study the same questions for evolution equations on manifolds with boundary. The bounds are sharp in several of the above cases. The method relies on the theory of nonlinear Markov semigroups ([9]) and on the connection between nonlinear ultracontractivity and logarithmic Sobolev inequality for the p–energy functional.

1. Introduction

Let (M,g) be a smooth, connected and compact Riemannian manifold without boundary, whose dimension is denoted by d and satisfies the condition $d \geq 3$. Let ∇ be the Riemannian gradient and m_g the Riemannian

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measure and consider, for $p > d$ (the *supercritical case*), the following functional:

$$\mathcal{E}_p(u) = \int_M |\nabla u|^p \, \mathrm{d}m_g \tag{1.1}$$

for any $u \in \mathrm{L}^2(M, m_g)$, where we adopt the convention that $\mathcal{E}_p(u)$ equals $+\infty$ if the distributional gradient of u does not belong to $\mathrm{L}^p(M, m_g)$. It is well–known that $\mathcal{E}_p$ is a convex, lower–semicontinuous functional. The subgradient of the functional $\mathcal{E}_p/p$, denoted by $\triangle_p$, generates a (nonlinear) strongly continuous nonexpansive semigroup $\{T_t : t \geq 0\}$ on $\mathrm{L}^2(M, m_g)$. On smooth functions, the operator $\triangle_p$ reads

$$\triangle_p u = \mathrm{div}\big(|\nabla u|^{p-2} \nabla u\big),$$

$|\cdot| = |\cdot|_x$ indicating the norm in the tangent space at x. We refer to [15] as a complete general reference for parabolic equations driven by operators of p–Laplace type in the Euclidean setting. It should be commented that the function $u(t,x) := (T_t u)(x)$ is also a weak solution in the sense of [15] of the equation $\dot{u} = \triangle_p u$.

Let us denote by $\overline{u}$ the mean of an integrable function u:

$$\overline{u} = \frac{1}{\mathrm{vol}(M)} \int_M u \, \mathrm{d}m_g.$$

Let finally $u(t) := T_t u$ be the time evolution associated to the semigroup at hand and to the initial datum $u(0) = u \in \mathrm{L}^1(M, m_g)$. Then $\overline{u(t)}$ does not depend upon time, so that it equals $\overline{u}$: we prove this fact by means of abstract semigroup theory in Lemma 3.1.

Our goal is to prove that the evolution enjoys an *instantaneous* L^q–L^∞ *smoothing property*, in the sense that the time–evolved $u(t)$ of an initial datum $u_0 \in \mathrm{L}^q$ ($q \geq 1$) is an essentially bounded function at each time $t > 0$. Moreover, $u(t)$ approaches $\overline{u}$ as time tends to infinity, and the quantitative estimate

$$||u(t) - \overline{u}||_\infty \leq \frac{C}{t^\beta} ||u(0) - \overline{u}||_q^\gamma \tag{1.2}$$

holds true at all times, where $\|\cdot\|_q$ denotes the norm in the space $\mathrm{L}^q(M, m_g)$ ($q \in [1, +\infty]$), β, γ are numerical constants depending only on p, q and d, and C is a constant depending on geometrical quantities like the injectivity radius of M, the sectional curvature of M, the volume of M and the diameter of M (or more precisely on the quantity

$$D(M) := \sup_{x \in M} \frac{1}{\mathrm{Vol}\, M} \int_M \varrho(x, \cdot) \mathrm{d}m_g,$$

ϱ being the Riemannian distance on M). By similar methods it is also possible to prove the following L^q–L^∞ Hölder continuity for the solutions to the evolution considered: if $u(t)$ and $v(t)$ are the solutions corresponding to the initial data $u(0)$, $v(0)$ *with common mean value*, then

$$\|u(t)-v(t)\|_\infty \le \frac{C}{t^\beta}\|u(0)-v(0)\|_q^\gamma$$

with β, γ as above.

Our approach in proving (1.2) is functional analytic in character and relies essentially only on the classical Sobolev inequality, valid for $p > d$:

$$\big\|u-\overline{u}\big\|_\infty \le C\big\|\nabla u\big\|_p. \tag{1.3}$$

A closer inspection of the proof reveals that, besides the Sobolev inequality, the property of $\mathcal{E}_p$ which makes the method work is the fact that such functional is constructed from a *derivation*, the gradient operator ∇. Our results can thus be generalized to a much wider setting involving functionals constructed from more general derivations (see [10] for such generalizations in the case $p < d$). We choose here to single out, in the final section, the particularly relevant case of subelliptic p–Laplacian operators associated to Hörmander vector fields on manifolds. This means that we consider the functional

$$\mathcal{E}_{p,X}(u) := \int_M |Xu|^p \,\mathrm{d}m_g,$$

where $\{X_i\}_{i=1}^m$ is a collection of Hörmander vector fields on M and

$$|Xu|^2 := \sum_{i=1}^m |X_i u|^2.$$

Then $\mathcal{E}_{p,X}$ is again a convex lower–semicontinuous functional on $\mathrm{L}^2(M,m_g)$. The subdifferential of $\mathcal{E}_{p,X}/p$ is denoted by $\triangle_{p,X}$ and generates a nonexpansive semigroup $\{T_t : t \ge 0\}$ on $\mathrm{L}^2(M,m_g)$. Formally one can write, on smooth functions,

$$\triangle_{p,X} = \sum_{i=1}^m X_i^*\big(|X_i u|^{p-2} X_i u\big),$$

X_i^* denoting the formal adjoint of X_i. Such an operator is usually referred to as the *subelliptic p–Laplacian*. An estimate of the same form given in (1.2) holds true, but the Euclidean dimension is replaced by the so–called *homogeneous dimension* associated to the vector fields. Even more general classes of vector fields can be considered, if a suitable Sobolev inequality holds.

The first step in proving the estimate (1.2) is the *Markov property* for the semigroups at hand. This is a property closely related to the maximum principle for the generator of the semigroup $\{T_t\}$ considered (see e.g. [4]), and amounts to requiring that $\{T_t\}$ is order preserving and can be extended to a nonexpansive semigroup on every $\mathrm{L}^p(M, m_g)$ space for $p \in [1, +\infty]$. We refer to [5] and [30] as complete general references for nonlinear semigroup theory.

It has been proven in [9] that the Markov property holds for the nonlinear evolutions considered here (even if the requirement $p > d$ does not hold), this being based upon the fact that $\mathcal{E}_p$ and $\mathcal{E}_{p,X}$ are *nonlinear Dirichlet forms* in the sense of [10] (see also [4]). The second step will consist in proving a new family of Gross' *logarithmic Sobolev inequalities* (in the spirit of [22]) involving the functionals $\mathcal{E}_p$ and $\mathcal{E}_{p,X}$, respectively. The proof of such an inequality will rely only upon the classical Sobolev inequality for the case $p > d$ (or upon its subelliptic generalization), and this is the point in which the geometrical objects appearing in the final inequalities appear. Finally, we shall derive a first–order differential inequality for the quantity $y(s) := \log(||u(s, \cdot)||_{r(s)})$ whenever $r : [0, t) : [1, +\infty)$ is any C^1 function. The coefficients of such an inequality depend only on time t, on geometrical quantities, on d and on p. Such an inequality will be afterwards integrated to yield the stated bounds. In fact, one obtains the required bounds first for a solution corresponding to essentially bounded data, and then for general L^q data by using the Markov property.

A comparison with some previous results is now given. While a discussion of similar problems *in the whole of* $\mathbb{R}^n$ has been given long ago in [17] by entirely different methods, and recently improved in [13], nothing seems to have appeared, apart from some estimates of a somewhat similar nature given in [15] (in any case the Neumann case and the compact manifold case are not discussed there) concerning asymptotics of evolution equations driven by the p–Laplacian in bounded domains before the recent work [8]. In this paper a similar discussion is given for the *Euclidean* p–Laplacian with Dirichlet boundary conditions on a bounded Euclidean domain: the solution approaches zero, instead of $\overline{u}$, in the course of time. In [10] a generalization of such results to a much larger class of operators is given, but Dirichlet boundary conditions are still assumed. Moreover, in both papers the assumption $p < d$ (*subcritical case*) is assumed throughout: such an assumption and the Dirichlet boundary conditions determine the form of the Sobolev inequalities on which our work relies, and thus the situation is different from the very beginning. We shall also comment later on the fact that the case of Dirichlet boundary conditions is much easier and can be dealt with in the present case as well, and that the case of Neumann boundary conditions displays exactly the same convergence properties discussed in Theorem 1.1. In the case of the present type of evolutions it seems that even the fact that $u(t)$ approaches $\overline{u}$ in the course of time is new. Nevertheless if Ω is a sufficiently regular Euclidean domain, considering the inhomogeneous *Dirichlet* problem, the $t \to +\infty$ limit of the solution of the corresponding p– Laplacian evolution equation has been characterized in terms of the elliptic problem with the same boundary values in [24].

