Singular solutions of fractional order conformal Laplacians

Maria del Mar González
Universitat Politecnica de Catalunya

Rafe Mazzeo
Stanford University

Yannick Sire
Université Paul Cézanne, Aix-Marseille III

Abstract

We investigate the singular sets of solutions of a natural family of conformally covariant pseudodifferential elliptic operators of fractional order, with the goal of developing generalizations of some well-known properties of solutions of the singular Yamabe problem.

1 Introduction

Let $(M, \overline{g})$ be a compact n-dimensional Riemannian manifold, $n \geq 3$. If $\Lambda \subset M$ is any closed set, then the 'standard' singular Yamabe problem concerns the existence and geometric properties of complete metrics of the form $g = u^{\frac{4}{n-2}}\overline{g}$ with constant scalar curvature. This corresponds to solving the partial differential equation

$$\Delta_{\overline{g}} u + \frac{n-2}{4(n-1)} R^{\overline{g}} u = \frac{n-2}{4(n-1)} R^g\, u^{\frac{n+2}{n-2}}, \qquad u > 0, \tag{1.1}$$

where R^g is constant and with a 'boundary condition' that $u \to \infty$ sufficiently quickly at Λ so that g is complete. (Note that in our convention, $\Delta_{\overline{g}}$ is an operator with nonnegative spectrum.) It is known that solutions with $R^g < 0$ exist quite generally if Λ is large in a capacitary sense [17], whereas for $R^g > 0$ existence is only known when Λ is a smooth submanifold (possibly with boundary) of dimension $k < (n-2)/2$, see [19], [8].

There are both analytic and geometric motivations for studying this problem. For example, in the positive case ($R^g > 0$), solutions to this problem are actually weak solutions across the singular set, so these results fit into the broader investigation of possible singular sets of weak solutions of semilinear elliptic equations. On the geometric side is a well-known theorem by Schoen and Yau [25] stating that if (M,h) is a compact manifold with a locally conformally flat metric h of positive scalar curvature, then the developing map D from the universal cover $\widetilde{M}$ to S^n, which by definition is conformal, is injective, and moreover, $\Lambda := S^n \setminus D(\widetilde{M})$ has Hausdorff dimension less than or equal to $(n-2)/2$. Regarding the lifted metric $\tilde{h}$ on $\widetilde{M}$ as a metric on Ω, this provides an interesting class of solutions of the singular Yamabe problem which are periodic with respect to a Kleinian group, and for which the singular set Λ is typically nonrectifiable. More generally, that paper also shows that if $\overline{g}$ is the standard round metric on the sphere and if $g = u^{\frac{4}{n-2}}\overline{g}$ is a complete metric with positive scalar curvature and bounded Ricci curvature on a domain $\Omega = S^n \setminus \Lambda$, then $\dim \Lambda \leq (n-2)/2$.

In the past two decades it has been realized that the conformal Laplacian, which is the operator appearing as the linear part of (1.1), fits into a holomorphic family of conformally covariant elliptic pseudodifferential operators. The operators in this family of positive even integer order are the GJMS operators, and these have a central role in conformal geometry. Just as the Yamabe problem is naturally associated to the conformal Laplacian, so too are there higher order Yamabe-type problems associated to the other GJMS operators, or more generally, also to these other conformally covariant operators with noninteger order. The higher (integer) order Yamabe problems have proved to be analytically challenging and provide insight into the GJMS operators themselves. Hence it is reasonable to hope that these fractional order (singular) Yamabe problems will have a similarly rich development and will bring out interesting features of these conformally covariant pseudodifferential operators. From a purely analytic perspective, little is known about regularity of solutions of semilinear pseudodifferential equations like these, and this family of geometric problems is a natural place to start. In fact, as we explain below, fractional powers of the Laplacian have also appeared recently in the work of Caffarelli and his collaborators as generalized Dirichlet to Neumann operators for certain singular divergence form elliptic equations, which further indicates the worth of studying such operators.

The present paper begins an investigation into these questions. Our goals here are limited: beyond presenting this set of problems as an interesting area of investigation, we prove a few results which indicate how certain properties of the fractional singular Yamabe problem extend some well-known results for the standard Yamabe equation.

To describe this more carefully, we first define the family of fractional conformal powers of the Laplacian. As we have already indicated, the linear operator which appears as the first two terms on the left in (1.1) is known as the conformal Laplacian associated to the metric $\overline{g}$, and denoted $P_1^{\overline{g}}$. It is conformally covariant in the sense that if f is any (smooth) function and $g = u^{\frac{4}{n-2}}\,\overline{g}$ for some $u > 0$, then

$$P_1^{\overline{g}}(uf) = u^{\frac{n+2}{n-2}} P_1^g(f). \tag{1.2}$$

Setting $f \equiv 1$ in (1.2) yields the familiar relationship (1.1) between the scalar curvatures $R^{\overline{g}}$ and R^g. P_1 is the first in a sequence of conformally covariant elliptic operators, P_k, which exist for all $k \in \mathbb{N}$ if n is odd, but only for $k \in \{1, \ldots, n/2\}$ if k is even. The first construction of these operators, by Graham-Jenne-Mason-Sparling [13] (for which reason they are known as the GJMS operators), proceeded by trying to find lower order geometric correction terms to Δ^k in order to obtain nice transformation properties under conformal changes of metric. Beyond the case $k = 1$ which we have already discussed, the operator

$$P_2 = \Delta^2 + \delta\,(a_n Rg + b_n Ric)\,d + \tfrac{n-4}{2} Q_2,$$

called the Paneitz operator (here Q_2 is the standard Q-curvature), had also been discovered much earlier than the operators P_k with $k > 2$.

This leads naturally to the question whether there exist any conformally covariant pseudodifferential operators of noninteger order. A partial result in this direction was given by Peterson [20], who showed that for any γ, the conformal covariance condition determines the full Riemannian symbol of a pseudodifferential operator with principal symbol $|\xi|^{2\gamma}$. Hence P_γ is determined modulo smoothing operators, but it is by no means clear that one can choose smoothing operators to make the conformal covariance relationships hold exactly. The breakthrough result, by Graham and Zworski [14], was that if $(M, [\overline{g}])$ is a smooth compact manifold endowed with a conformal structure, then the operators P_k can be realized as residues at the values $\gamma = k$ of the meromorphic family $S(n/2 + \gamma)$ of scattering operators associated to the Laplacian on any Poincaré-Einstein manifold (X, G) for which $(M, [\overline{g}])$ is the conformal infinity. These are the 'trivial' poles of the scattering operator, so-called because their location is independent of the interior geometry; $S(s)$ typically has infinitely many other poles, which are called resonances, the location and asymptotic distribution of which is a matter of considerable interest and ongoing study. Multiplying this scattering family by some Γ factors to regularize these poles, one obtains a holomorphic family of elliptic pseudodifferential operators $P_\gamma^{\overline{g}}$ (which patently depends on the filling (X, G)). An alternate construction of these operators has been obtained by Juhl, and his monograph [15] describes an intriguing general framework for studying conformally covariant operators, see also [16].

This realization of the GJMS operators has led to important new understanding of them, including for example the basic fact that $P_\gamma^{\overline{g}}$ is symmetric with respect to $dV_{\overline{g}}$, (something not obvious from the previous fundamentally algebraic construction). Hence even though the family $P_\gamma^{\bar{g}}$ is not entirely canonically associated to $(M, [\overline{g}])$ (as we explain in some detail below), its study can still illuminate the truly canonical operators which occur as special values at positive integers, i.e. the GJMS operators.

