Fractional Laplacian in Conformal Geometry

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Abstract

In this note, we study the connection between the fractional Laplacian operator that appeared in the recent work of Caffarelli-Silvestre and a class of conformally covariant operators in conformal geometry.

1   Introduction

In recent years, there has been independent study of fractional order operators by two different group of mathematicians. On one hand, there are extensive works that study properties of fractional Laplacian operators as non-local operators together with applications to free-boundary value problems and non-local minimal surfaces - by Caffarelli-Silvestre [6], and many others (see the related articles [5], [7], [4], [9]); on the other hand, there is the work of Graham-Zworski [17], (see also [20], [12], [27], for instance), that study a general class of conformally covariant operators (P_γ), parameterized by a real number γ and defined on the boundary of a conformally compact Einstein manifold, and which includes the fractional Laplacian operators as a special case when the boundary is the Euclidean space setting as boundary of the hyperbolic space. In this note, we will clarify the connection between the work of these two groups.

This paper is organized as follows: in section 2 of the paper we will briefly describe the the work of Graham-Zworski [17] and the notion of the fractional Paneitz operator P_γ.

In section 3, we will illustrate in theorem 3.1 that in the case when the fractional Laplacian operator $(-\Delta)^\gamma$ is defined on the Euclidean space $\mathbb{R}^n$ and $\gamma \in (0,1)$, the operator agrees with P_γ on the hyperbolic space. This is done by applying the extension theorem of Caffarelli-Silvestre [6] on a characterization of the fractional Laplacian operator $(-\Delta)^\gamma$. We then extend the work of Caffarelli-Silvestre [6] and the identification to P_γ as $(-\Delta)^\gamma$ on $\mathbb{R}^n$ for more general exponents $\gamma \in \left(0, \frac{n}{2}\right)$. To achieve this, we first show in theorem 3.2 that there are two different ways to define P_γ operator when $\gamma > 1$, one can define it through the scattering matrix as in the original work of Graham-Zworski, or one can define it through an iterated process based on the work of Caffarelli-Silvestre; and two definitions agree (the relationship is illustrated in the claim (3.16) in the proof of theorem 3.2 and equations (3.22) and (3.23)). Finally, we apply theorem 3.2 to generalize the extension theorem of Caffarelli-Silvestre from $\gamma \in (0,1)$ to $\gamma \in \left(0, \frac{n}{2}\right)$ (this is theorem 3.3).

In section 4 of the paper, we will discuss the extension theorem in the general setting of conformally compact Einstein manifolds. One point we make is that by choosing a suitable defining function (in lemma 4.5) which is related to the eigenfunctions of Laplacian of the

Einstein metric, equation (4.15) of the extension theorem on a general conformal compact Einstein manifold (theorem 4.7) is the same as the extension theorem on hyperbolic space studied in section 3; it is a pure divergence equation, degenerate elliptic, with a weight in the Muckenhoupt class A_2.

Finally, in section 5, we will discuss the extension theorem on a general asymptotically hyperbolic manifold; where the boundary manifold may not be the totally geodesic boundary of an asymptotically hyperbolic space. In particular, it provides a natural way to define the conformal fractional Laplacian on the boundary of any compact manifold, and a related fractional order curvature Q_γ. In this case, it is interesting to see that the statement of extension theorem (theorem 5.1) breaks down at $\gamma = \frac{1}{2}$ and at $\gamma > \frac{1}{2}$. In particular, there is an extra term in the the expression of $P_{\frac{1}{2}}$ which is the mean curvature of the boundary manifold. This dichotomy has already appeared in other settings (Caffarelli-Souganidis [7], for instance), and illustrates the fact that when $\gamma \in \left(0, \frac{1}{2}\right)$, the operator presents a stronger non-local behavior than when $\gamma \in \left(\frac{1}{2}, 1\right)$.

The authors wish that the results in this paper will lead to extension of the works of Caffarelli and others on fractional free boundary value problems and fractional mean curvature surfaces to general manifold settings. In particular, we pose the question of finding the relationship between the fractional order mean curvature very recently defined (see Caffarelli-Roquejoffre-Savin [4], Caffarelli-Valdinoci [8]), and our notion of Q_γ curvature coming from the fractional Paneitz operator. This seems to be a very interesting open question.

2 Background

First we review the connection between scattering theory on conformally compact Einstein manifolds and conformally invariant objects on their boundaries at infinity. The main references are Graham-Zworski [17] and Fefferman-Graham [11].

Let M be a compact manifold of dimension n with a metric $\hat{g}$. Let $\bar{X}^{n+1}$ be a compact manifold of dimension $n+1$ with boundary M, and denote by X the interior of $\bar{X}$. A function ρ is a defining function of ∂X in X if

$$\rho > 0 \text{ in } X, \quad \rho = 0 \text{ on } \partial X, \quad d\rho \neq 0 \text{ on } \partial X.$$

We say that g^+ is a conformally compact (c.c.) metric on X with conformal infinity $(M, [\hat{g}])$ if there exists a defining function ρ such that the manifold $(\bar{X}, \bar{g})$ is compact for $\bar{g} = \rho^2 g^+$, and $\bar{g}|_M \in [\hat{g}]$. If, in addition (X^{n+1}, g^+) is a conformally compact manifold and $Ric[g^+] = -ng^+$, then we call (X^{n+1}, g^+) a conformally compact Einstein manifold.

Given a conformally compact, asymptotically hyperbolic manifold (X^{n+1}, g^+) and a representative $\hat{g}$ in $[\hat{g}]$ on the conformal infinity M, there is a uniquely defining function ρ such that, on $M \times (0, \delta)$ in X, g^+ has the normal form $g^+ = \rho^{-2}(d\rho^2 + g_\rho)$ where g_ρ is a one parameter family of metrics on M satisfying $g_\rho|_M = \hat{g}$. Moreover, g_ρ has an asymptotic expansion which contains only even powers of ρ, at least up to degree n.

It is well known (c.f. Mazzeo-Melrose [23], Graham-Zworski [17]) that, given $f \in \mathbb{C}^\infty(M)$ and $s \in \mathbb{C}$, the eigenvalue problem

$$-\Delta_{g^+} u - s(n-s)u = 0, \quad \text{in } X \tag{2.1}$$

has a solution of the form

$$u = F\rho^{n-s} + H\rho^s, \quad F, H \in \mathcal{C}^\infty(X), \quad F|_{\rho=0} = f, \tag{2.2}$$

for all $s \in \mathbb{C}$ unless $s(n-s)$ belongs to the pure point spectrum of $-\Delta_{g^+}$. Now, the scattering operator on M is defined as $S(s)f = H|_M$, it is a meromorphic family of pseudo-differential operators in $Re(s) > n/2$. The values $s = n/2, n/2+1, n/2+2, \ldots$ are simple poles of finite rank, these are known as the trivial poles; $S(s)$ may have other

poles, however, for the rest of the paper we assume that we are not in those exceptional cases.

We define the conformally covariant fractional powers of the Laplacian as follows: for $s = \frac{n}{2} + \gamma$, $\gamma \in \left(0, \frac{n}{2}\right)$, $\gamma \notin \mathbb{N}$, we set

$$P_\gamma[g^+, \hat{g}] := d_\gamma S\left(\frac{n}{2} + \gamma\right), \quad d_\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)}. \tag{2.3}$$

With this choice of multiplicative factor, the principal symbol of P_γ is exactly the principal symbol of the fractional Laplacian $(-\Delta_{\hat{g}})^\gamma$, precisely, $|\xi|^{2\gamma}$. We thus have that $P_\gamma \in (-\Delta_{\hat{g}})^\gamma + \Psi_{\gamma-1}$, where we denote by Ψ_m to be the set of pseudo-differential operators on M of order m.