Theorem 1.1. *Let (M, g) be a smooth, connected and compact Riemannian manifold without boundary and with dimension $d > 2$. Consider, for any $t > 0$, the function $u(t) := T_t u(0)$, where $\{T_t : t \geq 0\}$ is the semigroup generated by the subdifferential of the functional $\mathcal{E}_p$ and $u(0) \in L^q(M)$ with $q \geq 1$. Then the following ultracontractive bound holds true for all $t \in (0, 1]$:*

$$\left\|u(t) - \overline{u}\right\|_\infty \leq \frac{A(p, q, d, M)}{t^\beta} \left\|u(0) - \overline{u}\right\|_q^\gamma \tag{1.4}$$

with

$$\beta = \frac{1}{q + p - 2}, \quad \gamma = \frac{q}{q + p - 2} \tag{1.5}$$

and

$$\begin{aligned} A(p, q, d, M) =& e^E (2^{p-1} C^p pq)^{1/(q+p-2)} \Big(\frac{q + p - 2}{q + 1}\Big)^{1/[(p-2)(p-3)]} \\ &\times b^\alpha \Big(\frac{q}{q + p - 2}\Big)^{q/[(p-2)^2(q+p-2)]}, \end{aligned} \tag{1.6}$$

where b and C are the constants appearing in the Sobolev inequality

$$\left\|u - \overline{u}\right\|_\infty \leq C b^{[1-(d/p)]} \left\|\nabla u\right\|_p,$$

$\alpha = (p - d)/(q + p - 2)$ and $E = p^p/[2^{2p-1} C^p b^{p-d} \mathrm{Vol}\, M]$. If $t > 2$ one has instead, for all data belonging to $L^2(M)$,

$$\|u(t) - \overline{u}\|_\infty \leq \frac{A(p, 2, d, M)}{\big(B(t - 1) + \|u(0) - \overline{u}\|_2^{2-p}\big)^{2/[p(p-2)]}} \tag{1.7}$$

and in particular, for any $\varepsilon \in [0, 1]$

$$\|u(t) - \overline{u}\|_\infty \leq A(p, 2, d, M) \frac{\|u(0) - \overline{u}\|_2^{2(1-\varepsilon)/p}}{[B(t - 1)]^{2\varepsilon/[p(p-2)]}}, \tag{1.8}$$

where

$$B = \frac{(p - 2) b^{d-p}}{C^p (\mathrm{Vol}\, M)^{p/2}}.$$

The proof will show that identical conclusions hold for the solutions to the equation $\dot{u} = \triangle_p u$ in bounded Euclidean domains, or in compact manifolds with smooth boundary, with homogeneous Neumann boundary conditions.

Corollary 1.2 (absolute bound)**.** *For all $t > 2$, all $\varepsilon \in (0,1)$ and all initial data u_0 in L^1 there exists $c_\varepsilon > 0$ such that*

$$\|u(t) - \overline{u}\|_\infty \leq c_\varepsilon t^{-(1-\varepsilon)/(p-2)} \tag{1.9}$$

independently of the initial datum u_0. Moreover, if the initial datum belongs to $L^r(M)$ with $\|u(0) - \overline{u}\|_r < 1$, then

$$\|u(t) - \overline{u}\|_\infty \leq c_\varepsilon \|u(0) - \overline{u}\|_r^\varepsilon t^{-(1-\varepsilon)/(p-2)}$$

for all $t \geq 2\|u(0) - \overline{u}\|_r^{2-p}$.

Remark 1.3 (sharpness of the bound)**.** 1) (compact manifold or Neumann cases). It is known from the results of [3] that a lower bound of the form

$$\|u(t)\|_2 \geq \frac{C}{t^{1/(p-2)}}$$

holds for any L^2 data and all t sufficiently large. A similar bound for the L^∞ norm thus holds as well. Hence, the bounds in Corollary 1.2 are close to the optimal ones for large time. For small times a comparison with the Baremblatt solutions ([15]) shows that the power of time is the correct one for data belonging to $\mathrm{L}^{p/d}$, while for data in L^q with $q > p/d$ the L^∞ our result is better in the sense that the norm diverges at a *slower* rate depending on q, a property which is familiar in the theory of linear ultracontractive semigroups but which seems not to have been explicitly stated so far in the nonlinear context.

2) (Dirichlet case). A similar result is shown on compact manifolds with smooth boundary, homogeneous Dirichlet boundary conditions being assumed. The main difference is in the fact that the solutions approach zero when t tends to infinity. The proof stems from the appropriate Sobolev inequality for functions in $W_0^{1,p}(M)$ and is easier than in the previous case. We shall deal with this situation in Section 5, but notice here that the time decay for $t \to +\infty$ proved there is sharp by comparison with the result of [28]. For short time remarks similar to the Neumann case hold. However, after completing the first draft of the present paper, we were acquainted with the preprints [14], [19]. By using their optimal logarithmic Sobolev inequality for the p–energy functional, bounds which are sharp also for general L^1 data and small times can be proved easily by the present methods in the Euclidean case.

Some words should now be said on the limit as $p \to \infty$ of our estimates. It is known that, in Euclidean domains, the solutions of the evolution equation associated $\dot{u}_p = \triangle_p u_p$, corresponding to a common initial datum u_0, converge in a suitable sense, in such limit, to the solution of the problem $\dot{u}_\infty \in L u_\infty$, $u(0) = u_0$, where L is the subdifferential of the functional which equals zero on the set of those $W_0^{1,\infty}$–functions such that $|\nabla u| \leq 1$ almost everywhere, and $+\infty$ otherwise. We refer to [2] for these results. Having at our disposal the above–mentioned bounds for general compact manifolds with boundary, one can gauge the sharpness of our bounds by taking limits as $p \to +\infty$. Our bounds will imply that $u_\infty(t)$ is bounded (for almost all t) by a geometrical constant, independently of the initial datum, as expected from the nature of the problem. To derive such a property we shall need the detailed expression of all constants appearing in (1.4); this is in fact the main reason for which we kept track of their value.

The paper is structured as follows: in Section 2, we prove some new logarithmic Sobolev inequalities which are of fundamental importance for the sequel. Section 3 contains several intermediate technical results preparatory to proving Theorem 1.1. The proof is completed in Section 4 by using the Markov property and a well–known scaling property for the evolution at hand. In Section 5, we deal with the case of homogeneous Dirichlet boundary conditions in manifolds with boundary and consider the limit as $p \to \infty$ in the Euclidean setting. In Section 6, we prove L^q–L^∞ Hölder continuity of the solutions and, in Section 7, we generalize the discussion to the case of subelliptic p–Laplacians.

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2. Sobolev and logarithmic Sobolev inequalities

Let us denote by δ_0 the injectivity radius of M. It may be useful to recall that, since M is compact, its injectivity radius is strictly positive (see [1]). Suppose that the sectional curvature K of M satisfies the bounds

$$-A^2 \leq K \leq B^2$$

for some A, B. Let also δ be a constant such that $\delta < \delta_0 \wedge (\pi/2B)$. We first recall the following well–known Sobolev inequality, valid in the case $p > d$ (see [1], p. 47):

$$\|u - \overline{u}\|_\infty \leq C b^{1-(d/p)} \|\nabla u\|_p, \tag{2.1}$$

where the constants C and b are the following, ω_d being the measure of the d–dimensional unit sphere:

$$C = \frac{2p}{p-d}\Big(\frac{2^d d}{\omega_{d-1}}\Big)^{1/p}\frac{\sinh(A\delta)}{A\delta}\Big(\frac{\pi}{2}\Big)^{(d-1)/p}; \tag{2.2}$$

$$b = \sup_{x\in M}\Big[\frac{1}{\operatorname{Vol} M}\int_M d(x,\cdot)^{1-\frac{d}{p}}\,\mathrm{d}m_g\Big]^{p/(p-d)}. \tag{2.3}$$

It is then clear than the constant b is always not larger than the diameter of M. Now we are ready to state and prove the following:

Proposition 2.1. *The logarithmic Sobolev inequality,*

$$\int_M |f|^p \log\Big(\frac{|f|}{\|f\|_p}\Big)^p\,\mathrm{d}x \le \varepsilon K b^{p-d}\mathcal{E}_p(f) + \|f\|_p^p\Big(\varepsilon\frac{2^{p-1}}{\operatorname{Vol} M} - \log\varepsilon\Big), \tag{2.4}$$

holds true for any $\varepsilon > 0$, for all $f \in W^{1,p}(M)$, $p > d > 2$. Here, $K = 2^{p-1}C^p$ and b, C are the constants appearing in the Sobolev inequality (2.1).

Proof. We shall prove the assertion, with no loss of generality, in the case in which f is such that $\|f\|_p = 1$, so that $\mu(x) = |f(x)|^p\,\mathrm{d}x$ is a probability measure. Then we compute

$$\begin{aligned}\int_M |f|^p\log(|f|^p)\,\mathrm{d}x &= \frac{p}{q}\int_M \log(|f|^q)|f|^p\,\mathrm{d}x \le \frac{p}{q}\log\int_M |f|^q|f|^p\,\mathrm{d}x\\ &= \frac{p}{q}\log\|f\|_{p+q}^{(p+q)p/p} = \frac{p}{q}\frac{p+q}{p}\log\|f\|_{p+q}^p = \Big(1+\frac{p}{q}\Big)\log\|f\|_{p+q}^p\end{aligned} \tag{2.5}$$

for any $q > 0$, where we have used in the first line Jensen's inequality for the probability measure μ. By letting $q \to +\infty$, we thus get

$$\int_M |f|^p\log(|f|^p)\,\mathrm{d}m_g \le \log\|f\|_\infty^p.$$

Using now the numerical inequality

$$\log t \le -\log\varepsilon + \varepsilon t \quad \forall\varepsilon, t > 0$$

it thus follows that

$$\int_D |f|^p\log(|f|^p)\,\mathrm{d}x \le -\log\varepsilon + \varepsilon\|f\|_\infty^p. \tag{2.6}$$

By the Sobolev inequality we have

$$\|f\|_\infty - \|\overline{f}\|_\infty \le \|f - \overline{f}\|_\infty \le Cb^{1-d/p}\|\nabla f\|_p \tag{2.7}$$

or

$$\|f\|_\infty^p \le (Cb^{1-d/p}\|\nabla f\|_p + |\overline{f}|)^p \le 2^{p-1}C^p b^{p-d}\|\nabla f\|_p^p + 2^{p-1}|\overline{f}|^p. \tag{2.8}$$

Here we used the numerical Young inequality $(a+b)^p \le 2^{p-1}(a^p+b^p)$. Combining the above inequality one gets

$$\int_D |f|^p \log(|f|^p)\,\mathrm{d}x \le -\log\varepsilon + \varepsilon 2^{p-1}\big(C^p b^{p-d}\|\nabla f\|_p^p + |\overline{f}|^p\big) \qquad (2.9)$$
$$\le -\log\varepsilon + \varepsilon K b^{p-d}\|\nabla f\|_p^p + \varepsilon\frac{2^{p-1}}{\mathrm{Vol\,M}}\|f\|_p^p$$

since $|\overline{f}|^p \le \|f\|_p^p/\mathrm{Vol\,M}$. This concludes the proof. □

3. Preliminary results

In this section we state and prove some preliminary results which prepare the ground for the proof of the main theorem. Clearly, the first two lemmas can be also proved directly, when thinking about weak solutions of the parabolic p–Laplacian, by a direct integration by parts. However, the following approach has a wider generality.

