FRACTIONAL POWERS OF THE WAVE OPERATOR VIA DIRICHLET-TO-NEUMANN MAPS IN ANTI-DE SITTER SPACES

ALBERTO ENCISO, MARÍA DEL MAR GONZÁLEZ AND BRUNO VERGARA

ABSTRACT. We show that the fractional wave operator, which is usually studied in the context of hypersingular integrals but had not yet appeared in mathematical physics, can be constructed as the Dirichlet-to-Neumann map associated with the Klein–Gordon equation in anti-de Sitter spacetimes. Several generalizations of this relation will be discussed too.

1. INTRODUCTION

It is a classical result in potential theory that the Dirichlet-to-Neumann map of the harmonic extension problem in the upper half-space is given by the square root of the Laplacian. This relation can be generalized to encompass all fractional powers of the Laplacian, and has recently made a major impact in the theory of nonlocal elliptic equations [3].

Indeed, for any $\alpha \in (0,1)$ this relation connects the multiplier

$$(1.1) \qquad [(-\Delta)^\alpha f]\hat{}(\xi) := |\xi|^{2\alpha} \widehat{f}(\xi)$$

on $\mathbb{R}^n$, which is a nonlocal operator of the form

$$(1.2) \qquad (-\Delta)^\alpha f(x) = c_{n,\alpha} \int_{\mathbb{R}^n} \frac{f(x) - f(x')}{|x - x'|^{n+2\alpha}} \, dx' \,,$$

with a local elliptic equation in $n + 1$ variables. Specifically, given a function f on $\mathbb{R}^n$ let us consider the function u on $\mathbb{R}^n \times \mathbb{R}_+$ that solves the boundary value problem

$$(1.3) \qquad \Delta_x u + \partial_{yy} u + \frac{1 - 2\alpha}{y} \partial_y u = 0 \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \,,$$
$$u(x, 0) = f(x) \,, \qquad \lim_{y \to \infty} f(x, y) = 0 \,;$$

it was shown in [3] that

$$(-\Delta)^\alpha f(x) = c_\alpha \lim_{y \to 0^+} y^{1-2\alpha} \partial_y u(x, y) \,.$$

This reduces to the ordinary derivative $\partial_y u(x, 0)$ in the case of the square root of the Laplacian ($\alpha = \frac{1}{2}$), and in fact for all values of α it provides the natural Neumann datum associated with the elliptic operator (1.3). Note that a generalized formula for higher powers $\alpha \in (0, \frac{n}{2})$ has been established in [4, 5].

Our first objective in this paper is to derive an analogous relation for fractional wave operators, by which we will always mean a fractional power of the usual wave operator $\Box := \partial_{tt} - \Delta$ and not an evolution equation driven by a fractional power of the Laplacian or, more generally, a generator of a suitable semigroup

(for fine information on the latter in various contexts, cf. [11, 19, 20]). Note that this cannot be seen as an analytic continuation of the elliptic case and that in fact several nontrivial choices need to be made, starting with the very definition of the fractional wave operator. This is due to the fact that the symbol of the wave operator, $|\xi|^2 - \tau^2$, is not positive definite, so one cannot immediately define $\Box^\alpha$ through this quantity to the power of α as in (1.1), and can also be seen in the integral formula (1.2), since formally replacing the squared Euclidean distance $|x - x'|^2$ by its Minkowskian counterpart

$$|x - x'|^2 - (t - t')^2$$

in the denominator leads to an integral that is too singular to be well defined. The analytic difficulties associated to dealing with fractional powers of hyperbolic operators will become apparent already in Section 2.

There is a considerable body of classical work on fractional wave operators, which have never appeared naturally in a physical problem but which are studied in detail in the theory of hypersingular integrals (see e.g. [18] and references therein). In particular, the fractional wave operator $\Box^\alpha$ is defined for all noninteger α as the multiplier

$$(1.4) \qquad \widehat{\Box^\alpha f}(\tau, \xi) := \sigma_\alpha(\tau, \xi)\, \widehat{f}(\tau, \xi)\,,$$

where the symbol σ_α is defined as

$$(1.5) \qquad \sigma_\alpha(\tau, \xi) := (|\xi|^2 - \tau^2)^\alpha \chi_+(\tau, \xi) + e^{i\pi\alpha\,\mathrm{sgn}(\tau)}(\tau^2 - |\xi|^2)^\alpha \chi_-(\tau, \xi),$$

with $\chi_\pm(\tau, \xi)$ denotes the indicator function of the set $\pm(|\xi|^2 - \tau^2) > 0$. Equivalently, choosing the principal branch of the complex logarithm one can write

$$(1.6) \qquad \sigma_\alpha(\tau, \xi) := \lim_{\epsilon \to 0^+} \left(|\xi|^2 - (\tau - i\epsilon)^2\right)^\alpha.$$

It should be noted that this is in fact a natural definition of $\Box^\alpha$ in the sense that $\sigma_\alpha(\tau, \xi)$ raised to the power of $1/\alpha$ gives $|\xi|^2 - \tau^2$, i.e., the symbol of the wave operator.

Our second objective in this paper is of a more geometric and physical nature. Specifically, we will show that the fractional wave operator, as considered in the theory of hypersingular integrals, does arise in gravitational physics as the Dirichlet-to-Neumann map of the Klein–Gordon equation (1.8) in anti-de Sitter spaces. Hyperbolic equations for fields in anti-de Sitter backgrounds with nontrivial data on their conformal boundaries have attracted much attention over the last two decades, especially in connection with the celebrated AdS/CFT correspondence in string theory [15, 25]. Indeed, this conjectural relation establishes a connection between conformal field theories in n dimensions and gravity fields on an $(n + 1)$-dimensional spacetime of anti-de Sitter type, to the effect that correlation functions in conformal field theory are given by the asymptotic behavior at infinity of the supergravity action. Mathematically, this involves describing the solution to the gravitational field equations in $(n + 1)$ dimensions (which, in the simplest case of a scalar field reduces to the Klein–Gordon equation) in terms of a conformal field, which plays the role of the boundary data imposed on the (timelike) conformal boundary.

To present the specifics of this connection in the simplest case, let us start by recalling the basic structure of the anti-de Sitter (AdS) of dimension $n + 1$, which is

a Lorentzian manifold of negative constant sectional curvature (which we set here to -1) and, as such, satisfies the Einstein equations with a negative cosmological constant. For concreteness, we will consider the AdS half-space (see e.g. [1] for a mathematical analysis of the problem and [2] for the physics of this space). The metric can be written using Poincaré coordinates $(t, x, y) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}_+$ as

$$(1.7) \qquad g^+ := \frac{dt^2 - dy^2 - |dx|^2}{y^2},$$

where $|dx|^2$ denotes the standard flat metric on $\mathbb{R}^{n-1}$.

Analytically, here one can take boundary conditions on the set $y = 0$ (which is conformal to the n-dimensional Minkowski space) and decay conditions at $y = \infty$. This makes it the obvious generalization of the half-space model of the $(n + 1)$-dimensional hyperbolic space, which one describes in terms of the Poincaré coordinates $(y, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ through the metric

$$g^+_{\mathbb{H}} := \frac{dy^2 + |dx|^2}{y^2},$$

which is the natural setting for (1.3) after a conformal change.

While we prefer to stick to the AdS half-space in this Introduction, let us mention that in Section 4 we will also consider in detail the problem for the usual AdS space, whose conformal timelike infinity is the cylinder $\mathbb{R} \times \mathbb{S}^{n-1}$. This is sometimes referred to as the global AdS space in the context of the AdS/CFT conjecture and can be covered by two half-space AdS charts. The results are qualitatively the same but the algebra is less transparent. We will also encounter the same behavior when we analyze more general stationary asymptotically anti-de Sitter metrics, again in Section 4.

