Classical Solutions for a nonlinear Fokker-Planck equation arising in Computational Neuroscience

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Abstract

In this paper we analyze the global existence of classical solutions to the initial boundary-value problem for a nonlinear parabolic equation describing the collective behavior of an ensemble of neurons. These equations were obtained as a diffusive approximation of the mean-field limit of a stochastic differential equation system. The resulting nonlocal Fokker-Planck equation presents a nonlinearity in the coefficients depending on the probability flux through the boundary. We show by an appropriate change of variables that this parabolic equation with nonlinear boundary conditions can be transformed into a non standard Stefan-like free boundary problem with a Dirac-delta source term. We prove that there are global classical solutions for inhibitory neural networks, while for excitatory networks we give local well-posedness of classical solutions together with a blow up criterium. Surprisingly, we will show that the spectrum for the operator in the linear case, that corresponding to a system of uncoupled networks, does not give any information about the large time asymptotic behavior.

1 Introduction

Collective behavior of a large ensemble of interacting neurons is commonly modeled by a system of stochastic differential equations. Each subsystem describes an individual neuron in the network as an electric circuit model with a choice of parameters such as the membrane potential v, the conductances, the proportion of open ion channels and their type. The individual description of each neuron includes a stochastic current, which describes the voltage rate change due to the electrical discharges (spikes) of the rest of the network neurons. We refer to the classical references [17, 12, 28] and the nice brief introduction [15] for a wider overview of this area and further references. As a result of the coupling network, the collective behavior of the stochastic differential system can lead to complicated dynamics such as existence of several stationary states, bifurcations and synchronization (see [1, 22, 23]).

The time evolution of the potential $v(t)$ through the cell membrane has been modeled by several authors [1, 2, 25, 7, 27, 24]. The neurons relax towards their resting potential v_L (leak potential) in the absence of any interaction. As mentioned above, all interactions of the neuron within the network are modeled by an incoming presynaptic current $I(t)$ given by an stochastic

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process to be specified below. Therefore, the evolution of the membrane potential is assumed to follow the equation

$$C_m \frac{dv}{dt} = -g_L(v - v_L) + I(t)\,, \tag{1.1}$$

where C_m is the capacitance of the membrane and g_L denotes the leak conductance. If the voltage achieves the so-called threshold voltage v_{th} (maximum voltage), then the neuron emits a spike transmitted to the network through $I(t)$ and its voltage is instantaneously reset to v_R (reset voltage). The mean firing rate produced by the network $N(t)$ is defined as the average number of spikes per unit time.

The network consists of $C = C_E + C_I$ neurons: C_I inhibitory and C_E excitatory, producing spikes of strength J_E and J_I respectively at their spike times. The total presynaptic current $I(t)$ in (1.1) is computed as the difference of the total spike strengths received through the synapses by a neuron at the network:

$$I(t) = J_E \sum_{i=1}^{C_E} \sum_j \delta(t - t^i_{Ej}) - J_I \sum_{i=1}^{C_I} \sum_j \delta(t - t^i_{Ij})\,,$$

where t^i_{Ej} and t^i_{Ij} are the times of the j^{th}-spike coming from the i^{th}-presynaptic excitatory and inhibitory neurons respectively. Most of the microscopic models for neuron dynamics assume that the spike times, t^i_{Ej} and t^i_{Ij}, follow independent discrete Poisson processes with probability of emitting a spike per unit time ν. This stochastic process $I(t)$ has mean given by $\mu_C = B\nu$ with $B := C_E J_E - C_I J_I$ and variance $\sigma_C^2 = (C_E J_E^2 + C_I J_I^2)\nu$. We will say that the network is excitatory if $B > 0$ (inhibitory respectively if $B < 0$). The model just described is known [12, 28, 1] as the Leaky Integrate & Fire (LIF) neuron model.

Dealing with these discrete Poisson processes can be difficult and thus an approximation was proposed in the literature. This approximation consists in substituting the stochastic process $I(t)$ by a drift-diffusion process with the same mean and variance $I(t)\,dt \approx \mu_C\,dt + \sigma_C\,d\mathcal{W}_t$, where $\mathcal{W}_t$ is the standard Brownian motion. We refer for more details of this approximation to [1, 2, 25, 7, 27, 24, 20]. The approximation to the original LIF neuron model (1.1) is given by

$$dv = (-v + v_L + \mu_C)\,dt + \sigma_C\,d\mathcal{W}_t\,, \tag{1.2}$$

where we choose the units such that $C_m = g_L = 1$, for $v \le v_{th}$ with the jump process: $v(t_o^+) = v_R$ whenever at t_o the voltage achieves the threshold value $v(t_o^-) = v_{th}$; with $v_L < v_R < v_{th}$. The last ingredient of the model is given by the probability ν of firing per unit time of the Poisson processes, i.e., the so-called total firing rate. The firing rate depends on the activity of the network and on some external stimuli and it is given by $\nu = \nu_{ext} + N(t)$, where $N(t)$ is the mean firing rate produced by the network and $\nu_{ext} \ge 0$ is the external firing rate. The value of $N(t)$ is then computed as the flux of neurons across the threshold or firing voltage v_{th}.