Lemma 3.1. *The formula* $\overline{T_t u} = \overline{u}$ *holds for any* $u \in L^1(M, m_g)$ *and any* $t \ge 0$.

Proof. We recall a result of [9]: a closed and convex set $K \subset \mathrm{L}^2(M, m_g)$ is left invariant by a semigroup $\{T_t\}_{t\ge 0}$ on $\mathrm{L}^2(M, m_g)$, associated to a convex lower–semicontinuous functional $\mathcal{E}$, if and only if

$$\mathcal{E}(P_K u) \le \mathcal{E}(u) \quad \forall u \in \mathrm{L}^2(M, m_g),$$

where P_K is the Hilbert projection onto K. Take here $K = K_c$ the set of functions whose mean value is a fixed $c \in \mathbb{R}$. The Hilbert projection onto K_c is $P_{K_c}(u) = u - \overline{u} + c$. Clearly,

$$\mathcal{E}_p(P_{K_c}(u)) = \mathcal{E}_p(u)$$

for all L^2 functions, so that each K_c is left invariant by the semigroup considered. This yields the assertion for L^2 data, and the same property follows by continuity and by the Markov property for all L^1 data. □

Lemma 3.2. *The semigroup* $\{T_t\}_{t\ge 0}$ *associated with the functional* $\mathcal{E}_p$ *satisfies the property* $T_t u = T_t(u - \overline{u}) + \overline{u}$ *for all* $u \in L^1(M, m_g)$.

Proof. As above it suffices to deal with L^2 data. Define

$$S_t u = T_t(u - \overline{u}) + \overline{u}.$$

We prove that $\{S_t\}_{t\ge 0}$ is a semigroup. Indeed, first notice that T_t preserves the space of functions with zero mean value (this follows from the previous lemma). Moreover,

$$S_{t_1}S_{t_2}(u) = T_{t_1}(S_{t_2}(u) - \overline{S_{t_2}(u)}) + \overline{S_{t_2}(u)} = T_{t_1}(S_{t_2}(u) - \overline{u}) + \overline{u}$$

$$= T_{t_1}(T_{t_2}(u - \overline{u}) + \overline{u} - \overline{u}) = T_{t_1+t_2}(u - \overline{u}) + \overline{u} = S_{t_1+t_2}(u).$$

Such a semigroup has clearly the same generator as T_t. □

In view of the above lemma it suffices to prove the statement of our main theorem for functions with zero mean. Indeed, if for all functions v with zero mean one has

$$\|T_t v\|_\infty \leq \text{const.}\, \|v\|_q,$$

then, for all functions u,

$$\|T_t u - \overline{u}\|_\infty = \|T_t(u - \overline{u})\|_\infty \leq \text{const.}\, \|u - \overline{u}\|_q.$$

The following result stems from the chain rule for the derivation ∇ and by the form of $\mathcal{E}_p$, by proceeding as in [10], since the proofs there did not depend upon the value of p. Below we shall use a crucial property for the semigroup considered here, proved in [9]: its *Markov property*, a property closely related to the maximum principle. It implies in particular that, if u is an essentially bounded function, then

$$\|T_t u\|_\infty \leq \|u\|_\infty$$

for all $t \geq 0$. In particular, $T_t u \in \mathrm{L}^\infty$ whenever $u \in \mathrm{L}^\infty$, so that all quantities below make sense because D has finite measure.

Lemma 3.3. *Let $T_t u$ be the semigroup associated to the convex lower–semicontinuous functional $\mathcal{E}_p/p$ and corresponding to an essentially bounded initial datum u_0 with zero mean. Let also $r : [0, +\infty) \to [2, +\infty)$ be a monotonically nondecreasing C^1 function. Then*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \log ||u(s)||_{r(s)} =& \frac{\dot{r}(s)}{r(s)} \int_M \frac{|u(s,x)|^{r(s)}}{||u(s)||_{r(s)}^{r(s)}} \log \frac{|u(s,x)|}{||u(s)||_{r(s)}} \,\mathrm{d}m_g \\ & - \Big(\frac{p}{r+p-2}\Big)^p \frac{(r(s)-1)}{||u(s)||_{r(s)}^{r(s)}} \mathcal{E}_p\Big(|u(s)|^{(r(s)+p-2)/p}\Big). \end{aligned} \tag{3.1}$$

We now use our logarithmic Sobolev inequalities to estimate the p–energy functional $\mathcal{E}_p$ in terms of an entropic integral. In fact, let us define $X := \cap_{p\geq 1} \mathrm{L}^p$ and consider the Young functional $J : [1, +\infty) \times X \to [0, +\infty]$ defined as follows:

$$J(q, u) := \int_M \frac{|u(x)|^q}{\|u\|_q^q} \log\Big(\frac{|u(x)|}{\|u\|_q}\Big) \,\mathrm{d}m_g.$$

Lemma 3.4. *If u is an essentially bounded initial datum with zero mean and $r : [0, +\infty] \to [1, +\infty]$ is a monotonically nondecreasing C^1 function, then the following inequality holds true for any $\varepsilon > 0$:*

$$\frac{\mathrm{d}}{\mathrm{d}s} \log \|u(s)\|_{r(s)} \leq \frac{\dot{r}(s)}{r(s)} J(r(s), u(s)) \tag{3.2}$$

$$- p(r(s)-1)\Big(\frac{p}{r(s)+p-2}\Big)^{p-1}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\times$$
$$\times\Big[\frac{1}{\varepsilon K b^{p-d}}\,J(r(s)+p-2,u(s))$$
$$+\frac{1}{(r(s)+p-2)Kb^{p-d}}\Big(\frac{\log\varepsilon}{\varepsilon}-\frac{2^{p-1}}{\operatorname{Vol}M}\Big)\Big],$$

where K and b are as in Proposition 2.1.

Proof. We can rewrite the family of logarithmic Sobolev inequalities of Section 2 as follows:

$$p\int_M \frac{|f(x)|^p}{\|f\|_p^p}\log\Big(\frac{|f(x)|}{\|f\|_p}\Big)\mathrm{d}m_g \le -\log\varepsilon+\mathcal{E}_p(f)\frac{\varepsilon K b^{p-d}}{\|f\|_p^p}+\frac{\varepsilon 2^{p-1}}{\operatorname{Vol}M}$$

or

$$pJ(p,f)\le -\log\varepsilon+\mathcal{E}_p(f)\frac{\varepsilon K b^{p-d}}{\|f\|_p^p}+\frac{\varepsilon 2^{p-1}}{\operatorname{Vol}M},$$

where $J(p,u)$ is the Young functional defined as above. Then the p–energy functional satisfies the following bound:

$$\mathcal{E}_p(f)\ge\frac{\|f\|_p^p}{\varepsilon K b^{p-d}}\big(\log\varepsilon+p\,J(p,f)\big)-\frac{2^{p-1}}{Kb^{p-d}\operatorname{Vol}M}\|f\|_p^p. \tag{3.3}$$

Choose now $f=|u(s)|^{r(s)+p-2/p}$, so that

$$\|f\|_p^p=\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}$$

and

$$pJ\big(p,|u(s\ )|^{r(s)+p-2/p}\big)=(r(s)+p-2)J\big(r(s)+p-2,u(s)\big).$$

Then the above inequality becomes

$$\mathcal{E}_p\big(|u(s)|^{\frac{r(s)+p-2}{p}}\big)\ge\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\varepsilon K b^{p-d}}(\log\varepsilon+pJ(p,|u(s)|^{r(s)+p-2/p}))$$
$$-\frac{2^{p-1}(r(s)-1)}{Kb^{p-d}\operatorname{Vol}M(r(s)+p-2)}\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}$$
$$=\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\varepsilon K b^{p-d}}(\log\varepsilon+(r(s)+p-2)\,J(r(s)+p-2,u(s)))$$
$$-\frac{2^{p-1}(r(s)-1)\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{Kb^{p-d}\operatorname{Vol}M(r(s)+p-2)}.$$