For various technical reasons, we focus here only on the operators P_γ when $\gamma \in \mathbb{R}$, $|\gamma| \leq n/2$. These have the following properties: first, $P_0 = \mathrm{Id}$, and more generally, P_k is the k^{th} GJMS operator, $k = 1, \ldots, n/2$; next, P_γ is a classical elliptic pseudodifferential operator of order 2γ with principal symbol $\sigma_{2\gamma}(P_\gamma^{\overline{g}}) = |\xi|_{\overline{g}}^{2\gamma}$, hence (since M is compact), P_γ is Fredholm on L^2 when $\gamma > 0$; if P_γ is invertible, then $P_{-\gamma} = P_\gamma^{-1}$; finally,

$$\text{if } g = u^{\frac{4}{n-2\gamma}}\overline{g}, \qquad \text{then } P_\gamma^{\overline{g}}(uf) = u^{\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{g}(f) \tag{1.3}$$

for any smooth function f. Generalizing the formulæ for scalar curvature ($\gamma = 1$) and the Paneitz-Branson Q-curvature ($\gamma = 2$), we make the definition that for any $0 < \gamma \leq n/2$, $Q_\gamma^{\overline{g}}$, the Q-curvature of order γ associated to a metric $\overline{g}$, is given by

$$Q_\gamma^{\overline{g}} = P_\gamma^{\overline{g}}(1). \tag{1.4}$$

Let us comment further on the choices involved in these definitions. First, Poincaré-Einstein fillings (X, G) of $(M, [\overline{g}])$ (which are defined at the beginning of §2), may not always exist, and when they exist, they may not be unique. The existence issue is not serious: the construction of [14] only uses that the metric G satisfy the Einstein equation to sufficiently high order, and one can even take $X = M \times [0, 1]$ with the conformal structure $[\overline{g}]$ at $M \times \{0\}$ and with the other boundary $M \times \{1\}$ a regular (incomplete) boundary for G. However, these comments indicate that the issue of lack of uniqueness is far worse, since there are always infinite dimensional families of asymptotically Poincaré-Einstein fillings. Any choice of one of these fixes a family of operators $P_\gamma^{\overline{g}}$, and for each such choice P_γ satisfies all the properties listed above. As already noted, the complete Riemannian symbol of $P_\gamma^{\overline{g}}$ is determined by the metric $\overline{g}$ and the conformal covariance; the choice of filling provides a consistent selection of smoothing terms in these pseudodifferential operators for which the same covariance properties hold. Hence the Q-curvatures $Q_\gamma^{\overline{g}}$ for noninteger values of γ are similarly ill-defined. In particular, except in certain special cases where there are canonical choices of fillings (e.g. the sphere), it is not clear that the existence of a metric $\overline{g}$ in a conformal class such that $Q_\gamma^{\overline{g}} > 0$ depends only on that conformal class. We leave open these significant problems, and in what follows, always make the tacit assumption that for any given $(M, [\overline{g}])$, we have fixed an approximately Poincaré-Einstein filling (X, G) and used this to define the family $P_\gamma^{\overline{g}}$. In other words, it is perhaps more sensible to think of P_γ and Q_γ as quantities determined by the pair $((M, [\overline{g}]), (X, G))$.

In any case, generalizing (1.1), consider the "fractional Yamabe problem": given a metric $\overline{g}$ on a compact manifold M, find $u > 0$ so that if $g = u^{4/(n-2\gamma)}\overline{g}$, then Q_γ^g is constant. This amounts to solving

$$P_\gamma^{\overline{g}} u = Q_\gamma^g u^{\frac{n+2\gamma}{n-2\gamma}}, \quad u > 0, \tag{1.5}$$

for $Q_\gamma^g = $ const. More generally, we can simply seek metrics g which are conformally related to $\overline{g}$ and such that $Q_\gamma^g \geq 0$ or $Q_\gamma^g < 0$ everywhere.

This fractional Yamabe problem has now been solved in many cases where the positive mass theorem is not needed [12], and further work on this is in progress.

As described earlier, it is is also interesting to construct complete metrics of constant (positive) Q_γ curvature on open subdomains $\Omega = M \setminus \Lambda$, or in other words, to find metrics $g = u^{4/(n-2\gamma)}\overline{g}$ which are complete on Ω and such that u satisfies (1.5) with Q_γ^g a constant. This is the fractional singular Yamabe problem. In the first few integer cases it is known that the positivity of the curvature places restrictions on $\dim \Lambda$ (see [25], [19] for the case $\gamma = 1$, [4] for $\gamma = 2$, and [11] for the analogous problem for the closely related σ_k curvature).

Although it is not at all clear how to define P_γ^g and Q_γ^g on a general complete open manifold, we can give a reasonable definition when Ω is an open dense set in a compact manifold M and the metric g is conformally related to a smooth metric $\overline{g}$ on M. Namely, we can define them by demanding that the relationship (1.3) holds. Note, however, that this too is not as simple as it first appears since, because of the nonlocal character of $P_\gamma^{\overline{g}}$, we must extend u as a distribution on all of M. We discuss this further below.

The purpose of this note is to clarify some basic features of this fractional singular Yamabe problem and to establish a few preliminary results about it. Our first result generalizes the Schoen-Yau theorem.

Theorem 1.1. *Suppose that $(M^n, \overline{g})$ is compact and $g = u^{\frac{4}{n-2\gamma}}\overline{g}$ is a complete metric on $\Omega = M \setminus \Lambda$, where Λ is a smooth k-dimensional submanifold.*

Assume furthermore that u is polyhomogeneous along Λ with leading exponent $-n/2+\gamma$. If $0 < \gamma \le \frac{n}{2}$, and if $Q^g_\gamma > 0$ everywhere for any choice of asymptotically Poincaré-Einstein extension (X, G) which defines $P^{\overline{g}}_\gamma$ and hence Q^g_γ, then n, k and γ are restricted by the inequality

$$\Gamma\left(\frac{n}{4} - \frac{k}{2} + \frac{\gamma}{2}\right) \Big/ \Gamma\left(\frac{n}{4} - \frac{k}{2} - \frac{\gamma}{2}\right) > 0, \tag{1.6}$$

where Γ is the ordinary Gamma function. This inequality holds in particular when $k < (n-2\gamma)/2$, and in this case then there is a unique distributional extension of u on all of M which is still a solution of (1.5).

Remark 1. *Recall that u is said to be polyhomogeneous along Λ if in terms of any cylindrical coordinate system (r, θ, y) in a tubular neighborhood of Λ, where r and θ are polar coordinates in disks in the normal bundle and y is a local coordinate along Λ, u admits an asymptotic expansion*

$$u \sim \sum a_{jk}(y, \theta) r^{\mu_j} (\log r)^k$$

where μ_j is a sequence of complex numbers with real part tending to infinity, for each j, a_{jk} is nonzero for only finitely many nonnegative integers k, and such that every coefficient $a_{jk} \in \mathcal{C}^\infty$. The number μ_0 is called the leading exponent if $\Re(\mu_j) > \Re(\mu_0)$ for all $j \neq 0$. We refer to [18] for a more thorough account of polyhomogeneity.