The stochastic problem (1.2) with the jump process specified above can be written in terms of a partial differential equation for the evolution of the probability density $p(v,t) \ge 0$ of finding neurons at a voltage $v \in (-\infty, v_{th}]$ at a time $t \ge 0$. This PDE has the structure of a backward Kolmogorov or Fokker-Planck equation with sources and is given by

$$\frac{\partial p}{\partial t}(v,t) = \frac{\partial}{\partial v}\big[(v - v_L - \mu_C)p(v,t)\big] + \frac{\sigma_C^2}{2}\frac{\partial^2 p}{\partial v^2}(v,t) + N(t)\,\delta_{v=v_R}, \qquad v \le v_{th}\,. \tag{1.3}$$

A Dirac delta source term in the right-hand side appears due to the firing at time $t \ge 0$ for neurons whose voltage is immediately reset to v_R. Imposing the condition that no neuron should have the firing voltage due to their instantaneous discharge, we complement (1.3) with Dirichlet and initial boundary conditions: $p(v_{th}, t) = 0$, $p(-\infty, t) = 0$, and $p(v, 0) = p_I(v) \ge 0$. The mean firing rate $N(t)$ is implicitly given by $N(t) := -\frac{\sigma_C^2}{2}\frac{\partial p}{\partial v}(v_{th}, t) \ge 0$, that is the flux of probability at v_{th}. The above definition for $N(t)$ formally implies that any solution to (1.3) is a probability density for all times, i.e.

$$\int_{-\infty}^{v_{th}} p(v,t)\,dv = \int_{-\infty}^{v_{th}} p_I(v)\,dv = 1\,, \quad \text{for all } t \ge 0.$$

Let us note that in most of the computational neuroscience literature [1, 20], equation (1.3) is specified on the intervals $(-\infty, v_R)$ or (v_R, v_{th}) with no Dirac delta source term but rather a boundary condition relating the values of the fluxes from the right and the left at $v = v_R$. The formulation presented here is equivalent and more suitable for mathematical treatment. Other more complicated microscopic models including the conductance, and leading to kinetic-like Fokker-Planck equations, have been studied recently, see [4] and the references therein. Finally, the nonlinear Fokker-Planck equation (1.3) can be rewritten as

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial v^2} + \frac{\partial}{\partial v}[(v-\bar{\mu})p] + N(t)\,\delta_{v=v_R}, \quad v \le v_{th}\,,$$

where $\sigma^2 = 2a_0^2 + a_1 N(t)$, with $a_0 > 0$, $a_1 \ge 0$ and $\bar{\mu} = B\nu_{ext} + BN(t)$. We will focus only on the simplest case in which the nonlinearity in the diffusion coefficient is neglected by assuming $a_1 = 0$. Without loss of generality, we can choose a new voltage variable $\tilde{v} \le 0$ and a scaled density $\tilde{p}$ defined by $\tilde{p}(t,\tilde{v}) = \beta p(t, \beta\tilde{v} + v_{th})$ for $\beta = a_0$. Then our main equation, after dropping the tildes, reads as

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial v^2} + \frac{\partial}{\partial v}\,[(v-\mu)p] + N(t)\,\delta_{v=v_R}, \quad v \le 0\,, \tag{1.4}$$

where the drift term, source of the nonlinearity, is given by

$$\mu = b_0 + bN(t) \qquad \text{with } N(t) = -\frac{\partial p}{\partial v}(0,t) \ge 0, \tag{1.5}$$

for $b_0 = (B\nu_{ext} - v_{th})/a_0$ and $b = B/a_0^3$, and complemented by the conditions

$$p(0,t) = 0, \qquad p(-\infty,t) = 0, \qquad p(v,0) = p_I(v) \ge 0\,. \tag{1.6}$$

Let us remark that the sign of b_0 determines if the neurons due only to external stimuli may produce a spike or not, therefore it controls the strength of the external stimuli.