Now we use this latter estimate in (3.1) above, which we can write as

$$\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} = \frac{\dot r(s)}{r(s)}J(r(s),u(s)) - \left(\frac{p}{r(s)+p-2}\right)^p \frac{r(s)-1}{\|u(s)\|_{r(s)}^{r(s)}}\mathcal{E}_p\left(u(s)^{\frac{r(s)+p-2}{p}}\right)$$

to get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} \le \frac{\dot r(s)}{r(s)}J(r(s),u(s)) - \left(\frac{p}{r(s)+p-2}\right)^p \frac{r(s)-1}{\|u(s)\|_{r(s)}^{r(s)}}\\
&\times \frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\varepsilon K b^{p-d}}(\log\varepsilon + (r(s)+p-2)\,J(r(s)+p-2,u(s)))\\
&- \left(\frac{p}{r(s)+p-2}\right)^p \frac{(r(s)-1)2^{p-1}}{\|u(s)\|_{r(s)}^{r(s)}}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{Kb^{p-d}\mathrm{Vol}\,M}\\
&\le \frac{\dot r(s)}{r(s)}J(r(s),u(s)) - (r(s)-1)\left(\frac{p}{r(s)+p-2}\right)^p\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\\
&\times\left(\frac{(r(s)+p-2)}{\varepsilon K b^{p-d}}\,J(r(s)+p-2,u(s)) + \frac{\log\varepsilon}{\varepsilon K b^{p-d}}\right)\\
&- \frac{(r(s)-1)}{(r(s)+p-2)^p}\frac{2^{p-1}p^p}{Kb^{p-d}\mathrm{Vol}\,M}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\\
&= \frac{\dot r(s)}{r(s)}J(r(s),u(s)) - p(r(s)-1)\left(\frac{p}{r(s)+p-2}\right)^{p-1}\times\\
&\times\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\left(\frac{1}{\varepsilon K b^{p-d}}\,J(r(s)+p-2,u(s)) + \frac{1}{r(s)+p-2}\frac{\log\varepsilon}{\varepsilon K b^{p-d}}\right)\\
&- \frac{(r(s)-1)}{(r(s)+p-2)^p}\frac{2^{p-1}p^p}{Kb^{p-d}\mathrm{Vol}\,M}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}. \qquad \square
\end{aligned}$$

Now we state and prove the crucial lemma:

Lemma 3.5. *Suppose that, besides the assumptions of the previous lemma, one also has* $\|u_0\|_\infty \le 1$. *Then for every* $s \ge 0$ *the following inequality holds* *true:*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)} &\le -\frac{\dot r(s)}{r(s)}\frac{p-2}{r(s)+p-2}\log\|u(s)\|_{r(s)}\\ &-\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\log\Big(\frac{r(s)}{\dot r(s)}\frac{p(r(s)-1)}{Kb^{p-d}}\Big(\frac{p}{r(s)+p-2}\Big)^{p-1}\Big)\\ &+\frac{p^p(r(s)-1)}{Kb^{p-d}\mathrm{Vol}\,M(r(s)+p-2)^p}\end{aligned} \tag{3.4}$$

Proof. Choose, in (3.2), ε as follows:

$$\varepsilon=\varepsilon(s)=\frac{r(s)}{\dot r(s)}\frac{p(r(s)-1)}{Kb^{p-d}}\Big(\frac{p}{r(s)+p-2}\Big)^{p-1}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}},$$

so that inequality (3.2) becomes

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)}\\ &\le\frac{\dot r(s)}{r(s)}(J(r(s),u(s))-J(r(s)+p-2,u(s)))-\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\\ &\times\log\Big(\frac{r(s)}{\dot r(s)}\frac{p(r(s)-1)}{Kb^{p-d}}\Big(\frac{p}{r(s)+p-2}\Big)^{p-1}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\Big)\\ &+\frac{p^p(r(s)-1)}{Kb^{p-d}\mathrm{Vol}\,M(r(s)+p-2)^p}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}.\end{aligned} \tag{3.5}$$

By the assumption $\|u_0\|_\infty\le 1$ we can deduce, by the Markov property satisfied by the evolution we are dealing with, that $\|u(t)\|_\infty\le 1$ for all positive t. Then for such data we have

$$\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\le 1$$

so that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)}\\ &\le\frac{\dot r(s)}{r(s)}(J(r(s),u(s))-J(r(s)+p-2,u(s)))-\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\\ &\times\log\Big(\frac{r(s)}{\dot r(s)}\frac{p(r(s)-1)}{Kb^{p-d}}\Big(\frac{p}{r(s)+p-2}\Big)^{p-1}\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}}\Big)\end{aligned}$$

$$+\frac{p^p(r(s)-1)}{Kb^{p-d}\mathrm{Vol}\,M(r(s)+p-2)^p}. \tag{3.6}$$

Define the function $N:[1,+\infty)\times X\to\mathbb{R}$ by $N(q,u)=\log\|u(s)\|_q^q$. For every fixed $u\in X$, N is a convex function of q, so that its derivative exists almost everywhere and

$$\frac{\mathrm{d}}{\mathrm{d}q}N(q,u)=\int_D\frac{|u|^q}{\|u\|_q^q}\log|u|=J(q,u)+\log\|u\|_q\quad\text{for a.e. }\ q.$$

Moreover, the convexity implies that the above derivative is a monotonically nondecreasing function, thus,

$$q_1\le q_2\quad\to\quad J(q_1,u)-J(q_2,u)\le\log\|u\|_{q_2}-\log\|u\|_{q_1}.$$

Recall that $p>d\ge 3$, so that $p-2>0$ and

$$J(r(s),u)-J(r(s)+p-2,u)\le\log\|u(s)\|_{r(s)+p-2}-\log\|u(s)\|_{r(s)}.$$

Now we use the latter inequality in (3.6) above to yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}s}\log\|u(s)\|_{r(s)}\le\frac{\dot r(s)}{r(s)}(J(r(s),u(s))-J(r(s)+p-2,u(s)))\\
&-\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\log(\frac{r(s)}{\dot r(s)}\frac{p(r(s)-1)}{Kb^{p-d}}(\frac{p}{r(s)+p-2})^{p-1})\\
&-\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\log(\frac{\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}}{\|u(s)\|_{r(s)}^{r(s)}})+\frac{p^p(r(s)-1)}{Kb^{p-d}\mathrm{Vol}\,M(r(s)+p-2)^p}\\
&\le\frac{\dot r(s)}{r(s)}[\log\|u(s)\|_{r(s)+p-2}-\log\|u(s)\|_{r(s)}\\
&-\frac{1}{r(s)+p-2}(\log\|u(s)\|_{r(s)+p-2}^{r(s)+p-2}-\log\|u(s)\|_{r(s)}^{r(s)})]\\
&-\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\log(\frac{r(s)}{\dot r(s)}\frac{p(r(s)-1)}{Kb^{p-d}}(\frac{p}{r(s)+p-2})^{p-1})\\
&+\frac{p^p(r(s)-1)}{Kb^{p-d}\mathrm{Vol}\,M(r(s)+p-2)^p}\\
&=-\frac{\dot r(s)}{r(s)}\frac{p-2}{r(s)+p-2}\log\|u(s)\|_{r(s)}\\
&-\frac{\dot r(s)}{r(s)}\frac{1}{r(s)+p-2}\log(\frac{r(s)}{\dot r(s)}\frac{p(r(s)-1)}{Kb^{p-d}}(\frac{p}{r(s)+p-2})^{p-1})\\
&+\frac{p^p(r(s)-1)}{Kb^{p-d}\mathrm{Vol}\,M(r(s)+p-2)^p}
\end{aligned}$$

which is the stated inequality. □

Lemma 3.6. *Define the following functions of* $s \geq 0$:

$$y(s) = \log \|u(s)\|_{r(s)}, \quad p(s) = \frac{\dot{r}(s)}{r(s)} \frac{p-2}{r(s)+p-2}$$
$$q(s) = \frac{\dot{r}(s)}{r(s)} \frac{1}{r(s)+p-2} \log(\frac{r(s)}{\dot{r}(s)} \frac{p(r(s)-1)}{Kb^{p-d}} (\frac{p}{r(s)+p-2})^{p-1}) - \frac{p^p(r(s)-1)}{Kb^{p-d} \mathrm{Vol}\, M (r(s)+p-2)^p}. \tag{3.7}$$

Then the following differential inequality holds true $\forall s \geq 0$:

$$\frac{\mathrm{d}y(s)}{\mathrm{d}s} + p(s)y(s) + q(s) \leq 0.$$

Thus, $y(s) \leq y_L(s)$*, provided* $y(0) \leq y_L(0)$*, where*

$$y_L(s) = \exp\Big(-\int_0^s p(\lambda)d\lambda\Big)\Big[y_L(0) - \int_0^s q(\lambda) \exp\Big(\int_0^\lambda p(\eta)d\eta\Big)d\lambda\Big]$$

is a solution of the following ordinary differential equation, $\forall s \geq 0$:

$$\frac{\mathrm{d}y(s)}{\mathrm{d}s} + p(s)y(s) + q(s) = 0.$$

Lemma 3.7. *Let us fix* $t > 0$*. Then the solution* y_L *to the linear equation of the previous lemma, with the choice* $r(s) = qt/(t-s)$ $(q \geq 1)$ *satisfies*

$$\omega(t) = \lim_{s \to t^-} y_L(s) = \frac{q}{q+p-2}[y_L(0) - \frac{1}{q}\log(t) - \frac{1}{q}\log(\frac{p}{Kb^{p-d}}) + M_{p,q}] + (\frac{p}{q})^{p-1} \frac{t}{Kb^{p-d}\mathrm{Vol}\, M} \tag{3.8}$$

with K, b *as above and* $M_{p,q} > 0$ *given by*

$$M_{p,q} = \frac{1}{q}\log(pq) - \frac{q+p-2}{q(p-2)(p-3)}\log(\frac{q+1}{q+p-2}) + \frac{1}{(p-2)^2}\log(\frac{q}{q+p-2}). \tag{3.9}$$

