A GLUING CONSTRUCTION OF SINGULAR SOLUTIONS FOR A FULLY NON-LINEAR EQUATION IN CONFORMAL GEOMETRY

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Abstract. In this paper we produce families of complete, non-compact Riemannian metrics with positive constant σ_2–curvature on the sphere $\mathbb{S}^n$, $n > 4$, with a prescribed singular set Λ given by a disjoint union of closed submanifolds whose dimension is positive and strictly less than $(n - \sqrt{n} - 2)/2$. The σ_2–curvature in conformal geometry is defined as the second elementary symmetric polynomial of the eigenvalues of the Schouten tensor, which yields a fully non-linear PDE for the conformal factor. We show that the classical gluing method of Mazzeo-Pacard (JDG 1996) for the scalar curvature still works in the fully non-linear setting. This is a consequence of the conformal properties of the σ_2 equation, which imply that the linearized operator has good mapping properties in weighted spaces. Our method could be potentially generalized to any σ_k, $2 \leq k < n/2$, nevertheless, the numerology becomes too involved.

Key Words: σ_k–curvature, fully nonlinear equations, conformal geometry, singular solutions, gluing methods, complete metrics

AMS subject classification: 35J75, 53C21, 58J60

1. Notations and preliminaries

Let (M, g) be a compact smooth n–dimensional Riemannian manifold without boundary and let $2 \leq 2k < n$. Taking advantage of this second assumption, we introduce the following formalism for a conformal change of metric

$$g_u := u^{\frac{4k}{n-2k}}\, g, \tag{1.1}$$

where the conformal factor $u > 0$ is a smooth positive function. In this context g will be referred as the background metric.

Let Ric_g, R_g be the Ricci tensor and scalar curvature of g, respectively. The Schouten tensor with respect to the metric g is given by

$$A_g = \tfrac{1}{n-2}\left(Ric_g - \tfrac{1}{2(n-1)} R_g g\right).$$

For the conformal metric (1.1), the Schouten tensor of g_u is related to the one of g by the conformal transformation law

$$A_{g_u} = A_g - \tfrac{2k}{n-2k} u^{-1} D^2 u + \tfrac{2kn}{(n-2k)^2} u^{-2} du \otimes du - \tfrac{2k^2}{(n-2k)^2} u^{-2} |du|^2 g, \tag{1.2}$$

where D^2 and $|\cdot|$ are computed with respect to the background metric g.

We define the σ_k*–curvature* as the k-th elementary symmetric function of the eigenvalues of the $(1,1)$-tensor $g^{-1}A_g$

$$\sigma_k(g^{-1}A_g) = \sum_{i_1<\ldots<i_k} \lambda_{i_1}\ldots\lambda_{i_k},$$

and the *positive cone* as set of metrics

$$\Gamma_k^+ = \{g \,:\, \sigma_1(g^{-1}A_g),\ldots,\sigma_k(g^{-1}A_g) > 0\}.$$

Fixed a background metric, the σ_k*–Yamabe problem* consists in finding a conformal metric in the positive cone Γ_k^+ of constant σ_k–curvature. This is a fully non-linear equation for the conformal factor u,

$$\sigma_k\big(\, g_u^{-1} A_{g_u}\big) = 2^{-k}\binom{n}{k}. \tag{1.3}$$

The objective of this paper is, given Λ a smooth, compact, closed p-dimensional submanifold of $\mathbb{S}^n$, to construct complete metrics on $\mathbb{S}^n\setminus\Lambda$ of positive constant σ_2–curvature (and in the positive Γ_2^+ cone) that are conformal to the canonical metric on $\mathbb{S}^n$ and become singular exactly on Λ. It is known that a restriction on the dimension p needs to be imposed - see the discussion below. In the proof we follow the classical gluing method of Mazzeo-Pacard [33] for the scalar curvature, which is a semilinear problem. Our main contribution here is to show that their scheme can be adapted to the fully non-linear equation (1.8) thanks to the conformal properties of the problem.

We remark that, even though this method could work for any $2 \leq k < n/2$, we have some computational difficulties that restrict our main Theorem to $k = 2$; however, we conjecture that it is true for other values of k. In any case, we will try to state the results as generally as possible.

The semilinear case, that is, for $k = 1$, is already well known in the literature. In addition to the aforementioned reference [33], we underline the classical result of Schoen-Yau [42], where they show that the Hausdorff dimension of the singular set Λ must satisfy $\dim_{\mathcal{H}}(\Lambda) \leq \frac{n-2}{2}$. We cite also Mazzeo-Smale [35], where they consider singular sets on the sphere that sufficiently close to an equatorial subsphere, Fakhi [15] when the singular set Λ is a submanifold with boundary, and Chan-González-Mazzeo [9] for singular sets that allow corners or edges.

For a general $k > 1$, a necessary condition for the existence of complete metrics on $\mathbb{S}^n \setminus \Lambda$ conformal to the standard one was given by one of the authors in [17]. More precisely, if

$$\sigma_1(B_{g_u}) \geq C_0 > 0 \quad \text{and} \quad \sigma_2(B_{g_u}), \ldots, \sigma_k(B_{g_u}) \geq 0$$

for some integer $1 \leq k < n/2$, then

$$\dim_{\mathcal{H}}(\Lambda) \leq \frac{n-2k}{2}.$$

The most representative example of such solutions is $\mathbb{S}^n \setminus \mathbb{S}^p$ with its canonical metric, which is conformal to the product $\mathbb{S}^{n-p-1} \times \mathbb{H}^{p+1}$ with its standard metric (a picture can be found in [3, Figure 1]). Its Schouten tensor is diagonal and, modulo a multiplicative factor of $1/2$, its eigenvalues are 1 and -1, with multiplicities $n-p-1$ and $p+1$, respectively. Then we can compute

(1.4)

$$2^k \sigma_k(\mathbb{H}^{p+1} \times \mathbb{S}^{n-p-1}) = \sum_{i=0}^{k} \binom{n-p-1}{i} \binom{p+1}{k-i} (-1)^{k-i} =: c_{n,p,k}.$$

For point singularities ($p = 0$) there is a rich geometry of Delaunay-type solutions (Chang-Han-Yang [12], Li-Han [27]). In this paper we will restrict to higher dimensional singularities, that is, $p > 0$. We set

(1.5)

$$\mathfrak{p}_k := \sup \left\{ p \geq 0 : \sigma_1(\mathbb{H}^{p+1} \times \mathbb{S}^{n-p-1}), \ldots, \sigma_k(\mathbb{H}^{p+1} \times \mathbb{S}^{n-p-1}) > 0 \right\}.$$

Exact formulas can be given for $k = 2, 3$. Indeed,

$$\mathfrak{p}_2 = \frac{n - \sqrt{n} - 2}{2}, \quad \mathfrak{p}_3 = \frac{n - 2 - \sqrt{3n-2}}{2}.$$

It was also shown in [17] that, for fixed $k > 1$, we have the asymptotic bound

$$\frac{n}{2} - C_1(k)\sqrt{n} \leq \mathfrak{p}_k < \frac{n}{2} - \frac{2 + \sqrt{n}}{2} \quad \text{as} \quad n \to \infty$$

for some constant $C_1(k)$.

We conjecture that a necessary condition for the existence of solutions to the σ_k–Yamabe problem in the positive cone with singular set a p–dimensional, compact, closed submanifold is

$$0 < p < \mathfrak{p}_k.$$

Our main result states that this is indeed sufficient in the particular case $k = 2$:

Theorem 1.1. *Let Λ be a subset of $\mathbb{S}^n$ which is a closed submanifold of dimension p, such that $0 < p < \mathfrak{p}_2$. Then there exists a complete metric g on $\mathbb{S}^n \setminus \Lambda$, conformal to the canonical metric g_c on $\mathbb{S}^n$, with constant σ_2–curvature (and in the positive Γ_2^+ cone), that is singular exactly on Λ.*

Remark 1.2. Since our proof is local, the same conclusion is true if Λ is a finite union of disjoint submanifolds with the specified restrictions.

Remark 1.3. Note that, for $k = 2$, the restriction of being in the Γ_2^+ cone is easily satisfied since

$$\sigma_2 \leq \frac{n-1}{2n}\sigma_1^2,$$

no matter what the sign of σ_1 is (see [48] for a proof of this classical inequality).

Let us make some comments on related bibliography. In the paper Mazzieri-Ndiaye [37] they construct constant σ_k metrics with isolated singularities by gluing Delaunay-type metrics. In the publication Silva-Santos [46], the authors use asymptotic matching to find solutions to the σ_2–Yamabe problem with isolated singularities. Another relevant paper is Guan-Lin-Wang [20], which showcases a gluing construction of manifolds of positive σ_k–curvature by connected sums of compact manifolds. A more recent paper is by Duncan-Wang [14], which contains complementary existence results.

Finally, we remark that our gluing method is very versatile and can be applied in many other settings, for instance, in non-local problems [1, 8].

1.1. **Scheme of the proof.**

As we have mentioned, our starting point is to use the gluing method of Mazzeo-Pacard [33] in order to construct positive solutions to (1.3).

First of all, note that by stereographic projection, it is equivalent to consider the problem in $\mathbb{S}^n \setminus \Lambda$ or $\mathbb{R}^n \setminus \Lambda$ with the canonical metrics $g_{\mathbb{S}^n}$ or $g_{\mathbb{R}^n}$, respectively.

For technical reasons, once a background metric g has been fixed, it is convenient to denote

$$B_{g_u} := \tfrac{n-2k}{2k}\, u^{\frac{2n}{n-2k}}\, g_u^{-1} A_{g_u} \tag{1.6}$$

so that

$$B_{g_u} = u^2 B_g + g^{-1}\left[-uD^2u + \tfrac{n}{n-2k} du \otimes du - \tfrac{k}{n-2k} |du|_g^2\, g\right], \tag{1.7}$$

where all derivatives are taken with respect to the background metric g. We can also write the positive cone as

$$\Gamma_k^+ = \{u > 0 \,:\, \sigma_1(B_{g_u}), \ldots, \sigma_k(B_{g_u}) > 0\}.$$

Now we reformulate the σ_k–equation (1.3) as

$$\mathcal{N}(u,g) := u^{-2k+1}\sigma_k\left(B_{g_u}\right) - cu^{q-2k+1} = 0, \tag{1.8}$$

where we have set

$$q := \frac{2kn}{n-2k}, \quad c := \binom{n}{k}\left(\frac{n-2k}{4k}\right)^k.$$

Remark that our arguments rely on the good conformal properties of the equation. Indeed, if two metrics g_1 and g are related by $u_1^{4k/(n-2k)}g_1 = u^{4k/(n-2k)}g$, then the non-linear operator from (1.8) enjoys the following *conformal equivariance property*

$$\mathcal{N}\left(u_1, g_1\right) = (u/u_1)^{-\frac{2kn}{n-2k}+2k-1}\,\mathcal{N}\left(u, g\right). \tag{1.9}$$

The linearized operator of $\mathcal{N}(\,\cdot\,, g)$ about u is defined as

$$\mathbb{L}(u,g)\left[\varphi\right] := \left.\frac{d}{ds}\right|_{s=0} \mathcal{N}\left(u + s\varphi, g\right). \tag{1.10}$$

As a direct consequence of the property (1.9), we have the following *conformal equivariance property* for the linearized operator

$$\mathbb{L}(u_1, g_1)[\varphi] \;=\; (u/u_1)^{-\frac{2kn}{n-2k}+2k-1}\,\mathbb{L}(u,g)\left[(u/u_1)\,\varphi\right]. \tag{1.11}$$

The normalization we have chosen for $\mathcal{N}$ may seem arbitrary at first sight. However, it is the one that makes this linearized operator resemble the Laplacian (see formula (4.4) in Section 4), thus recovering the original setting of [33].

The main idea in the proof of Theorem 1.1 to obtain a solution to (1.8) is, first to construct an approximate metric, written in the form

$$\bar{g}_\epsilon := \bar{u}_\varepsilon^{4k/(n-2k)} g,$$

with the right asymptotic behavior near the singularity, and then find a perturbation φ such that

$$\mathcal{N}(\bar{u}_\epsilon + \varphi, g) = 0. \tag{1.12}$$

For simplicity, once $\bar{u}_\epsilon$ and g have been fixed we will simply write

$$\mathbb{L}_\epsilon[\varphi] := \mathbb{L}(\bar{u}_\epsilon, g)[\varphi].$$

Then, equation (1.12) is equivalent to

$$\mathbb{L}_\epsilon[\varphi] + f_\epsilon + Q_\epsilon[\varphi] = 0, \tag{1.13}$$

where we have defined

$$f_\epsilon := \mathcal{N}(\bar{u}_\epsilon, g), \tag{1.14}$$

$$Q_\epsilon[\varphi] := \mathcal{N}(\bar{u}_\epsilon + \varphi, g) - \mathcal{N}(\bar{u}_\epsilon, g) - \mathbb{L}_\epsilon[\varphi]. \tag{1.15}$$

Most of the analysis in this paper is concerned with the study of the mapping properties of the linearized operator in weighted spaces. More precisely, if $\mathbb{L}_\epsilon$ has a right inverse, we can write (1.13) as

$$\varphi = -\mathbb{L}_\epsilon^{-1}(Q_\epsilon[u] + f_\epsilon).$$

A fixed point argument will show the existence of a solution φ.

Our reasoning will involve a delicate choice of parameters μ and ν. For future reference, we summarize our choices in Figure 1.1 below.

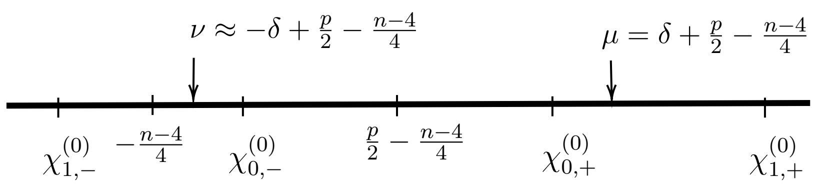

FIGURE 1. Our choice of parameters μ and ν.

We conclude the Introduction with some notation remarks. Denote the codimension of Λ by

$$N = n - p.$$

For one-variable functions φ_1,φ_2, we will write

$$\varphi_1 \sim \varphi_2 \quad \text{iff} \quad 0 < c_1 \leq \lim \frac{\varphi_1}{\varphi_2} \leq c_2$$

and

$$\varphi_1 \asymp \varphi_2 \quad \text{iff} \quad \lim \frac{\varphi_1}{\varphi_2} = 1.$$

The notation $\varphi = O(r^\beta)$ means, not only an estimate for the function φ, but also for its derivatives (with the corresponding order).

Acknowledgements. The authors would like to thank Rafe Mazzeo, Lorenzo Mazzieri, Frank Pacard and Mariel Sáez for their useful advice.

2. The model $\mathbb{R}^n \setminus \mathbb{R}^p$ in cylindrical coordinates

In this Section we consider the model $\mathbb{R}^n \setminus \mathbb{R}^p$, that is the building block in our construction. First note that $\mathbb{R}^n \setminus \mathbb{R}^p$ is diffeomorphic to $\mathbb{R}_t \times \mathbb{S}^{N-1}_\theta \times \mathbb{R}^p_z$, so we may write the Euclidean metric in polar-Fermi coordinates

$$g_E := dr^2 + r^2 g_\theta \,+\, \delta_{\alpha\beta}\, dz^\alpha \otimes\, dz^\beta, \tag{2.1}$$

where $\alpha, \beta = 1, \ldots, p$ will refer to coordinates $z \in \mathbb{R}^p$, and g_θ is the canonical metric on the sphere $\mathbb{S}^{N-1}$. Subindexes i, j will refer to coordinates $\theta \in \mathbb{S}^{N-1}$.

We describe now the spherical harmonic decomposition of $\mathbb{S}^{N-1}$. Let $\{\lambda_j\}_{j=1}^\infty$ be the eigenvalues for $-\Delta_\theta$ repeated according to multiplicity, with eigenfunctions $\{e_j(\theta)\}$, that is,

$$-\Delta_\theta e_j = \lambda_j e_j, \quad j = 0, 1, \ldots \tag{2.2}$$

In particular,

$$\lambda_0 = 0, \quad \lambda_j = N - 1 \quad \text{for } j = 1, \ldots, N, \quad \text{and so on.}$$

2.1. Cylindrical coordinates.

Our first observation is that it is more convenient to use a (conformal) cylinder-type metric as background. For this, we set

$$g_{cyl} = dt^2 + g_\theta + e^{2t}\delta_{\alpha\beta}\, dz^\alpha \otimes\, dz^\beta.$$

Then, denoting the radial coordinate by $r = e^{-t}$, we see that g_E and g_{cyl} are conformally related by

$$g_E = e^{-2t}\big(\, dt^2 \,+\, g_\theta\big) \,+\, \delta_{\alpha\beta}\, dz^\alpha \otimes\, dz^\beta = e^{-2t} g_{cyl}. \tag{2.3}$$

We also change the notation for the conformal factor from u to v where

$$u(r, \theta, z) = r^{-\frac{n-2k}{2k}} v(-\log r, \theta, z). \tag{2.4}$$

Thus we can record any conformal change by

$$g_u = u^{\frac{4k}{n-2k}} g_E = v^{\frac{4k}{n-2k}} g_{cyl}. \tag{2.5}$$

With some abuse of notation, we will denote this conformal metric as

$$g_v := v^{\frac{4k}{n-2k}} g_{cyl} \tag{2.6}$$

for a conformal factor $v > 0$.