The aim of this paper is to analyze the well-posedness of classical solutions to the initial-boundary value problem (1.4), (1.5), and (1.6). We first give a characterization of the maximal time of existence of classical solutions. We show that if the maximal existence time is finite, it coincides with the time at which the firing rate $N(t)$ diverges. Next, we show that classical solutions exist globally in time for inhibitory networks $b < 0$.

In a recent work [3], it was shown that the problem (1.4)-(1.6) can lead to finite-time blow up of solutions for excitatory networks $b > 0$ when the initial data is concentrated close enough to the threshold voltage. This result was obtained by a contradiction argument giving no information about the behavior at the blow-up time. Our theorem gives a characterization of this blow-up time when it occurs for $b > 0$. We do not have at the moment a complete understanding of the set of initial data leading to blow-up in finite time. This divergence in finite time of the firing rate has no clear biological significance; it could mean that some sort of synchronization of the whole network happens, see [3] for a deeper discussion. This is a scenario that does not show up in the typical reported applications [1, 2].

Our main theorem can be summarized as follows:

Theorem 1.1. *Let $p_I(x)$ be a non-negative $\mathcal{C}^1((-\infty, v_{th}]) \cap L^1(-\infty, v_{th})$ function such that $p_I(v_{th}) = 0$ and*

$$\lim_{x\to-\infty} \frac{\partial p_I}{\partial x} = 0.$$

There exists a unique classical solution to the problem (1.4)-(1.6) *in the time interval* $[0, T^*)$ *with $T^* > 0$. The maximal existence time $T^* > 0$ can be characterized as*

$$T^* = \sup\{t > 0 \,:\, N(t) < \infty\}\,.$$

Furthermore, when $b \le 0$ we have that $T^* = \infty$, while for $b > 0$ there exist classical solutions which blow up at finite time T^* and consequently have diverging mean firing rate as $t \nearrow T^*$.

A precise definition of classical solution will be discussed in the next section. The last statement in the Theorem 1.1 for $b > 0$ is obtained by combining the work in [3] with our result on existence of classical solutions and the characterization of the maximal time of existence.

As it is shown in Section 2, the main strategy of the proof is given by an equivalence. This equivalence, through an explicit time-space change of variables, transforms our problem into a Stefan-like free boundary problem with a Dirac delta source term, resembling price-formation models studied in [18, 13, 19, 6, 14]. Although the methods are based on finding an integral equation for the flux across the free boundary which allows to handle the Dirac delta source term, there is a crucial difference in the free boundary motion which in our case is similar to the Stefan law. In Section 3, we are able to use ideas and arguments in Stefan problems to show local existence of a solution, see [9, 10, 21].

In section 4 we prove global existence of classical solutions for inhibitory networks ($b < 0$) and give a characterization of the blow up time for excitatory networks ($b > 0$). The difference between the cases $b < 0$ and $b > 0$ corresponds to the classical dichotomy between the stable and the supercooled Stefan-problems, see [8, 16].

The final section is devoted to study the spectrum of the linear version of (1.4) ($b = 0$) that has some surprising features in contrast to the classical Fokker-Planck equation. In particular, we show that in most cases, the spectrum only contains the zero eigenvalue corresponding to the steady state. Thus we cannot extract any result on the asymptotic behavior of the time dependent solution via spectral gap and perturbation arguments.

2 Relation to the Stefan problem

The main aim of this section is to formulate equation (1.4) as a free boundary Stefan problem with a nonstandard right hand side. For this we recall a well known change of variables, [5], that transforms Fokker-Planck type equations into a non-homogeneous heat equation. This change of variables is given by $y = e^t v$, $\tau = (e^{2t} - 1)/2$, that yields

$$p(v,t) = e^t w\left(e^t v, \frac{1}{2}(e^{2t}-1)\right) \iff w(y,\tau) = (2\tau+1)^{-1/2} p\left(\frac{y}{\sqrt{2\tau+1}}, \frac{1}{2}\log(2\tau+1)\right).$$

In the sequel, to simplify the notation, we use $\alpha(\tau) = (2\tau+1)^{-1/2} = e^{-t}$. A straightforward computation shows that w satisfies

$$w_\tau = w_{yy} - \mu(\tau)\alpha(\tau)w_y + M(\tau)\delta_{y=\frac{v_R}{\alpha(\tau)}}, \qquad \text{where } M(\tau) = \alpha^2(\tau)N(t) = -\left. w_y\right|_{y=0}. \tag{2.1}$$