Let us then consider the Klein–Gordon equation with parameter μ in the AdS space (1.7),

$$(1.8) \qquad \Box_{g^+}\phi + \mu\phi = 0\,,$$

where $\Box_{g^+}$ is the wave operator associated with the AdS metric:

$$\Box_{g^+}\phi := y^2(\partial_{tt}\phi - \Delta_x\phi - \partial_{yy}\phi) - (1 - n)y\partial_y\phi.$$

Physically, μ is the mass of the particle modeled by the Klein–Gordon equation plus a negative contribution from the scalar curvature of the underlying space [22, Section 4.3]. For our purposes, it is convenient to assume that the parameter

$$\alpha := \left(\frac{n^2}{4} + \mu\right)^{1/2}$$

takes values in the interval $(0, \frac{n}{2})$.

We then have that Equation (1.8) can be rewritten as a wave equation with coefficients singular at $y = 0$,

$$(1.9) \qquad \partial_{tt}\phi - \Delta_x\phi - \partial_{yy}\phi - \frac{1 - n}{y}\partial_y\phi + \frac{4\alpha^2 - n^2}{4y^2}\phi = 0\,.$$

A simple look at the singularities of the equation reveals that the solutions are expected to scale at conformal infinity as $y^{\frac{n}{2} \pm \alpha}$. Then, since we are assuming $\alpha > 0$, the natural Dirichlet condition for this problem is

$$(1.10) \qquad \lim_{y \to 0^+} y^{\alpha - \frac{n}{2}}\phi(t, x, y) = f(t, x)\,.$$

If one prefers to prescribe a Neumann condition [23], one must instead impose

$$\lim_{y\to 0^+} y^{1-2\alpha}\,\partial_y\big(y^{\alpha-\frac{n}{2}}\phi(t,x,y)\big) = h(t,x)\,,$$

for $\alpha \in (0,1)$, and a generalized Neumann condition $\alpha \in (1,\frac{n}{2})$, as we will see below. With this notation in place, the main result of the paper is that the fractional wave operator $\Box^\alpha$ (in flat space) is the Dirichlet-to-Neumann map associated to switching on a nontrivial boundary datum in an anti-de Sitter space:

Theorem 1.1. *For any function $f \in C_0^\infty(\mathbb{R}^n)$, let ϕ be the solution of the Klein–Gordon equation* (1.8) *with this Dirichlet boundary condition (Eq.* (1.10)*) and trivial initial data at $-\infty$: $\phi(-\infty,x,y) = \phi_t(-\infty,x,y) = 0$. Assume moreover that the mass parameter α takes values in $(0,1)$. Then for any smooth enough function f one has the identity*

$$\Box^\alpha f(t,x) = c_\alpha \lim_{y\to 0^+} y^{1-2\alpha}\,\partial_y\big(y^{\alpha-\frac{n}{2}}\phi(t,x,y)\big)$$

for an explicit constant $c_\alpha = -2^{2\alpha-1}\Gamma(\alpha)/\Gamma(1-\alpha)$. More generally, if $\alpha \in (0,\frac{n}{2})$ is not an integer and we write $\alpha = m + \alpha_0$, with $m := \lfloor \alpha \rfloor$ the integer part, then

$$\Box^\alpha f(t,x) = c_\alpha \lim_{y\to 0^+} y^{2(1-\alpha_0)}\Big(\frac{1}{y}\partial_y\Big)^{m+1}\big(y^{\alpha-\frac{n}{2}}\phi(t,x,y)\big),$$

with $c_\alpha := (-1)^{m+1}2^{\alpha+\alpha_0-1}\dfrac{\Gamma(\alpha)}{\Gamma(1-\alpha_0)}$.

Several remarks are in order. First, an elementary observation is that this identity, which obviously applies to much for general data by the bounds that we establish in this paper, implies that the well-known estimates for the fractional wave operator immediately translate into assertions about the Dirichlet datum of the solution and its associated Neumann condition, and viceversa. Second, it is worth emphasizing that this relation can be generalized to more general asymptotically AdS metrics, as we will do it Section 4. A third comment is that Theorem 1.1 implies for $\alpha \in (1,\frac{n}{2})$ the operator $\Box^\alpha$ can be naturally interpreted as the scattering operator of the manifold. In Riemannian signature, this connection is discussed in detail in [4], and quite remarkably the interest in the the conformally covariant operators on the boundary that it defines (see e.g. [10, 16, 8]) was originally fueled by work of Newman, Penrose and LeBrun [14] on gravitational physics quite in the spirit of Maldacena's AdS/CFT correspondence.

Recall that, in the Lorentzian case, the construction of the scattering operator for a general asymptotically AdS metric was carried out in [21], but the resulting operator was not characterized.

The paper is organized as follows. In Section 2 we will recall some basic facts and definitions about fractional wave operators in a form that is particularly convenient for our purposes, and in particular, a convolution formula with a singular kernel. In Section 3 we will analyze the Klein–Gordon equation and establish the main result in the half-space region of anti-de Sitter space. This identity will be extended to more general asymptotically AdS spaces, and to the global AdS space, in Section 4.

2. Fractional powers of the wave operator

In this section we will recall some basic facts about fractional powers of the wave operator, as defined in (1.4). In the following we will see that, just as in the case

of the fractional Laplacian [13], one can represent $\Box^\alpha f$ as a regularized integral depending analytically on the power α when f is a sufficiently smooth function. To derive this result in full generality, we will need the analytic continuation of the classical hyperbolic Riesz potential [17], which will be explicitly constructed when the power is not a positive half-integer.

A careful analysis of the poles of the multiplier (1.5) shows that one can regard σ_α as a tempered distribution on $\mathbb{R}^n$, analytic in the parameter α for $\alpha \neq -\frac{n}{2} - k$ with k a non-negative integer (cf. e.g. [9, Chapter III]). This property is of crucial importance to recover the fractional wave operator as a convolution

$$\Box^\alpha f = k_\alpha * f,$$

where the kernel k_α coincides with the inverse Fourier transform of σ_α in a sense that will be specified later on. Notice that, while this relation holds true distributionally, it is not easy to transform it into a pointwise converging formula due to the singularities that the multiplier presents on the light cone.

To regularize the integrals that appear, we will use Riesz distributions. Let us recall that, for any complex parameter α with $\operatorname{Re}\alpha > \frac{n}{2} - 1$, R_α is the distribution whose action on a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ is given by the absolutely convergent integral

$$\langle R_\alpha, \varphi \rangle = C_{n,\alpha} \int_{\mathcal{K}_+^+} (t^2 - |x|^2)^{\alpha - \frac{n}{2}} \varphi(t,x)\, dt\, dx, \quad \varphi \in C_0^\infty(\mathcal{K}_+^+),$$

where we write the points in $\mathbb{R}^n$ as $(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1}$, the set

$$\mathcal{K}_+^+ := \{(t,x) \in \mathbb{R}^n : t^2 \geqslant |x|^2,\ t \geqslant 0\}$$

is the forward causal cone and the constant $C_{n,\alpha}$ takes the value

(2.1) $$C_{n,\alpha} := \frac{2^{1-2\alpha}\pi^{1-\frac{n}{2}}}{\Gamma(\alpha)\,\Gamma\!\left(\alpha + 1 - \frac{n}{2}\right)}.$$

A straightforward computation shows that the map $\alpha \mapsto \langle R_\alpha, \varphi \rangle$ is analytic in the half-plane $\operatorname{Re}\alpha > \frac{n}{2} - 1$. It is well-known (see e.g. [12]) that this mapping can be analytically continued to the whole complex plane by means of the identity $\Box R_{\alpha+1} = R_\alpha$. Hence for any complex number α it makes sense to consider the convolution of the distribution R_α a smooth compactly supported function $f \in C_0^\infty(\mathbb{R}^n)$, which we will denote by $I_\alpha f$ and call the *hyperbolic Riesz potential* of f. Notice that for $\operatorname{Re}\alpha > \frac{n}{2} - 1$ the Riesz potential simply reads as

(2.2) $$I_\alpha f(t,x) = \int_{\mathcal{K}_+^+} (s^2 - |y|^2)^{\alpha - \frac{n}{2}} f(t - s, x - y)\, ds\, dy.$$

In the following proposition we establish the relationship among the powers of the wave operator, the Riesz distribution and its associated potential, showing that, as in the Euclidean case, it is possible to understand $\Box^\alpha f$ as a regularized integral represented by the analytic extension of the Riesz potential. This relation was essentially stated in [18] using the results of [9] on the Fourier transform of analytically continued quadratic forms, but we prefer to sketch the proof here rather than to refer to vague variations on results from the above references.