Cylindrical coordinates provide a convenient framework for calculations. The Schouten tensor for g_{cyl} is diagonal and

$$\begin{aligned}\left(g_{cyl}^{-1}A_{g_{cyl}}\right)_t^t &= -\tfrac{1}{2},\\ \left(g_{cyl}^{-1}A_{g_{cyl}}\right)_j^i &= +\tfrac{1}{2}\,\delta_j^i,\\ \left(g_{cyl}^{-1}A_{g_{cyl}}\right)_\beta^\alpha &= -\tfrac{1}{2}\,\delta_\beta^\alpha.\end{aligned}$$

Define

$$B_{g_v} = \tfrac{n-2k}{2k}\,v^{\frac{2n}{n-2k}}\,g_v^{-1}A_{g_v}.$$

Then, since

$$A_{g_v} = A_{cyl} - \tfrac{2k}{n-2k}\,v^{-1}D^2v + \tfrac{2kn}{(n-2k)^2}\,v^{-2}\nabla v\otimes\nabla v - \tfrac{2k^2}{(n-2k)^2}\,v^{-2}\,|\nabla v|_{g_{cyl}}^2 g_{cyl},$$

where all the derivatives and norms are calculated with respect to the background metric g_{cyl}, we have

$$B_{g_v} = g_{cyl}^{-1}\left[\tfrac{n-2k}{2k}v^2B_{cyl} - vD^2v + \tfrac{n}{n-2k}\,\nabla v\otimes\nabla v - \tfrac{k}{n-2k}\,|\nabla v|_{g_{cyl}}^2 g_{cyl}\right],$$

and thus

$$\begin{aligned}\left(B_{g_v}\right)_t^t &= -\tfrac{n-2k}{4k}\,v^2 - v\partial_{tt}v + \tfrac{n}{n-2k}\,(\partial_t v)^2 - \tfrac{k}{n-2k}\,|\nabla v|_{g_{cyl}}^2,\\ \left(B_{g_v}\right)_j^i &= \tfrac{n-2k}{4k}\,v^2\,\delta_j^i - \tfrac{k}{n-2k}\,|\nabla v|_{g_{cyl}}^2\,\delta_j^i,\\ \left(B_{g_v}\right)_\beta^\alpha &= \left[-\tfrac{n-2k}{4k}\,v^2 - v\,\partial_t v - \tfrac{k}{n-2k}\,|\nabla v|_{g_{cyl}}^2\right]\delta_\beta^\alpha\\ &\quad + \left[-v\,\partial_{\alpha\beta}v + \tfrac{n}{n-2k}\,\partial_\alpha v\,\partial_\beta v\right]e^{-2t}.\end{aligned}$$

Notice that, for a function $v = v(t)$, the matrix B_{g_v} is diagonal and

$$\begin{aligned}\left(B_{g_v}\right)_t^t &= -\tfrac{n-2k}{4k}\,v^2 - v\ddot v + \tfrac{n-k}{n-2k}\,\dot v^2 =: \kappa_1,\\ \left(B_{g_v}\right)_j^i &= \left[\tfrac{n-2k}{4k}\,v^2 - \tfrac{k}{n-2k}\,\dot v^2\right]\delta_j^i =: \kappa_2\,\delta_j^i,\\ \left(B_{g_v}\right)_\beta^\alpha &= \left[-\tfrac{n-2k}{4k}\,v^2 - v\,\dot v - \tfrac{k}{n-2k}\,\dot v^2\right]\delta_\beta^\alpha =: \kappa_3\,\delta_\beta^\alpha.\end{aligned} \tag{2.7}$$

Its eigenvalues are $\kappa_1, \kappa_2, \kappa_3$ with multiplicities $1, N-1, p$, respectively.

Now, the covariance property (1.9) implies that

$$\mathcal{N}(u, g_E) = r^{\varrho}\mathcal{N}(v, g_{cyl}), \tag{2.8}$$

where we have defined

$$\varrho := -\tfrac{n-2k}{2k}\left(-\tfrac{2kn}{n-2k} + 2k - 1\right). \tag{2.9}$$

Thus we can rewrite original equation (1.3) in this notation as

$$\sigma_k(B_{g_v}) = cv^q. \tag{2.10}$$

Assume that we have an approximate solution $U_\epsilon(r)$ and define

$$U_\epsilon(r) = r^{-\frac{n-2k}{2k}} v_\epsilon(-\log r).$$

Then, the covariance property (1.11) yields

$$\mathcal{L}_\epsilon \varphi := \mathbb{L}(U_\epsilon, g_E)[\varphi] = r^{\varrho}\, \mathbb{L}(v_\epsilon, g_{cyl})[r^{\frac{n-2k}{2k}} \varphi]. \tag{2.11}$$

We observe that $\mathcal{L}_\epsilon$ resembles the Laplacian operator in polar coordinates, indeed, a more explicit formula will be given in equation (4.4) below. Nevertheless, the operator $\mathbb{L}(v_\epsilon, g_{cyl})$ has better mapping properties in weighted Sobolev spaces that we will explain in Section 6.

2.2. **The fast-decay ODE solution.** In the paper [19] the authors construct a very particular solution $U_1 = U_1(r)$ for our problem (1.8) on $\mathbb{R}^n \setminus \mathbb{R}^p$ that yields a complete metric near the singular set $\{r = 0\}$ and has fast decay as $r \to \infty$. Here is the first place where we encounter the restriction $k = 2$. More precisely:

Proposition 2.1. [19] *For each $0 < p < \mathfrak{p}_2$, there exists a positive solution $U_1 = U_1(r)$ for the equation*

$$\sigma_2\left(B_{g_u}\right) = cu^q \quad in \quad \mathbb{R}^n \setminus \mathbb{R}^p \tag{2.12}$$

satisfying:

- *When $r \to 0$, the solution has the precise asymptotic behavior*
$$U_1(r) \asymp v_\infty r^{-\frac{n-4}{4}}$$
where
$$(v_\infty)^{\frac{16}{n-4}} = c_{n,p,2} \binom{n}{2}^{-1} > 0. \tag{2.13}$$
and the constant $c_{n,p,2}$ is defined in (1.4).
- *When $r \to +\infty$,*
$$U_1(t) \sim r^{-\alpha_0 - \frac{n-4}{4}}$$
for some $\alpha_0 \in \left(0, \frac{n-4}{4}\right)$.
- *$r^{\frac{n-4}{4}} U_1$ is uniformly bounded for all $r > 0$.*
- *The metric $g_{U_1} := (U_1)^{\frac{8}{n-4}} g_E$ belongs to the positive cone Γ_2^+.*

Sketch of the proof. Although we will not give the full proof of this result, which is contained in [19], the key idea is to use the framework in cylindrical coordinates we have just presented and to find a solution v_1 to (2.10) that only depends on the radial variable t.

From the formulas in (2.7) one is able to write $\sigma_2(B_{g_v})$ in a reasonably simple form. Standard phase-plane analysis for the ODE (2.10) completes the proof. Now we recover U_1 by setting, in the notation (2.4),

$$U_1(r) = r^{-\frac{n-4}{4}} v_1(-\log r).$$

$\square$

Corollary 2.2. *For $\epsilon > 0$ we rescale*

$$U_\epsilon(r) = \epsilon^{-\frac{n-4}{4}} U_1\left(\frac{r}{\epsilon}\right). \tag{2.14}$$

Then U_ϵ is a solution of (1.8) *in $\mathbb{R}^n \setminus \mathbb{R}^p$ that only depends on the radial variable. Moreover,*

$$U_\epsilon(r) \asymp v_\infty r^{-\frac{n-4}{4}}, \quad \textit{when} \quad r << \epsilon \tag{2.15}$$

and

$$U_\epsilon(r) \sim \epsilon^{\alpha_0} r^{-\alpha_0 - \frac{n-4}{4}} \quad \textit{when} \quad r >> \epsilon. \tag{2.16}$$

In the following, we will set

$$\alpha_1 := \alpha_0 + \tfrac{n-4}{4},$$
$$U_\epsilon(r) = r^{-\frac{n-4}{4}} v_\epsilon(-\log r),$$

and

$$g_{v_\epsilon} = v_\epsilon^{\frac{8}{n-4}} g_{cyl}. \tag{2.17}$$

3. The approximate solution

Although in this Section we fix $k = 2$, we will try to state our results for any $k < n/2$ when possible. Indeed, this restriction is only necessary in Section 3.3.

Let Λ be a smooth, closed submanifold in $\mathbb{S}^n$ of dimension p, such that $0 < p < \mathfrak{p}_k$. Here we construct an approximate solution $\bar{u}_\epsilon$ to problem (1.8) on $\mathbb{S}^n \setminus \Lambda$ which is singular exactly at Λ with a precise blowup rate and, in addition, remains in the positive cone Γ_k^+. For this, we first need to look at the model case $\mathbb{R}^N \setminus \{0\}$ (this is, $p = 0$, $n = N$), which already contains the main ideas.

3.1. Matching asymptotics - an isolated singularity $\mathbb{R}^N \setminus \{0\}$.

Let $r = |x|$ be the radial variable in $\mathbb{R}^N$, $\theta = \frac{x}{r}$ the angular variable.

In the classical gluing paper of [33], the global approximate solution $\bar{u}_\epsilon$ is constructed from the model solution U_ϵ given in (2.14), extended to zero with the introduction of a cutoff in a ball of radius $r_0 >> \epsilon$. This does not work for general σ_k since such approximate solution may not be in the positive cone Γ_k^+. Instead, we need to make a more refined choice of this cutoff based on Lemma 7 from [22], where the authors are able to transplant an isolated singularity with given asymptotics to any conformally flat metric, while still remaining Γ_k^+ cone. We remark that this Lemma is the key ingredient in the construction from [20] of manifolds of positive σ_k–curvature by connected sums of compact manifolds. It is also the basic idea in the proof of our Proposition 3.2 below for matching asymptotics in the case of higher dimensional singularities, so we reproduce its proof in detail.

Lemma 3.1. [22] *Let $\mathcal{B}_1$ be the unit disk in $\mathbb{R}^N$ with the Euclidean metric $|dx|^2$. Let*

$$g_0 = u_0^{\frac{4k}{N-2k}}|dx|^2$$

be a smooth metric on $\mathcal{B}_1$ satisfying $g_0 \in \Gamma_k^+$ for $k < N/2$. Then, for any $0 < \alpha_1 < \frac{N-2k}{k}$, there exists a conformal metric

$$g = u^{\frac{4k}{N-2k}}|dx|^2 \quad \text{on} \quad \mathcal{B}_1 \setminus \{0\}$$

in the positive Γ_k^+ cone, satisfying the following:

- *$u(x) = u_0(x)$ for $|x| \in (r_0, 1]$,*
- *$u(x) = |x|^{-\alpha_1}$ for $|x| \in (0, r_3]$,*

for some $0 < r_3 < r_0 < 1$.

Proof. To follow the proof in [22], we change again the notation for the conformal factor, writing

$$g = e^{-2\omega}g_0 = e^{-2(\omega+\omega_0)}|dx|^2 = u^{\frac{2k}{N-2k}}|dx|^2, \tag{3.1}$$

where we have set $\omega_0 = -\frac{2k}{N-2k}\log u_0$. The transformation law for the Schouten tensor is

$$A_g = D^2\omega + \nabla\omega \otimes \nabla\omega - \tfrac{1}{2}|\nabla u|^2_{g_0}g_0 + A_{g_0}. \tag{3.2}$$

In this particular case, using $|dx|^2$ as the background metric,

$$\begin{aligned} A_g &= D^2(\omega+\omega_0) + \nabla(\omega+\omega_0) \otimes \nabla(\omega+\omega_0) - \tfrac{1}{2}|\nabla(\omega+\omega_0)|^2 I \\ &= D^2\omega + \nabla\omega \otimes \nabla\omega + \nabla\omega \otimes \nabla\omega_0 + \nabla\omega_0 \otimes \nabla\omega \\ &\quad - \left(\tfrac{1}{2}|\nabla\omega|^2 + \langle\nabla\omega, \nabla\omega_0\rangle\right) I + A_{g_0}. \end{aligned}$$

where all the derivatives are taken with respect to the Euclidean metric on $\mathbb{R}^N$.

The idea is to transplant an isolated singularity at the origin, staying inside the positive cone Γ_k^+, by doing a careful study of the transition region. Thus we seek ω so that

$$\begin{aligned} &\omega = 0 \quad \text{near} \quad r = 1, \\ &\omega = \alpha_2 \log r - \omega_0 \quad \text{near} \quad r = 0, \end{aligned}$$

and such that A_g remains in the positive cone Γ_k^+. Here we have defined $\alpha_2 = \frac{2k}{N-2k}\alpha_1 \in (0, 2)$.

As a first approximation, we impose

$$\omega'(r) = \frac{\phi(r)}{r}$$

for some suitable transition function $\phi(r)$ satisfying

$$\begin{aligned} \phi(r) &= 0 \quad \text{near} \quad r = 1, \\ \phi(r) &= \alpha_2 \quad \text{near} \quad r = 0. \end{aligned}$$

We calculate

$$A_g = \frac{2\phi - \phi^2}{2r^2} I_{N\times N} + \left(\frac{\phi'}{r} + \frac{\phi^2 - 2\phi}{r^2}\right)\theta\otimes\theta + A_{g_0} + O(|\nabla\omega_0|)\frac{\phi}{r}.$$

With some abuse of notation, we have

$$\sigma_k(g^{-1}A_g) = e^{-2k(\omega+\omega_0)}\sigma_k(A_g),$$

where the A_g in the right hand side is understood as a $(1,1)$-tensor, i.e., a matrix (this will not create confusion since the background metric is Euclidean).

We will take ϕ as follows: first, let $C_0 = \max|\nabla\omega_0|$, fix $\tau \in (0, 1/2)$ to be specified later, and choose

$$r_0 = \min\{\tfrac{1}{2}, C_0\tau\}. \tag{3.3}$$

Let $\phi(r) = 0$ for $r \in [r_0, 1)$, so that $g = g_0$. With this choice we make sure that it is enough to look at the asymptotic behavior when $r \to 0$.

Next, since A_{g_0} is in the positive cone, which is open, we can take $r_1 \in (0, r_0)$ and $\phi : [r_1, r_0) \to [0, \alpha_2)$ non-increasing such that A_g also belongs to the positive cone and $\phi(r_1) > 0$.

Let us figure out what conditions we need to impose in order to extend ϕ to the whole interval $(0, r_1)$. From our previous calculations we can estimate

$$A_g \geq \frac{2\phi - \phi^2}{2r^2} I_{N\times N} + \left(\frac{\phi'}{r} + \frac{\phi^2 - 2\phi}{r^2}\right)\theta\otimes\theta - \tau\frac{\phi}{r^2} + A_{g_0} =: J_\tau + A_{g_0}.$$

Consider first the unperturbed matrix

$$\hat{J} := \frac{2\phi - \phi^2}{2r^2} I_{N\times N} + \left(\frac{\phi'}{r} + \frac{\phi^2 - 2\phi}{r^2}\right)\theta\otimes\theta.$$

It has eigenvalues

$$\begin{aligned} \vartheta_1 &= \frac{-2\phi + \phi^2}{2r^2} + \frac{\phi'}{r}, \quad \text{of multiplicity } 1, \\ \vartheta_2 &= \frac{2\phi - \phi^2}{2r^2}, \quad \text{of multiplicity } N-1, \end{aligned}$$

so one easily sees that

$$\sigma_k(\hat{J}) = \frac{(N-1)!}{k!(N-k)!}\left(\frac{2\phi-\phi^2}{2r^2}\right)^k\left[N-2k+2k\frac{r\phi'}{2\phi-\phi^2}\right].$$

We find ϕ a solution of the ODE

$$2k\frac{r\phi'}{2\phi-\phi^2} = -\frac{1}{2},$$

with initial condition given by the above value for $\phi(r_1)$. This ODE is easily integrated, yielding

$$\phi(r) = \frac{2\delta}{\delta + r^{\frac{1}{2k}}},$$

for some small constant $\delta > 0$. Actually, it can be taken such that $\delta << r_1^{\frac{1}{2k}}$. With such a choice,

$$\vartheta_1 = -\frac{2k+1}{4k}\frac{2\phi-\phi^2}{r^2}, \quad \vartheta_2 = \frac{2\phi-\phi^2}{2r^2},$$

which means that the perturbation $\tau\frac{\phi}{r}$ is small compared to the value of the eigenvalues of $\hat{J}$, for some τ small enough depending only on k, N. We conclude that J_τ belongs to the positive cone in the region $(0, r_1)$. By the convexity of σ_k, also $A_g = J_\tau + A_{g_0} \in \Gamma_k^+$.

Then choose $r_2 \in (0, r_1)$ such that $\phi(r_2) = \alpha_2$, which is possible because $0 < \alpha_2 < 2$ by hypothesis. Define $\phi(r) = \alpha_2$ on $[0, r_2]$. The statement of the Lemma follows by smoothing out the conformal metric g, which is already $\mathcal{C}^{1,1}$.

□

3.2. **Matching asymptotics - higher dimensional singularities $\mathbb{R}^n \setminus \mathbb{R}^p$.** Now we refine Lemma 3.1 in order to account for higher dimensional singularities. Most of the proof works for any value of $k < n/2$. However, we need to restrict to the value $k = 2$ in order to avoid heavy numerology.

Assume that we are in the model case

$$\mathbb{R}^n\setminus\mathbb{R}^p = \mathbb{R}^+_r \times \mathbb{S}^{N-1}_\theta \times \mathbb{R}^p_z.$$

with the Euclidean metric in polar-Fermi coordinates g_E, given by (2.1). We write coordinates $y = (x, z)$, where $z \in \mathbb{R}^p$ and x the normal coordinate in $\mathbb{R}^N \setminus \{0\}$. Set $r = |x|$, $\theta = \frac{x}{r}$. For fixed $\rho > 0$, let $\mathcal{T}_\rho := \{r < \rho\}$ be a tubular neighborhood of $\mathbb{R}^p$ in $\mathbb{R}^n$.

Proposition 3.2. *Set $k=2$, $n\geq 5$ and $0<p<\mathfrak{p}_k$. Let $g_0=u_0^{\frac{4k}{n-2k}}g_E$ be a smooth metric on $\mathcal{T}_1$ satisfying $g_0\in\Gamma_k^+$. Then, for any $0<\alpha_1<\frac{n-2k}{k}$, there exists a conformal metric*

$$g=u^{\frac{4k}{n-2k}}g_E \quad on \quad \mathcal{T}_1\backslash\{r=0\}$$

in the Γ_k^+ cone, satisfying the following:

- *$u=u_0(r,\theta,z)$ for in $\mathcal{T}_1\backslash\mathcal{T}_{r_1}$,*
- *$u(r,\theta,z)=r^{-\alpha_1}$ in $\mathcal{T}_{r_3}\backslash\{r=0\}$,*

for some $0<r_3<r_0<1$.

Proof. As above, we change the notation for the conformal factor, taking the same conventions as in (3.1).

We seek ω so that

$$\begin{aligned}&\omega=0 \quad \text{near} \quad r=1,\\&\omega=\alpha_2\log r-\omega_0 \quad \text{near} \quad r=0,\end{aligned}$$

and such that A_g remains in the positive cone Γ_k^+. Here we also take $\alpha_2=\frac{2k}{n-2k}\alpha_1$. As a first approximation, we impose ω to be of the form

$$\omega'(r)=\frac{\phi(r)}{r},$$

for some transition function $\phi(r)$.

We will choose $r_0>0$ small enough, and set $\phi(r)=0$ for $r_0\leq r<1$. Next, since $A_{g_0}\in\Gamma_k^+$, an open set, we can take $r_1\in(0,r_0)$ and $\phi:[r_1,r_0)\to[0,\alpha_1)$ non-increasing such that A_g also belongs to the positive cone and $\phi(r_1)>0$ (but very small).