The operators $P_\gamma[g^+, \hat{g}]$ satisfy an important conformal covariance property (see [17]). Indeed, for a conformal change of metric

$$\hat{g}_v = v^{\frac{4}{n-2\gamma}} \hat{g}, \quad v > 0, \tag{2.4}$$

we have that

$$P_\gamma[g^+, \hat{g}_v]\phi = v^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma[g^+, \hat{g}](v\phi), \tag{2.5}$$

for all smooth functions ϕ.

We define the Q_γ curvature of the metric associated to the functional P_γ, to be

$$Q_\gamma[g^+, \hat{g}] := P_\gamma[g^+, \hat{g}](1). \tag{2.6}$$

In particular, for a change of metric as (2.4), we obtain the equation for the Q_γ curvature:

$$P_\gamma[g^+, \hat{g}](v) = v^{\frac{n+2\gamma}{n-2\gamma}} Q_\gamma[g^+, \hat{g}_v].$$

When γ is an integer, say $\gamma = k$, $k \in \mathbb{N}$, a careful study of the poles of $S(s)$ allows to define P_k. Indeed,

$$Res_{s=n/2+k} S(s) = c_k P_k, \quad c_k = (-1)^k [2^{2k} k!(k-1)!]^{-1}.$$

These are the conformally invariant powers of the Laplacian constructed by Graham-Jenne-Mason-Sparling [15], Fefferman-Graham [11], that are local operators, and satisfy

$$P_k = (-\Delta)^k + \text{ lower order terms.}$$

In particular, when $k = 1$ we have the conformal Laplacian,

$$P_1 = -\Delta + \frac{n-2}{4(n-1)} R,$$

and when $k = 2$, the Paneitz operator (c.f. [25])

$$P_2 = (-\Delta)^2 + \delta \left(a_n R g + b_n Ric\right) d + \frac{n-4}{2} Q_2.$$

Finally, we note that another realization of the conformal fractional powers of the Laplacian was given by Peterson in [26], through an analytic continuation argument from the differential operators P_k.

3   The extension problem on hyperbolic space

Given $\gamma \in \mathbb{R}$, the fractional Laplacian on $\mathbb{R}^n$, denoted as $(-\Delta_x)^\gamma$, for a function $f : \mathbb{R}^n \to \mathbb{R}$ is defined as a pseudo-differential operator by

$$\widehat{(-\Delta_x)^\gamma} f(\xi) = |\xi|^{2\gamma} \hat{f}(\xi),$$

i.e, its principal symbol is $|\xi|^{2\gamma}$. It can also be written as the singular integral (suitably regularized)

$$(-\Delta_x)^\gamma f(x) = C_{n,\gamma} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x - \xi|^{n+2\gamma}} \, d\xi.$$

Caffarelli-Silvestre have developed in [6] an equivalent definition, in the case $\gamma \in (0,1)$, using an extension problem to the upper half-space $\mathbb{R}^{n+1}_+$. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we construct the extension $U : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}$, $U = U(x,y)$, as the solution of the equation

$$\begin{cases} \Delta_x U + \dfrac{a}{y} \partial_y U + \partial_{yy} U = 0, & x \in \mathbb{R}^n, \ y \in [0,+\infty), \\[2mm] \qquad\qquad U(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where

$$\gamma = \frac{1-a}{2}.$$

Note that equation (3.1) can be written as a divergence equation

$$\operatorname{div}(y^a \nabla U) = 0 \quad \text{in } \mathbb{R}^{n+1}_+,$$

which is degenerate elliptic. Then the fractional Laplacian of f can be computed as

$$(-\Delta_x)^\gamma f = \frac{d_\gamma}{2\gamma} \lim_{y \to 0} y^a \partial_y U, \tag{3.2}$$

where the constant d_γ is defined in (2.3). We are indeed looking at a non-local Dirichlet to Neumann operator. To finish, just note that the Poisson kernel for the fractional Laplacian $(-\Delta)^\gamma$ in $\mathbb{R}^n$ is

$$K_\gamma(x,y) = c_{n,\gamma} \frac{y^{1-a}}{\left(|x|^2 + |y|^2 \right)^{\frac{n+1-a}{2}}},$$

and thus $U = K_\gamma *_x f$.

The main observation we make in this section is that, in the case $M = \mathbb{R}^n$ and $X = \mathbb{R}^{n+1}_+$ with coordinates $x \in \mathbb{R}^n$, $y > 0$, endowed the hyperbolic metric $g_{\mathbb{H}} = \frac{dy^2 + |dx|^2}{y^2}$, the scattering operator is nothing but the Caffarelli-Silvestre extension problem for the fractional Laplacian when $\gamma \in (0,1)$. After that, we give the generalization of the result by Caffarelli-Silvestre for other exponents $\gamma \in \left(0, \frac{n}{2}\right) \setminus \mathbb{N}$. For simplicity, we write $P_\gamma := P_\gamma[g_{\mathbb{H}}, |dx|^2]$, where $|dx|^2$ is the Euclidean metric on $\mathbb{R}^n$.

Theorem 3.1. *Fix $\gamma \in (0,1)$ and f a smooth function defined on $\mathbb{R}^n$. If U is a solution of the extension problem (3.1), then $u = y^{n-s}U$ is a solution of the eigenvalue problem (2.1) for $s = n/2 + \gamma$, and moreover,*

$$P_\gamma f = \frac{d_\gamma}{2\gamma} \lim_{y \to 0} (y^a \partial_y U) = (-\Delta_x)^\gamma, \tag{3.3}$$

where $a = 1 - 2\gamma$, $P_\gamma := P_\gamma[g_{\mathbb{H}}, |dx|^2]$, and the constant d_γ is defined in (2.3).

Proof. Fix f on $\mathbb{R}^n$ and let u be a solution of the scattering problem

$$-\Delta_{\mathbb{H}} u - s(n - s)u = 0 \text{ in } X. \tag{3.4}$$

We know that u can be written as

$$u = y^{n-s}F + y^s H, \tag{3.5}$$

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where $F|_{y=0} = f$ and $S(s)f = h$ for $h = H|_{y=0}$. Moreover,

$$F(x, y) = f(x) + f_2(x)y^2 + o(y^2) \quad \text{and} \quad H(x, y) = h(x) + h_2(x)y^2 + o(y^2). \tag{3.6}$$

On the other hand, the conformal Laplacian operator for a Riemannian metric g in a manifold X of dimension $N = n + 1$ is defined as

$$L_g = -\Delta_g + \frac{N-2}{4(N-1)}R_g.$$

For the hyperbolic metric, $R_{g_{\mathbb{H}}} = -n(n+1)$, so that

$$L_{g_{\mathbb{H}}} = -\Delta_{g_{\mathbb{H}}} - \frac{n^2-1}{4} \tag{3.7}$$

Then, from (3.4) we can compute

$$0 = -\Delta_{g_{\mathbb{H}}} u - s(n-s)u = L_{g_{\mathbb{H}}} u + \left(\gamma^2 - \tfrac{1}{4}\right) u = y^{\frac{n+3}{2}} L_{g_{eq}}\left(y^{-\frac{n+1}{2}} u\right) + \left(\gamma^2 - \tfrac{1}{4}\right) u. \tag{3.8}$$

where in the last equality we have used the conformal covariant property of the conformal Laplacian for the change of metric $g_{eq} = y^2 g_{\mathbb{H}}$:

$$L_{g_{\mathbb{H}}}(\psi) = y^{\frac{n+3}{2}} L_{g_{eq}}\left(y^{-\frac{n-1}{2}} \psi\right). \tag{3.9}$$