Proposition 2.1. *Let α be a complex number such that $\alpha + \frac{n}{2}$ is not a non-positive integer. Then the Fourier transform of R_α is the function*

(2.3) $$\widehat{R}_\alpha(\tau,\xi) = \sigma_{-\alpha}(\tau,\xi),$$

so, in particular, for any function $f \in C_0^\infty(\mathbb{R}^n)$ the αth power of the wave operator is given by

$$(2.4) \qquad \qquad \Box^\alpha f = I_{-\alpha} f.$$

Proof. Let us first assume that $\operatorname{Re}\alpha > \frac{n}{2} - 1$. The Fourier transform $\widehat{R}_\alpha$ is then the distribution given by

$$\langle \widehat{R}_\alpha, \varphi \rangle = \langle R_\alpha, \widehat{\varphi} \rangle = C_{n,\alpha} \int_{\mathcal{K}_+^+} (t^2 - |x|^2)^{\alpha - \frac{n}{2}} \widehat{\varphi}(t, x) \, dt \, dx$$

$$= C_{n,\alpha} \lim_{\epsilon \to 0^+} \int_{\mathcal{K}_+^+} (t^2 - |x|^2)^{\alpha - \frac{n}{2}} e^{-\epsilon t} \widehat{\varphi}(t, x) \, dt \, dx$$

$$= C_{n,\alpha} \int_{\mathbb{R}^n} \lim_{\epsilon \to 0^+} \left(\int_{\mathcal{K}_+^+} (t^2 - |x|^2)^{\alpha - \frac{n}{2}} e^{-i\xi x} e^{-t(\epsilon + i\tau)} \, dt \, dx \right) \varphi(\tau, \xi) \, d\tau \, d\xi,$$

where we have used dominated convergence and Fubini's theorem.

The inner integral is computed (for $n > 3$; the case $n = 3$ is much simpler) by first passing to polar coordinates $(r, \phi_1, \phi_2, \ldots, \phi_{n-2})$ with $r > 0$, $\phi_j \in [0, \pi]$ for $j = 1, \ldots, n - 3$ and $\phi_{n-2} \in [0, 2\pi]$:

$$x_1 = r \cos(\phi_1), \qquad x_j = r \cos(\phi_j) \prod_{i=1}^{j-1} \sin(\phi_i), \qquad x_{n-2} = r \prod_{i=1}^{n-2} \sin(\phi_i),$$

with $dx = r^{n-2} \sin^{n-3}(\phi_1) \sin^{n-4}(\phi_2) \cdots \sin(\phi_{n-3}) \, dr \, d\phi_1 \, d\phi_2 \cdots d\phi_{n-2}$ the volume element. On the other hand, we also need the well-known representation formulas [24] for the Bessel functions

(2.5)

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu + \frac{1}{2})\sqrt{\pi}} \int_{-1}^1 e^{izs}(1 - s^2)^{\nu - \frac{1}{2}} \, ds, \qquad \operatorname{Re}(\nu) > -\tfrac{1}{2}, \; z \in \mathbb{C}$$

$$K_\nu(z) = \frac{\sqrt{\pi} z^\nu}{2^\nu \Gamma(\nu + \frac{1}{2})} \int_1^\infty e^{-zw}(w^2 - 1)^{\nu - \frac{1}{2}} \, dw, \qquad \operatorname{Re}(\nu) > -\tfrac{1}{2}, \; |\operatorname{Arg}(z)| < \tfrac{\pi}{2},$$

and the Bessel integral

$$\int_0^\infty r^{\mu + \nu + 1} K_\mu(ar) J_\nu(br) dr = \frac{(2a)^\mu (2b)^\nu \Gamma(\mu + \nu + 1)}{(a^2 + b^2)^{\mu + \nu + 1}},$$

for $\operatorname{Re}\nu + 1 > |\operatorname{Re}\mu|$ and $\operatorname{Re} a > \operatorname{Im} b$. Making then the change $s = \cos(\phi_1)$ and using the above integral formulas, it is not difficult to check that in the half-plane $\operatorname{Re}\alpha > \frac{n}{2} - 1$,

$$\int_{\mathcal{K}_+^+} (t^2 - |x|^2)^{\alpha - \frac{n}{2}} e^{-i\xi x} e^{-t(\epsilon + i\tau)} \, dt \, dx$$

$$= \frac{2\pi^{\frac{n-2}{2}}}{\Gamma(\frac{n-2}{2})} \int_0^\infty dr \, r^{n-2} \int_{\{t^2 \geqslant r^2\}} (t^2 - r^2)^{\alpha - \frac{n}{2}} e^{-t(\epsilon + i\tau)} dt \int_{-1}^1 (1 - s^2)^{\frac{n}{2} - 2} e^{-i|\xi| r s} \, ds$$

$$= 2^\alpha \pi^{\frac{n}{2} - 1} \Gamma(\alpha + 1 - \tfrac{n}{2}) \cdot |\xi|^{\frac{3-n}{2}} (\epsilon + i\tau)^{\frac{n-1}{2} - \alpha}$$

$$\cdot \int_0^\infty dr \, r^\alpha K_{\alpha + \frac{1-n}{2}}(r(\epsilon + i\tau)) J_{\frac{n-3}{2}}(r|\xi|)$$

$$= \frac{1}{C_{n,\alpha}} (|\xi|^2 + (\epsilon + i\tau)^2)^{-\alpha},$$

where we have used in the second line that the surface of the $(n-3)$-sphere is given by $\frac{1}{\Gamma(\frac{n}{2}-1)} \cdot 2\pi^{\frac{n}{2}-1}$. Taking now the limit as $\epsilon \to 0^+$ we get (2.3).

To complete the proof we recall that, as pointed out right before introducing the concept of Riesz distribution, σ_α defines a tempered distribution that is analytic for any complex α except for $\alpha = -\frac{n}{2} - k$ with $k \in \mathbb{N}$. Therefore, although we have proved (2.3) when the parameter takes values in a certain open set of the complex plane, this relation must hold in the whole domain where R_α can be analytically continued. Moreover, since by definition $\widehat{\Box^\alpha f} = \sigma_\alpha \hat{f}$ for all α, it is also true that $\Box^\alpha f = R_{-\alpha} * f = I_{-\alpha} f$. The proposition then follows. $\qquad\square$

The above proposition is roughly the analog of the formula (1.2) for the fractional Laplacian, which for $0 < \alpha < n$ allows us to write the fractional Laplacian $(-\Delta)^{-\alpha} f$, up to a multiplicative constant, as the convolution of f with the locally integrable function $|x|^{2\alpha-n}$. Notice, however, that the singularities in the integral kernel are much stronger here.

For the benefit of the reader we shall next record an explicit formula for $\Box^\alpha f$ in terms of integrals regularized via suitable finite difference operators, which we borrow from [18, Eq. (9.93)], and connect it with the previous proposition. Before we can state the result, let us first introduce some further notation. For q a real parameter and $k, l \in \mathbb{N}$, let us denote the q-number of k, its q-factorial and the q-binomial coefficient as

$$[k]_q = \frac{1-q^k}{1-q}, \qquad [k]_q! = [1]_q[2]_q \cdots [k]_q, \qquad \binom{l}{k}_q = \frac{[l]_q!}{[k]_q! \, [l-k]_q!} \quad (k \leqslant l),$$

respectively. In addition, let us define the q-functions

$$(2.6) \qquad C_k^l = q^{k(\frac{k+1}{2}-l)}\binom{l}{k}_q, \qquad A_\mu^l = \sum_{k=0}^{l}(-1)^k q^{k\mu} C_k^l,$$

with $\mu \in \mathbb{C}$ and where we omit the dependence on q for notational simplicity.