Now,

$$\sigma_k(g^{-1}A_g)=e^{-2k(\omega+\omega_0)}\sigma_k(g_E^{-1}A_g).$$

By straightforward calculation from (3.2) we have that

$$A_g=A_{g_0}+\hat{J}+E(\phi), \tag{3.4}$$

where

$$\hat{J}=\begin{bmatrix}\hat{J}_0 & 0\\ 0 & \hat{J}_1\end{bmatrix}$$

for

$$\hat{J}_0=\frac{2\phi-\phi^2}{2r^2}I_{N\times N}+\left(\frac{\phi'}{r}+\frac{\phi^2-2\phi}{r^2}\right)\theta\otimes\theta \quad \text{and} \quad \hat{J}_1=-\frac{\phi^2}{2r^2}I_{p\times p},$$

and E is the perturbation term. We take ϕ exactly as above, i.e.,

$$\phi(r)=\frac{2\delta}{\delta+r^{\frac{1}{2k}}},$$

for some small constant $\delta > 0$ determined by the given $\phi(r_1)$. With such a choice, the eigenvalues of the matrix $\hat{J}$ are

$$\vartheta_1 = -\frac{2k+1}{4k}\frac{2\phi - \phi^2}{r^2}, \quad \vartheta_2 = \frac{2\phi-\phi^2}{2r^2}, \quad \vartheta_3 = -\frac{\phi^2}{2r^2}, \tag{3.5}$$

with multiplicities 1, $N-1$ and p, respectively. At this point we concentrate in the case $k = 2$. After some tedious but straightforward calculation, one has that for such ϕ, both

$$\begin{aligned}\sigma_2(\hat{J}) &= (N-1)\vartheta_1\vartheta_2 + p\vartheta_1\vartheta_3 + \binom{N-1}{2}\vartheta_2^2 + \binom{p}{2}\vartheta_3^2 \\ &\quad + p(N-1)\vartheta_2\vartheta_3 > 0, \\ \sigma_1(\hat{J}) &= \vartheta_1 + (N-1)\vartheta_2 + p\vartheta_3 > 0,\end{aligned}$$

which means that $\hat{J}$ is in the positive cone.

Moreover, the error term $E(\phi)$ can be estimated as above by

$$|E(\phi)| \leq \tau\frac{\phi}{r^2} \tag{3.6}$$

for some τ small enough but depending only on n, p, ω_0, which is only a small perturbation of the eigenvalues (3.5). Since the matrix $\hat{J}$ belongs to the positive cone, also does A_g in the region $0 < r < r_1$. Then choose $r_2 \in (0, r_1)$ such that $\phi(r_2) = \alpha_2$, which is possible because $0 < \alpha_2 < 2$ by hypothesis and define $\phi(r) = \alpha_2$ on $[0, r_2]$. The statement of the Proposition follows by smoothing out the conformal metric g, which is already $\mathcal{C}^{1,1}$ by construction. □

3.3. **Fermi coordinates.** Now we consider a general singular set Λ, which is taken to be a smooth compact, connected, closed, submanifold in $\mathbb{R}^n$ of dimension p.

Let $\mathcal{T}_\rho$ be the tubular neighbourhood of radius ρ around Λ. It is well known that $\mathcal{T}_\rho$ is a disk bundle over Λ; more precisely, it is diffeomorphic to the bundle of radius ρ in the normal bundle $\mathcal{N}\Lambda$. The Fermi coordinates will be constructed as coordinates in the normal bundle transferred to $\mathcal{T}_\rho$ via such diffeomorphism. More precisely, let r be the distance to Λ, which is well defined and smooth away from Λ for small ρ. Let also z be a local coordinate system on Λ and θ the angular variable on the sphere in each normal space $\mathcal{N}_z\Lambda$. We denote by $\mathcal{B}_\rho$ the ball of radius ρ in $\mathcal{N}\Lambda$ at a point $z \in \Lambda$. Finally we let x denote the rectangular coordinate in these normal spaces, so that $r = |x|$, $\theta = \frac{x}{|x|}$. Thus we can identify $\mathcal{T}_\rho$ with $\mathcal{B}_\rho \times \Lambda$, parameterized with coordinates $y = (x, z)$, $x \in \mathcal{B}_\rho$, $z \in \Lambda$. Moreover, any metric g on $\mathcal{T}_\rho$ can be compared to the canonical metric g_E from (2.1). Indeed,

$$g = \begin{pmatrix} 1 & 0 & O(r) \\ 0 & r^2 g_\theta + O(r^4) & O(r^2) \\ O(r) & O(r^2) & g_\Lambda + O(r) \end{pmatrix}, \tag{3.7}$$

where g_Λ is the metric on Λ. This expansion is classical, see, for instance, [16].

We can redo the arguments in Proposition 3.2 using Fermi coordinates near Λ (say, in a neighborhood $\mathcal{T}_1$), to obtain the following gluing result:

Proposition 3.3. *Set $k = 2$, $n > 5$, $0 < p < \mathfrak{p}_k$, $N = n - p$. Let $g_0 = u_0^{\frac{4k}{n-2k}} g_{\mathbb{R}^n}$ be a smooth metric on a tubular neighborhood $\mathcal{T}_1$ of Λ in $\mathbb{R}^n$ satisfying $g_0 \in \Gamma_k^+$. Fix any $0 < \alpha_1 < \frac{n-2k}{k}$. Then there exists a conformal metric*

$$g = u^{\frac{4k}{n-2k}} g_{\mathbb{R}^n} \quad \text{on} \quad \mathcal{T}_1 \setminus \{r = 0\}$$

that belongs to the positive cone Γ_k^+ and satisfies the following:

$$u(r,\theta,z) = u_0(r,\theta,z) \text{ in } \mathcal{T}_1 \backslash \mathcal{T}_{r_1}, \tag{3.8}$$

$$u(r,\theta,z) = r^{-\alpha_1} \text{ in } \mathcal{T}_{r_3} \backslash \{r = 0\}, \tag{3.9}$$

for some $0 < r_3 < r_0 < 1$.

Proof. As mentioned, we follow the proof of Proposition 3.2; the only difference is to keep track of the extra perturbation terms. More precisely, the Euclidean metric $g_{\mathbb{R}^n}$ in Fermi coordinates (given in the formula (3.7)) near the singular set differs from the model metric (2.1) in two ways:

- Perturbation terms of order (at least) $O(r)$ as $r \to 0$.
- In the z direction the flat metric $|dz|^2$ is replaced by g_Λ.

Similarly to the above, equation (3.4) is rewritten as

$$A_{\tilde{g}} = A_{g_0} + \hat{J} + E(\phi)$$

where

$$\hat{J} = \begin{bmatrix} \hat{J}_0 & 0 \\ 0 & \hat{J}_1 \end{bmatrix}$$

for

$$\hat{J}_0 = \frac{2\phi - \phi^2}{2r^2} I_{N\times N} + \left(\frac{\phi'}{r} + \frac{\phi^2 - 2\phi}{r^2} \right) \theta \otimes \theta \quad \text{and} \quad \hat{J}_1 = -\frac{\phi^2}{2r^2} g_\Lambda,$$

and E is the perturbation term.

To handle the block $\hat{J}_1$ we work in small enough neighborhoods of $z \in \Lambda$ and write g_Λ in normal coordinates. Now, for the term $E(\phi)$ we claim that we still have that $|E(\phi)| \leq \tau \frac{\phi}{r^2}$ for some small enough τ as in Proposition 3.2. To see this, one needs to control the $O(r)$ terms in the expansion of the metric $g_{\mathbb{R}^n}$ in Fermi coordinates from (3.7). For instance, for the inverse of the metric we have

$$(g_{\mathbb{R}^n})^{-1} = \begin{pmatrix} 1+O(r) & O(r) & O(r) \\ O(r) & r^{-2}g_\theta + O(r^{-1}) & O(1) \\ O(r) & O(1) & g_\Lambda^{-1} + O(r) \end{pmatrix},$$

the Christoffel symbols of $g_{\mathbb{R}^n}$

$$\begin{array}{lll} \Gamma^r_{rr} = O(r), & \Gamma^r_{r\theta} = O(r^2), & \Gamma^r_{rz} = O(r), \\ \Gamma^r_{\theta\theta} = rg_\theta + O(r^2), & \Gamma^r_{\theta z} = O(r), & \Gamma^r_{zz} = O(1), \end{array}$$

and the Hessian of a function $\omega = \omega(r)$

$$D^2\omega = \begin{pmatrix} \partial_{rr}\omega + O(r)\partial_r\omega & O(r^2)\partial_r\omega & O(r)\partial_r\omega \\ O(r^2)\partial_r\omega & -[rg_\theta + O(r^2)]\partial_r\omega & O(r)\partial_r\omega \\ O(r)\partial_r\omega & O(r)\partial_r\omega & O(1)\partial_r\omega \end{pmatrix}. \tag{3.10}$$

Therefore, the perturbation terms do not change the eigenvalues as $r \to 0$. The rest of the proof follows similarly as in Proposition 3.2. $\square$

3.4. **Construction of $\bar{u}_\epsilon$.** From the discussion in the previous Subsections we can build our approximate solution $\bar{u}_\epsilon$.

First we take any non-degenerate smooth, conformally flat metric on $\mathbb{R}^n$ that belongs to the positive cone Γ_k^+ and decays fast enough at infinity, denoted by

$$g_\dagger = u_\dagger^{\frac{4k}{n-2k}} g_{\mathbb{R}^n}.$$

Remark 3.4. In order to avoid complications at infinity, we will take the metric g_0 to be a compact metric on $\mathbb{R}^n \cup \{\infty\}$. For instance, a metric coming from a perturbation of the canonical metric on the sphere after stereographic projection. This is equivalent to working on a domain bounded $\Omega \setminus \Lambda$ with boundary conditions as in the original setting of Mazzeo-Pacard [33].

Proposition 3.3 allows to construct a globally defined metric

$$g_* = u_*^{\frac{4k}{n-2k}} g_{\mathbb{R}^n} \quad \text{on } \mathbb{R}^n \setminus \Lambda$$

that belongs to the positive cone Γ_k^+ and such that $u_1 \sim r^{-\alpha_1}$ as $r \to 0$. Observe that the quantities r_0, r_3 do not depend on ϵ, but they can be taken small enough when needed while keeping $r_0, r_3 >> \epsilon$.

Now we would like to glue this metric to the model singularity coming from considering the conformal factor $U_\epsilon(r)$ from (2.14) near the singular set. For this, we restrict to $k=2$ and rescale

$$g_\varepsilon = [\epsilon^{\alpha_0} u_*]^{\frac{4k}{n-2k}}\, g_{\mathbb{R}^n}. \tag{3.11}$$

Looking at the asymptotic behavior from (2.16), it is clear that we can patch both functions $\epsilon^{\alpha_0}u_1$ and U_ϵ in a tubular neighborhood $\mathcal{T}_\rho \setminus \mathcal{T}_{\rho'}$ with $\rho > \rho' >> \epsilon$ in order to construct a globally defined metric

$$\bar{g}_\varepsilon = \bar{u}_\epsilon^{\frac{4k}{n-2k}} g_{\mathbb{R}^n} \quad \text{on } \mathbb{R}^n \backslash \Lambda \tag{3.12}$$

in the positive cone Γ_k^+ with the required asymptotics. Indeed, the behavior of our approximate solution $\bar{u}_\epsilon$ near the singular set is given by

$$\bar{u}_\epsilon = U_\epsilon \asymp v_\infty r^{-\frac{n-2k}{2k}} \quad \text{when} \quad r << \epsilon,$$

which yields a complete metric near the singular set.

3.5. **Estimates.** In the following Lemma we check that, indeed, $\bar{u}_\epsilon$ is a good approximate solution. For this, it is better to switch to the v-notation in cylindrical coordinates. Thus, with some abuse of notation, we set $g_{\mathbb{R}^n,cyl}$ to be the Euclidean metric on $\mathbb{R}^n$ but using cylindrical Fermi coordinates Λ. The approximate $\bar{u}_\epsilon$ in this new setting will be denoted by $\bar{v}_\epsilon$, and the new metric by (3.12) by

$$\bar{g}_{\epsilon,cyl} = \bar{v}_\epsilon^{\frac{4k}{n-2k}} g_{\mathbb{R}^n,cyl} \quad \text{on } \mathbb{R}^n \backslash \Lambda.$$

Proposition 3.5. *Set $k=2$. There exists $\beta_0>0$ such that, for each*

$$\bar{\nu} < \beta_0 \tag{3.13}$$

we have the estimate

$$r^{-\bar{\nu}} \mathcal{N}(\bar{v}_\epsilon, g_{\mathbb{R}^n,cyl}) = O(\epsilon^{\beta_1}) \tag{3.14}$$

in a tubular neighborhood $\mathcal{T}_\rho$, for some $\beta_1 = \beta_1(\bar{\nu}, \alpha_0) > 0$.

Proof. First of all, away from the singular set, this is, $r >> \epsilon$, $\bar{v}_\epsilon$ is essentially a factor ϵ^{α_0}, thus by keeping track of this rescaling in our definition of the nonlinear operator (1.8) we have

$$\mathcal{N}(\epsilon^{\alpha_0}, g_{\mathbb{R}^n,cyl}) = O(\max\{\epsilon^{\alpha_0}, \epsilon^{\alpha_0(q-2k+1)}\}),$$

from where (3.14) follows.

The dangerous region is a tubular neighborhood $\mathcal{T}_{a\epsilon} = \{r < a\epsilon\}$ for some constant $a>0$. Here recall that $\bar{u}_\epsilon \asymp U_1$, and a similar relation holds also for all derivatives. Similarly, $\bar{v}_\epsilon \asymp v_1$. In order to calculate $\mathcal{N}(v_1, g_{\mathbb{R}^n,cyl})$ in this region we need to compare background metric $g_{\mathbb{R}^n}$

in Fermi coordinates given (3.7) to the model g_E from (2.1) as we did in the proof of Proposition 3.3.

To do so, we define the metrics

$$g_{U_1} = U_1^{\frac{4k}{n-2k}} g_{\mathbb{R}^n} \quad \text{and} \quad \hat{g}_{U_1} = U_1^{\frac{4k}{n-2k}} g_E, \quad \text{for } U_1 = U_1(r).$$

We will show that the eigenvalues of the perturbed matrix $B_{g_{U_1}}$ are very close to those of the unperturbed matrix $B_{\hat{g}_{U_1}}$. For this, recall the definition of the tensor B from (1.7) for a conformal change $g_u = u^{\frac{4k}{n-2k}} g$. The calculations in the proof of Proposition 3.3 imply that $B_{g_{\mathbb{R}^n}}$ and B_{g_E} are close as $r \to 0$ (one can also refer to the arguments in the proof of Proposition 2.19 of [35]). From the formula (3.10) for the Hessian $D^2 U_1$, we also obtain some perturbation terms of order $o(1)$ as $r \to 0$.

Similar arguments hold for the v-notation in cylindrical coordinates. Indeed, in this small neighborhood one has that

$$\mathcal{N}(v_1, g_{cyl}) = O(r^\beta)$$

for some $\beta > 0$. This completes the proof of the Proposition.

□

4. The linearized operator

We fix $\bar{u}_\epsilon$ the approximate solution from Section 3.2 above. Even though we need to restrict to $k = 2$, the results in this Section could potentially work for any k. We let $\mathbb{L}_\epsilon := \mathbb{L}(\bar{u}_\epsilon, g_{\mathbb{R}^n})$ be the linearized operator around $\bar{u}_\epsilon$, as defined in (1.10), and take a conformal perturbation of the metric (3.12), i.e, we set for $s \in \mathbb{R}$,

$$s \mapsto g_s := (\bar{u}_\epsilon + s\varphi)^{\frac{4k}{n-2k}} g_{\mathbb{R}^n}.$$

Let $\mathbb{B}_s$ be the symmetric $(1,1)$-tensor given by

$$\mathbb{B}_s := \tfrac{n-2k}{2k} (\bar{u}_\epsilon + s\varphi)^{\frac{2n}{n-2k}} g_s^{-1} A_{g_s}. \tag{4.1}$$

Recalling the definition (1.10), we have

$$\begin{aligned} \mathbb{L}_\epsilon[\varphi] : = &\left\{ (-2k+1)\bar{u}_\epsilon^{-2k} \sigma_k(\mathbb{B}_0) - c(q-2k+1)\bar{u}_\epsilon^{q-2k} \right\} \varphi \\ &+ \bar{u}_\epsilon^{-2k+1} \left.\frac{d}{ds}\right|_{s=0} \sigma_k(\mathbb{B}_s). \end{aligned} \tag{4.2}$$

We use a well known formula (see [48]) for the linearization of σ_k

$$\frac{d}{ds}\sigma_k(\mathbb{B}_s) = \operatorname{tr}\left(T^{k-1}(\mathbb{B}_s) \frac{d\mathbb{B}_s}{ds} \right), \tag{4.3}$$

where, for any integer $0 \leq m \leq k$, the m-th Newton tensor for a (1,1)-tensor $\mathbb{B}$ is defined as

$$T^m(\mathbb{B}) := \sigma_m I - \sigma_{m-1}\mathbb{B} + \ldots + (-1)^m \mathbb{B}^m.$$

Note that, for a metric in the positive cone Γ_k^+, its Newton tensor is positive definite for $m = 0, \ldots, k-1$. In particular, this implies that $\mathbb{L}_\epsilon$ is elliptic.

Remark 4.1. The fundamental observations about linearized operator $\mathbb{L}_\epsilon$ are the following:

- Away from the singularity, we have a usual uniformly elliptic operator. Note that we have carefully chosen the normalization in (1.8), so that, as $\epsilon \to 0$, $\mathbb{L}_\epsilon$ converges to $\mathbb{L}_{g_\dagger} := \mathbb{L}(u_\dagger, g_{\mathbb{R}^n})$ with the potential $-c(q-2k+1)u^{q-2k}$ removed (this term had a higher order in ϵ, thus vanishes at the limit). Here $g_\dagger$, $u_\dagger$ are given by our choice of background metric in Section 3.3.
- Near the singularity set, one can check that $\mathbb{L}_\epsilon$ is in the class of elliptic edge operators from [31]. More explicit formulas will be given below.

4.1. **The model linearization.** We first consider the linearized operator around the model solution U_ϵ on $\mathbb{R}^n \setminus \mathbb{R}^p$. For the convenience of the reader, we summarize here some results on the linear theory and postpone the proofs (which are mostly technical) until Section 5.