Remark 2. *As we have noted, inequality (1.6) is satisfied whenever $k < (n-2\gamma)/2$, and in fact is equivalent to this simpler inequality when $\gamma = 1$. When $\gamma = 2$, i.e. for the standard $Q-$curvature, this result is already known: it is shown in [4] that complete metrics with $Q_2 > 0$ and positive scalar curvature must have singular set with dimension less than $(n-4)/2$, which again agrees with (1.6).*

We also present a few special existence results. First, the following remark exhibits solutions coming from Kleinian group theory where Λ is nonrectifiable.

Remark 3. *Suppose that $\gamma \in [1, n/2)$. Let Γ be a convex cocompact subgroup of $SO(n+1, 1)$ with Poincaré exponent $\delta(\Gamma) \in [1, (n-2\gamma)/2)$. Let $\Lambda \subset S^n$ be the limit set of Γ. Then $\Omega = S^n \setminus \Lambda$ admits a complete metric g conformal to the round metric and with $Q^g_\gamma > 0$.*

As we explain below, this follows directly from the work of Qing and Raske [22].

Finally, one can also obtain existence of solutions when γ is sufficiently near 1 and Λ is smooth by perturbation theory.

Theorem 1.2. *Let $(M^n, [\bar{g}])$ be compact with nonnegative Yamabe constant and Λ a k-dimensional submanifold with $k < \frac{1}{2}(n-2)$. Then there exists an $\epsilon > 0$ such that if $\gamma \in (1-\epsilon, 1+\epsilon)$, there exists a solution to the fractional singular Yamabe problem* (1.5) *with $Q_\gamma > 0$ which is complete on $M \setminus \Lambda$.*

Our final result is a growth estimate for weak solutions that are singular on $S^n \backslash \Omega$. Our result is not very strong in the sense that we do need to require that u is a weak solution in the whole S^n. However, it provides the first insight into a general theory of weak solutions on subdomains of S^n.

Proposition 1.3. *Let g_c be the standard round metric on S^n, and (B^{n+1}, G) the Poincaré ball model of hyperbolic space, which has $(S^n, [g_c])$ as its conformal infinity. Let $g = u^{\frac{4}{n-2\gamma}} g_c$ be a complete metric on a dense subdomain of the sphere, $\Omega = S^n \setminus \Lambda$, with Q^g_γ equal to a positive constant, and such that u is a distributional solution to*

$$P^{g_c}_\gamma u = u^{\frac{n+2\gamma}{n-2\gamma}} \tag{1.7}$$

on S^n (with u finite only on Ω). Then, for all $z \in \Omega$,

$$u(z) \leq \frac{C}{d_{g_c}(z, \Lambda)^{\frac{n-2\gamma}{2}}},$$

where C depends only on n and γ.

There are many interesting questions not addressed here. For example, we point out again that there is not yet a good definition of the family P^g_γ on an arbitrary complete manifold (Ω, g). Provided one is able to make this definition, it would then be useful to compute the L^2-spectrum of P^g_γ, even for some specific examples such as $\mathbb{H}^n$ or $\mathbb{H}^{k+1} \times S^{n-k-1}$. Finally, it would also be important to obtain the correct generalization of the Schoen-Yau theorem for the operators P_γ. We hope to address these and other problems elsewhere.

2 Fractional conformal Laplacians

We now provide a more careful description of the construction of the family of conformally covariant operators P_γ, and also give two alternate definitions of these operators in the flat case to provide some perspective.

As we have described in the introduction, Graham and Zworski [14] discovered a beautiful connection between the scattering theory of the Laplacian on an asymptotically hyperbolic Einstein manifold and the GJMS operators on its conformal infinity. Let (M, g) be a compact n-dimensional

Riemannian manifold. Suppose that X is a smooth compact manifold with boundary, with $\partial X = M$, and denote by x a defining function for the boundary, i.e. $x \geq 0$ on X, $x = 0$ precisely on ∂X and $dx \neq 0$ there. A metric G on the interior of X is called conformally compact if $x^2 G = \overline{G}$ extends as a smooth nondegenerate metric on the closed manifold with boundary. It is not hard to check that G is complete and, provided that $|dx|_{\overline{G}} = 1$ at ∂X, the sectional curvatures of G all tend to -1 at 'infinity'. The metric G is called Poincaré-Einstein if it is conformally compact and also satisfies the Einstein equation $\mathrm{Ric}^G = -nG$. As we have explained, it is only necessary to consider asymptotically Poincaré-Einstein metrics; by definition, these are conformally compact metrics which satisfy $\mathrm{Ric}^G = -nG + \mathcal{O}(x^N)$ for some suitably large N (typically, $N > n$ is sufficient).

The conformal infinity of G is the conformal class of $\overline{G}\big|_{T\partial X}$; only the conformal class is well defined since the defining function x is defined up to a positive smooth multiple. If g is any representative of this conformal class, then there is a unique defining function x for M such that $G = x^{-2}(dx^2 + g(x))$ where $g(x)$ is a family of metrics on M (or rather, the level sets of x), with $g(0)$ the given initial metric.

We now define the scattering operator $S(s)$ for (X, G). Fix any $f_0 \in \mathcal{C}^\infty(M)$; then for all but a discrete set of values $s \in \mathbb{C}$, there exists a unique generalized eigenfunction u of the Laplace operator on X with eigenvalue $s(n-s)$. In other words, u satisfies

$$\begin{cases} (\Delta_G - s(n-s))u = 0 \\ u = f x^{n-s} + \tilde{f} x^s, \quad \text{for some } f, \tilde{f} \in \mathcal{C}^\infty(\overline{X}) \quad \text{with} \quad f|_{x=0} = f_0. \end{cases} \tag{2.8}$$

By definition, $S(s) f_0 = \tilde{f}|_{x=0}$. This is an elliptic pseudodifferential operator of order $2s - n$ which depends meromorphically on s; it is known to always have simple poles at the values $s = n/2, n/2+1, n/2+2, \ldots$. These locations are independent of (X, G), hence are called the trivial poles of the scattering operator. $S(s)$ has infinitely many other poles which are of great interest in other investigations, but do not concern us here. Letting $s = n/2 + \gamma$, we now define

$$P_\gamma^g = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)} S\left(\frac{n}{2} + \gamma\right); \tag{2.9}$$

because of these prefactors, one has that the principal symbol is

$$\sigma_{2\gamma}(P_\gamma^g) = |\eta|_g^{2\gamma}. \tag{2.10}$$

The scattering operator satisfies a functional equation, $S(s)S(n-s) = \mathrm{Id}$, which implies that

$$P_\gamma \circ P_{-\gamma} = \mathrm{Id}. \tag{2.11}$$

Finally, it is proved in [14] that the operators P^g_γ satisfy the conformal covariance equation (1.3).

This definition of the operators P_γ depends crucially on the choice of the Poincaré-Einstein filling (X, G). Graham and Zworski point out that it is only necessary that the metric G satisfy the Einstein equation to sufficiently high order as $x \to 0$ in order that the properties of the P_γ listed above be true (for γ in a finite range which depends on the order to which G satisfies the Einstein equation). As we have discussed in the introduction, it is always possible to find such metrics, and we suppose that one has been fixed.