We are now ready to write down the integral formula for $\Box^\alpha f$. For simplicity we will assume that $f \in C_0^\infty(\mathbb{R}^n)$, but the result it is still true e.g. for $C^l(\mathbb{R}^n)$ functions that decay fast enough at infinity, with $l > 2\alpha$.

Proposition 2.2. *Let us take a real $\alpha \in (0, \frac{l}{2})$, where $l \in \mathbb{N}$ and we assume that α is not a half-integer. Then for any $f \in C_0^\infty(\mathbb{R}^n)$ the fractional wave operator $\Box^\alpha$ is given by the absolutely convergent integral*

$$(2.7) \qquad \Box^\alpha f(t,x) = C_{n,-\alpha} \int_{\mathbb{R}_+ \times \mathbb{R}^{n-1}} \frac{\Delta_{s,y}^{l,\alpha} f(t,x)}{s^{\frac{n}{2}+\alpha}|y|^{n+2\alpha-1}} ds \, dy,$$

where $\Delta_{s,y}^{n,\alpha}$ stands for the difference operator

$$\Delta_{s,y}^{l,\alpha} f(t,x) = \frac{1}{A_{\frac{n}{2}-1+\alpha}^{l^*} A_{2\alpha}^l} \sum_{j=0}^{l^*}\sum_{k=0}^{l}(-1)^{j+k} C_j^{l^*} C_k^l \frac{(1+q^j s)^{2\alpha}}{(2+q^j s)^{\frac{n}{2}+\alpha}} f\left(t-q^k|y|, x-\frac{q^k y}{1+q^j s}\right),$$

l^* is the integer part of $\frac{n+l-1}{2}$, $q \neq 1$ is a positive constant, C_k^l and A_μ^l are the q-functions defined in (2.6), and the constant $C_{n,\alpha}$ is given by (2.1).

Proof. By Proposition 2.1 the result is equivalent to showing that the integral (2.7) represents the extension of $I_{-\alpha} f$ to the strip $0 < \alpha < l/2$. Firstly, observe that this integral converges absolutely on this interval except for integers and semi-integers

values of α. Indeed, a tedious but straightforward computation of the Taylor series of $\Delta_{s,y}^{l,\alpha} f$ about zero combined with the fact that $A_m^l = 0$ for $0 \leqslant m \leqslant l-1$ as a consequence of the well-known identity [6]

$$\sum_{k=0}^{l} q^{k\frac{(k-1)}{2}} z^k \binom{l}{k}_q = \prod_{k=0}^{l-1} (1 + zq^k)$$

applied to $z = -q^{1+m-l}$, reveal that

$$|\Delta_{s,y}^{l,\alpha} f| \leqslant C_q s^{l^*} |y|^l \|D^l f\|_\infty + \mathcal{O}\big(s^{l^*} |y|^{l+1} \|D^{(l+1)} f\|_\infty\big)$$

for (s,y) close to zero and where C_q is a real constant that depends only on q. From this bound and since f vanishes at infinity, it immediately follows that the integral is finite when $\alpha \in (0, \frac{l}{2})$ except for the zeros of $A_{\frac{n}{2}-1+\alpha}^{l^*}$ and $A_{2\alpha}^l$, which correspond to the points of the form $\alpha = \frac{k}{2}$, $k \in \mathbb{N}$ when $q \neq 1$ is a positive real number.

Now we need to prove that the formula above actually represents the analytic continuation of the Riesz potential from the half-plane $\operatorname{Re}\alpha > \frac{n}{2} - 1$ to the range $\alpha \in (0, \frac{l}{2})$. Note that by (2.4) and the form of the differences operator, it suffices to show that (2.7) coincides with the expression of the potential given in (2.2) replacing α by $-\alpha$. This assertion can be readily checked by introducing new variables

$$u := q^k |y|, \qquad v := q^k \frac{|y|}{1 + q^j s}$$

and changing to polar coordinates in y. Incidentally, since the above substitutions also remove the dependence of the operator $\square^\alpha f$ on the parameter q, then one can safely choose any positive q other than the limit value $q = 1$. $\qquad \square$

Remark 2.3. It is worth noticing that even in the simplest case, $0 < \alpha < 1$ and $n = 2$, the formula for the fractional wave operator is much more involved than its Euclidean counterpart and cannot be deduced from it. This reflects the different nature of the singularities of the corresponding kernels: the entire light cone in the hyperbolic case and a single point in the Euclidean setting. Remarkably, in the simple case that we are now discussing ($n = 2$, $0 < \alpha < 1$) there is another realization of $\square^\alpha$ (cf. [18, Theorem 9.30]) that is easier to compare with its fractional Laplacian analog:

$$(2.8) \qquad \square^\alpha f(t,x) = \frac{C_{2,-\alpha}}{2^{1+2\alpha}} \int_{\mathcal{K}_+^+} \frac{T_{s,y} f(t,x)}{(s^2 - y^2)^{1+\alpha}} ds\, dy,$$

where we are using the finite difference operator

$$T_{s,y} f(t,x) := f(t,x) - f\big(t - \tfrac{s+y}{2}, x - \tfrac{s+y}{2}\big) - f\big(t - \tfrac{s-y}{2}, x + \tfrac{s-y}{2}\big) + f(t - s, x - y).$$

Note that one can readily use this formula and the crude approximations

$$T_{s,y} f(t,x) = -sy\,\square f(t,x) + O(s^2 + y^2), \qquad \Gamma(-1 + \epsilon)^{-1} = -\epsilon + O(\epsilon^2)$$

to prove the pointwise convergence of $\square^\alpha f(t,x)$ to $\square f(t,x)$ as $\alpha \to 1$.

3. The Klein–Gordon equation in AdS spaces

In this section we will connect the fractional powers of the wave operator with the solutions to the mixed initial-boundary problem corresponding to the Klein–Gordon equation in an anti-de Sitter space. Our specific goal here is to give a local realization of the fractional wave operator as a Dirichlet-to-Neumann map

that is the Lorentzian counterpart of the relation for the fractional Laplacian derived in [3] and extended in [4]. For this purpose, we will use a Laplace–Fourier transform in order to transform our wave equation into an ODE that contains the relevant information about the solution at infinity, which will enable us to derive the Dirichlet-to-Neumann map in Fourier space.

Recall that our starting point was the Klein–Gordon equation

$$\Box_{g^+}\phi + \left(\alpha^2 - \frac{n^2}{4}\right)\phi = 0,$$

where $\Box_{g^+}$ denotes the wave operator associated to the AdS metric (1.7). As discussed in the Introduction, this equation can be thought as a wave equation with coefficients that are singular at conformal infinity and whose indicial roots are $n/2 \pm \alpha$. For simplicity, we will henceforth assume that α is not a half-integer (i.e., $2\alpha \notin \mathbb{N}$) in order to ensure that the solution does not have a logarithmic branch cut. The argument carries over to the case that $\alpha \notin \mathbb{N}$ with minor modifications.

Let us begin with the analysis of the wave equation (1.9). For this, is convenient to define the function

$$u(t, x, y) := y^{\alpha - \frac{n}{2}} \phi(t, x, y),$$

which satisfies the equation

(3.1) $$\partial_{tt} u - \Delta_x u - \partial_{yy} u - \frac{1 - 2\alpha}{y} \partial_y u = 0$$

with Dirichlet datum

(3.2) $$u(t, x, 0) = f(t, x).$$

We shall next prove a bound for u that will be needed later in this section. To state it, let us consider the weighted Lebesgue space

$$L_\alpha^2 := L^2(\mathbb{R}_+^n, y^{1-2\alpha} dx\, dy),$$

endowed with the norm

$$\|v\|_{L_\alpha^2}^2 := \int_{\mathbb{R}_+^n} v^2\, y^{1-2\alpha} dx\, dy,$$

and denote by $\dot{H}_\alpha^1$ (respectively, $\dot{H}_{\alpha,0}^1$) the closure of $C_0^\infty(\overline{\mathbb{R}_+^n})$ (respectively, $C_0^\infty(\mathbb{R}_+^n)$) with respect to the norm

$$\|v\|_{\dot{H}_\alpha^1}^2 := \int_{\mathbb{R}_+^n} \left(|\nabla_x v|^2 + (\partial_y v)^2\right) y^{1-2\alpha} dx\, dy.$$

In the following theorem we only state a qualitative result for boundary data $f \in C_0^\infty(\mathbb{R}^n)$, which is what we need in the paper, but in the proof we provide quantitative estimates that obviously extend the result to data in more general function spaces.