Proposition 4.2. *The operator $\mathcal{L}_\epsilon$ has the following expression:*
(4.4)
$$\mathcal{L}_\epsilon[\varphi] = \mathcal{A}_0^\epsilon(r)\varphi + \frac{\mathcal{A}_1^\epsilon(r)}{r}\partial_r\varphi + \mathcal{A}_2^\epsilon(r)\partial_{rr}\varphi + \frac{\mathcal{A}_3^\epsilon(r)}{r^2}\Delta_\theta\varphi + \mathcal{A}_4^\epsilon(r)\Delta_z\varphi,$$
for some smooth, bounded coefficients $\mathcal{A}_\ell^\epsilon(r)$, $\ell = 0,1,2,3,4$, satisfying

$$\mathcal{A}_2^\epsilon, \mathcal{A}_3^\epsilon, \mathcal{A}_4^\epsilon > 0.$$

In addition, the following limits exist: for $\ell = 0,1,2,3,4$,

$$\lim_{r\to 0} \mathcal{A}_\ell^\epsilon(r) = \beta_\ell^{(0)},$$
$$\lim_{r\to\infty} \mathcal{A}_\ell^\epsilon(r) = \beta_\ell^{(\infty)}.$$

Note that these limit values are independent of ϵ. This allows to define the limit operators

$$\mathcal{L}^{(0)}[\varphi] := \beta_0^{(0)}\varphi + \frac{\beta_1^{(0)}}{r}\partial_r\varphi + \beta_2^{(0)}\partial_{rr}\varphi + \frac{\beta_3^{(0)}}{r^2}\Delta_\theta\varphi + \beta_4^{(0)}\Delta_z\varphi, \tag{4.5}$$

as $r \to 0$, and

$$\mathcal{L}^{(\infty)}[\varphi] := \beta_0^{(\infty)}\varphi + \frac{\beta_1^{(\infty)}}{r}\partial_r\varphi + \beta_2^{(\infty)}\partial_{rr}\varphi + \frac{\beta_3^{(\infty)}}{r^2}\Delta_\theta\varphi + \beta_4^{(\infty)}\Delta_z\varphi, \tag{4.6}$$

as $r \to \infty$.

The crucial observation here is that, since coefficients $\beta_\ell^{(0)}, \beta_\ell^{(\infty)}$, $\ell = 0, 1, 2, 3, 4$, do not depend on ϵ, neither do the indicial roots for $\mathcal{L}_\epsilon$. Indeed, we may characterize all indicial roots as follows:

Proposition 4.3. *Assume $k = 2$ and fix $0 < p < \mathfrak{p}_2$. For each $j = 0, 1, \ldots$, there exist two indicial roots $\chi_{j,\pm}^{(0)}$ for the operator $\mathcal{L}_\epsilon$ as $r \to 0$. These satisfy:*

- *$\chi_{0,\pm}^{(0)}$ can be real or complex. In the former case,*

$$-\tfrac{n-4}{4} < \chi_{0,-}^{(0)} < \tfrac{p}{2} - \tfrac{n-4}{4} < \chi_{0,+}^{(0)},$$

  while in the latter, $\mathrm{Re}(\gamma_0^\pm) = \frac{p}{2} - \frac{n-4}{4}$.
- *For $j \geq 1$, $\chi_{j,\pm}^{(0)}$ are real numbers. In addition, we have the monotonicity*

$$\begin{aligned} \cdots \leq \chi_{2,-}^{(0)} \leq \chi_{1,-}^{(0)} < \mathrm{Re}(\chi_{0,-}^{(0)}) \leq \tfrac{p}{2} - \tfrac{n-4}{4} \\ \leq \mathrm{Re}(\chi_{0,+}^{(0)}) < \chi_{1,+}^{(0)} \leq \chi_{2,+}^{(0)} \leq \cdots \end{aligned} \tag{4.7}$$

- *It holds $\chi_{1,-}^{(0)} = -1 - \frac{n-4}{4}$.*

As $r \to \infty$, the picture is similar and, for each $j = 0, 1, \ldots$, there exist two indicial roots $\chi_{j,\pm}^{(\infty)}$ for $\mathcal{L}_\epsilon$ which satisfy:

- *For all j, $\chi_{j,\pm}^{(\infty)}$ are real numbers. In addition, we have the monotonicity*

$$\begin{aligned} \cdots \leq \chi_{2,-}^{(\infty)} \leq \chi_{1,-}^{(\infty)} < \chi_{0,-}^{(\infty)} \leq -\tfrac{n-4}{4} + \tfrac{p(n-3)}{2(n-1)} \\ \leq \chi_{0,+}^{(\infty)} < \chi_{1,+}^{(\infty)} \leq \chi_{2,+}^{(\infty)} \leq \cdots \end{aligned} \tag{4.8}$$

- *It holds*

$$-\tfrac{n-4}{4} + \tfrac{p}{2} > \chi_{0,-}^{(\infty)} = -\tfrac{n-4}{4} + \tfrac{p(n-3)}{2(n-1)} - \tfrac{\sqrt{4p+5p^2-5pn+pn^2-p^2n}}{2(n-1)}. \tag{4.9}$$

- *For $j = 1$,*

$$\chi_{1,-}^{(\infty)} = -1 - \alpha_0 - \tfrac{n-4}{4} \tag{4.10}$$

  and

$$\chi_{1,+}^{(\infty)} > 0. \tag{4.11}$$

Next, we give a "divergence-form" version for $\mathcal{L}_\epsilon$ with the introduction of an integrating factor. Indeed, let

$$\mathcal{H}_1^\epsilon(r) = \exp \int^r \frac{\mathcal{A}_1^\epsilon(s)}{s\mathcal{A}_2^\epsilon(s)}\, ds \quad \text{and} \quad (\mathcal{H}^\epsilon)^{-1}(r) = \frac{\mathcal{A}_2^\epsilon(r)}{\mathcal{H}_1^\epsilon(r)}.$$

Note that both functions are strictly positive. It holds
(4.12)

$$\mathcal{L}_\epsilon \varphi = (\mathcal{H}^\epsilon)^{-1}(r)\partial_r \left\{\mathcal{H}_1^\epsilon(r)\partial_r\varphi\right\} + \mathcal{A}_0^\epsilon(r)\varphi + \frac{\mathcal{A}_3^\epsilon(r)}{r^2}\Delta_\theta\varphi + \mathcal{A}_4^\epsilon(r)\Delta_z\varphi,$$

which shows that a natural space to work is $L^2(\mathbb{R}^n \setminus \mathbb{R}^p)$ with the (weighted) scalar product

$$\langle \varphi_1, \varphi_2 \rangle = \int_0^\infty \int_{\mathbb{S}^{N-1}} \int_{\mathbb{R}^p} \mathcal{H}^\epsilon(r)\varphi_1\varphi_2\, dz d\theta dr, \tag{4.13}$$

for which $\mathcal{L}_\epsilon$ is self-adjoint.

4.2. **A closer look at $\mathbb{L}_\epsilon$.** Let us go back to the general case $\mathbb{R}^n \setminus \Lambda$. Recall that $\mathbb{L}_\epsilon = \mathbb{L}(\bar{u}_\epsilon, g_{\mathbb{R}^n})$ is the linearized operator around the approximate solution $\bar{u}_\epsilon$. Note that, in a tubular neighborhood $\mathcal{T}_\rho(\Lambda)$ around the singular set the background metric is written as (3.7), which can be compared to the model (2.1). More precisely:

Proposition 4.4. *For functions supported in $\mathcal{T}_\rho(\Lambda)$ that do not depend on the z variable, we have that*

$$\mathbb{L}_\epsilon = \mathcal{L}_\epsilon + \mathcal{D}, \tag{4.14}$$

where $\mathcal{D}$ is a second order differential operator (at most) that satisfies, for functions of the form $\varphi = O(r^a)$ as $r \to 0$,

$$\mathcal{D}\varphi = O(r^{a-2+\sigma}),$$

for some $\sigma > 0$.

Remark 4.5. To handle the dependence on the variable z, we just need to recall that our arguments rely on localization and rescaling near a fixed point Λ, around which we use normal coordinates.

Proof of Proposition 4.4. As in [33, Section 4.2], in the neighborhood $\mathcal{T}_\rho(\Lambda)$, the difference between $\mathbb{L}_\epsilon$ and $\mathcal{L}_\epsilon$ comes from the extra terms of the metric in $\mathbb{R}^n \setminus \Lambda$ from (3.7) with respect to the model metric in $\mathbb{R}^n \setminus \mathbb{R}^p$ from (2.1) as we did in the proof of Proposition 3.3. Controlling the error terms in the metric yields the desired result. $\square$

5. Explicit calculations in the model case

In this Section we work with the model case $\mathbb{R}^n \setminus \mathbb{R}^p$ and give the proof of the statements in Section 4.1. Note that, even if we need to restrict to $k = 2$, many of our arguments are valid for any k once a model solution U_1 is available.

We will use the v-notation, using the metric g_{v_ϵ} given in (2.17), which is more convenient in the following calculations. Take now a conformal perturbation of the metric g_{v_ϵ} on $\mathbb{R}^n \backslash \mathbb{R}^p$, i.e, for $s \in \mathbb{R}$, set

$$s \mapsto g_s := (v_\epsilon + sw)^{\frac{4k}{n-2k}} \left[dt^2 + g_\theta + e^{2t}\,\delta_{\alpha\beta}\,dz^\alpha \otimes dz^\beta\right],$$

and consider B_s the symmetric $(1,1)$-tensor given by

$$B_s := \tfrac{n-2k}{2k}(v_\epsilon + sw)^{\frac{2n}{n-2k}} g_s^{-1} A_{g_s}. \tag{5.1}$$

Now let $L_\epsilon := \mathbb{L}(v_\epsilon, g_{cyl})$ be the linearized operator in this setting, that is,

$$\begin{aligned} L_\epsilon[w] :&= \frac{d}{ds}\bigg|_{s=0} \mathcal{N}(v_\epsilon + sw, g_{cyl}) \\ &= v_\epsilon^{-2k+1}\left\{ \frac{d}{ds}\bigg|_{s=0} \sigma_k(B_s) - cq(v_\epsilon)^{q-1} w \right\}, \end{aligned} \tag{5.2}$$

where we have used that v_ϵ is an exact solution of (2.10).

Proposition 4.2 in Section 4.1 follows from the following Lemma:

Lemma 5.1. *The linearized operator for the model in cylindrical coordinates is given by*

$$L_\epsilon[w] = a_0^\epsilon w + a_1^\epsilon \partial_t w + a_2^\epsilon \partial_{tt} w + a_3^\epsilon \Delta_\theta w + a_4^\epsilon e^{-2t} \Delta_z w, \tag{5.3}$$

where the coefficient functions $a_\ell^\epsilon = a_\ell^\epsilon(t)$ are given in (5.6)*. Moreover, $a_2^\epsilon, a_3^\epsilon, a_4^\epsilon$ have a sign, so this is an elliptic operator.*

Proof. We follow the calculations in [37] for the $\mathbb{R}^N \backslash \{0\}$ case. First recall from (2.7), for $B_0 (:= B_{g_{v_\epsilon}})$, that

$$\begin{aligned} \left(B_0\right)_t^t &= -\tfrac{n-2k}{4k}(v_\epsilon)^2 - v_\epsilon \ddot{v}_\epsilon + \tfrac{n-k}{n-2k}\dot{v}_\epsilon^{\,2} =: \kappa_1, \\ \left(B_0\right)_j^i &= \left[\tfrac{n-2k}{4k}(v_\epsilon)^2 - \tfrac{k}{n-2k}\dot{v}_\epsilon^{\,2}\right]\delta_i^j =: \kappa_2\,\delta_i^j, \\ \left(B_0\right)_\beta^\alpha &= -\tfrac{n-2k}{4k}\left(v_\epsilon + \tfrac{2k}{n-2k}\,\dot{v}_\epsilon\right)^2 \delta_\beta^\alpha =: \kappa_3\,\delta_\beta^\alpha, \end{aligned} \tag{5.4}$$

and the rest of the entries of the matrix vanish. Moreover, for the diagonal matrix B_0,

$$
\begin{aligned}
&[T^{k-1}(B_0)]_t^t = \sum_{m=0}^{k-1}(-1)^{k-1-m}\sigma_m(B_0)\kappa_1^{k-1-m} =: S_1,\\
&[T^{k-1}(B_0)]_j^i = \left[\sum_{m=0}^{k-1}(-1)^{k-1-m}\sigma_m(B_0)\kappa_2^{k-1-m}\right]\delta_j^i := S_2\,\delta_i^j,\\
&[T^{k-1}(B_0)]_\beta^\alpha = \left[\sum_{m=0}^{k-1}(-1)^{k-1-m}\sigma_m(B_0)\kappa_3^{k-1-m}\right]\delta_\beta^\alpha =: S_3\,\delta_\beta^\alpha.
\end{aligned} \tag{5.5}
$$

Note that, since g_{v_ϵ} belongs to the positive cone Γ_k^+, then its $(k-1)$-Newton tensor T^{k-1} is positive definite, so the quantities S_1, S_2, S_3 are strictly positive. We recall from (5.1) that

$$
\begin{aligned}
B_s = &\tfrac{n-2k}{2k}(v_\epsilon+sw)^2 g_{cyl}^{-1}\,A_{g_{cyl}} - (v_\epsilon+sw)g_{cyl}^{-1}\,D^2(v_\epsilon+sw)\\
&+\tfrac{n}{n-2k}g_{cyl}^{-1}\,d(v_\epsilon+sw)\otimes d(v_\epsilon+sw) - \tfrac{k}{n-2k}|d(v_\epsilon+sw)|^2_{g_{cyl}}I,
\end{aligned}
$$

from where it is easy to calculate its variation:

$$
\begin{aligned}
\frac{d\big(B_s\big)_t^t}{ds}\Big|_{s=0} &= -v_\epsilon\,\partial_{tt}w + \tfrac{2(n-k)}{n-2k}\,\dot v_\epsilon\,(\partial_t w) - \big(\tfrac{n-2k}{2k}\,v_\epsilon + \ddot v_\epsilon\big)\,w,\\
\frac{d\big(B_s\big)_j^i}{ds}\Big|_{s=0} &= -v_\epsilon\,g_\theta^{il}\,(D^2_\theta w)_{lj} - \tfrac{2k}{n-2k}\,\dot v_\epsilon\,\delta_j^i\,\partial_t w + \tfrac{n-2k}{2k}\,v_\epsilon\,\delta_j^i\,w,\\
\frac{d\big(B_s\big)_\beta^\alpha}{ds}\Big|_{s=0} &= -v_\epsilon\,e^{-2t}\partial_{\alpha\beta}w + \big[-\tfrac{2k}{n-2k}\,\dot v_\epsilon - v_\epsilon\big]\,\delta_\beta^\alpha\,\partial_t w\\
&\quad + \big[-\tfrac{n-2k}{2k}\,v_\epsilon + \dot v_\epsilon\big]\,w\delta_\beta^\alpha.
\end{aligned}
$$

Substituting (5.4) above one arrives at

$$
\begin{aligned}
\frac{d\big(B_s\big)_t^t}{ds}\Big|_{s=0} &= (v_\epsilon)^{-1}\,(B_0)_t^t\,w - \tfrac{n-k}{n-2k}\,(v_\epsilon)^{-1}(\dot v_\epsilon)^2 w - v_\epsilon\,\partial_{tt}w\\
&\quad + \tfrac{2(n-k)}{n-2k}\dot v_\epsilon\partial_t w - \tfrac{n-2k}{4k}v_\epsilon w,\\
\frac{d\big(B_s\big)_j^i}{ds}\Big|_{s=0} &= (v_\epsilon)^{-1}\,w\,(B_0)_j^i + \tfrac{k}{n-2k}\,(v_\epsilon)^{-1}(\dot v_\epsilon)^2\,\delta_j^i\,w + \tfrac{n-2k}{4k}v_\epsilon\delta_j^i w\\
&\quad - v_\epsilon(g_\theta)^{il}(D^2_\theta w)_{lj} - \tfrac{2k}{n-2k}\,\delta_j^i\,\dot v_\epsilon\,\partial_t w,\\
\frac{d\big(B_s\big)_\beta^\alpha}{ds}\Big|_{s=0} &= (v_\epsilon)^{-1}\,w\,(B_0)_\beta^\alpha + \tfrac{k}{n-2k}\,(v_\epsilon)^{-1}(\dot v_\epsilon)^2\delta_\beta^\alpha\,w - \tfrac{n-2k}{4k}v_\epsilon\delta_\beta^\alpha w\\
&\quad + \big[-v_\epsilon - \tfrac{2k}{n-2k}\,\dot v_\epsilon\big]\,\delta_\beta^\alpha\,\partial_t w - v_\epsilon\,e^{-2t}\,\partial_{\alpha\beta}w.
\end{aligned}
$$

Now, we can use the formula

$$k\sigma_k(B_0) = \mathrm{tr}\left(T^{k-1}(B_0)\cdot B_0\right)$$

to write

$$\begin{aligned}
\frac{d}{ds}\Big|_{s=0}\sigma_k(B_s) &= k\sigma_k(B_0)\,(v_\epsilon)^{-1}w \\
&+ S_1\left[-\tfrac{n-k}{n-2k}(v_\epsilon)^{-1}(\dot v_\epsilon)^2 w - \tfrac{n-2k}{4k}v_\epsilon w - v_\epsilon\,\partial_{tt}w + \tfrac{2(n-k)}{n-2k}\dot v_\epsilon\partial_t w\right] \\
&+ S_2\left[\tfrac{k}{n-2k}(N-1)(v_\epsilon)^{-1}(\dot v_\epsilon)^2 w + (N-1)\tfrac{n-2k}{4k}v_\epsilon w - v_\epsilon\Delta_\theta w\right. \\
&\qquad \left.-\tfrac{2k}{n-2k}(N-1)\dot v_\epsilon\partial_t w\right] \\
&+ S_3\left[\tfrac{k}{n-2k}p(v_\epsilon)^{-1}(\dot v_\epsilon)^2 w - p\tfrac{n-2k}{4k}v_\epsilon w + p\left(-v_\epsilon - \tfrac{2k}{n-2k}\dot v_\epsilon\right)\partial_t w\right. \\
&\qquad \left.-v_\epsilon e^{-2t}\Delta_z w\right].
\end{aligned}$$

Thus, using that v_ϵ is an exact solution to (2.10), we obtain that

$$L_\epsilon[w] = a_0 w + a_1\partial_t w + a_2\partial_{tt}w + a_3\Delta_\theta w + a_4 e^{-2t}\Delta_z w$$

for

(5.6)

$$\begin{aligned}
a_0 &= v_\epsilon^{-2k+1}\left\{(k-q)cv_\epsilon^{q-1} - \tfrac{n-k}{n-2k}S_1 v_\epsilon^{-1}\dot v_\epsilon^2 + \tfrac{k}{n-2k}(N-1)S_2 v_\epsilon^{-1}\dot v_\epsilon^2\right. \\
&\quad \left.+\tfrac{k}{n-2k}pS_3 v_\epsilon^{-1}\dot v_\epsilon^2 - \tfrac{n-2k}{4k}S_1 v_\epsilon + (N-1)\tfrac{n-2k}{4k}S_2 v_\epsilon - p\tfrac{n-2k}{4k}S_3 v_\epsilon\right\}, \\
a_1 &= v_\epsilon^{-2k+1}\left\{\tfrac{2(n-k)}{n-2k}\dot v_\epsilon S_1 - \tfrac{2k}{n-2k}(N-1)\dot v_\epsilon S_2 + p\left(-v_\epsilon - \tfrac{2k}{n-2k}\dot v_\epsilon\right)S_3\right\}, \\
a_2 &= -v_\epsilon^{-2k+2}S_1, \\
a_3 &= -v_\epsilon^{-2k+2}S_2, \\
a_4 &= -v_\epsilon^{-2k+2}S_3.
\end{aligned}$$

This completes the proof of the Lemma. □

Now we consider the proof of Proposition 4.3. Remark that the change of notation from u to v in (2.4) (and from φ to w) changes the indicial roots; nevertheless, their general structure ramians the same. For instance, compare the indicial roots for $\mathcal{L}_\epsilon$ from (4.7) to those of L_ϵ from (5.9).