Next, we change $u = y^{n-s}U$, and note that

$$L_{g_{eq}} = -\Delta = -\Delta_x - \partial_{yy}, \tag{3.10}$$

so it follows that

$$L_{g_{eq}}\left(y^{\frac{n+1}{2}-s}U\right) = -y^{\frac{n+1}{2}-s}\left[\Delta_x U + \partial_{yy}U + \frac{a}{y}\partial_y U + \left(\tfrac{n+1}{2} - s\right)\left(\tfrac{n+1}{2} - s - 1\right)\frac{U}{y^2}\right]. \tag{3.11}$$

Substituting (3.11) into (3.8), we observe that with the choice of $s = \frac{n}{2} + \gamma$ and $a = 1 - 2\gamma$ we arrive at

$$\Delta_x U + \partial_{yy}U + \frac{a}{y}\partial_y U = 0,$$

as we wished.

For the second part of the lemma, note that

$$P_\gamma f = d_\gamma S\left(\tfrac{n}{2} + \gamma\right) = d_\gamma h, \tag{3.12}$$

where h is given in (3.6). On the other hand, we also have that

$$U = y^{s-n}u = F + y^{2s-n}H,$$

and thus, looking at the orders of y in (3.6), we can conclude that the limit

$$\lim_{y\to 0} y^a \partial_y U \tag{3.13}$$

exists and equals h times the constant 2γ. The lemma is proven by comparing (3.13) together with (3.12), with the Caffarelli-Silvestre construction for the fractional Laplacian as given in (3.2). $\qquad\square$

The next step is to generalize theorem 3.1 to other non-integer exponents γ. To do this, we will first establish in theorem 3.2 below that there are two ways to define the operator P_γ when $\gamma > 1$, and the two definitions agree. We will then apply theorem 3.1 generalize theorem 3.1 to $\gamma > 1$.

The first way to define P_γ is the original definition of Graham Zworski [17], that is to define it using scattering matrix; thus when $\gamma > 1$, $P_\gamma f$ agrees with a higher order term in the power series expansion of the solution of a second order equation (e.g. equation (3.23)); the other is to define it by iterating the work of Caffarelli-Silvestre resulting in a PDE of order higher than 2, one can represent $P_\gamma f_0$ as a lower order term in the power series expansion of the solution of this PDE (e.g. equation (3.22)); we will show that the two definitions agree (see claim (3.16)).

First, fixed $\gamma \in \left(0, \frac{n}{2}\right) \backslash \mathbb{N}$, if $\gamma = m + \gamma_0$, for $m = [\gamma] \in \mathbb{N}$, $\gamma_0 \in (0,1)$, we can define the fractional Laplacian on $\mathbb{R}^n$ inductively as

$$(-\Delta_x)^\gamma = (-\Delta_x)^{\gamma_0} \circ (-\Delta_x)^m. \tag{3.14}$$

We have:

Theorem 3.2. *For any* $\gamma \in \left(0, \frac{n}{2}\right) \backslash \mathbb{N}$, *we have that*

$$P_\gamma[g_\mathbb{H}, |dx|^2] = (-\Delta_x)^\gamma,$$

where the fractional conformal Laplacian P_γ *on* $\mathbb{R}^n$ *is defined as in* (2.3).

Proof. The Proof is by induction on m. We set up the following notation:

$$\gamma_j = \gamma - (m - j) \quad \text{for} \quad j = 0, \ldots, m. \quad \text{Note that } j < \gamma_j < j+1, \quad \gamma_m = \gamma,$$
$$s_j = \frac{n}{2} + \gamma_j,$$
$$a_j = 1 - 2\gamma_j.$$

For each $j = 0, \ldots, m$, we set $f_j := (-\Delta_x)^{m-j} f$. Then the eigenvalue problem

$$-\Delta_{g_\mathbb{H}} u_{s_j} - s_j(n - s_j)u_j = 0, \quad \text{in } \mathbb{H}^{n+1},$$

has a unique solution $u_j = F_j y^{n-s_j} + H_j y^{s_j}$ satisfying $F_j|_{y=0} = f_j$. Set $h_j = H_j|_{y=0}$. Then the scattering operator is simply

$$P_{\gamma_j} f_j = d_{\gamma_j} S(s_j) f_j = h_j.$$

We set $U_j := y^{n-s_j} u_j$. The same computation as in the proof of theorem 3.1 gives that U_j is a solution of

$$\Delta U_j + \frac{a_j}{y} \partial_y U_j = 0. \tag{3.15}$$

On the other hand, since $\gamma_0 \in (0,1)$, theorem 3.1 applied to U_0 implies that

$$h_0 = \frac{1}{2\gamma_0} \lim_{y \to 0} \left(y^{a_0} \partial_y U_0\right),$$

and this limit is well defined.

We already know from theorem 3.1 that $P_{\gamma_0} = (-\Delta_x)^{\gamma_0}$. On the other hand, by construction $f_0 = (-\Delta_x)^m f$. Now we claim that

$$P_{\gamma_0} f_0 = P_\gamma f, \tag{3.16}$$

or equivalently, because of the definition of P_γ as in (2.3),

$$d_{\gamma_0} h_0 = d_\gamma h_m. \tag{3.17}$$

The proof of the theorem will be completed if we show claim (3.16).

We claim first that

$$\Delta U_m = -\frac{a_m}{1 + a_m} U_{m-1} \quad \text{in } \mathbb{R}^{n+1}_+. \tag{3.18}$$

Indeed, as mentioned above in (3.15), U_m is a solution of

$$\Delta U_m + \frac{a_m}{y}\partial_y U_m = 0. \tag{3.19}$$

Differentiating above expression with respect to y, we obtain

$$\Delta(\partial_y U_m) - \frac{a_m}{y^2}\partial_y U_m + \frac{a_m}{y}\partial_{yy} U_m = 0. \tag{3.20}$$

And taking the Laplacian of (3.19),

$$\Delta^2 U_m + a_m\left(\frac{1}{y}\Delta_x(\partial_y U_m) + \frac{1}{y^3}\partial_y U_m - \frac{2}{y^2}\partial_{yy} U_m + \frac{1}{y}\partial_{yyy} U_m\right) = 0. \tag{3.21}$$