Let us now address the issue of how to define P^g_γ and Q^g_γ when Ω is a dense open set in a compact manifold M and g is complete and conformal to a metric $\overline{g}$ which extends to all of M. (As usual, we assume that $(M, \overline{g})$ has an asymptotically Poincaré-Einstein filling). There is no difficulty in using the relationship (1.3) to define $P^g_\gamma f$ when $f \in \mathcal{C}^\infty_0(\Omega)$. From here one can use an abstract functional analytic argument to extend P^g_γ to act on any $f \in L^2(\Omega, dV_g)$. Indeed, it is straightforward to check that the operator P^g_γ defined in this way is essentially self-adjoint on $L^2(\Omega, dV_g)$ when γ is real. However, observe that $P_\gamma = \Delta^\gamma_g + K$, where K is a pseudo-differential operator of order $2\gamma - 1$. Furthermore, Δ^γ is self-adjoint by the functional calculus, so we can appeal to a classical theorem, see [23], which states that a lower order symmetric perturbation of a self-adjoint operator is essentially self-adjoint.

A separate, but also very interesting issue, is whether Q^g_γ is a positive constant implies that the conformal factor u is a weak solution of (1.5) on all of M. This is true (with some additional hypotheses) when $\gamma = 1$, cf. [25].

We conclude this section with two alternate definitions of the operators $P^{\overline{g}}_\gamma$ in the special case where $(M, [\overline{g}]) = \mathbb{R}^n$ with its standard flat conformal class.

The canonical Poincaré-Einstein filling in this case is the hyperbolic space $X = \mathbb{R}^{n+1}_+ = \mathbb{R}^+_x \times \mathbb{R}^n_y$ with metric $G = x^{-2}(dx^2 + |dy|^2)$.

Since $\overline{g}$ is flat, we have $P^{\overline{g}}_\gamma = \Delta^\gamma_{\overline{g}}$, and this can be written in either of the two equivalent forms:

$$\begin{aligned}\Delta^\gamma f(y) &= (2\pi)^{-n} \int_{\mathbb{R}^n} e^{iy\eta} |\eta|^{2\gamma} \hat{f}(\eta)\, d\eta \quad \text{or} \\ &= \text{P.V.} \int_{\mathbb{R}^n} \frac{f(y) - f(\tilde{y})}{|y - \tilde{y}|^{n+2\gamma}}\, d\tilde{y}.\end{aligned}$$

Both formulæ can be regularized so as to hold for any given γ.

One other way that Δ^γ arises is as a generalized Dirichlet to Neumann map; this definition is essentially the same as the one above involving the scattering operator (indeed, the point of view in geometric scattering theory is that the scattering operator *is* simply the Dirichlet to Neumann operator at infinity), but as recently rediscovered by Chang-Gonzalez [3] in relation to the work on (Euclidean) fractional Laplacians by Caffarelli and Silvestre [2], it is sometimes helpful to consider the equation in a slightly different form. In the following result, let (X, G) be an asymptotically Poincaré-Einstein filling of the compact manifold $(M^n, [\bar{g}])$. Fix a representative $\bar{g}$ of the conformal class on the boundary and let x be the boundary defining function on X such that $g = x^{-2}(dx^2 + \bar{g}_x)$ with $\bar{g}_0 = \bar{g}$. Also, write $\bar{G} = x^2 G$; this is an incomplete metric on $\overline{X}$ which is smooth (or at least polyhomogeneous) up to the boundary.

Proposition 2.1. *([3]) Let $U = x^{\frac{n}{2}-\gamma}u$ and*

$$E := \Delta_{\bar{G}}\left(x^{\frac{1-2\gamma}{2}}\right)x^{\frac{1-2\gamma}{2}} + \left(\gamma^2 - \tfrac{1}{4}\right)x^{-1-2\gamma} + \tfrac{n-1}{4n}R_{\bar{G}}x^{1-2\gamma}.$$

Then, for any $f_0 \in \mathcal{C}^\infty(M)$, the eigenvalue problem (2.8) *is equivalent to*

$$\begin{cases} -div\left(x^{1-2\gamma}\nabla U\right) + EU = 0 & on\ (X, \bar{G}), \\ \qquad\qquad\qquad U|_{x=0} = f_0 & on\ M, \end{cases} \tag{2.12}$$

where the divergence and gradient are taken with respect to $\bar{G}$. Moreover,

$$P_\gamma^{\bar{g}}(f_0) = d_\gamma \lim_{x\to 0} x^{1-2\gamma}\partial_x U$$

for some nonzero constant d_γ depending only on γ and n.

The Euclidean version of this result (where (X, G) is the hyperbolic upper half-space) was the one studied by Caffarelli and Silvestre. The main advantage in this reformulation is that certain estimates are more transparent from this point of view.

3 Proofs

We now turn to the proofs of Theorems 1.1 and 1.2, and Remark 3.

3.1 Dimensional restrictions on singular sets

The idea for the proof of Theorem 1.1 is straightforward: let u be a polyhomogeneous distribution on M with singular set along the smooth submanifold Λ. Suppose that the leading term in the expansion of u is $a(y)r^{-n/2+\gamma}$. Then by a standard result in microlocal analysis [6], the function $P_\gamma^{\overline{g}} u$ is again polyhomogeneous and has leading term $b(y)r^{-(2\gamma+n)/2}$, where $b(y) = \lambda a(y)$ for some constant λ. Now, if u is a conformal factor for which $g = u^{4/(n-2\gamma)}\overline{g}$ has $Q_\gamma^g > 0$, then $P_\gamma^{\overline{g}} u > 0$, which implies that $\lambda > 0$. So we must compute λ to obtain (1.6).

This microlocal argument states that if u is polyhomogeneous, then the leading term of $P_\gamma^{\overline{g}} u$ can be computed using the symbol calculus (for pseudodifferential operators and for polyhomogeneous distributions), and more specifically, that the principal symbol of $P_\gamma^{\overline{g}} u$ is equal to the product of the principal symbols of $P_\gamma^{\overline{g}}$ and that of u. (Note that the principal symbol of a distribution conormal to a submanifold Λ is computed in terms of the Fourier transform in the fibres of $N\Lambda$.) In the present setting, this implies that the constant λ is the same as for the model case when $M = S^n$ and Λ is an equatorial S^k, so we now focus on this special case.

Transform S^n to $\mathbb{R}^n$ by stereographic projection, so that Λ is mapped to a linear subspace $\mathbb{R}^k$ and $\overline{g}$ is the flat Euclidean metric (which we henceforth omit from the notation). Write $\mathbb{R}^n \ni y = (y', y'') \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, so that (in this model case) $u(y) = |y''|^{-n/2+\gamma}$ for the singular metric $u^{\frac{4}{n-2\gamma}}\overline{g}$; then

$$
\begin{aligned}
P_\gamma u(y) = \Delta^\gamma u(y) &= (2\pi)^{-n} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(y-\tilde{y})\cdot\eta} |\eta|^{2\gamma} |y''|^{-n/2+\gamma}\, d\tilde{y} d\eta \\
&= (2\pi)^{k-n} \int_{\mathbb{R}^{n-k} \times \mathbb{R}^{n-k}} e^{i(y''-\tilde{y}'')\cdot\eta''} |\tilde{y}''|^{-n/2+\gamma}\, d\tilde{y}'' d\eta''.
\end{aligned}
$$