5.1. **Indicial roots as** $r \to 0$**.** For this section we restrict to the case $k = 2$. Unfortunately the calculations become extremely messy, even if elementary, for larger values of k.

To study the behavior of L_ϵ as $r \to 0$, i.e., $t \to +\infty$, we define

$$b_\ell = \lim_{t\to+\infty} a_\ell^\epsilon(t), \quad \ell = 0,1,2,3,4.$$

We will give a precise formula for these coefficients below (in particular, they do not depend on ϵ).

Note that from (2.15) we have that $v_\epsilon \to v_\infty$ as $t \to +\infty$. Using (5.4) we can show that the matrix $B_0(t)$ converges to B_∞ for

$$(B_\infty)^t_t = -\tfrac{n-2k}{4k}(v_\infty)^2,$$
$$(B_\infty)^i_j = \tfrac{n-2k}{4k}(v_\infty)^2\delta^j_i,$$
$$(B_\infty)^\alpha_\beta = -\tfrac{n-2k}{4k}(v_\infty)^2\delta^\alpha_\beta.$$

Then from (5.5) we have

$$S_1^\infty := [T^1(B_\infty)]^t_t = \tfrac{n-4}{8}v_\infty^2(n-2p-1),$$
$$S_2^\infty := [T^1(B_\infty)]^i_j = \tfrac{n-4}{8}v_\infty^2(n-2p-3),$$
$$S_3^\infty := [T^1(B_\infty)]^\alpha_\beta = \tfrac{n-4}{8}v_\infty^2(n-2p-1),$$

which yields

$$\begin{aligned} b_0 = \Big[&(2-q)c(v_\infty)^{q-4} - \left(\tfrac{n-4}{8}\right)^2(n-2p-1) \\ &+ (N-1)\left(\tfrac{n-4}{8}\right)^2(n-2p-3) - p\left(\tfrac{n-4}{8}\right)^2(n-2p-1)\Big], \\ b_1 &= -p\tfrac{n-4}{8}(n-2p-1), \\ b_2 &= -\tfrac{n-4}{8}(n-2p-1), \\ b_3 &= -\tfrac{n-4}{8}(n-2p-3), \\ b_4 &= -\tfrac{n-4}{8}(n-2p-1). \end{aligned}$$

Recalling (2.13), we simplify b_0 to

$$\begin{aligned} b_0 &= \left(\tfrac{n-4}{8}\right)^2[(2-q)c_{n,p,2} - (p+1)(n-2p-1) + (N-1)(n-2p-3)] \\ &= -\left(\tfrac{n-4}{8}\right)(4p^2+8p-4np-5n+4+n^2). \end{aligned}$$

The behavior of L_ϵ when $t \to +\infty$, that is, $r \to 0$, is given by the normal operator

$$L^{(0)}[w] = b_0w + b_1\partial_t w + b_2\partial_{tt}w + b_3\Delta_\theta w + b_4 e^{-2t}\Delta_z w, \tag{5.7}$$

and the indicial operator

$$L^{(0)}_\natural[w] = b_0w + b_1\partial_t w + b_2\partial_{tt}w + b_3\Delta_\theta w, \tag{5.8}$$

We consider now the spherical harmonic decomposition of $\mathbb{S}^{N-1}$ and project the operators (4.6) and (5.8) over each eigenspace. For this, we set

$$L^{(0)}_j[w] = b_0w + b_1\partial_t w + b_2\partial_{tt}w - b_3\lambda_j w + b_4e^{-2t}\Delta_z w$$

and

$$L^{(0)}_{\natural j}[w] = b_0w + b_1\partial_t w + b_2\partial_{tt}w - b_3\lambda_j w.$$

We look for solutions of $L^{(0)}_{\natural j}[w]=0$ of the form $w(t)=e^{-\alpha t}$. Such α must satisfy the quadratic equation

$$f_j(\alpha):=b_0-b_3\lambda_j-b_1\alpha+b_2\alpha^2=0.$$

An elementary analysis of these parabolas yields that, for each $j=0,1,\ldots$, there exist two indicial roots $\gamma_j^\pm$ solution of $f_j(\alpha)=0$ satisfying:

- $\gamma_0^\pm$ can be real or complex. In the former case,
$$0<\gamma_0^-<\tfrac{p}{2}<\gamma_0^+,$$
while in the latter, $\operatorname{Re}\gamma_0^\pm=\frac{p}{2}>0$.
- For $j\geq 1$, $\gamma_j^\pm$ are real numbers. In addition, we have the monotonicity

$$\ldots\leq\gamma_2^-\leq\gamma_1^-<\operatorname{Re}\gamma_0^-\leq\tfrac{p}{2}\leq\operatorname{Re}\gamma_0^+<\gamma_1^+\leq\gamma_2^+\leq\ldots \tag{5.9}$$

- It holds $\gamma_1^-=-1$.

Indeed, the vertex of every parabola is located at $\alpha=-\frac{b_1}{2b_2}=\frac{p}{2}$. For the first statement, just note that $f_0(0)<0$ under the hypothesis $0<p<\mathfrak{p}_2$ (and this is sharp). For the second statement, note that $f_j(\alpha)$ is non-decreasing in j. In addition, for $j=1$, the polynomial $f_1(\alpha)$ has roots exactly at $\alpha=-1$ and $\alpha=p+1$, which shows our claim.

Observe that, similarly to the calculation in (4.12), it is possible to give a self-adjoint version of L_ϵ, denoted by $\widetilde{L}_\epsilon$,

$$\widetilde{L}_\epsilon[w]=\tilde{a}_0^\epsilon\widetilde{w}+\tilde{a}_2^\epsilon\partial_{tt}\widetilde{w}+\tilde{a}_3^\epsilon\Delta_\theta\widetilde{w}+\tilde{a}_4^\epsilon e^{-2t}\Delta_z\widetilde{w}, \tag{5.10}$$

for some coefficients $\tilde{a}_l$ which can be calculated from the original a_ℓ as in the proof of (4.12). We will not need its precise expression, only the limit operator as $r\to 0$: more precisely, the *conjugate* operator to $L^{(0)}$ is defined by

$$\widetilde{L}^{(0)}[\widetilde{w}]=e^{-\frac{p}{2}t}L^{(0)}[e^{\frac{p}{2}t}\widetilde{w}]=\tilde{b}_0\widetilde{w}+b_2\partial_{tt}\widetilde{w}+b_3\Delta_\theta\widetilde{w}+b_4e^{-2t}\Delta_z\widetilde{w}, \tag{5.11}$$

and similarly, for $L^{(0)}_\natural$,

$$\widetilde{L}^{(0)}_\natural[\widetilde{w}]=e^{-\frac{p}{2}t}L^{(0)}_\natural[e^{\frac{p}{2}t}\widetilde{w}]=\bar{b}_0\widetilde{w}+b_2\partial_{tt}\widetilde{w}+b_3\Delta_\theta\widetilde{w},$$

where we have defined

$$\tilde{b}_0=b_0-\frac{b_1^2}{4b_2}.$$

Remark 5.2. The advantage of the conjugate operator $\widetilde{L}_\epsilon$ is that it is self-adjoint and the indicial roots as $r \to 0$ (which will be denoted by $\delta_j^\pm$) are centered at the origin, fact that makes the Hilbert space analysis more clear.

5.2. **Indicial roots as** $r \to \infty$**.** We now calculate the indicial roots of L_ϵ as $r \to \infty$, i.e., $t \to -\infty$. For this, we define

$$d_\ell = \lim_{t \to +\infty} a_\ell^\epsilon(t), \quad \ell = 0, 1, 2, 3, 4.$$

In this limit, v_ϵ behaves as $\epsilon^{\alpha_0} e^{\alpha_0 t}$. We have that $B_0(t)$ converges to $B_{-\infty}$ as $t \to -\infty$, where

$$\begin{aligned}
(B_{-\infty})_t^t &= v_\epsilon^2 \left(-\tfrac{n-2k}{4k} + \tfrac{k}{n-2k}\alpha_0^2\right),\\
(B_{-\infty})_j^i &= v_\epsilon^2 \left(\tfrac{n-2k}{4k} - \tfrac{k}{n-2k}\alpha_0^2\right)\delta_i^j,\\
(B_{-\infty})_\beta^\alpha &= v_\epsilon^2 \left(-\tfrac{n-2k}{4k}\right)\left(1 + \tfrac{2k}{n-2k}\alpha_0\right)^2 \delta_\beta^\alpha,
\end{aligned}$$

from where

$$\begin{aligned}
S_1^{-\infty} &= v_\epsilon^2 \left(\tfrac{n-4}{8}(n-2p-1) - \alpha_0 p + \alpha_0^2 2\tfrac{-n+1}{n-4}\right) =: v_\epsilon^2 s_1,\\
S_2^{-\infty} &= v_\epsilon^2 \left(\tfrac{n-4}{8}(n-2p-3) - \alpha_0 p + \alpha_0^2 2\tfrac{-n+3}{n-4}\right) =: v_\epsilon^2 s_2\\
S_3^{-\infty} &= v_\epsilon^2 \left(\tfrac{n-4}{8}(n-2p-1) - \alpha_0(p-1) + \alpha_0^2 2\tfrac{-n+3}{n-4}\right) =: v_\epsilon^2 s_3.
\end{aligned}$$

Then, the expressions in (5.6) imply

(5.12)
$$\begin{aligned}
d_0 &= \Big\{-\tfrac{n-2}{n-4}s_1\alpha_0^2 + \tfrac{2}{n-4}(n-p-1)s_2\alpha_0^2\\
&\quad + \tfrac{2}{n-4}ps_3\alpha_0^2 - \tfrac{n-4}{8}s_1 + (n-p-1)\tfrac{n-4}{8}s_2 - p\tfrac{n-4}{8}s_3\Big\},\\
d_1 &= \left\{\tfrac{2(n-2)}{n-4}\alpha_0 s_1 - \tfrac{4}{n-4}(n-p-1)\alpha_0 s_2 + p\left(-1 - \tfrac{4}{n-4}\alpha_0\right)s_3\right\},\\
d_2 &= -\left(\tfrac{n-4}{8}(n-2p-1) - \alpha_0 p + \alpha_0^2 2\tfrac{-n+1}{n-4}\right),\\
d_3 &= -\left(\tfrac{n-4}{8}(n-2p-3) - \alpha_0 p + \alpha_0^2 2\tfrac{-n+3}{n-4}\right),\\
d_4 &= -\left(\tfrac{n-4}{8}(n-2p-1) - \alpha_0(p-1) + \alpha_0^2 2\tfrac{-n+3}{n-4}\right).
\end{aligned}$$

Indicial roots are calculated as the roots the quadratic polynomial

$$d_0 - d_3\lambda_j - d_1\alpha + d_2\alpha^2 = 0, \quad j = 0, 1, \ldots$$

Even though we could proceed as in Subsection 5.1, the expressions of the coefficients from (5.12) are so involved that we prefer to take a different path. In any case, for each $j = 0, 1, \ldots$, there exist two indicial roots $\vartheta_j^\pm$, with similar monotonicity.

To prove the remaining statements for in Proposition 4.3 on the behavior of the indicial roots at infinity we need to switch back to the

u-notation. Full details of the proof will be postponed until Section 8, however, let us give here the main ideas:

We start with the $j = 0$ mode, that corresponds to the radially symmetric solutions. The fundamental observation is that knowing an exact solution for the non-linear ODE (2.12), one can produce radially symmetric solutions of its linearized equation using dilation invariance, as explained in Remark 8.4. But we know precisely the behavior as $r \to \infty$ of the fast-decaying solution U_1 for (2.12); indeed, in Proposition 2.1 we showed such solutions must satisfy $U \sim r^{-\alpha_0^\pm}$. The precise value of $\alpha_0^\pm$ was calculated in the paper [19] and it is given by

$$\alpha_0^\pm = \frac{n-4}{4} - \frac{p(n-3)}{2(n-1)} \pm \frac{\sqrt{4p+5p^2-5pn+pn^2-p^2n}}{2(n-1)}. \tag{5.13}$$

(we denote $\alpha_0 = \alpha_0^-$). From here we conclude that $-\alpha_0^\pm$ are the two indicial roots for $j = 0$.

Note also that the expression inside the square root in formula (5.13) is always positive in our range of p. This implies that, for all j, all the indicial roots are real numbers. Summarizing our discussion, we have (4.8).

We can also give explicit formulas for $j = 1$. Using rotational invariance from Remark 8.6 one can find solutions in the kernel for $j = 1$, and this yields (4.10). Finally, to show (4.11), one just needs to check that

$$-\tfrac{n-4}{4} + p\tfrac{n-3}{n-1} + 1 + \alpha_0 > 0,$$

which follows from simple algebra for $n \geq 5$.

6. Function spaces

The objective of this section is to set up the functional analytic framework on $\mathbb{R}^n \setminus \Lambda$.

Remark 6.1. In general, it would be necessary to control the asymptotic behavior of a function both near the singular set and at infinity. Thus one should work with weighted spaces on $\mathbb{R}^n \setminus \Lambda$ with two parameters (one near Λ, another one at infinity). However, thanks to Remark 3.4 we are only considering metrics with a very precise behavior at infinity that comes an initial compact manifold and it is not necessary to introduce a weight at infinity, which is equivalent working on a domain $\Omega \setminus \Lambda$ with zero Dirichlet condition (compare the statements of Theorems 2 and 3 in [33]).

In order to simplify our presentation, we will write the function domain as $\Omega \setminus \Lambda$ with this understanding and only introduce a weight near the singular set Λ.

6.1. **Weighted Hölder spaces.** We now define the weighted Hölder spaces $\mathcal{C}^{2,\alpha}_{\mu}(\Omega \setminus \Lambda)$, following the notations and definitions in Section 3 of [33]. Intuitively, these spaces consist of functions which are products of powers of the distance to Λ with functions whose Hölder norms are invariant under homothetic transformations centered at an arbitrary point on Λ.

Let u be a function in a tubular neighbourhood $\mathcal{T} := \mathcal{T}_\rho$ of Λ and define

$$\|u\|^{\mathcal{T}}_{0,\alpha,0} = \sup_{y\in\mathcal{T}} |u| + \sup_{y,y'\in\mathcal{T}} \frac{(r+\tilde{r})^\alpha |u(y)-u(y')|}{|r-r'|^\alpha + |z-z'|^\alpha + (r+r')^\alpha|\theta-\theta'|^\alpha},$$

where y, y' are two points in $\mathcal{T}$ and $(r,\theta,z),(r',\theta',z')$ their Fermi coordinates.

Definition 6.2. The space $\mathcal{C}^{l,\alpha}_0(\Omega\setminus\Lambda)$ is defined to be the set of all $u \in \mathcal{C}^{l,\alpha}(\Omega\setminus\Lambda)$ for which the norm

$$\|u\|_{\mathcal{C}^{l,\alpha}_0} = \|u\|_{\mathcal{C}^{l,\alpha}(\Omega\setminus\mathcal{T}_{\rho/2})} + \sum_{j=0}^{l} \|\nabla^j u\|_{\mathcal{C}^{0,\alpha}(\mathcal{T}_\rho)}$$

is finite.

Now we consider a function d behaving as the Fermi coordinate r in a tubular neighborhood $\mathcal{T}_\rho$ of Λ and a positive constant (say, identically 1) away from the singular set. Then a function u belongs to $\mathcal{C}^{l,\alpha}_\mu(\Omega\setminus\Lambda)$ if and only if

$$u = d^\mu \hat{u} \quad \text{for some} \quad \hat{u} \in \mathcal{C}^{l,\alpha}(\Omega\setminus\Lambda).$$

This space is endowed with the natural norm

$$\|u\|_{\mathcal{C}^{l,\alpha}_\mu} := \|d^{-\mu}u\|_{\mathcal{C}^{l,\alpha}_0}.$$

Basic properties of these norms can be found in [33, Section 3].

In the particular setting of $\mathbb{R}^n \setminus \mathbb{R}^p$ it will be necessary to introduce weighted Hölder spaces with respect to the r variable for functions having different behaviors near $r=0$ and $r=\infty$.

First, in the case of an isolated singularity, this is, $\mathbb{R}^N\setminus\{0\}$, given any $\mu_1,\mu_2\in\mathbb{R}$, for $R>0$ fixed we set

$$\begin{aligned}\mathcal{C}^{l,\alpha}_{\mu_1}(B_R\setminus\{0\}) &= \{u = r^{\mu_1}\phi \,:\, \phi\in\mathcal{C}^{l,\alpha}_0(B_R\setminus\{0\})\},\\ \mathcal{C}^{l,\alpha}_{\mu_2}(\mathbb{R}^N\setminus B_R) &= \{u = r^{\mu_2}\phi \,:\, \phi\in\mathcal{C}^{l,\alpha}_0(\mathbb{R}^N\setminus B_R)\},\end{aligned}$$

and thus we can define:

Definition 6.3. The space $\mathcal{C}^{l,\alpha}_{\mu_1,\mu_2}(\mathbb{R}^N \setminus \{0\})$ consists of all functions u for which the norm

$$\|u\|_{\mathcal{C}^{l,\alpha}_{\mu_1,\mu_2}} = \sup_{B_R\setminus\{0\}} \|r^{-\mu_1}u\|_{l,\alpha,0} + \sup_{\mathbb{R}^N\setminus B_R} \|r^{-\mu_2}u\|_{l,\alpha,0}$$

is finite. The spaces $\mathcal{C}^{l,\alpha}_{\mu,\nu}(\mathbb{R}^n \setminus \mathbb{R}^p)$ are defined similarly, in terms of the (global) Fermi coordinates (r,θ,z) and the weights r^{μ_1}, r^{μ_2}.

6.2. **Weighted Sobolev spaces.** As we have discussed, $\mathbb{L}_\epsilon$ is second order linear, elliptic operator, uniformly elliptic away from the singular set Λ where it has the structure of an edge operator from [31]. Here we try to make a an effort to present Mazzeo's theory of edge operators in a more transparent way.

A possibility is to work with the "self-djoint" version of $\mathbb{L}_\epsilon$, which near the singular set is essentially (4.12). Another approach is to pass to the Fermi variable $t = -\log r$ and introduce the "conjugate" operator $\widetilde{\mathbb{L}}_\epsilon$ which, near the singular set, is written as (5.10). This operator is fully characterized both near Λ and near infinity (and it is independent of ϵ at both places).