Substitute (3.20) into (3.21) and take into account that $a_m = a_{m-1} - 2$. This proves that

$$\Delta^2 U_m + \frac{a_{m-1}}{y}\partial_y(\Delta U_m) = 0. \tag{3.22}$$

Next, note from (3.15) that the function U_{m-1} solves the equation

$$\Delta U_{m-1} + \frac{a_{m-1}}{y}\partial_y U_{m-1} = 0. \tag{3.23}$$

Thus we see from (3.22) and (3.23) that ΔU_m and U_{m-1} satisfy the same equation in $\mathbb{R}^{n+1}_+$. Let us compare now the boundary values. First of all, by hypothesis,

$$U_{m-1}|_{y=0} = -\Delta_x f. \tag{3.24}$$

On the other hand, we claim that

$$\Delta U_m|_{y=0} = \frac{a_m}{1 + a_m}\Delta_x f. \tag{3.25}$$

The proof of this fact is a simple computation: we know that $U_m = F_m + y^{2s_m - n}H_m$ where $F_m = f + b_m y^2 + O(y^3)$ and $H_m = h_m + O(y^2)$. Then we can compute

$$\Delta U_m|_{y=0} = \Delta_x f + 2b_m \tag{3.26}$$

and $\frac{1}{y}\partial_y U_m|_{y=0} = 2b_m$. From (3.19) we must have

$$b_m = \frac{\Delta_x f}{2(-1 - a_m)}.$$

Equation (3.26) implies then

$$\Delta U_m|_{y=0} = \frac{a_m}{1 + a_m}\Delta_x f.$$

We have shown from (3.24) and (3.25) that ΔU_m and U_{m-1} have the same boundary values (modulo a multiplicative constant). Since, as we have mentioned, they satisfy the same second order elliptic equation, they must coincide in the whole $\mathbb{R}^{n+1}_+$. This is,

$$\Delta U_m = -\frac{a_m}{1 + a_m}U_{m-1}, \tag{3.27}$$

as claimed in (3.27).

Now we are ready to complete the proof of the theorem. From (3.27) and (3.19) we conclude that

$$U_{m-1} = \frac{1 + a_m}{y}(\partial_y U_m). \tag{3.28}$$

We look at the asymptotic expansions for U_m and U_{m-1}:

$$U_m = F_m + y^{2s_m-n}H_m, \quad F_m = f_m + O(y^2), \quad H_m = h_m + O(y^2),$$

so that

$$\frac{1}{y}\partial_y U_m = \frac{1}{y}\partial_y F_m + (2s_m - n)\,y^{2s_m-n-2}H_m + y^{2s_m-n}\left[\frac{1}{y}\partial_y H_m\right],$$

while

$$U_{m-1} = F_{m-1} + y^{2s_{m-1}-n}H_{m-1}, \quad F_{m-1} = f_{m-1} + O(y^2), \quad H_{m-1} = h_{m-1} + O(y^2).$$

Since we have the relation (3.28), comparing the coefficients of the term $y^{2s_m-n-2} = y^{2s_{m-1}-n}$ we obtain that

$$(2s_m - n)(1 + a_m)h_m = h_{m-1},$$

which is

$$(2\gamma_m)h_m = h_{m-1}. \tag{3.29}$$

We set

$$A_m = 2^m \gamma_m \ldots \gamma_1, \text{ if } m > 0, \quad A_0 = 1,$$

and

$$c_m = \prod_{j=1}^m (a_j + 1) \text{ if } m > 0, \quad c_0 = 1, \tag{3.30}$$

which will be necessary later. Applying (3.29) inductively we arrive at

$$A_m c_m h_m = h_0. \tag{3.31}$$

But because

$$c_m A_m = \frac{d_\gamma}{d_{\gamma_0}},$$

then we have actually shown (3.17) and the theorem is proved. $\qquad\square$

Moreover, we have the following characterization for $P_\gamma := P_\gamma[g_{\mathbb{H}}, |dx|^2]$ as a Dirichlet-to-Neumann operator, thus generalizing the result of Caffarelli-Silvestre for exponents $\gamma \in \left(0, \frac{n}{2}\right) \backslash \mathbb{N}$ not necessarily less than one:

Theorem 3.3. *Fix $\gamma \in \left(0, \frac{n}{2}\right) \backslash \mathbb{N}$, $a = 1 - 2\gamma$. Let f a smooth function defined on $\mathbb{R}^n$, and let $U = U(x, y)$ be the solution of the boundary value problem*

$$\begin{cases} \Delta_x U + \dfrac{a}{y}\partial_y U + \partial_{yy} U = 0 & in \quad \mathbb{R}^{n+1}_+, \\ \\ U(x, 0) = f(x) & for\ all \quad x \in \mathbb{R}^n. \end{cases}$$

Then function $u := y^{n-s}U$ is the solution of the Poisson equation on the hyperbolic space $\mathbb{H}^{n+1}$

$$\begin{cases} -\Delta_{g_{\mathbb{H}}} u - s(n-s)u = 0 & in \quad \mathbb{H}^{n+1}, \\ u = Fy^{n-s} + Hy^s, \\ F(x, 0) = f(x). \end{cases}$$

And, the following limit exists and we have the equality

$$P_\gamma f = \frac{d_\gamma}{2\gamma_0} A_m^{-1} \lim_{y \to 0} y^{a_0}\partial_y \left[y^{-1}\partial_y \left(y^{-1}\partial_y \left(\ldots y^{-1}\partial_y U\right)\right)\right], \tag{3.32}$$

where we are taking $m+1$ derivatives in the above expression, and the constant is given by

$$A_m = 2^m(\gamma - 1)\ldots(\gamma - m + 1). \tag{3.33}$$

We are using the notation $m = [\gamma] \in \mathbb{N}$, $\gamma_0 = \gamma - m$, $a_0 = 1 - 2\gamma_0$, and the constant d_γ as defined in (2.3).

Proof. We keep the same notation as in the previous theorem. In this construction, U is precisely U_m.

We would like to show first first that

$$h_0 = \frac{c_m}{2\gamma_0} \lim_{y \to 0} y^{a_0} \overbrace{\partial_y \left[y^{-1} \partial_y (y^{-1} \ldots (y^{-1} \partial_y U_m)) \right]}^{m+1 \text{ derivatives}}, \quad (3.34)$$

where the constant c_m is defined in (3.30), and $P_{\gamma_0} f_0 = h_0$. The proof goes by induction on m. The case $m = 0$ is precisely the conclusion of theorem 3.1. Assume that it is true for $m - 1$, i.e.,

$$h_0 = \frac{c_{m-1}}{2\gamma_0} \lim_{y \to 0} y^{a_0} \overbrace{\partial_y \left[y^{-1} \partial_y (y^{-1} \ldots (y^{-1} \partial_y U_{m-1})) \right]}^{m \text{ derivatives}}. \quad (3.35)$$

Now substitute (3.28) in (3.35) and use that $c_m = c_{m-1}(1 + a_m)$. We immediately obtain (3.34).

Next, we recall relation (3.31) between h_m and h_0. The definition of the operator $P_\gamma f = d_\gamma h_m$ and (3.34) complete the proof of the theorem. $\qquad\square$

4 The extension problem on c.c. Einstein manifolds

We fix $\gamma \in \left(0, \frac{n}{2}\right) \backslash \mathbb{N}$. In this section we discuss the generalization of theorems 3.1, 3.2 and 3.3 from hyperbolic space to any conformally compact Einstein manifold (X, g^+).

First we write the extension problem analogous to (3.1) in a conformally compact Einstein metric. The resulting problem (4.2) is still of divergence-type, degenerate elliptic, and with a weight in the Muckenhoupt class A_2 (c.f. [24]), but some lower order terms appear - they depend on the underlying geometry. This is the content of lemma 4.1.

Next, we point out in lemma 4.5 that, by choosing a suitable defining function ρ^*, which is related to the eigenfunction of the Laplacian of the Einstein metric, the equation (4.15) of the extension theorem on general conformal compact Einstein manifolds is the same as the extension theorem on hyperbolic space studied in section 3, of pure divergence form.