Lemma 3.1. *Let k be the lowest integer such that $k > \frac{1+\alpha}{2}$. Given a boundary datum $f \in C_0^\infty(\mathbb{R}^n)$ and an integer $j < k$, define*

$$u_j(t, x, y) := y^{2j} \chi(y) \Box^j f(t, x),$$

where $\chi(y)$ is a fixed smooth cutoff function identically 1 in $y < 1$ and 0 in $y > 2$, and $\Box^j$ denotes the j-th power of the wave operator $\Box = \partial_{tt} - \Delta_x$. Then there are

real numbers c_j and a function $v \in L^\infty(\mathbb{R}, \dot{H}_\alpha^1)$ such that

$$(3.3) \qquad u(t, x, y) := \sum_{j=0}^{k-1} c_j u_j(t, x, y) + v(t, x, y) \,,$$

is the unique solution of Equation (3.1) with trivial initial data $u(-\infty, x, y) = u_t(-\infty, x, y) = 0$ and boundary condition $u(t, x, 0) = f(t, x)$.

Proof. Let us consider the initial condition

$$(3.4) \qquad u(t_0, \cdot) = u_t(t_0, \cdot) = 0 \,,$$

where t_0 is any number such that $f(t, x) = 0$ for all $t < t_0$. We shall see in the proof that the solution is independent of the choice of t_0, so it is equivalent to imposing $u(-\infty, \cdot) = u_t(-\infty, \cdot) = 0$.

We shall start by considering an auxiliary non-homogeneous Cauchy problem of the form

$$(3.5a) \qquad \partial_{tt} v - \Delta_x v - \partial_{yy} v - \frac{1 - 2\alpha}{y} \partial_y v = F(t, x, y),$$

$$(3.5b) \qquad v(t_0, \cdot) = v_t(t_0, \cdot) = 0.$$

We shall next show that if $F \in L^1(\mathbb{R}, L_\alpha^2)$, there is a unique solution

$$v \in L_{\text{loc}}^2(\mathbb{R}, \dot{H}_{\alpha,0}^1) \cap H_{\text{loc}}^1(\mathbb{R}, L_\alpha^2)$$

to this equation. Notice that the fact that $v(t, \cdot)$ takes values in $\dot{H}_{\alpha,0}^1$ means that we are imposing the boundary condition $v|_{y=0} = 0$.

To prove this, we will use an a priori estimate for the energy associated to the solution v, which we define as

$$E_v(t) := \frac{1}{2} \int_{\mathbb{R}_+^{n+1}} (v_t^2 + |\nabla_x v|^2 + v_y^2) \, y^{1-2\alpha} \, dx \, dy.$$

To prove this estimate, it is standard that by a density argument one can assume that v is in $C_0^\infty(\mathbb{R}_+^{n+1})$, differentiate under the integral sign and integrate by parts to find that

$$\frac{d}{dt} E_v(t) = \int_{\mathbb{R}_+^n} (v_t v_{tt} + \nabla_x v_t \cdot \nabla_x v + v_y v_{yt}) \, y^{1-2\alpha} dx \, dy$$

$$= \int_{\mathbb{R}_+^n} v_t \left(v_{tt} - \Delta_x v - v_{yy} - \frac{1 - 2\alpha}{y} v_y \right) y^{1-2\alpha} dx \, dy$$

$$= \int_{\mathbb{R}_+^n} F \, v_t \, y^{1-2\alpha} dx \, dy$$

$$\leqslant C \|F(t, \cdot)\|_{L_\alpha^2} \, E_v(t)^{1/2} \,.$$

Using now Grönwall's inequality, we arrive at

$$E_v(t)^{1/2} \leqslant E_v(t_0)^{1/2} + C \left| \int_{t_0}^t \|F(t', \cdot)\|_{L_\alpha^2} \, dt' \right|,$$

which, by the trivial initial conditions, readily implies the estimate

$$(3.6) \qquad \sup_{t \in \mathbb{R}} \left(\|v(t, \cdot)\|_{\dot{H}_\alpha^1} + \|v_t(t, \cdot)\|_{L_\alpha^2} \right) \leqslant C \int_{-\infty}^\infty \|F(t', \cdot)\|_{L_\alpha^2} \, dt',$$

thereby ensuring that $v \in L^\infty(\mathbb{R}, \dot{H}^1_{\alpha,0})$. It is standard that this estimate leads to the existence of a unique solution $v \in L^2_{\mathrm{loc}}(\mathbb{R}, \dot{H}^1_{\alpha,0}) \cap H^1_{\mathrm{loc}}(\mathbb{R}, L^2_\alpha)$ to the problem (3.5).

To apply the estimate (3.6) in our problem, let us set

$$v(t,x,y) := u(t,x,y) - \sum_{j=0}^{k-1} \frac{(-1)^j y^{2j} \chi(y)}{\prod_{l=1}^{j} 4l(l-\alpha)} \Box^j f(t,x),$$

where the product is to be taken as 1 when $l = 0$. Using now that

$$\left(\partial_{yy} + \frac{1-2\alpha}{y}\partial_y\right) y^j = j(j-2\alpha)\, y^{j-2},$$

a direct calculation shows that

$$(3.7) \quad \partial_{tt} v - \Delta_x v - \partial_{yy} v - \frac{1-2\alpha}{y}\partial_y v = \frac{(-1)^k y^{2k-2}\chi(y)}{\prod_{l=1}^{k-1} 4l(l-\alpha)}\Box^k f + \sum_{j=0}^{k-1} \chi_j(y)\,\Box^j f,$$

where $\chi_j(y)$ is a smooth function whose support is contained in the interval $[1,2]$. Moreover, by construction v satisfies the initial and boundary conditions

$$v(t_0,x,y) = v_t(t_0,x,y) = v(t,x,0) = 0.$$

The point now is that, as the right hand side of (3.7) behaves as y^{2k-2} and vanishes for $y > 2$, it is easy to see that it is in $L^1(\mathbb{R}, L^2_\alpha)$, so the estimate (3.6) ensures that u written as in (3.3) is the unique solution of (3.1) satisfying the boundary condition $u|_{y=0} = f$ and vanishing initial data. $\Box$

Remark 3.2. Arguing as in [7, Proposition 5.2], we could have proved higher regularity estimates for the solution in suitable weighted spaces, but we will not need that result. The global L^∞ bound in time, on the contrary, will be essential and is not proved in the aforementioned paper, as it does not hold for the more general equations there considered.

Given a positive real parameter α that is not a half-integer, let us write it as $\alpha = \alpha_0 + m$, where m denotes its integer part and $\alpha_0 \in (0,1)$. The generalized Dirichlet-to-Neumann map Λ_α is then defined as in [4, Theorem 3.3]

$$(3.8) \qquad \Lambda_\alpha f(t,x) = c_\alpha \lim_{y \to 0^+} y^{2(1-\alpha_0)} \left(\frac{1}{y}\partial_y\right)^{m+1} u(t,x,y),$$

with

$$c_\alpha := (-1)^{m+1} 2^{\alpha+\alpha_0-1} \frac{\Gamma(\alpha)}{\Gamma(1-\alpha_0)}$$

an inessential normalizing factor.