Moreover, since the weight in the function space is not relevant unless we are near the singular set (recall that we are ignoring the contribution from infinity, as explained in Remark 6.1), for our purposes it is enough to consider, for $\delta \in \mathbb{R}$, the norm

$$\|\varphi\|^2_{L^2_\delta(\Omega\setminus\Lambda)} = \int_{\Omega\setminus\mathcal{T}_\rho} \varphi^2\,dy + \int_0^\rho \int_{\mathbb{S}^{N-1}} \int_\Lambda \varphi^2 r^{\frac{n-4}{2}-p-2\delta-1}\,dzd\theta dr. \tag{6.1}$$

The last term in the expression (6.1) above has a more user friendly expression in the variable $t = -\log r$. Indeed, using the same notation as in (5.11), if we set

$$\varphi = d^{-\frac{n-4}{4}} w = d^{-\frac{n-4}{4}+\frac{p}{2}}\widetilde{w} \tag{6.2}$$

then, taking into account that $d = r$ on near the singular set, it simplifies to

$$\int_{-\log\rho}^{+\infty} \int_{\mathbb{S}^{N-1}} \int_\Lambda \widetilde{w}^2 e^{2\delta t}\,dzd\theta dt.$$

Finally, weighted Sobolev spaces $W^{k,2}_\delta$ are defined similarly.

6.3. **Duality.** We will consider the spaces $L^2_\delta(\Omega\setminus\Lambda)$ and $L^2_{-\delta}(\Omega\setminus\Lambda)$ to be dual with respect to the natural pairing

$$L^2_\delta \times L^2_{-\delta} \ni (\widetilde{w}_1, \widetilde{w}_2) \mapsto \int_{\Omega\setminus\Lambda} \widetilde{w}_1\widetilde{w}_2. \tag{6.3}$$

Fixed $\delta \in \mathbb{R}$, the dual of

$$\widetilde{L}^{(0)} : L^2_\delta(\mathbb{R}^n \setminus \mathbb{R}^p) \to L^2_\delta(\mathbb{R}^n \setminus \mathbb{R}^p)$$

is given by

$$(\widetilde{L}^{(0)})^* = r^{-2\delta}\widetilde{L}^{(0)} r^{2\delta} : L^2_{-\delta}(\mathbb{R}^n \setminus \mathbb{R}^p) \to L^2_{-\delta}(\mathbb{R}^n \setminus \mathbb{R}^p). \tag{6.4}$$

We do not claim that $\mathbb{L}_\epsilon$ is self-adjoint. Nevertheless, relative to the pairing (6.3), the adjoint of

$$\widetilde{\mathbb{L}}_\epsilon : L^2_\delta(\Omega \setminus \Lambda) \to L^2_\delta(\Omega \setminus \Lambda)$$

is still a second order elliptic operator with the same structure acting on

$$(\widetilde{\mathbb{L}}_\epsilon)^* : L^2_{-\delta}(\Omega \setminus \Lambda) \to L^2_{-\delta}(\Omega \setminus \Lambda).$$

We will show in Proposition 7.1 that $\widetilde{\mathbb{L}}_\epsilon$ (and thus, the original $\mathbb{L}_\epsilon$) is semi-Fredholm when $\delta > 0$ not an indicial root. This implies that

$$\ker((\widetilde{\mathbb{L}}_\epsilon)^*)^\perp = \mathrm{Rg}(\widetilde{\mathbb{L}}_\epsilon). \tag{6.5}$$

Thus an easy way to prove that such $\widetilde{\mathbb{L}}_\epsilon$ is surjective is to check that its adjoint is injective.

7. A priori estimates and L^2 semi-Fredholm properties

Let $\mathbb{L}_\epsilon$ be the linearized operator around $\bar{u}_\epsilon$ in $\mathbb{R}^n \setminus \Lambda$, that is, $\mathbb{L}(\bar{u}_\epsilon, g_{\mathbb{R}^n})$. Fredholm properties for this type of operators were shown in Mazzeo [31] in great generality (using the theory of edge operators) and we refer to this paper for the complete proofs. Here, instead, we consider a simple PDE approach for the L^2 theory which was presented in the lecture notes [39].

For the sake of clarity, as mentioned in Remark 6.1, we will work with functions that are supported in a domain Ω in order to avoid the complications as $r \to \infty$.

Fixed $\epsilon > 0$, the operator $\widetilde{\mathbb{L}}_\epsilon : L^2_\delta(\Omega \setminus \Lambda) \to L^2_\delta(\Omega \setminus \Lambda)$ is linear and unbounded, densely defined and has closed graph. Our main result in this Section proves (semi)-Fredholm properties, encoded in the a-priori estimate from Proposition 7.1 for solutions of the equation

$$\widetilde{\mathbb{L}}_\epsilon \widetilde{w} = \widetilde{h} \quad \text{in} \quad \Omega \setminus \Lambda. \tag{7.1}$$

Using the notation in Remark 5.2, we will denote by $\delta_j^\pm$ the indicial roots of $\widetilde{\mathbb{L}}_\epsilon$ as $r \to 0$ (recall that they do not depend on ϵ).

Proposition 7.1. *Let $\delta \neq \delta_j^{\pm}$, $\delta > 0$ and take $\widetilde{w} \in L^2_\delta$, $\widetilde{h} \in L^2_\delta$ satisfying* (7.1)*. Then*

$$\|\widetilde{w}\|_{L^2_\delta(\Omega\setminus\Lambda)} \leq C\left(\|\widetilde{h}\|_{L^2_\delta(\Omega\setminus\Lambda)} + \|\widetilde{w}\|_{L^2(V)}\right) \tag{7.2}$$

for V any compact set in $\Omega\setminus\Lambda$, and some constant $C(V)$ not depending on $\widetilde{w}$.

Note that there is no dependence on ϵ in the conclusion of the Proposition. The proof follows from a series of Lemmas:

Lemma 7.2. *(Localization) It is sufficient to prove the Proposition for functions in L^2_δ supported in $\mathcal{T}_\rho(\Lambda)$ for some small ρ.*

Proof. Introduce a cutoff χ identically one on $\mathcal{T}_{\rho/2}(\Lambda)$, vanishing outside $\mathcal{T}_\rho(\Lambda)$. Then

$$\widetilde{h}_1 := \widetilde{\mathbb{L}}_\epsilon(\widetilde{w}\chi) = \chi\widetilde{\mathbb{L}}_\epsilon\widetilde{w} + [\widetilde{\mathbb{L}}_\epsilon, \chi]\widetilde{w}.$$

Thus if inequality (7.2) is true for $\widetilde{w}\chi$, it is also true for $\widetilde{w}$ by adding a compactly supported term. □

Lemma 7.3. *(Reduction to the model case) For functions supported on $\mathcal{T}_\rho(\Lambda)$ for some small ρ, it is enough to prove* (7.2) *for the model operator $\widetilde{L}^{(0)}$ instead of $\widetilde{\mathbb{L}}_\epsilon$.*

Proof. First recall Proposition 4.4 (and Remark 4.5) to reduce the problem to study the model operator L_ϵ (or its conjugate $\widetilde{L}_\epsilon$). Next, because of the ODE study from Proposition 2.1, we have for some $\sigma > 0$, in a small enough neighborhood $\{r < a\epsilon\}$ for some $a > 0$,

$$\left\|D^\ell\left(v_\epsilon - v_\infty\right)\right\|_{L^\infty(\mathcal{T}_\rho(\Lambda))} \leq C_\ell\, e^{-\sigma t}, \quad \ell = 0, 1, \ldots$$

Thus it is clear that

$$\widetilde{L}_\epsilon = \widetilde{L}^{(0)}(1 + O(e^{-\sigma' t})).$$

Now, for points a bit further away from Λ, say, for $r >> \epsilon$, the operator $\widetilde{L}_\epsilon$ does not depend on ϵ (at least, up to lower order terms), and it is a regular uniformly elliptic operator so standard Sobolev estimates hold. □

Now we give the proof of Proposition 7.1 for the model $\widetilde{L}^{(0)}$. Assume, for now, that $\delta_j := \delta_j^+ > 0$ for all j. Recall the definition of the (conjugate) operator from (5.11); after projection over spherical harmonics, it becomes

$$\widetilde{L}^{(0)}_j\widetilde{w} = (\tilde{b}_0 - b_3\lambda_j)\widetilde{w} + b_2\partial_{tt}\widetilde{w} + b_4 e^{-2t}\Delta_z\widetilde{w}, \quad b_2, b_4 < 0.$$

Take the Fourier transform in the variable z, and set

$$K_j\omega := \tilde{b}_{0,j}\omega + b_2\partial_{tt}\omega - b_4 e^{-2t}|\zeta|^2\omega,$$

where we have defined $\tilde{b}_{0,j} := \tilde{b}_0 - b_3\lambda_j$, ζ the Fourier variable and ω the Fourier transform of $\widetilde{w}$. Now we make the change $b_4|\zeta|^2e^{-2t} = b_2e^{-2\tau}$ and work in the variable τ. This operator reduces (up to a negative multiplicative constant) to

$$K_j\omega = \partial_{\tau\tau}\omega - \delta_j^2\omega - e^{-2\tau}\omega, \quad \text{for } \omega = \omega(\tau). \tag{7.3}$$

Define the space $L^2_\delta(d\tau)$ the weighted space with respect to the variable τ and weight $e^{2\delta\tau}$, and let us study the mapping properties of K_j in $L^2_\delta(d\tau)$.

Note that for functions supported on $\tau \in (\tau_0, \infty)$ for τ_0 big enough, the term $-e^{-2\tau}$ is just a perturbation and can be ignored. Thus, for each fixed j, we consider the equation

$$\mathfrak{K}_j\omega = \psi \quad \text{for} \quad \mathfrak{K}_j\omega := \partial_{\tau\tau}\omega - \delta_j^2\omega. \tag{7.4}$$

For simplicity, we take $\tau_0 = 0$ in the next Lemma. The dependence on τ_0 will be retaken in Lemma 7.5, in order be able to go back to the variable t.

Such $\mathfrak{K}_j$ is a totally characteristic operator and has good Fredholm properties.

Lemma 7.4. *If δ not an indicial root, for every solution $\omega(\tau)$ of (7.4) supported on $(0,\infty)$ we have*

$$\|\omega\|_{L^2_\delta(d\tau)} \leq C\|\psi\|_{L^2_\delta(d\tau)}. \tag{7.5}$$

Proof. First we show that the estimate is true if $-\delta_j < \delta < \delta_j$. Multiply equation (7.4) by $e^{2\delta\tau}\omega$:

$$-\int_0^\infty \omega(\partial_{\tau\tau}\omega)e^{2\delta\tau}\,d\tau + \delta_j^2\int_0^\infty \omega^2e^{2\delta\tau}\,d\tau = -\int_0^\infty \psi\omega e^{2\delta\tau}\,d\tau. \tag{7.6}$$

Integration by parts, noting that the boundary terms vanish, yields

$$-\int_0^\infty \omega(\partial_{\tau\tau}\omega)e^{2\delta\tau}\,d\tau = \int_0^\infty (\partial_\tau\omega)^2e^{2\delta\tau}\,d\tau + 2\delta\int_0^\infty \omega(\partial_\tau\omega)e^{2\delta\tau}\,d\tau. \tag{7.7}$$

For the last term in (7.7),

$$2\delta\int_0^\infty \omega(\partial_\tau\omega)e^{2\delta\tau}\,d\tau = \delta\int_0^\infty \partial_\tau(\omega^2)e^{2\delta\tau}\,d\tau = -2\delta^2\int_0^\infty \omega^2e^{2\delta\tau}\,d\tau. \tag{7.8}$$

Substitute the two expressions above into (7.6) to obtain

$$\begin{aligned}\int_0^\infty (\partial_\tau\omega)^2 e^{2\delta\tau}\,d\tau + (\delta_j^2 - 2\delta^2)\int_0^\infty \omega^2 e^{2\delta\tau}\,d\tau &= -\int_0^\infty \psi\omega e^{2\delta\tau}\,d\tau \\ &\leq \left(\int_0^\infty \psi^2 e^{2\delta\tau}\,d\tau\right)^{\frac{1}{2}}\left(\int_0^\infty \omega^2 e^{2\delta\tau}\,d\tau\right)^{\frac{1}{2}}.\end{aligned} \tag{7.9}$$

On the other hand, Holder estimates in (7.8) above will give

$$\delta\int_0^\infty \omega^2 e^{2\delta\tau}\,d\tau \leq \left(\int_0^\infty \omega^2 e^{2\delta\tau}\,d\tau\right)^{\frac{1}{2}}\left(\int_0^\infty (\partial_\tau\omega)^2 e^{2\delta\tau}\,d\tau\right)^{\frac{1}{2}}$$

and thus,

$$\delta^2\int_0^\infty \omega^2 e^{2\delta\tau} \leq \int_0^\infty (\partial_\tau\omega)^2 e^{2\delta\tau}\,d\tau. \tag{7.10}$$

Substituting (7.10) into (7.9) implies

$$(\delta_j^2 - \delta^2)\int_0^\infty \omega^2 e^{2\delta\tau}\,d\tau \leq \left(\int_0^\infty \psi^2 e^{2\delta\tau}\,d\tau\right)^{\frac{1}{2}}\left(\int_0^\infty \omega^2 e^{2\delta\tau}\,d\tau\right)^{\frac{1}{2}}.$$

To finish, just note that $\delta_j^2 - \delta^2 > 0$ because of our hypothesis, so that

$$\int_0^\infty \omega^2 e^{2\delta\tau}\,d\tau \leq \frac{1}{(\delta_j^2 - \delta^2)^2}\int_0^\infty \psi^2 e^{2\delta\tau}\,d\tau,$$

as desired.

Now we prove estimate (7.5) if $\delta > \delta_j$ (the remaining case $\delta < -\delta_j$ is very similar). First remark that problem (7.4) is an ODE, which has a unique solution. Using the variation of constants formula, it is written as

$$\omega = \frac{1}{W}\left(B^+\int_\tau^{+\infty} B^-\psi - B^-\int_\tau^{+\infty} B^+\psi\right) := \frac{1}{W}(\omega_1 + \omega_2),$$

where $B^+ = e^{\delta_j\tau}$ and $B^- = e^{-\delta_j\tau}$ and W the Wronskian. We proceed as follows: first, for the term $\omega_2 := B^-\int B^+\psi$, use integration by parts

$$\begin{aligned}\|\omega_1\|^2_{L^2_\delta(d\tau)} &= \int_0^\infty e^{-2\delta_j\tau}\left(\int_\tau^\infty e^{\delta_j\tau}\psi\right)^2 e^{2\tau\delta}\,d\tau \\ &= \int_0^\infty \partial_\tau\left(\frac{1}{2\delta - 2\delta_j}e^{-2\delta_j\tau + 2\delta\tau}\right)\left(\int_\tau^\infty e^{\delta_j\tau}\psi\right)^2 d\tau \\ &= \frac{1}{\delta - \delta_j}\int_0^\infty \psi\omega_1 e^{2\delta\tau}\,d\tau.\end{aligned}$$

To finish, just use Holder inequality:

$$\|\omega_1\|^2_{L^2_\delta(d\tau)} \lesssim \left(\int_0^\infty f^2 e^{2\delta\tau} d\tau\right)^{\frac{1}{2}} \|\omega_1\|_{L^2_\delta(d\tau)}.$$

For the first term of ω_1 the inequality is proved similarly. □

Now we go back to the problem

$$K_j\omega = \psi \quad \text{for} \quad K_j\omega = \partial_{\tau\tau}\omega - \delta_j^2\omega - e^{-2\tau}\omega, \tag{7.11}$$

in order to understand the dependence on τ_0. While estimate (7.5) should still be true, the constant C would depend on τ_0. This is not enough to go back to the variable t (recall that we are working in general with functions supported on $\mathcal{T}_\rho(\Lambda)$. The strongest assumption $\delta > 0$ will provide this extra control, as we will show in the following Lemma:

Lemma 7.5. *Fix $\delta > 0$ not an indicial root. Let $\omega(\tau)$ be a solution of (7.11) supported on $\tau \in (\tau_0, \infty)$, $\tau_0 \in \mathbb{R}$. Then*

$$\|\omega\|_{L^2_\delta(d\tau)} \leq C\|\psi\|_{L^2_\delta(d\tau)}, \tag{7.12}$$

for a constant C independent of ω and τ_0.

Proof. The proof goes similarly to that of Lemma 7.4. First, in the case $0 < \delta < \delta_j$, one can repeat the proof line by line, noting that the additional term $-\int \omega^2 e^{2\tau} e^{2\delta\tau}$ has the right sign and can be dropped while keeping the inequality.

The case $\delta > \delta_j$ is more delicate, since involves Bessel functions. We can still write the solution to problem (7.4) as

$$\omega = \frac{1}{W}\left(B^+\int_\tau^{+\infty} B^-\psi - B^-\int_\tau^{+\infty} B^+\psi\right) =: \frac{1}{W}(\omega_1 + \omega_2),$$

where

$$B_+(\tau) := K_{\delta_j}(e^{-\tau}), \quad B_-(\tau) := I_{\delta_j}(e^{-\tau}),$$

where $I_{\delta_j}, K_{\delta_j}$ are the modified Bessel functions of the second kind. Their asymptotic behavior is well known and, indeed, when $\tau \to +\infty$, $B^+(\tau) \sim e^{\delta_j\tau}$ and $B^-(\tau) \sim e^{-\delta_j\tau}$. The Wronskian W is well known and has constant value (see [50], Chapter III, formula (80)).