Finally, in theorems 4.7 and 4.8, we show how the extension problem with respect to this new defining function allows to compute the fractional Paneitz operator P_γ.

Lemma 4.1. *Let (X, g^+) be any conformally compact Einstein manifold with boundary M. For any defining function ρ of M in X, not necessarily geodesic, the equation*

$$-\Delta_{g^+} u - s(n-s)u = 0 \quad in \ (X, g^+), \quad (4.1)$$

is equivalent to

$$-div\left(\rho^a \nabla U\right) + E(\rho)U = 0 \quad in \ (X, \bar{g}), \quad (4.2)$$

where

$$\bar{g} = \rho^2 g^+, \quad U = \rho^{s-n} u$$

and the derivatives in (4.2) are taken with respect to the metric $\bar{g}$. The lower order term is given by

$$E(\rho) = -\Delta_{\bar{g}}\left(\rho^{\frac{a}{2}}\right)\rho^{\frac{a}{2}} + \left(\gamma^2 - \frac{1}{4}\right)\rho^{-2+a} + \frac{n-1}{4n} R_{\bar{g}}\rho^a, \quad (4.3)$$

or writing everything back in the metric g^+,

$$E(\rho) = -\Delta_{g^+}(\rho^{\frac{n-1+a}{2}})\rho^{\frac{-n-3+a}{2}} - \left(\frac{n^2}{4} - \gamma^2\right)\rho^{-2+a}. \quad (4.4)$$

Here we denote $s = \frac{n}{2} + \gamma$, $a = 1 - 2\gamma$.

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Remark 4.2. For the model case $X = \mathbb{R}^{n+1}_+$, $M = \mathbb{R}^n$, $g^+ = \frac{dy^2 + |dx|^2}{y^2}$, with the defining function $y > 0$, $\bar{g} = dy^2 + |dx|^2$, it automatically follows from (4.3) that

$$E(y) \equiv 0.$$

Proof. Note that (4.1) is equivalent to

$$L_{g^+} u + \left(\gamma^2 - \tfrac{1}{4}\right) u = 0, \tag{4.5}$$

using the fact that for an Einstein metric g^+,

$$L_{g^+} = -\Delta_{g^+} - \frac{n^2 - 1}{4}.$$

On the other hand, the invariance of the conformal Laplacian reads:

$$L_{g^+}(\phi) = \rho^{\frac{n+3}{2}} L_{\bar{g}}\left(\rho^{-\frac{n-1}{2}} \phi\right)$$

for the change of metric $\bar{g} = \rho^2 g^+$. Thus, writing everything in terms of $\bar{g}$ and the new function $U = \rho^{s-n} u$, then (4.5) is just

$$L_{\bar{g}}\left(\rho^{\frac{a}{2}} U\right) + \left(\gamma^2 - \tfrac{1}{4}\right) \rho^{-2 + \frac{a}{2}} U = 0. \tag{4.6}$$

Next, it is a straightforward computation to check that:

$$\rho^{\frac{a}{2}} \Delta_{\bar{g}}\left(\rho^{\frac{a}{2}} U\right) = \text{div}_{\bar{g}}\left(\rho^a \nabla_{\bar{g}} U\right) + \Delta_{\bar{g}}\left(\rho^{\frac{a}{2}}\right) \rho^{\frac{a}{2}} U.$$

Substituting the above expression into (4.6) gives the desired result (4.2).

$\square$

Let (X, g^+) be a conformally compact Einstein metric with boundary $(M, [\hat{g}])$. Then, lemma 2.1 in [14] gives that, fixed a metric $\hat{g}$ on the boundary M, there exists a unique defining function y in X such that in the neighborhood $M \times (0, \delta)$, the metric splits as

$$g^+ = \frac{dy^2 + g_y}{y^2}, \tag{4.7}$$

where g_y is a one-parameter family of metrics on M with $g_y|_{y=0} = \hat{g}$. Moreover,

$$g_y = g^{(0)} + \frac{g^{(2)}}{2} y^2 + \dots \tag{4.8}$$

only contains even terms up to order n. We write

$$g^{(0)} := \hat{g}, \quad g^{(1)} := \partial_y g_y|_{y=0} = 0, \quad g^{(2)} := \partial_{yy} g_y|_{y=0}. \tag{4.9}$$

Set $\bar{g} = y^2 g^+$ and

$$\Psi := \partial_y \log \det(g_y) = \sum_{i,j} g_y^{ij} \partial_y (g_y)_{ij}. \tag{4.10}$$

Note that

$$\Psi_0 := \frac{1}{2n} \Psi|_{y=0} = \frac{1}{2n} trace_{\hat{g}}(g^{(1)})$$

is the mean curvature of M as hypersurface of $(X, \bar{g})$, which is zero by (4.9).

Lemma 4.1 is true for any defining function independent of the behavior near the boundary. However, in the specific case that we have the splitting (4.7), then we can have a more explicit expression for the lower order terms $E(\rho)$:

Theorem 4.3. *If the defining function (denoted by y in the following) is chosen such that metric splits as (4.7) in a neighborhood $M \times (0, \delta)$, then*

$$E(y) = \frac{-n+1+a}{4} \Psi y^{a-1} = \frac{n-1-a}{4n} R_{\bar{g}} y^a \quad in \ M \times (0, \delta).$$

And, in particular,

$$\lim_{y \to 0} \frac{E(y)}{y^a} = \frac{-n+1+a}{4} trace_{g^{(0)}} g^{(2)}.$$

Moreover, formula (3.32) for the calculation of the conformal fractional Laplacian is still true, i.e.,

$$P_\gamma[g^+, \hat{g}]f = \frac{d_\gamma}{2\gamma_0} A_m^{-1} \lim_{y \to 0} y^{a_0} \partial_y \left[y^{-1} \partial_y \left(y^{-1} \partial_y \left(\ldots y^{-1} \partial_y U \right) \right) \right],$$

where are $m + 1$ derivatives in the formula above, and U is the solution of the extension problem

$$\begin{cases} -div\left(y^a \nabla U\right) + E(y)U = 0 & in \ (X, \bar{g}), \\ \qquad\qquad\qquad\qquad\ U = f & on \ M; \end{cases} \tag{4.11}$$

here the derivatives are taken with respect to the metric $\bar{g} = y^2 g^+$, and the constants are $M = [\gamma]$, $\gamma_0 = \gamma - m$, $a_0 = 1 - 2\gamma_0$, A_m is given in (3.33) and d_γ in (2.3).