The following theorem, which is the central result of this paper, is essentially a rewording of Theorem 1.1:

Theorem 3.3. *Given a positive real number α that is not an integer and $f \in C_0^\infty(\mathbb{R}^n)$, let u be the solution of the initial-boundary problem in the Poincaré-AdS space (3.1) with Dirichlet datum (3.2) and initial data $u(-\infty, \cdot) = u_t(-\infty, \cdot) = 0$. Then the map Λ_α defined in (3.8) is given by the αth power of the wave operator:*

$$(3.9) \qquad\qquad\qquad \Lambda_\alpha f = \Box^\alpha f.$$

Proof. To begin with, let us start by noticing that the solution with the initial condition $u(-\infty, \cdot) = u_t(-\infty, \cdot) = 0$ is well defined, and it can be equivalently defined by the condition $u(t_0, \cdot) = u_t(t_0, \cdot) = 0$ where t_0 is any number such that $f(t, x) = 0$ for all $t < t_0$. In what follows, t_0 will denote a number with this property.

Consider the Laplace transform of $u(t, \cdot)$,

$$U(s, \cdot) = \int_{t_0}^{\infty} e^{-s(t-t_0)} \, u(t, \cdot) \, dt,$$

where $s = \epsilon + i\tau$ with $\epsilon > 0$ and $\tau \in \mathbb{R}$. Notice this expression converges as a vector-valued function because u is a Banach-space valued L^{∞} function of time by Lemma 3.1. (When $\alpha < 1$, the norm is simply that of $\dot{H}_{\alpha}^1$, while for $\alpha \geqslant 1$ the norm must also control the terms u_j appearing in (3.3), for example by decomposing

$$u(t, x, y) =: \chi(y) \sum_{j=0}^{k-1} y^{2j} \, \bar{u}_j(t, x) + v(t, x, y)$$

and adding the L^{∞} norm of the functions $\bar{u}_j$ and the $L^{\infty}(\mathbb{R}, \dot{H}_{\alpha}^1)$ norm of v).

Furthermore, observe that one can then recover u through the inverse Laplace transform formula

$$u(t, \cdot) = \frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{s(t-t_0)} \, U(s, \cdot) \, ds,$$

where the integration contour is the vertical line of numbers whose real part is $\epsilon > 0$.

We tackle the problem of finding explicit solutions to problem (3.1)-(3.2) as follows. First, we apply the Laplace transform on the equation for u to remove the time derivatives by integration by parts and then use the trivial initial conditions in the LHS above,

$$\int_{t_0}^{\infty} e^{-s(t-t_0)} \, u_{tt}(t, \cdot) \, dt = s^2 U(s, \cdot) - u_t(t_0, \cdot) - s u(t_0, \cdot) = s^2 U(s, \cdot)$$

to find that $U(s, x, y)$ satisfies the equation

$$\partial_{yy} U(s, x, y) + \frac{1-2\alpha}{y} \partial_y U(s, \xi, y) + (\Delta_x - s^2) U(s, x, y) = 0.$$

Next we take the Fourier transform in space with respect to the variable x, which here is denoted with a tilde to avoid confusions with the space-time Fourier transform. This yields the ODE

$$\partial_{yy} \widetilde{U}(s, \xi, y) + \frac{1-2\alpha}{y} \partial_y \widetilde{U}(s, \xi, y) - (|\xi|^2 + s^2) \widetilde{U}(s, \xi, y) = 0.$$

The general solution of this equation can be written as a linear combination of Bessel functions multiplied by a certain power of y, and spans a two-dimensional vector space. However, by Lemma 3.1 the solution $u(t, \cdot)$ is given by the sum of a function bounded in $\dot{H}_{\alpha}^1$ and other terms that are uniformly bounded in y, so we may discard the independent solution that grows exponentially in y as $y \to \infty$ to arrive at the formula

$$(3.10) \qquad \widetilde{U}(s, \xi, y) = \frac{2^{1-\alpha}}{\Gamma(\alpha)} y^{\alpha} (|\xi|^2 + s^2)^{\frac{\alpha}{2}} K_{\alpha}(y\sqrt{|\xi|^2 + s^2}) \widetilde{F}(s, \xi),$$

where K_α denotes the modified Bessel function of the second kind defined in (2.5). The constant (which depends on s and ξ) has been chosen to ensure that the Dirichlet condition is satisfied:

$$\widetilde{U}(s,\xi,0) = \widetilde{F}(s,\xi).$$

Here and in what follows, $F(s,x)$ denotes the Laplace transform of $f(t,x)$, computed as above, and the tilde denotes the Fourier transform in space.

Therefore, using the identities

$$\left(\frac{1}{z}\frac{d}{dz}\right)^k (z^\nu K_\nu(z)) = (-1)^k z^{\nu-k} K_{\nu-k}(z), \qquad k = 0, 1, 2, \ldots$$

$$K_\nu(z) = \frac{\Gamma(\nu)}{2}\left(\frac{z}{2}\right)^{-\nu} + O(z^{-\nu+1}), \qquad 0 < |z| \ll \sqrt{\nu+1},\ \nu > 0,$$

for modified Bessel functions (cf. [24]), we readily infer that the Dirichlet-to-Neumann map (3.8) reads in the Laplace-Fourier space as

$$\widetilde{\Lambda_\alpha F}(s,\xi) := c_\alpha \lim_{y\to 0^+} y^{2(1-\alpha_0)}\left(\frac{1}{y}\partial_y\right)^{m+1}\widetilde{U}(s,\xi,y) = (|\xi|^2 + s^2)^\alpha \widetilde{F}(s,\xi).$$

The key point now is that one can relate the inverse Laplace transform with the inverse Fourier transform in time using the freedom in the choice of the parameter ϵ. To show this, notice that by the Laplace inversion formula, we have that the Dirichlet-to-Neumann map in the variables (t,ξ) can be written as

$$\widetilde{\Lambda_\alpha f}(t,\xi) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} e^{s(t-t_0)}(|\xi|^2+s^2)^\alpha \widetilde{F}(s,\xi)\ ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{t_0}^{\infty} e^{(\epsilon+i\tau)(t-t')}(|\xi|^2 + (\epsilon+i\tau)^2)^\alpha \widetilde{f}(t',\xi)\ d\tau\ dt'.$$

Since the latter integral does not depend on the value of ϵ and f is smooth with compact support in $\{t \geqslant t_0\}$, by the dominated convergence theorem one can take the limit as $\epsilon \to 0^+$ inside the integral to find that

$$\widetilde{\Lambda_\alpha f}(t,\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau t}\sigma_\alpha(\tau,\xi)\widehat{f}(\tau,\xi)\ d\tau,$$

with σ_α as in (1.6) and $\widehat{f}(\tau,\xi)$ denoting the Fourier transform of f with respect to both time and space variables as in Section 2. Taking now the Fourier transform with respect to the time, we obtain

$$\widehat{\Lambda_\alpha f}(\tau,\xi) = \sigma_\alpha(\tau,\xi)\widehat{f}(\tau,\xi),$$

proving our claim. $\qquad\qquad\square$

A consequence of the proof is an explicit formula for the spacetime energy of the solution in terms of its boundary datum that is analogous to the result in Euclidean signature from [3]:

Corollary 3.4. *With u and f be as in Theorem 3.3, the total energy of u is*

$$\int_{\mathbb{R}^{n+1}_+} \left[(\partial_y u)^2 + |\nabla_x u|^2 + (\partial_t u)^2\right] y^{1-2\alpha}\ dt\ dx\ dy = C_\alpha \int_{\mathbb{R}^n} \sigma_\alpha(\tau,\xi)\,|\widehat{f}(\tau,\xi)|^2\ d\tau\ d\xi,$$

with C_α a nonzero constant.