We will give the proof for the term $\omega_1 := B_+ \int_\tau^\infty B_-\psi$. An analogous argument yields the estimate for ω_2. First use integration by parts

$$
\begin{aligned}
&\|\omega_1\|^2_{L^2_\delta(d\tau)} \\
&= \int_{\tau_0}^\infty B_+(\tau)^2 \left(\int_\tau^\infty B_-(\sigma)\psi(\sigma)\, d\sigma \right)^2 e^{2\tau\delta}\, d\tau \\
&= \int_{\tau_0}^\infty \partial_\tau \left(\frac{1}{2\delta + 2\delta_j} e^{2\delta\tau + 2\delta_j\tau} \right) e^{-2\delta_j\tau} B_+(\tau)^2 \left(\int_\tau^\infty B_-(\sigma)\psi(\sigma)\, d\sigma \right)^2 d\tau \\
&=: J_1 + J_2,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \frac{1}{\delta + \delta_j} \int_{\tau_0}^\infty e^{2\delta\tau} B_+(\tau)^2 B_-(\tau)\psi(\tau) \int_\tau^\infty B_-(\sigma)\psi(\sigma)\, d\sigma\, d\tau \\
&= \frac{1}{\delta + \delta_j} \int_{\tau_0}^\infty e^{2\delta\tau} B_+(\tau) B_-(\tau)\psi(\tau)\omega_1(\tau)\, d\tau
\end{aligned}
$$

just nothing that $B_+(\tau)B_-(\tau)$ is a uniformly bounded positive function. Finally, Hölder's inequality yields

$$
\begin{aligned}
J_1 &\le \frac{C}{\delta + \delta_j} \left(\int_{\tau_0}^\infty \psi(\tau)^2 e^{2\delta\tau}\, d\tau \right)^{1/2} \left(\int_{\tau_0}^\infty \omega_1(\tau)^2 e^{2\delta\tau}\, d\tau \right)^{1/2} \\
&= \frac{C}{\delta + \delta_j} \|\psi\|_{L^2_\delta(d\tau)} \|\omega_1\|_{L^2_\delta(d\tau)}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
J_2 &= -\frac{1}{2\delta + 2\delta_j} \int_{\tau_0}^\infty e^{2\delta\tau + 2\delta_j\tau} \partial_\tau \left(e^{-2\delta_j\tau} B_+(\tau)^2 \right) \int_\tau^\infty B_-(\sigma)\psi(\sigma)\, d\sigma\, d\tau \\
&= -\frac{1}{2\delta + 2\delta_j} \int_{\tau_0}^\infty e^{2\delta\tau} \partial_\tau \log \left(e^{-2\delta_j\tau} B_+(\tau)^2 \right) \omega_1(\tau)^2\, d\tau.
\end{aligned}
$$

Calculate, for $s = e^{-\tau}$,

$$
\begin{aligned}
\partial_\tau \log \left(e^{-2\delta_j\tau} B_+(\tau)^2 \right) &= 2 \left[-\delta_j + \frac{\partial_\tau B_+(\tau)}{B_+(\tau)} \right] = 2 \left[-\delta_j - \frac{s\partial_s K_{\delta_j}(s)}{K_{\delta_j}(s)} \right] \\
&= -2 \frac{s^{1-\delta_j} \partial_s (s^{\delta_j} K_{\delta_j}(s))}{K_{\delta_j}(s)} \ge 0,
\end{aligned}
$$

using Property 3.71 in [50] which implies $\partial_s(s^{\delta_j} K_{\delta_j}(s)) \le 0$. By the crucial hypothesis $\delta > 0$, the term J_2 has a sign and can be dropped. From the estimate for J_1 we have that

$$
\|\omega_1\|^2_{L^2_\delta(d\tau)} \le C \|\psi\|^2_{L^2_\delta(d\tau)},
$$

for functions supported in $(\tau_0, +\infty)$ but now this constant C is independent of τ_0, as desired.

□

Finally, taking Fourier transform back will complete the proof of Proposition 7.1. If $\operatorname{Re}\delta_0 = 0$, then one needs to consider Bessel functions with complex argument. Nevertheless, modifications are minimal.

8. Injectivity of the model operator

Let μ be a weight satisfying

$$\tfrac{p}{2} - \tfrac{n-4}{4} \leq \operatorname{Re}(\chi_{0,+}^{(0)}) < \mu < \chi_{1,+}^{(0)}. \tag{8.1}$$

We let $\mathcal{L}_1$ to be the variable coefficient operator which is given by (4.4) evaluated at $\epsilon = 1$, that is,

$$\mathcal{L}_1[\varphi] = \mathcal{A}_0^1(r)\varphi + \frac{\mathcal{A}_1^1(r)}{r}\partial_r\varphi + A_2^1(r)\partial_{rr}\varphi + \frac{\mathcal{A}_3^1(r)}{r^2}\Delta_\theta\varphi + \mathcal{A}_4^1(r)\Delta_z\varphi. \tag{8.2}$$

The aim of this Section is to prove:

Proposition 8.1. *The only solution $\varphi \in \mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^n \setminus \mathbb{R}^p)$ of*

$$\mathcal{L}_1\varphi = 0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}^p \tag{8.3}$$

is $\varphi \equiv 0$.

8.1. The normal operators $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(\infty)}$.

We study first the constant coefficient operators $\mathcal{L}^{(0)}$ and $\mathcal{L}^{(\infty)}$ on $\mathbb{R}^n \setminus \mathbb{R}^p$; precise formulas are given in (4.5) and (4.6).

Proposition 8.2. *Any solution $\varphi \in \mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^n \setminus \mathbb{R}^p)$ of $\mathcal{L}^{(0)}\varphi = 0$ must vanish identically.*

Proof. First remark that it is enough to study injectivity of the projected operators

$$\mathcal{L}_j^{(0)}[\varphi] = \beta_0^{(0)}\varphi + \frac{\beta_1^{(0)}}{r}\partial_r\varphi + \beta_2^{(0)}\partial_{tt}\varphi - \lambda_j\frac{\beta_3^{(0)}}{r^2}\varphi + \beta_4^{(0)}\Delta_z\varphi = 0.$$

Recalling the discussion in Section 7, this can be reduced to proving injectivity for each equation

$$\partial_{\tau\tau}\omega - \delta_j^2\omega - e^{-2\tau}\omega = 0, \quad \text{for } \omega = \omega(\tau), \quad \tau \in \mathbb{R}, \quad j = 0, 1, \ldots \tag{8.4}$$

under the assumption that ω has the behavior

$$\omega(r) = \begin{cases} O(r^\delta) & \text{as} \quad r \to 0, \\ O(r^{-\frac{p}{2}+\frac{n-4}{4}}) & \text{as} \quad r \to \infty. \end{cases}$$

Here we have defined

$$\delta := \mu - \tfrac{p}{2} + \tfrac{n-4}{4} > 0.$$

Let us look then at equation (8.4). For any j, this is a Bessel ODE which has two linearly independent solutions, given by the modified Bessel functions of second kind $K_{\delta_j}(r)$ and $I_{\delta_j}(r)$ in the variable $r = e^{-t}$. Since the asymptotic behavior of the Bessel functions is well known, any solution with such behavior as $r \to 0$ cannot be bounded as $r \to \infty$, which is not possible because the choice of function space. □

Proposition 8.3. *Similarly, any solution $\varphi \in \mathcal{C}^{2,\alpha}_{\mu,0}(\mathbb{R}^n \setminus \mathbb{R}^p)$ of*

$$\mathcal{L}^{(\infty)}\varphi = 0$$

must vanish identically.

Proof. It is the same proof as in Proposition 8.2, using the asymptotics of the Bessel functions, but with the new indicial roots. First, for each j, there are two solutions. However, the one that is not exponentially growing as $r \to \infty$ is not in the kernel thanks to condition (4.9). □

8.2. **Beginning of the proof of Proposition 8.1.** The first observation is that, since U_ϵ only depends on the radial variable, the coefficients $\mathcal{A}^1_\ell$, $\ell = 0, 1, 2, 3, 4$ only depend on r as well, so one can project over spherical harmonics and show injectivity for each operator

$$\mathcal{L}_{1,j}\varphi = 0 \quad \varphi = \varphi(r, z),\ r > 0, z \in \mathbb{R}^p,$$

for $j = 0, 1, \ldots$. Here

$$\mathcal{L}_{1,j}\varphi = \mathcal{A}^1_0(r)\varphi + \frac{\mathcal{A}^1_1(r)}{r}\partial_r\varphi + A^1_2(r)\partial_{tt}\varphi - \lambda_j\frac{\mathcal{A}^1_3(r)}{r^2}\varphi + \mathcal{A}^1_4(r)\Delta_z\varphi. \tag{8.5}$$

Next, as we did in Section 7, Fourier transform in the variable z reduces to the problem to study the operator

$$J_j\omega := \mathcal{A}^1_0(r)\omega + \frac{\mathcal{A}^1_1(r)}{r}\partial_r\omega + A^1_2(r)\partial_{rr}\omega - \lambda_j\frac{\mathcal{A}^1_3(r)}{r^2} - \mathcal{A}^1_4(r)|\zeta|^2\omega = 0.$$

This is an ODE in the variable $r \in \mathbb{R}$ for each fixed ζ. Indicial roots for this problem were given in Proposition 4.3. We will consider the different values of j in the following paragraphs.

The first observation is that, for $j = 0$, our choice of weight $\mu > \mathrm{Re}(\chi^{(0)}_{0,+})$ prevents having any solution in the kernel with behavior $O(r^\mu)$ as $r \to 0$.

Next, in Sections 8.3 and 8.4 we try to understand the effect of symmetries of the equation. It will be useful to consider the reduced operator (when $p = 0$ so there is no variable z) and $\epsilon = 1$, which is given by

$$\mathfrak{L}_1\varphi = \mathcal{A}_0^1(r)\varphi + \frac{\mathcal{A}_1^1(r)}{r}\partial_r\varphi + \mathcal{A}_2^1(r)\partial_{rr}\varphi + \frac{\mathcal{A}_3^1(r)}{r^2}\Delta_\theta\varphi,$$

and its spherical harmonic projection

$$\mathfrak{L}_{1,j}\varphi = \mathcal{A}_0^1(r)\varphi + \frac{\mathcal{A}_1^1(r)}{r}\partial_r\varphi + \mathcal{A}_2^1(r)\partial_{rr}\varphi - \lambda_j\frac{\mathcal{A}_3^1(r)}{r^2}\varphi.$$

8.3. **Non-degeneracy in the radial direction.** Even if it is not needed in our discussion, let us take a detour to characterize the kernel of the operator $\mathfrak{L}_{1,0}$ (that is, for $j = 0$), given by

$$\mathfrak{L}_{1,0}[\varphi] = \mathcal{A}_0^1(r)\varphi + \frac{\mathcal{A}_1^1(r)}{r}\partial_r\varphi + \mathcal{A}_2^1(r)\partial_{rr}\varphi, \quad \varphi = \varphi(r),$$

and prove that it is non-degenerate. We start with an immediate observation:

Remark 8.4. The σ_k-equation is dilation invariant. This implies that the function

$$\varphi_\sharp := r\partial_r U_1 + \tfrac{n-4}{4}U_1$$

is a solution to the linear problem $\mathfrak{L}_{1,0}\varphi_\sharp = 0$.

We will show in the next Lemma that this is actually the only possible solution. For this, it is better to go back to the tilde-notation consider the conjugate operator $\widetilde{\mathfrak{L}}_{1,0}$ and the corresponding solution $\widetilde{w}_\sharp$.

Lemma 8.5. *Any other solution to*

$$\mathfrak{L}_{1,0}\widetilde{w} = 0 \tag{8.6}$$

that decays to zero as $t \to \infty$ must be a multiple of $\widetilde{w}_\sharp$.

Proof. Let $\widetilde{w}_1, \widetilde{w}_2$ be two solutions of (8.6) that decay to zero as $t \pm \infty$, that is, of the form $\widetilde{w}_i = \alpha_i(1 + o_i(1))e^{\varsigma_i t}$, $i = 1, 2$, as $t \to +\infty$, $\alpha_i \neq 0$. Define its Wronskian $W(t) := \omega_1'\omega_2 - \omega_1\omega_2'$

$$W(t) = \widetilde{w}_1'\widetilde{w}_2 - \widetilde{w}_1\widetilde{w}_2' = \alpha_1\alpha_2(\varsigma_1 - \varsigma_2 + o(1))e^{(\varsigma_1+\varsigma_2)t}.$$

Since the Wronskian is constant, then we must have $\varsigma_1 = \varsigma_2$. Let us assume, by rescaling, that $\alpha_1 = \alpha_2 = 1$. Now look at the next order. For this, we write $\omega_i = e^{\varsigma_i t} + \alpha_i^1(1 + o_i(1))e^{\varsigma_i^{(1)}t}$, $i = 1, 2$, as $t \to +\infty$, $\alpha_i^{(1)} \neq 0$. The same argument will yield that $\varsigma_1^{(1)} = \varsigma_2^{(1)}$. Inductively, we will be able to conclude that $\omega_1 \equiv \omega_2$, since we have analytic continuation for the solutions of this ODE. $\square$

8.4. **Rotational invariance implies non-degeneracy.** Now we use rotational invariance to show injectivity for the first non-zero mode, this is, for $\lambda_1 = \ldots = \lambda_N = N-1$ (recall the notation in (2.2)). Assume that $p=0$ for now and study $\mathfrak{L}_{1,j}$ for $j=1,\ldots,N$.

Remark 8.6. Notice first that rotational invariance yields that

$$\mathfrak{L}_1(\partial_j U_1) = 0, \quad j = 1,\ldots,N.$$

Since $\partial_j = e_j\partial_r$, then $\varphi_\diamond := \partial_r U_1$ belongs to the kernel of $\mathfrak{L}_{1,j}$ for each $j=1,\ldots,N$.

Recall that, in addition,

$$\varphi_\diamond(r) \asymp r^{-\frac{n-4}{4}-1} \quad \text{as} \quad r\to 0,$$

and

$$\varphi_\diamond(r) \sim r^{-\alpha_0-\frac{n-4}{4}-1} \quad \text{as} \quad r\to\infty.$$

By contradiction, assume that φ_j is a solution to $\mathfrak{L}_{1,j}\varphi_j = 0$ in the space $\mathcal{C}^{2,\alpha}_{\mu,0}$. Looking at (4.11), one knows that it behaves like

$$\varphi_j(r) \sim r^{-\alpha_0-\frac{n-4}{4}-1} \quad \text{as} \quad r\to\infty.$$

We also have, by our choice of μ in (8.1), that

$$\varphi_j \asymp r^{\chi^{(0)}_{j+}} \quad \text{as} \quad r\to 0.$$

Then there is a (non-trivial) linear combination of $\varphi_\diamond$ and φ_j that decays faster than $r^{-\alpha_0-\frac{n-4}{4}-1}$ as $r\to\infty$. Looking at the different behaviors as $r\to 0$, we see that this combination is non-vanishing. Looking at the indicial roots $\chi^{(\infty)}_{j\pm}$, this is a contradiction since no (non-trivial) solutions can decay faster at $r\to\infty$.

Now, to pass from $\mathfrak{L}_{1,j}$ to $\mathcal{L}_{1,j}$, $j=1,\ldots,N$ we need to take Fourier transform in z and use the same continuity argument as in [33, Proposition 4], considering the Fourier variable $|\zeta|^2$ as a parameter.

8.5. **The higher modes.** To complete the proof of Proposition 8.1 it remains to study the case $j>N$.

We consider first the eigenvalue problem for $\mathcal{L}_{1,0}\varphi = \eta\varphi$. Although we know that 0 is an eigenvalue, we have no information on its Morse index. Let η_0 be the first eigenvalue. It is well known (thanks to self-adjointness with respect to the scalar product (4.13)), that its corresponding eigenfunction φ_0 is strictly positive.

Now, let φ_j be a solution to $\mathcal{L}_{1,j}\varphi_j = 0$, for $j > N$, which can be written as $\mathcal{L}_{1,0}\varphi_j = \lambda_j\varphi_j$ after we have taken Fourier transform in the variable z. Combining this equation with $\mathcal{L}_{1,0}\varphi_0 = \eta_0\varphi_0$ we arrive to

$$(\lambda_j - \eta_0)\varphi_j\varphi_0 = \varphi_0\mathcal{L}_{1,0}\varphi_j - \varphi_j\mathcal{L}_{1,0}\varphi_0.$$

Integrate this expression in the set where $\{\varphi_j > 0\}$ with respect to the weighted L^2 space from (4.13), and use the divergence theorem to obtain that

$$(\lambda_j - \eta_0)\int_{\{\varphi_j>0\}} \varphi_j\varphi_0\mathcal{H}^1\,dr = \int_{\{\varphi_j=0\}} \mathcal{H}^1\left\{\varphi_0\partial_{\vec{\nu}}\varphi_j - \varphi_j\partial_{\vec{\nu}}\varphi_0\right\}\,ds.$$

Here $\vec{\nu}$ is the exterior normal to the integration set. It is easy to check that $\partial_{\vec{\nu}}\varphi_j < 0$. Then, from the above formula we reach a contradiction unless

$$\int_{\{\varphi_j>0\}} \varphi_j\varphi_0\mathcal{H}^1\,dr = 0.$$

In particular, this implies that $\varphi_j \equiv 0$, as desired.

9. Injectivity of $\mathbb{L}_\epsilon$

Let μ be a weight as in (8.1). We rely on the results of the previous Section to show injectivity in weighted Hölder spaces for small ϵ.

Looking back at (2.11) and Proposition 4.4, we know that

$$\mathbb{L}_\epsilon : \mathcal{C}^{2,\alpha}_\mu(\Omega\setminus\Lambda) \to \mathcal{C}^{0,\alpha}_{\mu-2}(\Omega\setminus\Lambda).$$

Proposition 9.1. *There exists ϵ_0 such that for all $0 < \epsilon < \epsilon_0$, the operator $\mathbb{L}_\epsilon$ is injective in $\mathcal{C}^{2,\alpha}_\mu(\Omega\setminus\Lambda)$.*

The idea is to use a contradiction argument as $\epsilon \to 0$ which, after rescaling, reduces the problem to the model operator $\mathcal{L}_1$. This is a rather standard argument by now (see [33] or [15, Proposiion 3.1] for the scalar curvature case). Thus, assume that there exists a sequence $\{\epsilon_l\}$, $\epsilon_l \to 0$ such that $\mathbb{L}_l := \mathbb{L}_{\epsilon_l}$ is not injective, i.e., there exists $\varphi_l \in \mathcal{C}^{2,\alpha}_\mu$ with $\mathbb{L}_l\varphi_l = 0$. Rescaling, we can always assume that

$$\|\varphi_l\|_{\mathcal{C}^0_\mu} = 1.$$

Then there exists $y_l \in \Omega\setminus\Lambda$ such that

$$1 \geq d_l^{-\mu}\varphi_l(y_l) > \tfrac{1}{2}, \tag{9.1}$$

where $d_l := d(y_l)$ and d is the function defined in Definition 6.2. Now, since Ω is taken to be a compact manifold, up to a subsequence we can find $y_0 \in \Omega$ such that $y_l \to y_0$.

We will need a preliminary convergence Lemma:

Lemma 9.2. *Consider $\mu > \frac{p}{2} - \frac{n-4}{4}$ not an indicial root. Given a sequence $\{\varphi_l\}$ in $\mathcal{C}^{2,\alpha}_\mu$, if $\|\varphi_l\|_{\mathcal{C}^{2,\alpha}_\mu} \leq C$, then it has a convergent subsequence. This is still true if we only have $\|\varphi_l\|_{\mathcal{C}^{\alpha}_\mu} \leq C$ but each φ_l is a solution of the homogeneous equation $\mathbb{L}_l\varphi = 0$.*

Proof. We use elliptic estimates and Ascoli's theorem in compact sets. Note that, if we have convergence of a subsequence in a compact set, the estimate (7.2) will convergence in weighted L^2-spaces. Elliptic estimates again will yield the desired conclusion. □

There are several possibilities according to the position of y_0:

Case 1: we first assume that $y_0 \notin \Lambda$. Note that y_0 could be the point at infinity. However, this point is not distinguished in Ω as we have pointed out in Remarks 3.4 and 6.1.