Now recall a well-known formula for the Fourier transform of homogeneous distributions in $\mathbb{R}^N$:

$$
\int_{\mathbb{R}^N} e^{-iz\cdot\zeta} |z|^{-N+\alpha}\, dz = c(N,\alpha)|\zeta|^{-\alpha},
$$

where

$$
c(N,\alpha) = \pi^{\alpha - N/2} \frac{\Gamma(\alpha/2)}{\Gamma((N-\alpha)/2)}.
$$

Applying this formula with $N = n-k$ (and replacing y'' by y and η'' by η, for simplicity) yields first that

$$
\int_{\mathbb{R}^N} e^{-iy\cdot\eta} |y|^{-n/2+\gamma}\, dy = c\left(n-k, \frac{n}{2} - k + \gamma\right) |\eta|^{-\frac{n}{2}+k-\gamma},
$$

then, multiplying by $|\eta|^{2\gamma}$ and taking inverse Fourier transform we obtain

$$\frac{1}{(2\pi)^{n-k}} c\left(n-k, \frac{n}{2}-k+\gamma\right) c\left(n-k, \frac{n}{2}+\gamma\right) |y|^{-\frac{n}{2}-\gamma}.$$

Altogether then, the multiplicative factor λ is equal to

$$2^{k-n} \pi^{k-n+2\gamma} \frac{\Gamma\left(\frac{1}{2}(\frac{n}{2}-k+\gamma)\right)}{\Gamma(\frac{1}{2}\left(\frac{n}{2}-\gamma\right))} \frac{\Gamma\left(\frac{1}{2}(\frac{n}{2}+\gamma)\right)}{\Gamma\left(\frac{1}{2}(\frac{n}{2}-k-\gamma)\right)}.$$

Discarding the factors which are always positive (which includes $\Gamma(n/4-\gamma/2)$ since $\gamma < n/2$), we obtain (1.6).

It is unfortunately slightly messy to write down the entire set of values of k and γ for which (1.6) holds. However, if $k < (n-2\gamma)/2$, then both $\frac{n}{2} - k \pm \frac{1}{2}\gamma > 0$. Furthermore, if $\gamma = 1$ and $A := \frac{n}{2} - k < \frac{1}{2}$, then the Γ function always takes on values with different signs at $A+\gamma/2$ and $A-\gamma/2$. More generally, if we fix n and k and let γ increase from 0 to $n/2$, then $\Gamma(A+\gamma/2)/\Gamma(A-\gamma/2) = 1$ when $\gamma = 0$; $\Gamma(A-\gamma/2)$ changes sign every time γ increases by 2, whereas $\Gamma(A+\gamma/2)$ also changes similarly, but only for γ in the range $(0, -2A)$, where $A < 0$.

To prove the final statement of the theorem, note that if $-\gamma - n/2$, the leading exponent of $P_\gamma^{\overline{g}} u$, is greater than $k - n$, the codimension of Λ, then $P_\gamma^{\overline{g}} u$ cannot have any mass supported on Λ, which means that u is a weak solution of $P_\gamma^{\overline{g}} u = Q_\gamma^g u^{(n+2\gamma)/(n-2\gamma)}$ on all of M.

3.2 Kleinian groups

We now turn to a special case where this problem has a direct relationship to hyperbolic geometry. Let Γ be a convex cocompact group of motions acting on $\mathbb{H}^{n+1}$. Thus Γ acts discretely and properly discontinuously on hyperbolic space, is geometrically finite and contains no parabolic elements. Its domain of discontinuity is the maximal open set $\Omega \subset S^n$ on which the action of Γ extends to a discrete and properly discontinuous action; by definition of convex cocompactness, the quotient $\Omega/\Gamma = Y$ is a compact manifold with a locally conformally flat structure. The complement $S^n \setminus \Omega = \Lambda$ is the limit set of Γ. Furthermore, the manifold $X = \mathbb{H}^{n+1}/\Gamma$ with its hyperbolic metric is Poincaré-Einstein with conformal infinity Y with the conformal structure induced from S^n. We use these canonical fillings to define P_γ and Q_γ.

In [22], Qing and Raske attack the problem of finding metrics of constant Q_k curvature (with $k < n/2$ an integer). Their method involves finding metrics of constant Q_γ curvature for all $1 \leq \gamma \leq k$. They rephrase the problem

$P_\gamma u = Q_\gamma u^{(n+2\gamma)/(n-2\gamma)}$ in the equivalent form $u = P_{-\gamma}(Q_\gamma u^{(n+2\gamma)/(n-2\gamma)})$. The advantage of this modification is that $P_{-\gamma}$ is a pseudodifferential operator of negative order, and its Schwartz kernel can be obtained by summing the translates of the Schwartz kernel of $P_{-\gamma}$ on S^n over the group Γ. This sum converges provided the Poincaré exponent of Γ is less than $\frac{n-2\gamma}{2}$, and because this is a convergent sum one directly obtains explicit control and positivity of this operator. Fixing $1 \leq \gamma < n/2$ and restricting to convex cocompact groups Γ with Poincaré exponent in this range, they are able to prove that if $Y = \Omega/\Gamma$ has positive Yamabe type, then it admits a metric of constant positive Q_γ curvature.

The proof of Remark 3 follows directly from this by lifting the conformal factor and solution metric to $\Omega \subset S^n$. Namely, by the theorem of Schoen and Yau, the developing map of Y is injective from the universal cover $\tilde{Y}$ to Ω, and the solution metric g on Y lifts to a complete metric $\tilde{g}$ on Ω of the form $u^{(n+2\gamma)/(n-2\gamma)}\overline{g}$, where $\overline{g}$ is the standard round metric. Using the bound on the Poincaré exponent and the compactness of Y, standard lattice point counting arguments show that $u(p) \leq c\,\mathrm{dist}_{\overline{g}}\,(p,\Lambda)^{(2\gamma-n)/2}$. This shows that not only is u a solution of the modified integral equation, but is also a weak solution of (1.5) on all of S^n and that Q_γ^g is constant. Finally, by Patterson-Sullivan theory, the dimension of the limit set Λ is precisely the Poincaré exponent $\delta(\Gamma)$. In other words, we have produced a solution to the fractional singular Yamabe problem with exponent $\gamma \in [1, n/2)$ and with singular set of dimension less than $n/2 - \gamma$.

The Qing-Raske theorem is not stated for the remaining cases $\gamma \in (0,1)$; it is plausible that their proof may be adapted to work then, and hence the lifted solution would also give a solution to our problem also for γ in this range, but we do not claim this. Note, however, the results in §4 below concerning growth estimates for solutions of this equation for this range of γ.

3.3 Perturbation methods

We come at last to the perturbation result. We deduce existence of solutions for the fractional singular Yamabe problem for values of γ near 1 from the the general existence result in [19] for the singular Yamabe problem with $\gamma = 1$. Let $(M, \overline{g})$ and Λ be a submanifold of dimension k as in the statement of the theorem. (Slightly more generally, we could let the different components of Λ have different dimensions, but for simplicity we assume that Λ is connected.) Then there is a function u on $M \setminus \Lambda$ such that $g = u^{4/(n-2)}\overline{g}$ is complete and its scalar curvature Q_1^g is a positive constant. Moreover, it is known that the linearization of the equation (1.1) at any one of the solutions u constructed in [19] is surjective on appropriate weighted Hölder spaces.