Proof. With the present choice of $r(s)$ we have

$$\frac{\dot{r}(s)}{r(s)} = \frac{1}{t-s}$$

so we get

$$p(s) = \frac{p-2}{qt + (p-2)(t-s)} \tag{3.10}$$

$$
\begin{aligned}
q(s) &= \frac{1}{qt+(p-2)(t-s)}\log\Big(\frac{p}{Kb^{p-d}}(qt-(t-s))\Big(\frac{p(t-s)}{qt+p-2}\Big)^{p-1}\Big) \\
&\quad - \frac{p^p(r(s)-1)}{Kb^{p-d}\mathrm{Vol}\,M(r(s)+p-2)^p} \\
&= \frac{1}{qt+(p-2)(t-s)}\log\Big(\frac{p}{Kb^{p-d}}\Big) + \frac{1}{qt+(p-2)(t-s)}\log(qt-(t-s)) \\
&\quad + \frac{p-1}{qt+(p-2)(t-s)}\log\Big(\frac{p(t-s)}{qt+p-2}\Big) - \frac{p^p(r(s)-1)}{Kb^{p-d}\mathrm{Vol}\,M(r(s)+p-2)^p}.
\end{aligned}
$$

Then we have

$$
P(s) := \int_0^s p(\lambda)d\lambda = \int_0^s \frac{p-2}{qt+(p-2)(t-\lambda)}d\lambda = \log\frac{qt+(p-2)t}{qt+(p-2)(t-s)}
$$

so that

$$
e^{P(s)} = \frac{qt+(p-2)t}{qt+(p-2)(t-s)}.
$$

It shall also be useful to notice that

$$
e^{P(s)} = \frac{q+p-2}{q}\frac{r(s)}{r(s)+p-2}.
$$

Moreover, we compute

$$
Q(s) := \int_0^s q(\lambda)e^{P(\lambda)}d\lambda = \int_0^s q(\lambda)e^{P(\lambda)}d\lambda = Q_1(s)+Q_2(s)+Q_3(s)+Q_4(s)
$$

where

$$
\begin{aligned}
Q_1(s) &= \int_0^s \frac{1}{qt+(p-2)(t-\lambda)}\log\Big(\frac{p}{Kb^{p-d}}\Big)\frac{qt+(p-2)t}{qt+(p-2)(t-\lambda)}\,\mathrm{d}\lambda \\
Q_2(s) &= \int_0^s \frac{1}{qt+(p-2)(t-\lambda)}\log(qt-(t-\lambda))\frac{qt+(p-2)t}{qt+(p-2)(t-\lambda)}\,\mathrm{d}\lambda \\
Q_3(s) &= \int_0^s \frac{p-1}{qt+(p-2)(t-\lambda)}\log\Big(\frac{p(t-\lambda)}{qt+(p-2)(t-\lambda)}\Big)\times \\
&\quad \times\frac{qt+(p-2)t}{qt+(p-2)(t-\lambda)}\,\mathrm{d}\lambda \\
Q_4(s) &= -\frac{p^p}{Kb^{p-d}\mathrm{Vol}\,M}\int_0^s \frac{r(\lambda)(r(\lambda)-1)}{[r(\lambda)+p-2]^{p+1}}\,\mathrm{d}\lambda.
\end{aligned}
$$

Explicit calculations yield, in the limit as $s\to t^-$:

$$
\begin{aligned}
Q_1(s) &\to \frac{1}{q}\log\Big(\frac{p}{Kb^{p-d}}\Big) = \frac{1}{q}\log\Big(\frac{1}{Kb^{p-d}}\Big) + \frac{1}{q}\log(p) \\
Q_2(s) &\to \frac{1}{q}\log(qt) - \frac{q+p-2}{q(p-2)(p-3)}\log\Big(\frac{q+1}{q+p-2}\Big) + \frac{1}{(p-2)^2}\log\Big(\frac{q}{(q+1)}\Big)
\end{aligned}
$$

$$Q_3(s) \to \frac{(p-1)}{q}(\log(\frac{q+p-2}{p})+1), \quad Q_4(s) \to Q_4(t) \le Et$$

with $E = \frac{(q+p-2)p^{p-1}}{q^p K b^{p-d} \mathrm{Vol}\, M}$, where we have used the bound

$$\frac{r(\lambda)(r(\lambda)-1)}{[r(\lambda)+p-2]^{p+1}} \le \frac{1}{r(\lambda)^{p-1}}.$$

Finally, we get

$$\begin{aligned}\omega(t) &= \lim_{s\to t^-} y_L(s) = \lim_{s\to t^-} e^{P(s)}[y_L(0)+Q(s)] \\ &= \frac{q}{q+p-2}[y_L(0) - \frac{1}{q}\log(t) - \frac{1}{q}\log(\frac{1}{Kb^{p-d}}) + M_{p,q} + Et],\end{aligned}$$

where

$$M_{p,q} = \frac{1}{q}\log(pq) - \frac{q+p-2}{q(p-2)(p-3)}\log(\frac{q+1}{q+p-2}) + \frac{1}{(p-2)^2}\log(\frac{q}{q+p-2}).$$

4. The main theorem

If u is a solution corresponding to an essentially bounded initial datum with $\|u_0\|_\infty \le 1$ and zero mean we notice that, by the Markov property, the following contractivity property holds true for all $0 \le s \le t$:

$$\|u(t)\|_r \le \|u(s)\|_r.$$

Therefore, by the previous results one has, for all such s and t ,

$$\|u(t)\|_{r(s)} \le \|u(s)\|_{r(s)} = \exp(\log \|u(s)\|_{r(s)}) = e^{y(s)} \le e^{y_L(s)},$$

whence, letting $s \to t^-$, and recalling that $r(s) \to +\infty$ as $s \to t^-$, we deduce

$$\begin{aligned}\|u(t)\|_\infty &= \lim_{s\to t^-} \|u(t)\|_{r(s)} \le \lim_{s\to t^-} \|u(s)\|_{r(s)} \\ &= \lim_{s\to t^-} e^{y(s)} \le \lim_{s\to t^-} e^{y_L(s)} = e^{\omega(t)}.\end{aligned}$$

By the explicit form for $e^{\omega(t)}$ we have

$$\begin{aligned}e^{\omega(t)} &= \exp(\frac{q}{q+p-2}[y_L(0) - \frac{1}{q}\log(t) - \frac{1}{q}\log(\frac{1}{Kb^{p-d}}) + M_{p,q} + Et]) \\ &= (e^{y_L(0)})^{q/(q+p-2)} t^{-1/(q+p-2)} b^{(p-d)/(q+p-2)} \\ &\times K^{1/(q+p-2)} \exp(\frac{q}{q+p-2}M_{p,q}) e^{Et} \\ &= (e^{y_L(0)})^\gamma \frac{b^\alpha}{t^\beta} K^{1/(q+p-2)} \exp(\frac{q}{q+p-2}M_{p,q}) e^{Et}\end{aligned} \tag{4.1}$$

with

$$\alpha = \frac{p-d}{q+p-2}, \quad \beta = \frac{1}{q+p-2}, \quad \gamma = \frac{q}{q+p-2}. \tag{4.2}$$

Since $y_L(0) = \log \|u(0)\|_{r(0)} = \log \|u(0)\|_q$, recalling that $r(0) = q$ by hypothesis: $(e^{y_L(0)})^\gamma = \|u(0)\|_q^\gamma$. Moreover, the explicit form for $M_{p,q}$ allows us to write

$$\begin{aligned}
&\exp(\frac{q}{q+p-2} M_{p,q}) \\
&= \exp\{\frac{q}{q+p-2}[\frac{1}{q}\log(pq) - \frac{q+p-2}{q(p-2)(p-3)}\log(\frac{q+1}{q+p-2}) \\
&+ \frac{1}{(p-2)^2}\log(\frac{q}{q+p-2})]\}.
\end{aligned}$$

Finally, we get

$$\begin{aligned}
&\|u(t)\|_\infty \le e^{\omega(t)} = \frac{b^\alpha}{t^\beta}\|u(0)\|_q^\gamma (Kpq)^{1/(q+p-2)} \\
&\times (\frac{q+p-2}{q+1})^{1/[(p-2)(p-3)]}(\frac{q}{q+p-2})^{q/[(p-2)^2(q+p-2)]} e^{Et} \\
&\le \frac{1}{t^\beta}\|u(0)\|_q^\gamma A(p,q,d,M),
\end{aligned}$$

with $A(p,q,d,M)$ as in the statement, at least for essentially bounded initial data.

The assumption that the initial datum u satisfies $\|u\|_\infty \le 1$ can be removed by virtue of the fact that u is a solution to the equation $\dot{u} = \triangle_p u$ with initial datum u_0 if and only if $v_\lambda(t,x) = \lambda u(\lambda^{p-2}t, x)$ is a solution to $\dot{v_\lambda} = \triangle_p v_\lambda$ with initial datum $v_\lambda(0,x) = \lambda u_0(x)$, for arbitrary $\lambda > 0$. It thus suffices to choose $\lambda = \|u\|_\infty^{-1}$.

To deal with the case of general L^q–data, it suffices to refer to the discussion given in [8], which does not depend either upon the value of p or on the Euclidean setting but uses the Markov property. This concludes the proof for small times.