For ϵ_l small enough, the operator $\mathbb{L}_\epsilon$ coincides with the operator $\mathbb{L}_{g_\dagger}$ from Remark 4.1. By Lemma 9.2, the sequence $\{\varphi_l\}$ converges (up to passing to a subsequence) to some φ_0 in $\Omega \setminus \Lambda$ satisfying

$$\mathbb{L}_{g_\dagger}\varphi_0 = 0 \quad \text{in } \Omega \setminus \Lambda. \tag{9.2}$$

Then, since Λ has Hausdorff dimension $p < n-2$ and $u = O(d^\mu)$, we can apply classical removability of singularity results for quasilinear equations (see, for instance, Chapter 3.1.3 in [30], or the original [44]) to conclude that φ_0 can be extended to a weak solution of $\mathbb{L}_{g_\dagger}\varphi_0 = 0$ in the whole Ω. Then we must have $\varphi_0 \equiv 0$ by our non-degeneracy hypothesis on g_1. This yields a contradiction with (9.1).

Case 2: Assume now that $y_0 \in \Lambda$. Let $r_l = \mathrm{dist}(x_l, \Lambda)$ to be the radial Fermi coordinate, and rescale

$$\hat{\varphi}_l(y) = \epsilon_l^{-\mu}\varphi_l(y_l + \epsilon_l y).$$

Then Lemma 9.2 implies that, up to a subsequence, $\hat{\varphi}_l$ converges to $\hat{\varphi}_0 \in \mathcal{C}^0_{\mu,0}(\mathbb{R}^n \backslash \mathbb{R}^p)$, a solution of $L\hat{\varphi}_0 = 0$ in $\mathbb{R}^n \backslash \mathbb{R}^p$ for some linear operator L.

There are three possibilities for L according to the behavior of ϵ_l/r_l, since the approximate solution $\bar{u}_\epsilon$ has a different behavior in each regime (recall Corollary 2.2):

First, if $\epsilon_l/r_l \to +\infty$, then we are in the situation (2.15), and L reduces to be the operator $\mathcal{L}^{(0)}$ from (4.5) in $\mathbb{R}^n \setminus \mathbb{R}^p$. Because of Proposition 8.2, $\hat{\varphi}_0$ must be identically zero. Contradiction again with (9.1).

Second, assume $\epsilon_l/r_l \to C \neq 0$. Without loss of generality, we can take $C = 1$, otherwise rescale. Then, taking into account the scaling

(2.14), L coincides with the operator $\mathcal{L}_1$ in expression (8.2). We can use Proposition 8.1 to reach a contradiction as above.

Finally, if $\epsilon_l/r_l \to 0$, the operator L is $\mathcal{L}^{(\infty)}$ from (4.6) defined in $\mathbb{R}^n \setminus \mathbb{R}^p$, and we argue as in the previous cases, using Proposition 8.3 to conclude. This completes the proof of Proposition 9.1. $\square$

Remark 9.3. Proposition 9.1 could be rewritten line by line with the operator $\widetilde{\mathbb{L}}_\epsilon$ replaced by $(\widetilde{\mathbb{L}}_\epsilon)^*$. Indeed, the main ingredient is the relation (6.4) and the characterization of $\mathbb{L}_\epsilon$ in terms of $\mathcal{L}_\epsilon$ from Proposition 4.4.

For injectivity in Lebesgue spaces, first set

$$\delta = \mu - \tfrac{p}{2} + \tfrac{n-4}{4}. \tag{9.3}$$

Then:

Corollary 9.4. *For every $\epsilon \in (0, \epsilon_0)$, $(\widetilde{\mathbb{L}}_\epsilon)^*$ is injective in $L^2_\delta(\Omega \setminus \Lambda)$, and thus, $\widetilde{\mathbb{L}}_\epsilon$ is surjective in $L^2_{-\delta}(\Omega \setminus \Lambda)$.*

Proof. First use Remark 9.3 to obtain injectivity in Hölder spaces. Then, classical regularity estimates allow to pass from Lebesgue spaces to Hölder spaces. This implies, in particular, that injectivity holds in weighted L^2_δ spaces if we choose a parameter by (9.3).

For the second assertion, simply recall the relation (6.5). $\square$

Finally, we will prove an auxiliary result that will be needed in the next Section:

Lemma 9.5. *Assume μ satisfies* (8.1). *There exists $\epsilon_0 > 0$ and $C > 0$ such that, for every $\epsilon \in (0, \epsilon_0)$, if $\psi \in \mathcal{C}^{2,\alpha}_\mu(\Omega \setminus \Lambda)$ is a solution to*

$$(\mathbb{L}_\epsilon)^* \psi = h$$

for $h \in \mathcal{C}^{0,\alpha}_{\mu-2}(\Omega \setminus \Lambda)$, then

$$\|\psi\|_{\mathcal{C}^{2,\alpha}_\mu} \le C \|h\|_{\mathcal{C}^{0,\alpha}_{\mu-2}}. \tag{9.4}$$

Proof. This a contradiction argument very similar to the proof of Proposition 9.1(see also Remark 9.3) and thus, we omit it. $\square$

10. Uniform surjectivity of $\mathbb{L}_\epsilon$

We have just shown that for each fixed ϵ, the operator

$$\widetilde{\mathbb{L}}_\epsilon : L^2_{-\delta}(\Omega \setminus \Lambda) \to L^2_{-\delta}(\Omega \setminus \Lambda)$$

is surjective. Now we would like to construct a right inverse for $\mathbb{L}_\epsilon$. For that we set

$$\widetilde{\mathbb{L}}_\epsilon := \widetilde{\mathbb{L}}_\epsilon \circ d^{2\delta} \circ \widetilde{\mathbb{L}}_\epsilon^* : L^2_{-\delta}(\Omega \setminus \Lambda) \to L^2_{-\delta}(\Omega \setminus \Lambda).$$

This $\widetilde{\mathbb{L}}_\epsilon$ is an isomorphism (with suitable boundary conditions (or growth at infinity) and thus, it has a bounded two-sided inverse

$$\widetilde{\mathbb{G}}_\epsilon : L^2_{-\delta}(\Omega \setminus \Lambda) \to L^2_{-\delta}(\Omega \setminus \Lambda).$$

In particular, $\widetilde{\mathbb{L}}_\epsilon \circ \widetilde{\mathbb{G}}_\epsilon = I$, which means that

$$\widetilde{\mathbb{G}}_\epsilon := d^{-2\delta}(\widetilde{\mathbb{L}}_\epsilon)^* d^{2\delta} \widetilde{\mathbb{G}}_\epsilon : L^2_{-\delta}(\Omega \setminus \Lambda) \to L^2_{-\delta}(\Omega \setminus \Lambda) \tag{10.1}$$

is a bounded right inverse for $\widetilde{\mathbb{L}}_\epsilon$ that maps into the range of $d^{-2\delta}(\widetilde{\mathbb{L}}_\epsilon)^*$.

In particular, an analogous result is true for $\mathbb{G}_\epsilon$ with the usual shift of indexes. Now we choose

$$\nu < \tfrac{p}{2} - \tfrac{n-4}{4} \quad \text{slightly larger than } \nu' := -\delta + \tfrac{p}{2} - \tfrac{n-4}{4} \tag{10.2}$$

and restrict this inverse to the smaller set $\mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Lambda)$. Then

$$\mathbb{G}_\epsilon : \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Lambda) \to L^2_{\nu'}(\Omega \setminus \Lambda).$$

Let us improve the regularity of this inverse:

Proposition 10.1. *If $\varphi \in L^2_{\nu'}(\Omega \setminus \Lambda)$ is a solution of $\mathbb{L}_\epsilon \varphi = h$ for $h \in \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Lambda)$, then we have that $\varphi \in \mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Lambda)$ for ν close enough to ν', $\nu' < \nu$.*

Proof. Rescaled Schauder estimates immediately imply that the solution $\varphi \in \mathcal{C}^{2,\alpha}_{\nu'}(\Omega \setminus \Lambda)$. However, the main statement in this Proposition is an improvement of weight from ν' to ν. This fact follows from the work of Mazzeo [31, Theorem 7.14], where they show that a change in the asymptotics of φ is created only by the crossing of an indicial root, and this cannot happen if ν and ν' are close enough. □

In summary, there exists a (bounded) right inverse for

$$\mathbb{L}_\epsilon : \mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Lambda) \to \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Lambda),$$

given by

$$\mathbb{G}_\epsilon : \mathcal{C}^{0,\alpha}_{\nu-2}(\Omega \setminus \Lambda) \to \mathcal{C}^{2,\alpha}_{\nu}(\Omega \setminus \Lambda).$$

The main result in this Section is the following:

Proposition 10.2. *There exists $\epsilon_0 > 0$ such that, for all $\epsilon \in (0, \epsilon_0)$, the norm of $\mathbb{G}_\epsilon$ does not depend on ϵ.*

Proof. The proof is very similar to [33, Theorem 6] and [15, Proposition 4.4]. As in the proof of Proposition 9.1, we argue by contradiction. Assume that there exist sequences $\{\epsilon_l\}$, $\{h_l\}$, $\{\varphi_l\}$ satisfying $\varphi_l = \mathbb{G}_{\epsilon_l} h_l$ and

$$\sup_{\Omega\setminus\Lambda}\{d^{-\nu}|h_l|\} = 1 \quad \text{but} \quad \sup_{\Omega\setminus\Lambda}\{d^{-\nu}|\varphi_l|\} =: m_l \to \infty$$

as $\epsilon_l \to 0$.

We denote $\mathbb{L}_l := \mathbb{L}_{\epsilon_l}, \mathbb{G}_l := \mathbb{G}_{\epsilon_l}$ and recall that, by the duality relation (10.1), $\varphi_l = d^{2\nu}(\mathbb{L}_\epsilon)^*\psi_l$ for $\psi_l \in \mathcal{C}^{2,\alpha}_{\mu'}$ with μ' very close to μ. Now rescale

$$\hat{\varphi}_l := \frac{\varphi_l}{m_l}, \quad \hat{h}_l := \frac{h_l}{m_l}, \quad \hat{\psi}_l := \frac{\psi_l}{m_l}.$$

Choose a point $y_l \in \Omega \setminus \Lambda$ where

$$\tfrac{1}{2} \leq d(y_l)^{-\nu}\hat{\varphi}_l(y_l) \leq 1.$$

By compactness, we can show that, up to a subsequence, $y_l \to y_0$ for some $y_0 \in \Omega$. There are two cases depending on the location of y_0:

Case 1: Suppose that $y_0 \in \Omega \setminus \Lambda$. Arguing as in Case 1 in the proof of Proposition 9.1 we reach a contradiction using non-degeneracy.

Case 2: Now assume that $y_0 \in \Lambda$, and let $r_l := \mathrm{dist}(y_l, \Lambda)$ to be the radial Fermi coordinate, and rescale

$$\check{\varphi}_l(y) := r_l^{-\nu}\hat{\varphi}_l(y_l + r_l y),$$

and similarly with the other functions. Since $\|\check{\varphi}_l\|_{\mathcal{C}^{2,\alpha}_\nu} \leq C$, we can find a convergent subsequence to a function $\check{\varphi}_0$ satisfying

$$L\check{\varphi}_0 = 0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}^p \tag{10.3}$$

for some operator L that we will study below. One may also check that $\check{\psi}_l$ converges to a function $\check{\psi}_0$, thanks to the bounds in Lemma 9.5. Moreover, this limit satisfies

$$\check{\varphi}_0 = L^*(\check{\psi}_0), \tag{10.4}$$

Take Fourier transform of equations (10.3) and (10.4) in the variable z, denoting the Fourier variable by ζ, and the transformed operator by K_ζ. Then the both equations reduce to

$$\begin{aligned} &K_\zeta\omega_0 = 0, \\ &\omega_0 = K_\zeta^*\varpi_0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}^p. \end{aligned}$$

Hence $0 = K_\zeta(\omega_0) = K_\zeta K_\zeta^*(\varpi_0)$. Multiply this equation by ϖ_0 and integrate by parts to obtain, for each fixed ζ,

$$\int |K_\zeta^* \varpi_0|^2 r^2 \, dr = 0.$$

This implies that $\varpi_0 \equiv 0$, which yields a contradiction.

The linear operator L (and its Fourier version K) will depend on the behavior of the quotient ϵ_l / r_l. First, if $\epsilon_l / r_l \to \infty$, we have that $L = \mathcal{L}^{(0)}$ because of (2.15). Second, assume that $\epsilon_l / r_l \to 1$; then $L = \mathcal{L}_1$ defined in $\mathbb{R}^n \backslash \mathbb{R}^p$. Finally, if $\epsilon_l / r_l \to 0$, it holds that $L = \mathcal{L}^{(\infty)}$. But in all these cases the above argument works. $\square$

Remark 10.3. In addition to (10.2), we will impose further restrictions on ν. Indeed, we need to ask that ν also satisfies

$$-\tfrac{n-4}{4} < \nu < \min\{-\tfrac{n-4}{4} + 1, -\operatorname{Re}(\chi_{0,-}^{(0)})\}. \tag{10.5}$$

Recalling the values of μ and δ in (8.1) and (9.3), respectively, it is clear that these are compatible choices. A summary can be found in Figure 1.

11. Nonlinear analysis

Now we go back to our original equation (1.13). We restrict to $k = 2$ and set $\bar{u}_\epsilon$ to be the approximate solution constructed in Section 3.3.

For this part, it is better to switch back again to the v-notation in cylindrical coordinates and consider the equation

$$\mathcal{N}(v, g_{\mathbb{R}^n, cyl}) := v^{-2k+1} \sigma_k \left(B_{g_v} \right) - c v^{q-2k+1} = 0, \tag{11.1}$$

which is equivalent to

$$\bar{L}_\epsilon[\varphi] + \bar{f}_\epsilon + \bar{Q}_\epsilon[\varphi] = 0,$$

where we have defined

$$\begin{aligned} &\bar{f}_\epsilon := \mathcal{N}(\bar{v}_\epsilon, g_{\mathbb{R}^n, cyl}), \\ &\bar{Q}_\epsilon[w] := \mathcal{N}(\bar{v}_\epsilon + w, g_{\mathbb{R}^n, cyl}) - \mathcal{N}(\bar{v}_\epsilon, g_{\mathbb{R}^n, cyl}) - \bar{L}_\epsilon[w]. \end{aligned}$$

We fix

$$\bar{\nu} = \nu + \tfrac{n-4}{4}.$$

From the discussion in the previous Section we have that the operator

$$\bar{L}_\epsilon : \mathcal{C}^{2,\alpha}_{\bar{\nu}}(\Omega \setminus \Lambda) \to \mathcal{C}^{0,\alpha}_{\bar{\nu}}(\Omega \setminus \Lambda),$$

has a right inverse

$$\bar{G}_\epsilon : \mathcal{C}^{0,\alpha}_{\bar{\nu}}(\Omega \setminus \Lambda) \to \mathcal{C}^{2,\alpha}_{\bar{\nu}}(\Omega \setminus \Lambda)$$

with norm independent of ϵ (see Proposition 10.2). Now we can define the operator

$$\bar{T}_\epsilon(w) := -\bar{G}_\epsilon(\bar{Q}_\epsilon[w] + \bar{f}_\epsilon). \tag{11.2}$$

A fixed point of $\bar{T}_\epsilon$ will yield a solution to equation (11.1).

Our first claim is that, for the error term $\bar{f}_\epsilon$ there exists a uniform constant C such that for all $\epsilon \in (0, \epsilon_0)$ it holds

$$\|\bar{f}_\epsilon\|_{\mathcal{C}^{0,\alpha}_{\bar{\nu}}} \leq C\epsilon^{\beta_1}$$

as long as $\bar{\nu} < \beta_0$. This follows from Proposition 3.5, since a $\mathcal{C}^\alpha$ estimate follows from a (weighted) L^∞ bound of $\bar{f}_\epsilon$. Note that this imposes a further restriction $\nu < \beta_0 - \frac{n-4}{4}$, which is still compatible with (10.5).

Now we give an estimate for the quadratic term. Let m be the uniform bound for $\|\bar{G}_\epsilon\|$. Then

$$\|\bar{G}_\epsilon \bar{f}_\epsilon\|_{\mathcal{C}^{2,\alpha}_{\bar{\nu}}} \leq mC\epsilon_0^{\beta_1}.$$

Define the set

$$\mathcal{B}(\epsilon_0, \sigma) := \{w \in \mathcal{C}^{2,\alpha}_{\bar{\nu}}(\Omega \setminus \Lambda) \,:\, \|w\|_{\mathcal{C}^{2,\alpha}_{\bar{\nu}}} \leq \sigma\epsilon_0^{\beta_1}\},$$

and choose σ large enough so that $\bar{G}_\epsilon \bar{f}_\epsilon \in \mathcal{B}(\epsilon_0, \sigma/2)$.

Lemma 11.1. *For ϵ_0 small enough, we have*

$$\|\bar{Q}(w_2) - \bar{Q}(w_1)\|_{\mathcal{C}^{0,\alpha}_{\bar{\nu}}} \leq \frac{1}{2m}\|w_2 - w_1\|_{\mathcal{C}^{2,\alpha}_{\bar{\nu}}}$$

for all $w_1, w_2 \in \mathcal{B}(\epsilon_0, \sigma)$, $\epsilon \in (0, \epsilon_0)$.

Proof. This is just Lemma 5.2 in [15] or Lemma 9 in [33]. □

Recalling the definition of the operator $\bar{T}_\epsilon$ from (11.2), we have just seen that is a contraction on $\mathcal{B}(\epsilon_0, \sigma)$ for ϵ_0 small enough. This yields a solution w to (11.1), as desired.

Now we can go back to the u-notation. We have just produced a solution $\bar{u}_\epsilon + \varphi$ that yields a complete metric near Λ. Indeed, one the one hand, $\bar{u}_\epsilon$ behaves like $v_\infty r^{-\frac{n-4}{4}}$ as $r \to 0$ (see (2.15)). On the other hand, thanks to the choice $\nu > -\frac{n-4}{4}$ from (10.5), $\varphi = O(r^\nu)$ has a less singular behavior which is not seen near Λ.

In order to complete the proof of Theorem 1.1, we need to prove that $\bar{u}_\epsilon + \varphi$ is positive. First, looking at its asymptotic behavior near Λ, this is the case for points near singular set. Away from Λ we have a uniformly elliptic semi-linear equation (recall that our initial metric

$g_\dagger$ was already in the positive cone). Then, positivity holds from the application of the maximum principle in the positive cone ([29]).

Financial support. M.d.M. González acknowledges financial support from the Spanish Government, grant numbers MTM2017-85757-P, PID2020-113596GB-I00; additionally, Grant RED2018-102650-T funded by MCIN/AEI/ 10.13039/501100011033, and the "Severo Ochoa Programme for Centers of Excellence in R&D" (CEX2019-000904-S).

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