In the following we phrase this rigorously and parlay this surjectivity into an existence theorem for γ near 1 using the implicit function theorem. Let $\mathcal{T}^\sigma\Lambda$ denote the tube of radius σ (with respect to $\overline{g}$) around Λ; this is canonically diffeomorphic to the neighbourhood of radius σ in the normal bundle $N\Lambda$ for σ sufficiently small, and we use this to transfer cylindrical coordinates $(r, y, \theta) \in [0, \sigma) \times \mathcal{U}_y \times S^{n-k-1}$ in a local trivialization of $N\Lambda$ to Fermi coordinates in $\mathcal{T}^\sigma\Lambda$.

We use these coordinates to define weighted Hölder spaces with a certain dilation covariance property. For $w \in \mathcal{C}^0(\mathcal{T}^\sigma\Lambda)$, let

$$\|w\|_{e,0,\alpha,0} = \sup_{z\in\mathcal{T}^\sigma\Lambda} |w| + \sup_{z,\tilde{z}\in\mathcal{T}^\sigma\Lambda} \frac{(r+\tilde{r})^\alpha |w(z)-w(\tilde{z})|}{|r-\tilde{r}|^\alpha + |y-\tilde{y}|^\alpha + (r+\tilde{r})^\alpha|\theta-\tilde{\theta}|^\alpha}.$$

and denote by $\mathcal{C}^{0,\alpha}_e(M\backslash\Lambda)$ the space of all functions $w \in \mathcal{C}^0(T^\sigma\Lambda)$ such that this norm is finite. The initial subscript e in the norm signifies that these are 'edge' Hölder spaces. Next, $\mathcal{C}^{k,\alpha}_e(M\backslash\Lambda)$ denotes the subspace of $\mathcal{C}^k(M\backslash\Lambda)$ on which the norm

$$\|w\|_{k,\alpha,0} = \|w\|_{k,\alpha,M_{\sigma/2}} + \sum_{j=0}^{k} \|\nabla^j w\|^{\mathcal{T}^\sigma\Lambda}_{e,0,\alpha}$$

is finite, where $M_{\sigma/2} = M\backslash\mathcal{T}^\Lambda_{\sigma/2}$. Finally, for $\nu\in\mathbb{R}$, let

$$\mathcal{C}^{k,\alpha}_\nu(M\backslash\Lambda) = \left\{ w = r^\nu\overline{w} :\ \overline{w} \in \mathcal{C}^{k,\alpha}_e(M\backslash\Lambda)\right\},$$

with corresponding norm $||\cdot||_{e,k,\alpha,\nu}$.

Fixing $Q^g_\gamma = 1$, the linearization of $u \mapsto P^{\overline{g}}_\gamma u - u^{(n+2\gamma)/(n-2\gamma)}$ is the operator

$$v \mapsto L_\gamma v := P^{\overline{g}}_\gamma v - \frac{n+2\gamma}{n-2\gamma} u^{\frac{4\gamma}{n-2\gamma}} v.$$

Let u be one of the solutions to the singular Yamabe problem ($\gamma = 1$) on $M\setminus\Lambda$ constructed in [19]. It is proved there that the solution u has the form $u = c_1 r^{1-n/2}(1+v)$, where $v \in \mathcal{C}^{2,\alpha}_\nu$ for any $0 < \nu < k/2$ and $c_1 > 0$ depends only on the dimensions k and n; furthermore, the mapping

$$L_1 : \mathcal{C}^{2,\alpha}_\nu(M\setminus\Lambda) \longrightarrow \mathcal{C}^{0,\alpha}_{\nu-2}(M\setminus\Lambda)$$

is surjective for ν in this same range.

We claim that for γ sufficiently close to 1, and for $\nu \in (\eta, k/2-\eta)$, where $\eta > 0$ is some small fixed number, the mapping

$$L_\gamma : \mathcal{C}^{2,\alpha}_\nu(M \setminus \Lambda) \longrightarrow \mathcal{C}^{0,\alpha+2(1-\gamma)}_{\nu-2\gamma}(M \setminus \Lambda)$$

is also bounded and surjective. The boundedness follows by an interpolation argument. Indeed, the spaces $\mathcal{C}^{k,\alpha}_0$ have interpolation properties which are identical to those for the ordinary Hölder spaces since they *are* just the standard Hölder spaces for the complete metric $\tilde{g} = \overline{g}/r^2$; a minor adjustment shows that the addition of the weight factor behaves as expected. The assertion about the boundedness of L_γ is clearly true for $\gamma = 0, 1, 2$, and hence by interpolation is true for all γ close to 1. (It is true for the full range of $\gamma \in (0, 2)$ if one makes the standard change, replacing the Hölder space by a Zygmund space, when $\alpha + 2(1-\gamma)$ is an integer.) This also follows from [18] because $r^{2\gamma} L_\gamma$ is a pseudodifferential edge operator of order 2γ. Similarly, surjectivity follows from the construction of a parametrix for L_γ in the edge calculus, from [18] again. This proves that L_γ is Fredholm, and since it is surjective at $\gamma = 1$, it must remain surjective for values of γ which are close to 1. We write its right inverse as G_γ.

Now consider the mapping

$$(\gamma, c, v) \longmapsto N(\gamma, c, v) := G_\gamma(P^{\overline{g}}_\gamma c r^{\gamma-n/2}(1+v) - (c r^{\gamma-n/2}(1+v))^{\frac{n+2\gamma}{n-2\gamma}}).$$

If $u_1 = c_1 r^{1-n/2}(1+v_1)$ is the solution to the singular Yamabe problem from [19], then $N(1, c_1, v_1) = 0$. Let $c \in (c_1 - \varepsilon, c_1 + \varepsilon)$, and similarly, $v - v_1$ lie in a ball of radius ε about 0 in $\mathcal{C}^{2,\alpha}_\nu$. Clearly $D_v N|_{(1,c_1,v_1)} = G_1 L_1 = \mathrm{Id}$. The implicit function theorem now applies to show that for every (γ, c) near to $(1, c_1)$, there exists a unique $v_\gamma \in \mathcal{C}^{2,\alpha}_\nu$ with norm less than ε such that $u_\gamma = c r^{\gamma-n/2}(1+v_\gamma)$ is a solution of the fractional singular Yamabe problem with singular set Λ.

4 Growth estimates for weak solutions on S^n

In this final section we furnish the proof of Proposition 1.3: if $\gamma \in (0, 1)$ and $\Omega \subset S^n$ is dense, then any weak solution of the fractional singular Yamabe problem

$$P^{g_c}_\gamma(u) = u^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{in } S^n, \quad u > 0, \quad u \text{ singular along } S^n \setminus \Omega \tag{4.13}$$

satisfies a general growth estimate. This is a direct adaptation of Schoen's proof (which is written out in full in [21]) for the case $\gamma = 1$.

We first comment on the local regularity for solutions of (1.7). There are several ways to deduce the necessary estimates. The path we follow uses the equivalence, as described in 2.1, of (1.7) with the extension problem (2.12):

$$\begin{cases} -\operatorname{div}\left(x^{1-2\gamma}\nabla U\right) + E(x)U = 0, & \text{in } (X, \bar{G}), \\ -y^{1-2\gamma}\partial_x U = c_{n,\gamma} U^{\frac{n+2\gamma}{n-2\gamma}}, & \text{on } x = 0; \end{cases} \tag{4.14}$$

here $U = x^{\frac{n}{2}-\gamma}u$, $\bar{G} = x^2 G$.