To deal with the case of large times we first prove an L^2–L^2 time decay estimate for arbitrary time. We compute, for initial data with zero mean

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_2^2 &= -2\|\nabla u\|_p^p \le -2C^{-p}b^{d-p}\|u(t)\|_\infty^p \\
&\le -2C^{-p}b^{d-p}(\mathrm{Vol}\, M)^{-p/2}\|u(t)\|_2^p,
\end{aligned}$$

where we have used the Sobolev inequality in the first step and the constants C, b are those appearing in (2.1). Thus, setting $f(t) = \|u(t)\|_2^2$ we have proved that

$$\dot{f}(t) \leq -2C^{-p}b^{d-p}(\mathrm{Vol}\, M)^{-p/2}f(t)^{p/2}.$$

This yields the bound, valid for all positive t,

$$\|u(t)\|_2 \leq \frac{1}{(Bt + \|u(0)\|_2^{2-p})^{1/(p-2)}},$$

where we have set

$$B = \frac{(p-2)b^{d-p}}{C^p(\mathrm{Vol}\, M)^{p/2}}.$$

The L^2–L^∞ smoothing property for small time and the above L^2–L^2 time decay can be used, together with the semigroup property to yield, for all positive t,

$$\begin{aligned}\|u(t)\|_\infty &\leq A(p,2,d,M)\|u(t-1)\|_2^{2/p}\\ &\leq \frac{A(p,2,d,M)}{(B(t-1) + \|u(0)\|_2^{2-p})^{2/[p(p-2)]}}.\end{aligned}$$

The latter statement is obtained from the numerical inequality

$$a + b \geq a^\varepsilon b^{1-\varepsilon}$$

valid for all positive a, b and all $\varepsilon \in (0,1)$. Putting $a = B(t-1)$ and $b = \|u(0)\|_2^{2-p}$ we thus get, for all $t > 2$,

$$\|u(t)\|_\infty \leq A(p,2,d,M)\frac{\|u(0)\|_2^{(p-2)(1-\varepsilon)}}{[B(t-1)]^{2\varepsilon/[p(p-2)]}}.$$

Proof of Corollary 1.2. We shall start from the bound

$$\|u(t)\|_2 \leq At^{-1/(p-2)}$$

proved above. Now we prove a similar statement for any $r > 1$, and for $r = 1$ by a limiting argument. First notice that, proceeding as before,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_r^r = -c\int_M |\nabla |u|^{(r+p-2)/p}|^p \mathrm{d}m_g.$$

Hereafter we adopt the convention that numerical constants may change from line to line. Using the Poincaré and the triangle inequality we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_r^r &\leq -c|\||u(t)|^{(r+p-2)/p}\|_p - \|\overline{|u(t)|^{(r+p-2)/p}}\|_p|^p\\ &= -c|\|u(t)\|_{r+p-2}^{(r+p-2)/p} - (\mathrm{Vol}\, M)^{(1-p)/p}\|u(t)\|_{(r+p-2)/p}^{(r+p-2)/p}|^p.\end{aligned}$$

Hölder's inequality implies that the difference in the right-hand side of the latter formula is positive. Using again the numerical inequality $(a+b)^p \le c(a^p+b^p)$ for positive a, b we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_r^r \le -c_1\|u(t)\|_{r+p-2}^{r+p-2} + c_2\|u(t)\|_{(r+p-2)/p}^{r+p-2}.$$

Choosing now $r = p+2$ so that $(r+p-2)/p = 2$ and using now the optimal decay estimate for the L^2 norm stated in the proof of the previous theorem and Hölder's inequality we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_{p+2}^{p+2} \le -c_1\|u(t)\|_{p+2}^{2p} + \frac{c_2}{t^{2p/(p-2)}}.$$

It is immediate to check that the ordinary differential equation

$$\dot{a} = -c_1 a^{2p/(p+2)} + c_2 t^{-2p/(p-2)}$$

has a solution of the form $a^*(t) = At^{-(p+2)/(p-2)}$ for a suitable $A > 0$. This implies a similar decay property for $\|u(t)\|_{p+2}^{p+2}$ and hence that $\|u(t)\|_{p+2} \le At^{-1/(p-2)}$ for data in L^{p+2}. We now iterate the procedure using the latter result, thus proving first an identical estimate for the L^r norm with $r = 2+p+p^2$, and then for a sequence of L^{r_n} norms with $r_n = 2+p+\cdots+p^n \to +\infty$ with an n–dependent proportionality constant. Finally, first for L^∞ data,

$$\|u(t)\|_\infty \le c\|u(t-1)\|_r^{r/(r+p-2)} \le c_r t^{-r/[(p-2)(r+p-2)]}$$

for $t \ge 2$. The instantaneous L^1–L^∞ smoothing of the evolution allows us to prove this inequality for general L^1 data as well. The final statement follows by applying the bound (1.9) to the scaled solution $v(t) = cu(c^{p-2}t)$ for $c = \|u(0) - \overline{u}\|_r^{-1}$. □

5. Manifolds with boundary, Sobolev constants and the limit as $p \to \infty$.

In this section we assume that (M, g) is a smooth, connected and compact Riemannian manifold *with smooth boundary*, whose dimension d satisfies $d \ge 3$. The parameter p is still supposed to satisfy the condition $p > d$, so that we can consider the Sobolev inequality

$$\|u\|_\infty \le \widetilde{C} b^{1-(d/p)}\|\nabla u\|_p \quad \forall u \in W_0^{1,p}(M). \tag{5.1}$$

We shall comment later on about the value of $\widetilde{C}$ and b in the particular case of Euclidean domains.

We aim at considering a version of the evolution equation $\dot{u} = \triangle_p u$ with homogeneous Dirichlet boundary condition at ∂M. To this end we introduce the functional

$$\widetilde{\mathcal{E}}_p(u) := \begin{cases} \int_M |\nabla u|^p \mathrm{d}m_g & \text{if } u \in W_0^{1,p}(M) \\ +\infty & \text{otherwise} \end{cases} \tag{5.2}$$

and consider the subdifferential of $\widetilde{\mathcal{E}}_p$. We can proceed exactly as above, but the simpler form of the Sobolev inequality satisfied by the functions for which $\widetilde{\mathcal{E}}_p < +\infty$ makes things easier. As expected from the structure of the problem the evolution with Dirichlet boundary conditions will drive the system to the equilibrium solution $u \equiv 0$. In fact we shall not bother the reader with all details (which would be a step–by–step repetition of the previous ones) but only notice that from (5.1) the following family of logarithmic Sobolev inequalities can easily be deduced:

$$\int_M |u|^p \log(\frac{|u|}{\|u\|})^p \mathrm{d}m_g \leq \varepsilon \tilde{K} b^{p-d} \|\nabla u\|_p^p - \|u\|_p^p \log \varepsilon \tag{5.3}$$

for all $\varepsilon > 0$ and for all $u \in W_0^{1,p}$. Here $\tilde{K} = \widetilde{C}^p > 0$ where b and $\widetilde{C}$ are as the Sobolev inequality (5.1). The point here is the absence of a term behaving like $\varepsilon \|u\|_p^p$ in the right-hand side of the logarithmic Sobolev inequality. This allows us to treat simultaneously the short- and long–time regime.

In fact we have the following result:

Theorem 5.1. *Let (M, g) be a smooth, connected and compact Riemannian manifold with smooth boundary and with dimension $d > 2$. Consider, for any $t > 0$, the function $u(t) := T_t u(0)$, where $\{T_t : t \geq 0\}$ is the semigroup associated to the subdifferential of the functional $\widetilde{\mathcal{E}}_p$ and $u(0) \in L^q(M)$ with $q \geq 1$. Then the following ultracontractive bound holds true for all $t > 0$:*

$$\|u(t)\|_\infty \leq \widetilde{C}(p, q, d) \frac{b^\alpha}{t^\beta} \|u(0)\|_q^\gamma, \tag{5.4}$$

where α, β and γ are as in Theorem 1.1 *and*

$$\begin{aligned} \widetilde{C}(p, q, d) =& (\widetilde{C}^p pq)^{1/(q+p-2)} (\frac{q+p-2}{q+1})^{1/[(p-2)(p-3)]} \\ & \times (\frac{q}{q+p-2})^{q/[(p-2)^2(q+p-2)]} \end{aligned} \tag{5.5}$$

b and $\widetilde{C}$ being the constants appearing in the Sobolev inequality (5.1).

For large times the bound

$$\|u(t)\|_\infty \leq \frac{c}{t^{1/(p-2)}}$$

holds true for all data $u_0 \in \mathrm{L}^1(M)$, $t \geq 1$ and for a suitable $c > 0$ depending only on p, d and on geometric quantities, but independent of u_0.

In fact the only point to prove is the large–time estimate, which uses the fact, proved exactly as in the proof of the main theorem, that the L^2 norm of solutions (and hence the L^1 norm as well) decay for large t as $t^{-1/(p-2)}$. More precisely, letting $\beta = \beta|_{q=1} = 1/(p-1)$ and $\gamma = \gamma|_{q=1} = 1/(p-1)$ be the constants appearing in the main theorem and evaluated when $q = 1$,

$$\|u(t)\|_\infty \leq \frac{c}{t^\beta}\|u(t/2)\|_1^\gamma \leq \frac{c}{t^{\beta+\gamma/(p-2)}} = \frac{c}{t^{1/(p-2)}}.$$

We shall discuss now the limit $p \to +\infty$ in the equation at hand, in the case in which M is a bounded Euclidean domain with smooth boundary. In addition to its own interest, such discussion will allow us to gauge the sharpness of our main bounds in such a limit also. We shall need in this connection to discuss briefly the value of the Sobolev constant in (5.1).