The additional change of variables $u(x,\tau) = w(y,\tau)$ with

$$x = y - \int_0^\tau \mu(s)\alpha(s)\,ds = y - b_0\left(\sqrt{1+2t}-1\right) - b\int_0^\tau M(s)\alpha^{-1}(s)\,ds,$$

removes the term with w_y in (2.1). Let $s_I = v_{th} = 0$. We have the following equivalent equation, whose proof is straightforward by the changes of variables specified above:

Lemma 2.1. *System* (1.4)-(1.6) *is equivalent to the following problem*

$$\begin{cases} u_t = u_{xx} + M(t)\delta_{x=s_1(t)}, & x < s(t), t > 0, \\ s(t) = s_I - b_0\left(\sqrt{1+2t}-1\right) - b\displaystyle\int_0^t M(s)\alpha^{-1}(s)\,ds, & t > 0, \\ s_1(t) = s(t) + \dfrac{v_R}{\alpha(t)}, & t > 0, \\ M(t) = -\left. u_x\right|_{x=s(t)}, & t > 0, \\ u(-\infty,t) = 0, \ \ u(s(t),t) = 0, & t > 0, \\ u(x,0) = u_I(x), & x < s_I. \end{cases} \tag{2.2}$$

We now give a definition of classical solution for the Stefan-like free boundary problem (2.2). It is immediate to translate this to a notion of classical solution to the original problem (1.4)-(1.6) by substituting u by p, x by v, $M(t)$ by $N(t)$, $s_1(t)$ by v_R, and $s(t)$ by v_{th}.

Throughout the paper we will make the following assumptions **(H1)** on the initial data u_I: $u_I(x)$ is a non-negative $\mathcal{C}^1((-\infty, v_{th}]) \cap L^1(-\infty, v_{th})$ function such that $u_I(v_{th}) = 0$ and

$$\lim_{x \to -\infty} \frac{\partial u_I}{\partial x} = 0.$$

Definition 2.2. *We say that* $(u(x,t), s(t))$ *is a classical solution to* (2.2) *in the time interval* $J = [0, T)$ *or* $J = [0, T]$ *for a given* $0 < T \leq \infty$ *and with initial data* $u_I(x)$ *satisfying* **(H1)***, if the following conditions are satisfied:*

1. $M(t)$ *is a continuous function for all* $t \in J$,
2. u *is continuous in the region* $\{(x,t) : -\infty < x \leq s(t), t \in J\}$,
3. u_{xx} *and* u_t *are continuous in the region* $\{(x,t) : -\infty < x < s_1(t), t \in J\backslash\{0\}\} \cup \{(x,t) : s_1(t) < x < s(t), t \in J\backslash\{0\}\}$,
4. $u_x(s_1(t)^-, t)$, $u_x(s_1(t)^+, t)$, $u_x(s(t)^-, t)$ *are well defined,*
5. $u_x(x,t) \to 0$ *when* $x \to -\infty$,
6. *Problem problem* (2.2) *is satisfied (in the classical sense).*

The next lemma presents some of the a priori properties of the solution to (2.2).

Lemma 2.3. *Let* $u(x,t)$ *be a solution to* (2.2) *in the sense of Definition* 2.2. *It holds:*

i) the mass is conserved, i.e., for all $t > 0$

$$\int_{-\infty}^{s(t)} u(x,t)\, dx = \int_{-\infty}^{s_I} u_I(x)\, dx\,,$$

ii) the flux across the free boundary s_1 *is exactly the strength of the source term:*

$$M(t) := -u_x(s(t), t) = u_x(s_1(t)^-, t) - u_x(s_1(t)^+, t),$$

iii) for $b_0 < 0$ *and* $b < 0$ *(resp.* $b_0 > 0$ *and* $b > 0$*) the free boundary* $s(t)$ *is a monotone increasing (resp. decreasing) function of time.*

Proof. Mass conservation follows by straightforward integration by parts. To establish the jump across the free boundary, i.e. part *ii)*, integrate the first equation in (2.2) over the interval $(-\infty, s_1(t))$, yielding

$$\int_{-\infty}^{s_1(t)} u_t dx - \int_{-\infty}^{s_1(t)} u_{xx} dx = 0.$$

Hence,

$$\frac{\partial}{\partial t} \int_{-\infty}^{s_1(t)} u(x,t) dx = u_x(s_1(t)^-, t) + \dot{s}_1(t) u(s_1(t), t). \tag{2.3}$$