From this point of view, we can use the linear regularity theorem [18, Theorem 7.14] to prove that U is smooth up to $x = 0$ away from Λ. This can also be deduced using standard elliptic estimates for the pseudodifferential operator $P_\gamma^{g_c}$, but we refer also to more classical sources from which this can also be deduced, in particular the paper by [7]; we also refer to more recent references [1] (where many properties of the solution are written down), and [12] (which holds for more general ambient metrics). In particular, from these last papers, one has that Schauder and local $L^p \to L^\infty$ estimates hold, and the equation also satisfies the standard maximum principles.

Fix $z_0 \notin \Lambda$ and choose $\sigma < \operatorname{dist}_{g_c}(z_0, \Lambda)$. For simplicity, write $\rho(z) := \operatorname{dist}_{g_c}(z, z_0)$. Now define

$$f(z) := (\sigma - \rho(z))^{\frac{n-2\gamma}{2}}\, u(z);$$

note that $f = 0$ on $\partial B_\sigma(x_0)$.

It suffices to show that $f(z) \le c$ for some $c > 0$ and for all $z \in B_\sigma(z_0)$ since if we choose $\sigma = \operatorname{dist}(z_0, \Lambda)/2$, then $f(z_0) = \sigma^{\frac{n-2\gamma}{2}} u(z_0)$, and hence

$$u(z_0) \le \frac{c}{d(z_0, \Lambda)^{\frac{n-2\gamma}{2}}},$$

which would finish the proof.

We prove this claim by contradiction. Assume that no such c exists. Then there exists a sequence $\{u_m, \Lambda_m, \sigma_m, z_{0,m}, z_m\}$ such that for all m, f_m attains its maximum in $B_{\sigma_m}(z_{0,m})$ at z_m, and

$$f(z_m) := (\sigma_m - \operatorname{dist}(z_m, z_{0,m}))^{\frac{n-2\gamma}{2}} u_m(z_m) > m.$$

Since $(\sigma_m - \operatorname{dist}(z_m, z_{0,m}))^{\frac{n-2\gamma}{2}} \le \sigma_m^{\frac{n-2\gamma}{2}} \le C$ for all m, we see that necessarily $u_m(z_m) \to \infty$.

Let z be a system of Riemann normal coordinates centered at z_m, so that the corresponding metric coefficients satisfy $(g_c)_{ij} = \delta_{ij} + O(|z|^2)$. (As m

varies, these coordinate systems also vary, but there is no reason to include this in the notation.) Set $\lambda_m = (u_m(z_m))^{\frac{2}{n-2\gamma}}$; we consider the dilated coordinate system $\zeta = \lambda_m z$, the corresponding sequence of metrics $\hat{g}_m$, where

$$g_c = \lambda_m^{-2} \sum_{i,j=1}^{n} (g_c)_{ij}(\zeta/\lambda_m) d\zeta^i d\zeta^j := \lambda_m^{-2} \hat{g}_m,$$

and finally the dilated family of solutions

$$v_m(\zeta) := \lambda_m^{-\frac{n-2\gamma}{2}} u_m\left(\frac{\zeta}{\lambda_m}\right).$$

By construction, $v_m(0) = 1$ for all m, and

$$g_m := u_m^{\frac{4}{n-2\gamma}} g_c = v_m^{\frac{4}{n-2\gamma}} \hat{g}_m.$$

We show below that $\hat{g}_m$ and v_m are defined on an expanding sequence of balls on $\mathbb{R}^n$, and it is then clear that $\hat{g}_m$ converges to the Euclidean metric uniformly in $\mathcal{C}^\infty$ on any compact subset.

Let $r_m = \frac{1}{2}(\sigma_m - \rho(z_m))$, or equivalently, $\rho_m(z_m) + 2r_m = \sigma_m$. Then

$$\sigma_m - \rho(z_m) \geq \sigma_m - \rho_m(z_m) - r_m = r_m$$

on the ball $\mathrm{dist}_{g_c}(z, z_m) < r_m$, and hence on this same ball,

$$u_m(z) \leq \left(\frac{\sigma_m - \rho_m(z_m)}{\sigma_m - \rho_m(z)}\right)^{\frac{n+2\gamma}{n-2\gamma}} u_m(z_m) \leq c\, u_m(z_m), \quad c = 2^{\frac{n-2\gamma}{2}}.$$

The corresponding ball in rescaled coordinates contains $\{\zeta : |\zeta| < m^{\frac{2}{n-2\gamma}}\}$, hence has radius tending to infinity. By construction, $v_m(z) \leq c$ on this entire ball, and $v_m(0) \equiv 1$. Since these functions are uniformly bounded and satisfy the converging set of elliptic pseudodifferential equations

$$P_\gamma^{\hat{g}_m} v_m = v_m^{\frac{n+2\gamma}{n-2\gamma}}, \tag{4.15}$$

we conclude using the local regularity theory (which is straightforward since v_m is bounded) that v_m is bounded in $\mathcal{C}^{2,\alpha}$ of every compact set, and hence we can extract a convergent subsequence. We thus obtain a smooth solution v to the 'flat' equation

$$-(\Delta_{\mathbb{R}^n})^\gamma v = v^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{in } \mathbb{R}^n. \tag{4.16}$$

Since each $v_m > 0$, we see that $v \geq 0$, but $v \not\equiv 0$ since $v(0) = 1$. There is a maximum principle for this equation [12, Corollary 3.6] when $0 < \gamma \leq 1$, so we conclude that $v > 0$ on all of $\mathbb{R}^n$.

There is a complete characterization of positive solutions of (4.16), [12, §5]). They are the extremal functions for the embedding $H^\gamma(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2\gamma}}(\mathbb{R}^n)$, and are necessarily of the form

$$v(z) = C \left(\frac{\mu}{|z - z_0|^2 + \mu^2} \right)^{\frac{n-2\gamma}{2}},$$

for some $\mu, c > 0$ and $z_0 \in \mathbb{R}^n$ (these are well known "bubbles").

The argument is completed using Theorem 4.1 below, which states that a small ball in $\Omega_m = S^n \setminus \Lambda_m$ must have a concave boundary with respect to to g_m for m sufficiently large. This is a contradiction to the already known limiting form of the v_m. The proof of Proposition 1.3 is thus completed.

□

Note that the previous arguments do not require that u is a weak solution in the whole S^n. The only place where this strong hypothesis is required is in the following convexity claim:

Theorem 4.1. *In the same hypothesis as in Proposition 1.3, any open ball B (with respect to g_c) with $\bar{B} \subset \Omega$, has boundary ∂B which is geodesically convex with respect to g.*

This result was proved in [24] for constant scalar curvature metrics, and also in the case $\gamma \in (1, n/2)$ for locally conformally flat manifolds satisfying some extra conditions by [22]. The crucial step is the application of the Alexandroff moving plane method. As we show here, the same ideas work in the fractional case. The moving plane method has been successfully applied to fractional order operators in [22] and [5], at least when the equation is rewritten as an integral equation. However, the proof in the present seting is simpler because of the equivalent formulation (4.14) and the precise asymptotics (4.19), so we include the details for the reader's convenience. Our proof follows the classical arguments for the Laplacian by Gidas-Ni-Nirenberg in [10], [9].