We first recall that the limiting behaviour as $p \to +\infty$ has been discussed in [2] in the case $D = \mathbb{R}^d$, where it is shown in particular that the solutions u_p to the equation $\dot{u} = \triangle_p u$ converge, in a suitable sense, as $p \to +\infty$, to the solution u_∞ of the evolution equation driven by the subdifferential of the functional I_∞ which equals zero on those L^2 functions whose distributional gradient satisfies $|\nabla u| \leq 1$ almost everywhere, and $+\infty$ elsewhere. We notice here that, although the discussion of that paper is given in the case $D = \mathbb{R}^d$, there is no essential difference in dealing with the case of a bounded smooth domain, denoted again by D, as we keep on assuming. In fact, we shall still denote by I_∞ the functional equaling zero on $\mathrm{L}^2(D)$ functions which, when extended to be zero outside D, are such that the distributional gradient satisfies $|\nabla u| \leq 1$ almost everywhere, and $+\infty$ otherwise. It has been shown in [9] that such a functional is lower semicontinuous. We may say that abstract homogeneous Dirichlet boundary conditions are assumed.

Our main result implies a uniform bound on u_∞ in terms of geometric quantities only; in particular, there is no time decay nor dependence on the initial data at all. This is not so surprising first in view of the existence of stationary solutions, and then of the fact (noticed in [2]) that mass transfer occurs instantly. Our calculations will be possible since we have explicit expressions for the numerical constants appearing in our main result. Again, the following bound has an a–priori nature.

Corollary 5.2. *Let $D \subset \mathbb{R}^d$ $(d \geq 3)$ be a smooth, bounded Euclidean domain. Let u_∞ be a weak solution to the problem $\dot{u}_\infty \in \partial I_\infty$ for almost every positive t, with homogeneous Dirichlet boundary conditions, corresponding to the initial datum u_0, which is assumed to be a Lipschitz–continuous function. Then*

$$\|u_\infty(t)\|_\infty \leq b$$

for almost every positive t, where b is the constant appearing in the family of Sobolev inequalities (5.1) *for the domain D.*

Proof. Let us consider the family $\{u_p\}$ of weak solutions to the family of equations

$$\dot{u} = \triangle_p u$$

with homogeneous Dirichlet boundary conditions, corresponding to the (common) initial datum u_0. We have proved above that

$$\|u_p(t)\|_\infty \le C(p,d,q)\frac{b^\alpha}{t^\beta}\|u(0)\|_q^\gamma$$

with explicit expressions for the constants involved. We first notice that, as $p \to +\infty$, $\alpha \to 1$, $\beta \to 0$, $\gamma \to 0$. Moreover, an easy computation also shows that the constant

$$C(p,q,d) = (\widetilde{C}^p pq)^{1/(q+p-2)} \times (\frac{q+p-2}{q+1})^{1/[(p-2)(p-3)]}(\frac{q}{q+p-2})^{q/[(p-2)^2(q+p-2)]} \tag{5.6}$$

tends to 1 in such a limit. This uses the explicit form of $\widetilde{C}$ which can be taken to be (see below) $\widetilde{C} = [(p-1)/(p-d)]^{1-(1/p)}$. It thus remain to discuss the relation between $\|u_p(t)\|_\infty$ and $\|u_\infty(t)\|_\infty$. To this end, we notice that in [2] it has been proved that the set $\{u_p\}_{p\ge d+1}$ is bounded in $\mathrm{L}^\infty(D\times[0,T])$ given any positive T. Then we can extract a sequence $\{u_{p_k}\}$ converging in the w^* topology of such space to a function u. In [2] it is also proved that u is the (unique) solution to the problem $\dot{u} \in \partial I_\infty$ (for almost every t) corresponding to the initial datum u_0. The statement follows from the weak* lower semicontinuity of the L^∞ norm. □

It has thus some relevance to have some information about the value of the constant b above.

We then state a version of the Morrey theorem.

Theorem 5.3. (Morrey). *Let $D \subset \mathbb{R}^d$ be an open and connected bounded set and $p > d$. Then in the Sobolev inequality*

$$\|u\|_\infty \le \widetilde{C} b^{1-(d/p)}\|\nabla u\|_p \quad \forall u \in W_0^{1,p}(D)$$

one can choose $\widetilde{C} = [(p-1)/(p-d)]^{1-(1/p)}$, and b equaling the radius of the smallest closed ball of $\mathbb{R}^d$ which contains D. Moreover, one has the estimate $b \le \sqrt{3}\,\mathrm{diam}\,(D)/2$.

The proof of this theorem is quite similar to the one in the book of Davies [12], so it is omitted here, but we remark that it depends on the Median Lemma (see e.g. [29], p. 387). In some cases it is clearly possible to choose $b =$

diam $(D)/2$. This holds for those bounded domains D having the following geometrical property: there exists a point $x_D \in D$ such that the ball of radius $b = \operatorname{diam}(D)/2$ includes D.

In particular, the latter fact holds in any ball, and thus our limiting estimate cannot be improved on such domains, since the stationary radial function $u(x,t) = r - |x|$ is a solution to the equation at hand.

It is also possible to prove that one can choose $b = \operatorname{diam}(D)/2$ when D satisfies an alternative geometrical condition: there exists $r > 0$ such that all points $x, y \in D$ with (geodesic) distance l can be joined by a chain of at most l/r intersecting closed cubes. We omit the details.

6. L^q–L$^\infty$ Hölder continuity

We return now to the situation in which M has no boundary, although a similar discussion could be given in the setting of Section 5.

Theorem 6.1. *Under the same assumptions and with the notation of Theorem* 1.1, *let* $u(0), v(0) \in L^q(M)$ $(q \geq 1)$ *have common mean value. Then the following estimate* (L^q–L^∞ *Hölder continuity*) *holds true for all* $t \in (0,1]$:

$$\|u(t) - v(t)\|_\infty \leq \frac{c_1}{t^\beta} \|u(0) - v(0)\|_q^\gamma \tag{6.1}$$

for all $t > 0$ *and for a suitable* $c_1 > 0$, *where* β, γ *are as in Theorem* 1.1.

If $t > 2$ *one has instead, for all data belonging to* $L^2(M)$:

$$\|u(t) - v(t)\|_\infty \leq \frac{c_2}{(c_3(t-1) + \|u(0) - v(0)\|_2^{2-p})^{2/[p(p-2)]}} \tag{6.2}$$

for suitable $c_2, c_3 > 0$, *and in particular, for any* $\varepsilon \in (0,1)$,

$$\|u(t) - v(t)\|_\infty \leq c_2 \frac{\|u(0) - v(0)\|_2^{2(1-\varepsilon)/p}}{[c_3(t-1)]^{2\varepsilon/[p(p-2)]}}. \tag{6.3}$$

Proof. (sketch). The proof follows closely the proof of Theorem 1.1, so that we shall only stress the relevant differences. We consider the quantity

$$f_r(s) = \int_M |u(s) - v(s)|^r \mathrm{d}m_g$$

and compute the time derivative of $f_r(s)$. This yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} f_r(s) = -r(r-1) \int_M |u(s) - v(s)|^{r-2} \\ \times \langle \nabla u(s) - \nabla v(s), |\nabla u(s)|^{p-2} u(s) - |\nabla v(s)|^{p-2} v(s) \rangle \mathrm{d}m_g, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in the tangent space at the corresponding point.

We now use the well–known inequality (cf. [15])

$$\langle |a|a^{p-2} - |b|b^{p-2}, a-b\rangle \geq c|a-b|^p$$

to yield

$$\frac{\mathrm{d}}{\mathrm{d}s} f_r(s) \leq -cr(r-1)\int_M |u(s)-v(s)|^{r-2}|\nabla(u(s)-v(s))|^p \mathrm{d}m_g.$$

Having at our disposal an inequality in which only the *difference* $T_s u - T_s v$ is involved, one can prove as in [10], by straightforward calculations, first that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s} f_{r(s)}(s) \leq \dot{r}(s)\int_M |u(s)-v(s)|^{r(s)} \log|u(s)-v(s)| \mathrm{d}m_g\\ &- cr(s)(r(s)-1)\int_M |u(s)-v(s)|^{r(s)-2}|\nabla(u(s)-v(s))|^p \mathrm{d}m_g,\end{aligned}$$

for any nonincreasing function of class C^1 $r : [0, +\infty) \to [2, +\infty)$, and then that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}s} \log\|u(s)-v(s)\|_{r(s)}\\ &\leq \frac{\dot{r}(s)}{r(s)}\int_M \Big(\frac{|u(s)-v(s)|}{\|u(s)-v(s)\|_{r(s)}}\Big)^{r(s)} \log\Big(\frac{|u(s)-v(s)|}{\|u(s)-v(s)\|_{r(s)}}\Big)\mathrm{d}m_g\\ &- c\Big(\frac{p}{r(s)+p-2}\Big)^p \frac{r(s)-1}{\|u(s)-v(s)\|^{r(s)}} \mathcal{E}_p(|u(s)-v(s)|^{(r(s)+p-2)/p}).\end{aligned}$$

Such an inequality is identical, apart from the numerical factor c in the coefficient of $\mathcal{E}_p$ to the conclusion of Lemma 3.3, and $u(s)-v(s)$ has zero mean at all times. The subsequent calculations follow then along the same lines. □

The above proof and its close analogy to the proof of Theorem 1.1 show that the constant c_1 can be taken to be

$$\begin{aligned}c_1(p,q,d,c,M) =& e^E b^\alpha \Big(\frac{2^{p-1}C^p}{c} pq\Big)^{1/(q+p-2)} \Big(\frac{q+p-2}{q+1}\Big)^{1/[(p-2)(p-3)]}\\ &\times \Big(\frac{q}{q+p-2}\Big)^{q/[(p-2)^2(q+p-2)]}\end{aligned}$$

while c_2 can be taken to be

$$c_2 = c_1(p,2,d,c,M) e^{\frac{p^p}{2^{2p-1}C^p b^{p-d} \mathrm{Vol}\, M}} \quad \text{and} \quad c_3 = \frac{(p-2)b^{d-p}}{C^p(\mathrm{Vol}\, M)^{p/2}}.$$

Clearly similar calculations show identical time–decay estimates for solutions corresponding to data with *different* mean, but in such a case $u(t) - v(s)$ approaches (at the same rate) $\overline{u} - \overline{v}$.