Proof. We first note that the Laplace transform of the function $u(s, \cdot)$ at $s = \epsilon + i\tau$,

$$U(\epsilon + i\tau) = \int_0^\infty e^{-i\tau t} e^{-\epsilon t} u(t, \cdot)\, dt,$$

is the Fourier transform of the function $e^{-\epsilon t} u(t, \cdot)$. Thus by Plancherel theorem we can write

$$(3.11) \qquad \int_{\epsilon-i\infty}^{\epsilon+i\infty} |U(s, \cdot)|^2\, ds = \int_{-\infty}^{+\infty} |U(\epsilon + i\tau, \cdot)|^2\, d\tau = \int_0^\infty e^{-2\epsilon t} |u(t, \cdot)|^2\, dt.$$

Now we consider the energy for equation (3.1), given by

$$E = \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_0^\infty \left[(\partial_y u)^2 + |\nabla_x u|^2 + (\partial_t u)^2 \right] dt\, dx\, dy$$

$$= \lim_{\epsilon \to 0} \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_0^\infty \left[(\partial_y u)^2 + |\nabla_x u|^2 + (\partial_t u)^2 \right] e^{-2\epsilon t}\, dt\, dx\, dy,$$

which, using Plancherel identity (3.11) for the Laplace transform, becomes

$$E = \lim_{\epsilon \to 0} \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \left[|\partial_y U(s, x, y)|^2 + |\nabla_x U(s, x, y)|^2 \right.$$
$$\left. + s^2 |U(s, x, y)|^2 \right] ds\, dx\, dy.$$

Now we take Fourier transform in the variable x, yielding

$$E = \lim_{\epsilon \to 0} \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \left[|\partial_y \widetilde{U}(s, \xi, y)|^2 + (|\xi|^2 + s^2)|\widetilde{U}(s, \xi, y)|^2 \right] ds\, d\xi\, dy.$$

Substituting the explicit expression (3.10) we arrive at

$$E = \lim_{\epsilon \to 0} \int_0^\infty y^{1-2\alpha} \int_{\mathbb{R}^{n-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} (|\xi|^2 + s^2) \left[|K_1'(y\sqrt{|\xi|^2 + s^2})|^2 \right.$$
$$\left. + |K_1(y\sqrt{|\xi|^2 + s^2})|^2 \right] |\widetilde{F}(s, \xi)|^2\, ds\, d\xi\, dy,$$

where, for simplicity, we have set $K_1(y) := \dfrac{2^{1-\alpha}}{\Gamma(\alpha)} y^\alpha K_\alpha(y)$. A change of variable allows us to integrate in y, obtaining

$$E = \lim_{\epsilon \to 0} C_\alpha \int_{\mathbb{R}^{n-1}} \int_{\epsilon-i\infty}^{\epsilon+i\infty} (|\xi|^2 + s^2)^\alpha |\widetilde{F}(s, \xi)|^2\, ds\, d\xi$$

with C_α an explicit constant, as claimed. $\qquad \square$

4. Application to other asymptotically AdS metrics

Our point in this section is to show that the identities established in Theorem 1.1 remain valid in considerably more general situations. We will illustrate this fact by connecting other fractional wave operators with the Dirichlet-to-Neumann map (or, more generally, the scattering operator) of two simple classes of static asymptotically AdS manifolds:

Fractional waves in product spaces. Consider a compact Riemannian manifold $\mathcal{M}$ of dimension $n-1$ endowed with a Riemannian metric g_0 and take the natural wave operator on $\mathbb{R} \times \mathcal{M}$, which is

$$\Box_0 := \partial_{tt} - \Delta_{g_0},$$

where Δ_{g_0} stands for the Laplace-Beltrami operator on $\mathcal{M}$.

Since $\mathcal{M}$ is compact, we can take an orthonormal basis $\{Y_j\}_{j \in \mathbb{N}}$ of eigenfunctions of the Laplacian Δ_{g_0}, which satisfy

$$-\Delta_{g_0} Y_j = \lambda_j^2 Y_j \,,$$

and write any L^2 function f on M (depending on t as a parameter) as the L^2-convergent series $f(t, \cdot) = \sum_j f_j(t) Y_j(\cdot)$. Let us denote by $(\Box_0 f)_j(t)$ the jth component of the function $\Box_0 f$ in this basis. Taking the Fourier transform with respect to t we obtain that

$$\widehat{\Box_0 f_j}(\tau) = (\lambda_j^2 - \tau^2)\widehat{f}_j(\tau)$$

and, given a real parameter α, we can define here the αth power of the wave operator as the pseudo-differential operator that in the Fourier space reads as

$$(4.1) \qquad \widehat{\Box_0^\alpha f_j}(\tau) := \sigma_\alpha(\tau, \lambda_j)\widehat{f}_j(\tau),$$

with σ_α the function defined in (1.6).

Consider now an $(n+1)$-dimensional Lorentzian spacetime with the metric

$$(4.2) \qquad g^+ := \frac{dt^2 - dy^2 - g_0}{y^2},$$

where $t \in \mathbb{R}$ is the time coordinate and $y \in \mathbb{R}_+$ is a spatial coordinate. Comparing with the metric defined in (1.7), it is clear that the Klein-Gordon equation with parameter $\mu := (\alpha^2 - n^2/4)$ associated to the metric (4.2) then takes the form

$$\partial_{tt}\phi - \Delta_{g_0}\phi - \partial_{yy}\phi - \frac{1-n}{y}\partial_y\phi + \frac{4\alpha^2 - n^2}{4y^2}\phi = 0 \,,$$

where one must prescribe some suitable initial-boundary conditions for the scalar field ϕ. As in the last section, the above equation can be rewritten in terms of the rescaled function $u := y^{\alpha - \frac{n}{2}}\phi$ as

$$(4.3) \qquad \partial_{tt}u = \Delta_{g_0}u + \frac{1-2\alpha}{y}\partial_y u + \partial_{yy}u,$$

where we take trivial initial data at time $-\infty$ and prescribe the boundary condition at timelike conformal infinity: $u|_{y=0} = f$.

We can next define the (generalized) Dirichlet-to-Neumann map through its coefficients

$$(4.4) \qquad (\Lambda_\alpha f)_j = c_\alpha \lim_{y \to 0^+} y^{2(1-\alpha_0)}\left(\frac{1}{y}\partial_y\right)^{m+1} u_j,$$

with c_α as before, $\alpha = \alpha_0 + m$, $m = \lfloor\alpha\rfloor$ the integer part of α and $\alpha_0 \in (0,1)$.

By means of expansion in eigenfunctions of the Laplacian Δ_{g_0} and the Laplace transform in time (together with the vanishing initial conditions), the equation (4.3) can be transformed into our well known ordinary equation

$$\partial_{yy}U_j(s, y) + \frac{1-2\alpha}{y}\partial_y U_j(s, y) - (\lambda_j^2 + s^2)U_j(s, y) = 0,$$

with boundary condition $F_j(s) = U_j(s,0)$, where

$$U_j(s,y) := \int_{t_0}^{\infty} e^{-s(t-t_0)} u_j(s,y)\, dt$$

is the Laplace transform of the coefficient u_j with $s = \epsilon + i\tau$ and ϵ a fixed positive constant. Notice that U_j can be shown to be well defined for $\epsilon > 0$ by an L^∞ bound in time that goes exactly as in Lemma 3.1.

Arguing just as in the previous section, we find that the Dirichlet-to-Neumann map in the transformed space reads as

$$\Lambda_\alpha F_j := c_\alpha \lim_{y \to 0^+} y^{2(1-\alpha_0)} \left(\frac{1}{y} \partial_y \right)^{m+1} U_j(s,y) = (\lambda_j^2 + s^2)^\alpha F_j(s),$$

and therefore, by the inverse Laplace transform formula and the Fourier transform in time,

$$\widehat{\Lambda f}_j(\tau) = \sigma_\alpha(\tau, \lambda_j) \widehat{f}_j(\tau).$$

Thereby, we can identify the Dirichlet-to-Neumann map in the Lorenztian space with metric (4.2) with the powers of the wave operator $\Box_0$.

The global anti-de Sitter space. Consider now the global anti-de Sitter mentioned in the introduction, which is diffeomorphic to $\mathbb{R}^{n+1}$ and one can describe through spherical coordinates

$$(t, r, \theta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}^{n-2},$$

which cover the whole manifold modulo the usual abuse of notation at the origin. In these coordinates the metric of AdS_{n+1} reads as

(4.5) $$g^+ := (1+r^2)dt^2 - \frac{1}{1+r^2}dr^2 - r^2 g_{\mathbb{S}^{n-2}},$$

where $g_{\mathbb{S}^{n-2}}$ is the canonical metric on the unit $(n-2)$-dimensional sphere, associated with the coordinate θ.