For simplicity, we denote $P_\gamma := P_\gamma^{|dx|^2}$. Let v be a distributional solution of

$$P_\gamma v = v^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{in } \mathbb{R}^n. \tag{4.17}$$

We will apply the Alexandroff reflection with respect to the planes $S_\lambda := \{x \in \mathbb{R}^n : x^n = \lambda\}$. Let $\Sigma_\lambda := \{x \in \mathbb{R}^n : x^n > \lambda\}$ be the hyperplane lying above S_λ. Given $x = (x^1, \dots, x^n) \in \Sigma_\lambda$, define x^λ to be the reflection of x with respect to the hyperplane S_λ, i.e., $x^\lambda := (x^1, \dots, x^{n-1}, 2\lambda - x^n)$. We also define $v_\lambda(x) := v(x^\lambda)$ and

$$w_\lambda(x) := v_\lambda(x) - v(x).$$

Note that the equation satisfied by v_λ is the same as the satisfied by v. Although this fact is not clear for non-local operators, it is easily seen to be true in the Caffarelli-Silvestre extension (4.14). Then, by linearity,

$$P_\gamma w_\lambda = v_\lambda^{\frac{n+2\gamma}{n-2\gamma}} - v^{\frac{n+2\gamma}{n-2\gamma}}, \text{ weakly.} \tag{4.18}$$

We will need a couple of preliminary results:

Lemma 4.2. *Let v be any function with asymptotics*

$$v(x) = |x|^{2\gamma-n} \left(a + \sum_{i=1}^{n} \frac{b_i x^i}{|x|^2} + O\left(|x|^2\right) \right) \quad \text{when } |x| \to \infty, \tag{4.19}$$

for some $a > 0$. Then there exists $\lambda_0 > 0$ such that for all $\lambda \geq \lambda_0$,

$$w_\lambda(x) > 0 \quad \text{for all } x \in \Sigma_\lambda.$$

Proof. This is just Lemma 2.2. in [10], and it does not use (4.17). □

Lemma 4.3. *Let v a weak solution of (4.17). If for some $\lambda < \lambda_0$ we have that $w_\lambda(x) \geq 0$ but $w_\lambda \not\equiv 0$ in Σ_λ, then*

$$w_\lambda(x) > 0 \text{ in } \Sigma_\lambda \quad \text{and} \quad \partial_n v(x) < 0 \text{ on } S_\lambda. \tag{4.20}$$

Proof. When v solves the constant scalar curvature equation, this is just Lemma 2.2. and Lemma 4.3 in [9]. In our case, we need a strong maximum principle and Hopf's lemma for the operator P_γ (see [12], [1]). We know from (4.18) that $P_\gamma w_\lambda \geq 0$. Since $w_\lambda \geq 0$ in Σ_λ (and is not identically zero), and since w_λ vanishes on the boundary S_λ, the strong maximum principle gives that $w_\lambda > 0$ in all of Σ_λ. On the other hand, Hopf's lemma implies that $\partial_n w_\lambda > 0$ on S_λ. Then, $\partial_n w_\lambda = \partial_n v_\lambda - \partial_n v = -2\partial_n v$, so we immediately have $\partial_n v < 0$ along S_λ.

□

Proof of Theorem 4.1: Let g be a complete metric of constant positive scalar Q_γ curvature on $\Omega \subset S^n$ of the form $g = u^{\frac{4}{n-2\gamma}} g_c$, and B an open ball in Ω. Let $S = \partial B$ be the boundary sphere. Fix any point $p \in S$. Use stereographic projection to map Ω into $\tilde{\Omega} \subset \mathbb{R}^n$ so that p is mapped to infinity. Then S is transformed to a hyperplane $\tilde{S}$, and the projected $\partial\tilde{\Omega}$ lies on one side of $\tilde{S}$, say below. Use linear coordinates $(x^1, \ldots, x^n) \in \mathbb{R}^n$ with $\tilde{S} = \{x^n = 0\}$.

By stereographic projection, the metric g transforms to a conformally flat metric on $\mathbb{R}^n$, $g_v = v^{\frac{4}{n-2\gamma}} |dx|^2$. Since the scalar curvature equation is conformally covariant, we also have

$$\Delta^\gamma v = v^{\frac{n+2\gamma}{n-2\gamma}},$$

on $\tilde{\Omega} \subset \mathbb{R}^n$. Note that v is a weak solution of this equation on all of $\mathbb{R}^n$.

Since the function u is smooth and strictly positive at p, the function v is regular at infinity, i.e. has the asymptotics (4.19) for some $a > 0$ as $|x| \to \infty$.

Step 1. Starting the reflection: Thanks to Lemma 4.2, we can initiate the reflection argument when λ is sufficiently large. Note that the equation satisfied by v is not needed here since we have the precise behavior (4.19).

Step 2. Continuation: We now move the plane S_λ, so long as it does not touch the singular set. Suppose that at some $\lambda_1 > 0$ we have $w_{\lambda_1}(x) > 0$ for all $x \in \Sigma_{\lambda_1}$, but $w_{\lambda_1} \not\equiv 0$ in Σ_{λ_1}. Then the plane can be moved further; more precisely, there exists some $\epsilon > 0$ not depending on λ_1 such that $w_\lambda \geq 0$ in Σ_λ for all $\lambda \in [\lambda_1 - \epsilon, \lambda_1]$.

We observe first that because of Lemma 4.3 we must have

$$\partial_n v < 0 \quad \text{on } \Sigma_{\lambda_1}. \tag{4.21}$$

Next, the proof of our claim follows as in Lemma 2.3. in [10] by contradiction. Thus, assume that there is a sequence $\lambda_j \to \lambda_1$ and a sequence of points $\{x_j\}$, $x_j \in \Sigma_{\lambda_j}$ such that

$$w_{\lambda_j}(x_j) \leq 0. \tag{4.22}$$

Either a subsequence, which we again call $\{x_j\}$, converges to $x_\infty \in \Sigma_{\lambda_1}$ or else $x_j \to \infty$. In the first case, because of (4.22) we must have $\partial_n v(x_\infty) \leq 0$, thus contradicting (4.21). So $x_j \to \infty$. But in this second case may use the asymptotics for v from (4.19), that imply

$$\frac{|x_j|^n}{\lambda_j - x_j^n} w_{\lambda_j}(x) \to -(n-2\gamma)a < 0.$$

This is a contradiction to (4.22).

Finally, note that in this process we never have $w_\lambda \equiv 0$ since the existence of the singularity of v implies that it has no plane of symmetry. Hence the moving plane can be moved all the way to $\lambda = 0$.

Step 3. Conclusions: We have shown that Σ_λ can be moved to $\lambda = 0$, and then $w_\lambda(x) > 0$ for all $x \in \Sigma_0$ and

$$\partial_n v(x) < 0 \quad \text{for all } x \in \overline{\Sigma_0}.$$

Since $g = v^{\frac{4}{n-2\gamma}}|dx|^2$, the second fundamental form of any plane S_λ, $\lambda \geq 0$, with respect to g is given by

$$\left(-\frac{4}{n-2\gamma} v^{-1} \frac{\partial v}{\partial x^n} I \right).$$

The sign of $\partial_n v$ therefore implies that S_λ is locally geodesically convex for all $\lambda \geq 0$. When transferred back to the sphere, this shows that any round ball contained in Ω has locally geodesically convex boundary, as claimed. $\square$

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