The proof of the following corollary is identical to that of Corollary 1.2 as soon as the considerations given in the proof of the previous theorem have been taken into account.

Corollary 6.2. *For all $t > 2$, all $\varepsilon \in (0,1)$ and all initial data u_0, v_0 with common mean and in L^1 there exists $c_\varepsilon > 0$ such that*

$$\|u(t) - v(t)\|_\infty \le c_\varepsilon t^{-(1-\varepsilon)/(p-2)} \tag{6.4}$$

independently of the initial datum u_0. Moreover, if the initial datum belongs to $L^r(M)$ with $\|u(0) - \overline{u}\|_r < 1$, then

$$\|u(t) - v(t)\|_\infty \le c_\varepsilon \|u(0) - v(0)\|_r^\varepsilon t^{-(1-\varepsilon)/(p-2)}$$

for all $t \ge 2\|u(0) - v(0)\|_r^{2-p}$.

7. THE p–SUB–LAPLACIAN ASSOCIATED TO A COLLECTION OF VECTOR FIELDS

We shall present a generalization of our main results, which will be based upon the fact that essentially all our calculations depend on the validity of a suitable Sobolev inequality only.

We now deal with the following setting: M is a smooth and connected Riemannian manifold without boundary, and $\{X_i\}_{i=1}^m$ is a collection of vector fields with locally Lipschitz coefficients on M. Notice that the integer m may or may not be equal to the dimension d of M. We aim at proving a result similar to that of the previous subsection for evolution equations driven by possibly degenerate operators similar to the p–Laplacian but constructed in terms of the vector fields $\{X_i\}$. Our aim is to show that our approach depends almost entirely on a suitable Sobolev inequality, and not on the details of the generator itself. Although some further generalization could be given (e.g. to sub-Riemannian structures in the sense of [31]) we feel that the present example will suffice in this connection.

We first recall some well–known facts. First, the *intrinsic quasi–metric* relative to the family of vector fields at hand is defined as follows:

$$\varrho(x,y) = \inf\{T > 0 : \text{there exists a sub–unit curve } \gamma : [0,T] \to M \\ \text{with } \gamma(0) = x, \gamma(T) = y\},$$

where γ is said to be sub–unit if

$$|\langle \dot\gamma(t), \xi\rangle|^2 \le \sum_i |\langle X_i(\gamma(t)), \xi\rangle|^2$$

for any $\xi \in T_{\gamma(t)}M$, for almost every $t \in [0,T]$.

We *assume* without further comment that ϱ is always finite, so that it is a true metric on M. We shall assume that the Riemannian measure μ is doubling in the sense that, denoting by $B(x,r)$ the open intrinsic ball of center $x \in M$ and radius $r > 0$ one has

$$\mu(B(x,2r)) \leq \delta\, \mu(B(x,r)) \tag{7.1}$$

for any $x \in M$. In particular (M, ϱ, μ) is a homogeneous space in the sense of [11]. It is well–known that (7.1) implies that the inequality

$$\frac{\mu(B)}{\mu(B_0)} \geq \text{const.}(\frac{r}{r_0})^s \tag{7.2}$$

holds whenever B_0 is an intrinsic ball of radius r_0 and $B = B(x,r)$ with $x \in B_0$ and $r \leq r_0$, where $s = \log_2 \delta$ (see e.g. Lemma 14.6 in [23]). We refer e.g. to [27] and [32] and references quoted as general references on these topics.

It will moreover be assumed that a p–Poincaré inequality holds. By this we mean the following: there exists c such that, for all Lipschitz functions,

$$\int_M |u - \overline{u}|^p \mathrm{d}m_g \leq c(\int_M |Xu|^p \mathrm{d}m_g). \tag{7.3}$$

For a proof of this inequality in a very general context see [21]. See also [7] for the discussion of local properties of solution to subelliptic equations associated to the p–sub-Laplacian.

We shall next consider the operator formally given by

$$L_X u := \sum_{i=1}^m X_i^*(|Xu|^{p-2} X_i),$$

where $|Xu|^2 := \sum_{i=1}^m |X_i u|^2$ and X_i^* is the formal adjoint of X_i. To give sense to such an operator again we consider the convex lower–semicontinuous functional $\mathcal{E}_{p,X}$ defined as

$$\mathcal{E}_{p,X}(u) := \int_M |Xu|^p \mathrm{d}m_g$$

whenever the integral is finite, and $+\infty$ elsewhere. We shall still denote by L_X its subdifferential. We again refer to [9] for a proof of the fact that $\mathcal{E}_{p,X}$ is a nonlinear Dirichlet form for all $p > 1$. Such a *subelliptic p–Laplacian* has appeared, e.g. in [6], [25], at least in the case of vector fields satisfying the Hörmander condition, i.e. such that the Lie algebra generated by the vector fields at hand equals the whole tangent space at each point.

We shall be interested in certain Sobolev inequalities involving $\mathcal{E}_{p,X}$. For various versions of the Sobolev inequalities we refer, of course with no claim of completeness, e.g. to [16], [20], [23], [26] and references quoted.

In fact, it is known (see [23], Theorem 5.1, part 3, for this fact in an even more general setting) that the inequality

$$\|u-\overline{u}\|_\infty \le C\widetilde{b}^{[1-(s/p)]}\mathcal{E}_{p,X}(u)^{1/p} \tag{7.4}$$

holds provided the vector fields at hand satisfy, besides (7.2), a p–Poincaré inequality with $p > s$. The constant $\widetilde{b}$ can be taken to equal the diameter of M in the metric ϱ.

We can now state our final result.

Theorem 7.1. *Let us suppose that the vector fields at hand satisfy the doubling condition* (7.1) *for some* $\delta > 0$ *and the p–Poincaré inequality* (7.3) *for some* $p > s := \log_2 \delta$. *Consider the nonlinear semigroup* T_t *driven by the subdifferential of the nonlinear Dirichlet form* $\mathcal{E}_{p,X}$. *Then the following estimate holds true for the time–evolved function* $u(t) = T_t u(0)$ *of an initial datum* $u(0) \in L^q(M)$:

$$\|u(t)-\overline{u}\|_\infty \le \frac{C}{t^\beta}\|u-\overline{u}\|_q^\gamma \tag{7.5}$$

for all $t \in (0,1)$, *for all* $u, v \in L^q(M)$, *where* β, γ *are as in Theorem* 1.1, *but with the Euclidean dimension d replaced by the homogeneous dimension s.*

If $t > 2$ *one has instead, for all data belonging to* $L^2(M)$,

$$\|u(t)-\overline{u}\|_\infty \le \frac{c_1 b^{(p-d)/p}}{(c_2(t-1)+\|u(0)-\overline{u}\|_2^{2-p})^{2/[p(p-2)]}} \tag{7.6}$$

for suitable $c_1, c_2 > 0$ *and in particular, for any* $\varepsilon \in [0,1]$,

$$\|u(t)-\overline{u}\|_\infty \le c_3\frac{\|u(0)-\overline{u}\|_2^{2(1-\varepsilon)/p}}{[(t-1)]^{2\varepsilon/[p(p-2)]}}. \tag{7.7}$$

The proof of such a result follows as in Theorem 1.1, since they are only based upon the appropriate ordinary, and hence logarithmic, Sobolev inequalities, as soon as the fact that the originating functionals are nonlinear Dirichlet forms have been established. Similar two–function estimates in the spirit of Theorem 6.1 follow similarly.

Corollary 7.2. *For all* $t > 2$, *all* $\varepsilon \in (0,1)$ *and all initial data* u_0 *in* L^1 *there exists* $c_\varepsilon > 0$ *such that*

$$\|u(t)-\overline{u}\|_\infty \le c_\varepsilon t^{-(1-\varepsilon)/(p-2)}, \tag{7.8}$$

independently of the initial datum u_0*. Moreover, if the initial datum belongs to* $L^r(M)$ *with* $\|u(0)-\overline{u}\|_r < 1$*, then*

$$\|u(t)-\overline{u}\|_\infty \leq c_\varepsilon \|u(0)-\overline{u}\|_r^\varepsilon t^{-(1-\varepsilon)/(p-2)}$$

for all $t \geq 2\|u(0)-\overline{u}\|_r^{2-p}$*.*

Remark 7.3. The class of vector fields satisfying the assumptions of Theorem 7.1 is large. Indeed, we mention only that vector fields satisfying the Hörmander condition as well as vector fields of Grushin type, fall within such a class. We refer e.g. to [18] and references quoted for such results. In such a paper one can also find both a detailed discussion of Grushin–type vector fields, and some more relevant examples of vector fields satisfying our running assumptions.

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