In this space we can picture the spatial limit $r \to +\infty$ as the cylinder $\mathbb{R}_t \times \mathbb{S}^{n-2}$ with the standard metric

$$g_0 := dt^2 - g_{\mathbb{S}^{n-2}},$$

which defines the timelike conformal infinity of the spacetime.

Let us now focus on the Klein–Gordon equation on this anti-de Sitter space with the natural Dirichlet datum

$$\lim_{r \to \infty} r^{\frac{n}{2}-\alpha} \phi(t,r,\theta) = f(t,\theta),$$

$f \in C_0^\infty(\mathbb{R}^{n+1})$ and trivial initial conditions at time $-\infty$. Upon expanding the operator $\Box_{g^+}$ associated to the metric (4.5), we obtain the Klein–Gordon equation

(4.6)
$$\partial_{tt}\phi = \frac{1+r^2}{r^2}\Delta_\theta \phi + (1+r^2)^2 \partial_{rr}\phi$$
$$+ (1+r^2)\left(\frac{n-1}{r} + (n+1)r \right)\partial_r \phi - (1+r^2)\left(\alpha^2 - \frac{n^2}{4} \right)\phi.$$

As before, in order obtain the scattering operator we take a basis $\{\mathcal{Y}_j\}_{j \in \mathbb{N}}$ of spherical harmonics of energy $\lambda_j^2 := j(j+n-3)$. They satisfy the equation

$$-\Delta_\theta \mathcal{Y}_j = \lambda_j^2 \mathcal{Y}_j,$$

with Δ_θ the Laplace-Beltrami operator on $\mathbb{S}^{n-2}$. Introducing the coefficients

$$\phi_j(t,r) := \int_{\mathbb{S}^{n-2}} \phi(t,r,\theta)\,\mathcal{Y}_j(\theta)\,d\theta\,,$$

one can apply the Laplace transform and expand the Klein–Gordon equation in the basis of spherical harmonics to obtain an ODE in the variable r,

$$(4.7)\quad (1+r^2)\partial_{rr}\Phi_j + \Big(\frac{n-1}{r}+(n+1)r\Big)\partial_r\Phi_j + \Big(\frac{s^2}{1+r^2}+\frac{\lambda_j^2}{r^2}+\alpha^2-\frac{n^2}{4}\Big)\Phi_j = 0\,,$$

where

$$\Phi_j(s,r) := \int_{t_0}^{\infty} e^{-s(t-t_0)}\,\phi_j(t,r)\,dt$$

is the Laplace transform of $\phi_j(r,\theta)$.

The explicit solution of this equation is a combination of certain powers of r multiplied by ordinary hypergeometric functions. Discarding the solution that is not locally in H^1 at the origin, $r=0$, we then obtain

$$\Phi_j(s,r) = c_j(s)r^\beta e^{-\frac{i}{2}\log(1+r^2)s}\,{}_2F_1\Big(\tfrac{1}{2}(\beta-is+\tfrac{n}{2}-\alpha),\tfrac{1}{2}(\beta-is+\tfrac{n}{2}+\alpha),\beta+\tfrac{n}{2},-r^2\Big),$$

where

$$\beta := \frac{1}{2}\Big(2-n+\sqrt{4\lambda^2+(n-2)^2}\Big).$$

The coefficient $c_j(s)$ is readily computed using that $F_j(s) := \lim_{r\to\infty} r^{\frac{n}{2}-\alpha}\Phi_j(s,r)$ must be the Laplace transform of the jth component of the boundary datum.

The transformed Dirichlet-to-Neumann operator is then readily shown to be

$$\Lambda_\alpha F_j(s) := \lim_{r\to\infty} r^{1+2\alpha}\partial_r(r^{\frac{n}{2}-\alpha}\Phi_j(s,r)).$$

Using now the explicit solution, we obtain that

$$\Lambda_\alpha F_j(s) = \frac{\Gamma(-\alpha)\Gamma\big(\tfrac{1}{2}(\beta-is+\tfrac{n}{2}+\alpha)\big)\Gamma\big(\tfrac{1}{2}(\beta+is+\tfrac{n}{2}+\alpha)\big)}{\Gamma(\alpha)\Gamma\big(\tfrac{1}{2}(\beta-is+\tfrac{n}{2}-\alpha)\big)\Gamma\big(\tfrac{1}{2}(\beta+is+\tfrac{n}{2}-\alpha)\big)}(\beta-is+\tfrac{n}{2}-\alpha)F_j(s),$$

which can be written using the Fourier transform in time as

$$\widehat{\Lambda f}_j(\tau) := \lim_{\epsilon\to 0^+} \Lambda_\alpha F_j(\epsilon+i\tau).$$

It should be noticed that in the limit of large frequencies of the multiplier $\sigma_\alpha(\tau,\lambda_j)$ is, up to some numerical factor, the principal symbol of the scattering operator in this globally defined AdS space, as one would have expected.

Acknowledgements

References

[1] A. Bachelot, The Klein–Gordon equation in the anti-de Sitter cosmology, J. Math. Pures Appl. 96 (2011) 527–554.

[2] C.A. Ballón, N. Braga, Anti-de Sitter boundary in Poincaré coordinates, Gen. Rel. Grav. 39 (2007) 1367-1379.

[3] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial Differential Equations 32 (2007) 1245–1260.

[4] S.-Y. A. Chang, M.d.M. González, Fractional Laplacian in conformal geometry, Adv. Math. 226 (2011) 1410–1432.

[5] S.-Y. A. Chang, M.d.M. González, On a class of non-local operators in conformal geometry, preprint, 2016.

[6] J. Cigler, Operatormethoden für q-Identitäten, Monatsh. Math. 88 (1979) 87–105.

[7] A. Enciso, N. Kamran, A singular initial-boundary value problem for nonlinear wave equations and holography in asymptotically anti-de Sitter spaces, J. Math. Pures Appl. 103 (2015) 1053–1091.

[8] C. Fefferman and C. R. Graham, *The ambient metric*, Princeton University Press, Princeton, 2012.

[9] I.M. Gelfand, G.E. Shilov, *Generalized Functions I*, Academic Press, New York, 1964.

[10] C. R. Graham, M. Zworski, Scattering matrix in conformal geometry, Invent. Math., 152 (2003) 89–118.

[11] M. Kemppainen, P. Sjögren, J.L. Torrea, Wave extension problem for the fractional Laplacian. Discrete Contin. Dyn. Syst. 35 (2015) 4905–4929.

[12] J.A.C. Kolk, V.S. Varadarajan, Riesz distributions, Math. Scand. 68 (1991) 273–291.

[13] N.S. Landkoff, *Foundations of Modern Potential Theory*, Springer-Verlag, Berlin, 1972.

[14] C. LeBrun, H-space with a cosmological constant, Proc. Roy. Soc. London Ser. A 380 (1982) 171–185.

[15] J. Maldacena, The large N limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 (1998) 231–252.

[16] R.R. Mazzeo, R.B. Melrose, Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature, J. Funct. Anal. 75 (1987) 260–310.

[17] M. Riesz, L'intégrale de Riemann-Liouville et le problème de Cauchy, Acta Math. 81 (1951) 1–223.

[18] S.G. Samko, *Hypersingular integrals and their applications*, CRC Press, Boca Ratón, 2001.

[19] P.R. Stinga, J.L. Torrea, Extension problem and Harnack's inequality for some fractional operators. Comm. Partial Differential Equations 35 (2010) 2092–2122.

[20] P. R. Stinga, J. L. Torrea. Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation, arXiv:1511.01945.

[21] A. Vasy, The wave equation on asymptotically anti de Sitter spaces. Anal. PDE 5 (2012) 81–144.

[22] R.M. Wald, *General relativity*, University of Chicago Press, Chicago, 1984.

[23] C. Warnick, The massive wave equation in asymptotically AdS spacetimes, Comm. Math. Phys. 321 (2013) 85–111.

[24] G.N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, Cambridge, 1966.

[25] E. Witten, Anti de Sitter space and holography, Adv. Theor. Math. Phys. 2 (1998) 253–291.