Proof. The first assertion is a straightforward calculation from (4.3): we know that near $\{y = 0\}$, the metric $\bar{g}$ can be split as $\bar{g} = dy^2 + g_y$. Then

$$\Delta_{\bar{g}} = \partial_{yy} + \tfrac{1}{2} \Psi \partial_y + \Delta_{g_y}$$

for Ψ as given in (4.10). Moreover, for the conformal change $g^+ = y^{-2} \bar{g}$, we can write the scalar curvature equation

$$-\Delta_{\bar{g}}\left(y^{-\frac{N-2}{2}}\right) + c_N R_{\bar{g}} y^{-\frac{N-2}{2}} = c_N \left(y^{-\frac{N-2}{2}}\right)^{\frac{N+2}{N-2}} R_{g^+}, \quad c_N = \frac{N-2}{4(N-1)}, \quad N = n + 1.$$

Since $R_{g^+} = -N(N-1)$ we quickly obtain that

$$R_{\bar{g}} = -n\Psi y^{-1}.$$

With all these ingredients, computing all the terms in (4.3), we can show that near M,

$$E(y) = \frac{-n+1+a}{4} \Psi y^{a-1} = \frac{n-1-a}{4n} R_{\bar{g}} y^a.$$

Now we compute the asymptotic behavior of $E(y)$ when $y \to 0$. Note that for (even) Poincaré metrics, we have the expansion (4.8), and thus,

$$\lim_{y \to 0} \frac{E(y)}{y^a} = \frac{-n+1+a}{4} trace_{g^{(0)}} g^{(2)}.$$

$\square$

Remark 4.4. Before we continue, we remind the reader of how to compute the $Q_\gamma[g^+, \hat{g}]$ curvature, as defined in (2.6), for $\gamma \in \left(0, \frac{n}{2}\right) \setminus \mathbb{N}$, $s = \frac{n}{2} + \gamma$. We set $f \equiv 1$, and find the solution to the Poisson problem $-\Delta_{g^+} v - s(n-s)v = 0$ such that

$$v = F y^{n-s} + H y^s, \quad F|_{y=0} = 1, \quad H|_{y=0} = h.$$

Then,

$$Q_\gamma[g^+, \hat{g}] = d_\gamma h.$$

Next, we show that it is possible to find a special defining function satisfying that the zero-th order term $E(\rho)$ in equation (4.2) vanishes so that the extension problem is a pure divergence equation similar to the Euclidean one (3.1) studied in section 3. We recover the conformal powers of the Laplacian as the Dirichlet-to-Neumann operator from theorem 3.1 (or theorems 3.2, 3.3), plus a curvature term.

Lemma 4.5. *Let (X, g^+) be a conformally compact Einstein manifold with conformal infinity $(M, [\hat{g}])$. Fixed a metric $\hat{g}$ on M, assume that y is the defining function on X such that on a neighborhood $M \times (0, \delta)$, the metric splits as $g^+ = \frac{dy^2 + g_y}{y^2}$, where g_y is a one-parameter family of metrics on M with $g_y|_{y=0} = \hat{g}$, and Taylor expansion (4.8). For each $\gamma \in (0, 1)$, there exists another (positive) defining function ρ^* on $M \times (0, \delta)$, satisfying $\rho^* = y + O(y^{2\gamma+1})$, and such that for the term E defined in (4.3) we have*

$$E(\rho^*) \equiv 0.$$

Moreover, the metric $g^* = (\rho^*)^2 g^+$ satisfies $g^*|_{y=0} = \hat{g}$ and has asymptotic expansion

$$g^* = (d\rho^*)^2 \left[1 + O((\rho^*)^{2\gamma}) \right] + \hat{g} \left[1 + O((\rho^*)^{2\gamma}) \right]. \tag{4.12}$$

Proof. We solve the eigenvalue problem (2.1) with Dirichlet condition (2.2) given by $f \equiv 1$, and $s = \frac{n}{2} + \gamma$. The solution can be written as $v = y^{n-s}F + y^s H$ for

$$F = 1 + O(y^2), \quad H = h + O(y^2).$$

We set

$$\rho^* := v^{1/(n-s)}; \tag{4.13}$$

we emphasize that ρ^* is chosen as a power of the eigenfunction of Δ_{g^+} at $s = \frac{n}{2} + \gamma$. We check now that this ρ^* satisfies all the properties stated in the lemma.

First note that v solves the equation $-\Delta_{g^+} v - s(n-s)v = 0$. This quickly implies the vanishing of $E(\rho^*)$ if we use the equivalent formula (4.4). Next, the asymptotic behavior of ρ^* is precisely

$$\rho^*(y) = y \left[1 + \frac{1}{n-s} h y^{2\gamma} + O(y^2) \right]. \tag{4.14}$$

Moreover, because of remark 4.4, we precisely have $d_\gamma h = Q_\gamma[g^+, \hat{g}]$. On the other hand, the asymptotic expansion (4.12) for g^* can be easily checked from (4.14) and the asymptotic behavior of $\bar{g} = dy^2 + g_y(x)$, $g_y = \hat{g} + O(y^2)$. $\square$

Lemma 4.6. *The ρ^* constructed in the previous lemma can be defined in the whole X and it is indeed positive and smooth.*

Proof. A summary of properties for degenerate elliptic equations, necessary to deal with (4.1) or (4.2) can be found in González-Qing [13]. The classical reference on degenerate elliptic equations with Muckenhoupt weights is Fabes-Kenig-Serapioni [10], while Cabré-Sire [3] has retaken the study in relation to fractional Laplacians on Euclidean space.

Note that v is strictly positive in the whole X, thanks to the maximum principle and uniqueness for equation (4.1). This shows that ρ^* is an acceptable defining function on the whole X. $\square$

Theorem 4.7. *Assume the same hypothesis as in lemma 4.5. Fix $\gamma \in (0, 1)$ and let ρ^* be the special defining function constructed in lemma 4.5. For each smooth function f on M, let U solve the extension problem*

$$\begin{cases} -div\left((\rho^*)^a \nabla U\right) = 0 & in\ (X, g^*), \\ \qquad\qquad U = f & on\ M, \end{cases} \tag{4.15}$$

where the derivatives are taken with respect to the metric $g^* = (\rho^*)^2 g^+$. Then

$$P_\gamma[g^+, \hat{g}] f = \frac{d_\gamma}{2\gamma} \lim_{\rho^* \to 0} (\rho^*)^a \partial_{\rho^*} U + f Q_\gamma[g^+, \hat{g}]. \tag{4.16}$$

The fractional order curvature $Q_\gamma[g^+, \hat{g}]$ is defined in remark 4.4.

Proof. We set up the same notation as in the previous results. On one hand, in order to compute the scattering operator with Dirichlet data f we need to consider the equation

$$-\Delta_{g^+} u - s(n-s)u = 0 \quad \text{in } (X, g^+). \tag{4.17}$$

There exists a solution $u = y^{n-s}F + y^s H$ with $F = f + 0(y^2)$, $H = h + O(y^2)$. Then $P_\gamma f := P_\gamma[g^+, \hat{g}]f$ is defined as $P_\gamma f = d_\gamma h$.

On the other hand, the Q_γ curvature can be computed as indicated in remark 4.4. Let v be the solution of the Poisson equation (4.17), but with Dirichlet data data identically $\tilde{f} \equiv 1$. It can be written as

$$v = y^{n-s}\tilde{F} + y^s \tilde{H} \quad \text{with} \quad \tilde{F} = 1 + 0(y^2), \quad \tilde{H} = \tilde{h} + O(y^2).$$

Then $Q_\gamma := P_\gamma 1 = d_\gamma \tilde{h}$. In addition, $\rho^* = v^{1/(n-s)}$.

Now we claim that $U := (\rho^*)^{s-n}u$ satisfies (4.15). First, lemma 4.1 applied to the defining function ρ^* gives that U is a solution of (4.2). Second, for our special choice of defining function, $E(\rho^*) = 0$ thanks to lemma 4.5. Finally, note that by construction,

$$U = \frac{u}{v} = \frac{F + y^{2s-n}H}{\tilde{F} + y^{2s-n}\tilde{H}}.$$

In particular, $U|_{y=0} = f$. This shows that U is a solution of (4.15), as claimed.

Next, let us compute $\partial_y U$. It is easy to check that

$$\lim_{y \to 0} y^a \partial_y U = (2\gamma)h - 2\gamma f\tilde{h},$$

and (4.16) follows. This completes the proof of theorem 4.7. $\square$

Before we state the next result, we need to introduce some notations: for any smooth function w, we denote $w = O_E(1)$ if the function w has only even terms in the expansion (up to order n), i.e,

$$w = w_0 + w_1 y^2 + w_2 y^4 + \ldots$$

We also define the operator $B := y^{-1}\partial_y$. We claim that for any $k = 1, 2, \ldots$, it is true that:

1. If $w = O_E(1)$, then also $B^k w = O_E(1)$.

2. If $w = hy^l O_E(1)$ for some function $h = h(x)$ and $l \in \mathbb{N}$, then

$$B^k w = l(l-2)\ldots(l - 2(k-1))y^{l-2k}h + O(y^{l-2k+1}).$$

and

$$\partial_y B^k w = l(l-2)\ldots(l - 2(k-1))(l - 2k)y^{l-2k-1}h + O(y^{l-2k}).$$

We show now that theorem 4.7 is is still valid for any exponent $\gamma \in \left(0, \frac{n}{2}\right) \setminus \mathbb{N}$, thus generalizing theorem 3.3 on hyperbolic space to any conformally compact Einstein manifold:

Theorem 4.8. *Fix $s = \frac{n}{2} + \gamma$ for $\gamma \in \left(0, \frac{n}{2}\right) \setminus \mathbb{N}$. Set $\gamma = m + \gamma_0$, $m = [\gamma]$, $\gamma_0 \in (0, 1)$, $a = 1 - 2\gamma$. In the hypothesis of theorem 4.7, let U be a solution of (4.15) Then*

$$P_\gamma[g^+, \hat{g}]f = \frac{d_\gamma}{2\gamma_0} A_m^{-1} \left[\lim_{\rho^* \to 0} (\rho^*)^{a_0} \partial_{\rho^*} B^m U \right] + f Q_\gamma[g^+, \hat{g}],$$

where $a_0 = 1 - 2\gamma_0$ and A_m is given in (3.33).

Proof. Let us compute

$$Lim := \lim_{y \to 0} y^{a_0} \partial_y (B^m U)$$

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for $U = (\rho^*)^{s-n}u$. As before, we can write

$$U = \frac{F + y^{2s-n}H}{\tilde{F} + y^{2s-n}\tilde{H}} =: \frac{V}{W},$$

where

$$F = f + y^2 O_E(1), \tilde{F} = 1 + y^2 O_E(1), \quad H = h + y^2 O_E(1), \tilde{H} = \tilde{h} + y^2 O_E(1).$$

We apply the product formula

$$B^m(VW^{-1}) = \sum_{k=0}^{m} c_{m,k}(B^k V)(B^{m-k}(W^{-1})).$$

From this sum, only the terms $k = 0$, $k = m$ are important, since the rest are of higher order in y. Note that the term $k = m$ is just

$$(B^m V)W^{-1} = (2s - n)(2s - n - 2)\dots(2s - n - 2(m-1))y^{2s-n-2m}[h + y^2 O_E(1)]$$
$$+ O_E(1) + h.o.t.$$

so that

$$\partial_y \left[(B^m V)W^{-1}\right] = O(y) + 2\gamma_0 A_m y^{2\gamma_0 - 1} h + h.o.t.$$

On the other hand, for the $k = 0$ term we observe that

$$B^m(W^{-1}) = -B^m(W)/W^2 + h.o.t.,$$

and thus

$$\partial_y \left[VB^m(W^{-1})\right] = -y^{2\gamma_0 - 1}2\gamma_0 A_m \tilde{h}f + h.o.t.$$

Consequently

$$Lim = 2\gamma_0 A_m h - 2\gamma_0 A_m \tilde{h}f = \frac{2\gamma_0 A_m}{d_\gamma}\left[P_\gamma f - fP_\gamma 1\right],$$

and the the proof of the proposition is completed after taking into account that $P_\gamma 1 = Q_\gamma$. □

5 The general case

Given a compact manifold $\bar{X}$ with boundary M, and a smooth metric $\bar{g}$ on $\bar{X}$, there exists an asymptotically hyperbolic metric with constant scalar curvature in the interior X, in the same conformal class of $\bar{g}$. This is the well known singular Yamabe problem, that has been well understood in a series of papers. We should cite Aviles-MacOwen [2], Mazzeo [22], Andersson-Chrusciel-Friedrich [1], in the case of negative constant scalar curvature.

We remark now that the construction of scattering operator $S(s)$ can be generalized to manifolds which are not Einstein, but just asymptotically hyperbolic (c.f. Mazzeo-Melrose [23] for most of the values of the parameter s, and Guillarmou [18] for a closer look at the remaining poles). In this section we try to understand how many of the results in the previous sections generalize to a compact manifold with boundary $(\bar{X}, \bar{g})$, not necessarily conformally compact Einstein. For the rest of the section, $\gamma \in (0, 1)$.

The case $\gamma = \frac{1}{2}$ was studied by Gillarmou-Gillopé [19]. They considered the scattering operator of asymptotically hyperbolic manifold, and its relation to mean curvature. Note that $\gamma = \frac{1}{2}$ is a splitting point for the behavior of P_γ, as we will see in theorem 5.1.

Let $(\bar{X}, \bar{g})$ be a smooth $(n+1)$-dimensional compact manifold with boundary $M^n = \partial X$. As we have mentioned above, there exists an asymptotically hyperbolic metric g^+ in the interior X, conformal to $\bar{g}$, and that has negative constant scalar curvature

$R_{g^+} = -n(n+1)$. Moreover, g^+ has a very specific polyhomogeneous expansion. More precisely, let ρ be a geodesic boundary defining function of $(\partial \bar{X}, \bar{g})$, i.e.,

$$\bar{g} = d\rho^2 + \bar{g}_\rho$$

for some one-parameter family of metrics $\bar{g}_\rho$ on M, then we have that

$$g^+ = \frac{\bar{g}(1 + \rho\alpha + \rho^n\beta)}{\rho^2}, \tag{5.1}$$

where $\alpha \in \mathcal{C}^\infty(\bar{X})$, $\beta \in \mathcal{C}^\infty(X)$ and β has a polyhomogeneous expansion

$$\beta(\rho, x) = \sum_{i=0}^\infty \sum_{j=0}^{N_i} \beta_{ij} \rho^i (\log \rho)^j \tag{5.2}$$

near the boundary, $N_i \in \mathbb{N} \cup \{0\}$ and $\beta_{ij} \in \mathcal{C}^\infty(\bar{X})$. Here we note that the log terms do not appear in the first terms of the expansion, and they can be ignore in our setting, because $\gamma \in (0,1)$ and we will not need such high orders. We define

$$\frac{1}{\hat{\rho}^2} := \frac{1 + \rho\alpha + \rho^n\beta}{\rho^2}, \tag{5.3}$$

so that (5.1) is rewritten as

$$g^+ = \frac{\bar{g}}{\hat{\rho}^2}, \tag{5.4}$$

On the other hand, note that $\bar{g}_\rho$ may not only have even terms in its expansion. However, by the work of Graham-Lee [16], fixed the boundary metric $\hat{g} := \bar{g}_\rho|_{\rho=0} = \bar{g}|_M$, we can find a boundary defining function $y = \rho + O(\rho^2)$ such that

$$g^+ = \frac{dy^2 + g_y}{y^2} \tag{5.5}$$

near M, where g_y is a one-parameter family of metrics on M such that $g_y|_{y=0} = \hat{g}$, with the regularity of $\rho\alpha + \rho^n\beta$. The main property of g_y is that, if we make the expansion

$$g_y = g^{(0)} + g^{(1)}y + O(y^2),$$

then

$$g^{(0)} = \hat{g}, \quad trace_{g^{(0)}} g^{(1)} = 0. \tag{5.6}$$

We set $\tilde{g} = dy^2 + g_y$ so that

$$g^+ = \frac{\tilde{g}}{y^2}. \tag{5.7}$$

The scattering operator for (X, g^+) is computed as follows: first, solve the Poisson equation $-\Delta_{g^+} u - s(n-s)u = 0$. For each $f \in \mathcal{C}^\infty(M)$, there exists a solution of the form

$$u = y^{n-s}F + y^s H, \quad F = f + O(y^2), \quad H = h + O(y). \tag{5.8}$$

Then, for $s = \frac{n}{2} + \gamma$, we define the conformal fractional Laplacian in this setting as

$$P_\gamma[g^+, \hat{g}]f = d_\gamma h. \tag{5.9}$$

In our case, we do have some log terms in the expansion (5.2). However, they do appear at order n, and consequently, they do not change the first terms in the asymptotic expansion for u.

We write

$$\hat{\rho} = y(1 + y\phi + O(y^2)) \tag{5.10}$$

for some $\phi \in \mathcal{C}^\infty(\partial X)$ which will be made precise later. Because the metrics (5.4) and (5.7) are equal, we can write $y^2 \bar{g} = \hat{\rho}^2 \tilde{g}$. Then, restricting to the tangential direction, we get that

$$(1 + 2y\phi + O(y^2))g_y = \bar{g}_\rho. \tag{5.11}$$

We write the Taylor expansions for g_y and $\bar{g}_\rho$ in the variable y, taking into account that $\rho = y + O(y^2)$:

$$\begin{aligned} g_y &= g^{(0)} + g^{(1)}y + O(y^2), \quad g^{(0)} = \hat{g}, \quad g^{(1)} = 0, \\ \bar{g}_\rho &= \bar{g}^{(0)} + \bar{g}^{(1)}y + O(y^2), \quad \bar{g}^{(0)} = \hat{g}, \end{aligned} \tag{5.12}$$

then looking at the orders of y in (5.11), we obtain

$$g^{(1)} + \phi\hat{g} = \bar{g}^{(1)}.$$

Taking trace above with respect to $\hat{g}$, and using (5.6), we are able to find a formula for ϕ. Indeed,

$$\phi = -\frac{1}{2n} trace_{\hat{g}}(\bar{g}^{(1)}). \tag{5.13}$$

This is nothing but the mean curvature of ∂X as a boundary of the $(n+1)$-manifold $(\bar{X}, \bar{g})$, with a minus sign. We denote it by

$$\Psi_0 := \frac{1}{2n} trace_{\hat{g}}(\bar{g}^{(1)}). \tag{5.14}$$

We have shown from (5.10) and (5.13) that

$$\hat{\rho} = y(1 - \Psi_0 y + O(y^2)). \tag{5.15}$$

We will need at a later section the relation between ρ and y, so we indicate it here. First, from (5.3) we know that

$$\rho = \hat{\rho}\left(1 + \frac{\alpha}{2}\hat{\rho} + O(\hat{\rho}^2)\right). \tag{5.16}$$

If we substitute (5.15), then

$$\rho = y\left[1 + \left(-\Psi_0 + \frac{\alpha}{2}\right)y + O(y^2)\right]. \tag{5.17}$$

In the following, we show in what cases lemma 4.1 for the calculation of $P_\gamma[g^+, \hat{g}]$ is still true:

Theorem 5.1. *Let $(\bar{X}, \bar{g})$ be a smooth, $(n+1)$-dimensional smooth, compact manifold with boundary, and let $\hat{g}$ be the restriction of the metric $\bar{g}$ to the boundary $M := \partial X$. Let ρ be a geodesic boundary defining function. Then there exists an asymptotically hyperbolic metric g^+ on X of the form (5.1), with respect to which the conformal fractional Laplacian $P_\gamma[g^+, \hat{g}]$ can be defined as (5.9), and it can be computed through the following extension problem: for each smooth given function f on M, consider*

$$\begin{cases} -div(\rho^a \nabla U) + E(\rho) = 0 & in \ (\bar{X}, \bar{g}), \\ \qquad\qquad\qquad U = f & on \ M, \end{cases} \tag{5.18}$$

where the derivatives are taken with respect to the original metric $\bar{g}$. Then, there exists a unique solution U and moreover,

1. For $\gamma \in (0, \frac{1}{2})$,

$$P_\gamma[g^+, \hat{g}]f = \frac{d_\gamma}{2\gamma} \lim_{\rho \to 0} \rho^a \partial_\rho U, \tag{5.19}$$

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2. *For $\gamma = \frac{1}{2}$, we have an extra term*

$$P_{\frac{1}{2}}[g^+, \hat{g}]f = \lim_{\rho \to 0} \partial_\rho U + \left(\frac{n}{2} - \frac{1}{2}\right) \Psi_0 f,$$

where Ψ_0 is the mean curvature of M as defined in (5.14).

3. *If $\gamma \in \left(\frac{1}{2}, 1\right)$, the limit in the right hand side of (5.19) exists if and only if the the mean curvature Ψ_0 of the boundary ∂X in $(\bar{X}, \bar{g})$ vanishes identically, in which case, (5.19) holds too.*

Proof. We follow the notations in the previous paragraphs. It is possible to find an asymptotically hyperbolic metric g^+ on X, of the form (5.1). We define the new functions $\hat{\rho}$ as (5.3), and y satisfying (5.5). Then we have seen that it is possible to define the scattering operator for the metric (X, g^+), as follows: let u be the solution of the eigenvalue problem $-\Delta_{g^+} u - s(n-s)u = 0$, with given data f. Its asymptotic expansion is given in (5.8), $u = y^{n-s}F + y^s H$, $F_{y=0} = f$, $H|_{y=0} = h$, while the scattering operator is just $P_\gamma = d_\gamma h$.

Second, lemma 4.1 applied to the defining function $\hat{\rho}$ shows that

$$U := \hat{\rho}^{s-n} u$$

is the solution of (5.18), since $\bar{g} = \hat{\rho}^2 g^+$.

Next, we have shown in (5.15) that

$$\hat{\rho} = y(1 - y\Psi_0 + O(y^2)),$$

so that we can expand U as

$$U = \left[f - (s-n)\Psi_0 y + O(y^2)\right] + y^{2s-n}\left[h + O(y)\right].$$

Then

$$y^a \partial_y U = -(s-n)\Psi_0 f y^a + (2s-n)h + o(1). \tag{5.20}$$

Now we take the limit above when $y \to 0$. It is clear that the limit exists if and only if $a \geq 0$, unless $\Psi_0 = 0$. In particular, if $\gamma < \frac{1}{2}$, then

$$\lim_{y \to 0} y^a \partial_y U = (2s-n)h,$$

as desired. For the case $\gamma = \frac{1}{2}$, just note that $d_{\frac{1}{2}} = -1$. $\qquad\square$

Remark 5.2. The second conclusion of theorem 5.1, i.e., the case $\gamma = \frac{1}{2}$, was shown by Guillarmou-Gillopé in [19].

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