Similarly, an integration of the first equation in (2.2) in the interval $(s_1(t), s(t))$ gives

$$\frac{\partial}{\partial t} \int_{s_1(t)}^{s(t)} u(x,t) dx + \dot{s}_1(t) u(s_1(t), t) - \dot{s}(t) u(s(t), t) = u_x(s(t), t) - u_x(s_1(t)^+, t).$$

If we substitute $u(s(t), t) = 0$ in the previous line we get

$$\frac{\partial}{\partial t} \int_{s_1(t)}^{s(t)} u dx + \dot{s}_1(t) u(s_1(t), t) = u_x(s(t), t) - u_x(s_1(t)^+, t). \tag{2.4}$$

Adding (2.3) to (2.4) and recalling that the mass is preserved we get

$$0 = \frac{\partial}{\partial t}\int_{-\infty}^{s(t)} u(x,t)dx = u_x(s_1(t)^-,t) + u_x(s(t),t) - u_x(s_1(t)^+,t).$$

It follows that

$$u_x(s(t),t) = u_x(s_1(t)^+,t) - u_x(s_1(t)^-,t),$$

as desired. The free boundary is an increasing function of time since $b_0 < 0$, $b < 0$, $\alpha > 0$, and

$$s(t) = s_I - b_0\left(\sqrt{1+2t} - 1\right) - b\int_0^t M(s)\alpha^{-1}(s)\,ds, \quad t > 0.$$

The fact that $M(t)$ is strictly positive follows by the classical Hopf's lemma. □

3 Local existence and uniqueness

In this section we prove local existence of solution. Our method is inspired by the theory developed by Friedman in [9, 10] for the Stefan problem. We first derive an integral formulation for the problem. A derivative with respect to x yields an integral equation for the flux M, where a fixed point argument can be used to obtain short time existence. Once $M(t)$ is known, the function u the solution of a linear problem.

Theorem 3.1. *Let $u_I(x)$ satisfy* **(H1)**. *Problem* (2.2) *has an unique classical solution* (u,s) *in the sense of Definition* 2.2 *for any* $t \in [0,T]$, $T > 0$. *The existence time T is an inversely proportional function of*

$$\sup_{-\infty < x \le s_I}\left|\frac{\partial u_I}{\partial x}\right|.$$

The proof of Theorem 3.1 will be divided in several steps. The first step deals with an integral formulation of the solution, which is used to show the existence of $M(t)$.

3.1 The integral formulation

Let G be the Green's function for the heat equation on the real line:

$$G(x,t,\xi,\tau) = \frac{1}{[4\pi(t-\tau)]^{1/2}}\exp\left\{-\frac{|x-\xi|^2}{4(t-\tau)}\right\}.$$

To obtain an integral formulation of the solution u of (2.2), recall the following Green's identity

$$\frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi} - u\frac{\partial G}{\partial \xi}\right) - \frac{\partial}{\partial \tau}(Gu) = 0. \tag{3.1}$$

To recover u we first integrate the identity (3.1) in the two regions

$$-\infty < \xi < s_1(\tau),\ 0 < \tau < t, \quad \text{and} \quad s_1(\tau) < \xi < s(\tau),\ 0 < \tau < t,$$

and then add up the results from the integration. We split the resulting expression into the following four terms; the only problematic one is the one containing $u_{\xi\xi}$:

$$I = \int_0^t\int_{-\infty}^{s_1(\tau)}\frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi}\right)d\xi d\tau, \qquad II = \int_0^t\int_{s_1(\tau)}^{s(\tau)}\frac{\partial}{\partial \xi}\left(G\frac{\partial u}{\partial \xi}\right)d\xi d\tau,$$

$$III = \int_0^t\int_{-\infty}^{s(\tau)}\frac{\partial}{\partial \xi}\left(u\frac{\partial G}{\partial \xi}\right)d\xi d\tau, \qquad IV = \int_0^t\int_{-\infty}^{s(\tau)}\frac{\partial}{\partial \tau}(Gu)\,d\xi d\tau.$$

Each term will be analyzed separately. Note that u and G have enough decay as $|\xi| \to \infty$ to justify the following computations due to Definition 2.2. Since $G(x,t,-\infty,\tau) = 0$ it holds

$$I = \int_0^t G\,\frac{\partial u}{\partial \xi}\bigg|_{\xi=-\infty}^{\xi=s_1(\tau)} d\tau = \int_0^t G(x,t,s_1(\tau),\tau)\,\frac{\partial u}{\partial \xi}\bigg|_{s_1(\tau)^-} d\tau. \tag{3.2}$$

Next, we obtain

$$II = \int_0^t \left\{ G\frac{\partial u}{\partial \xi}\bigg|_{\xi=s(\tau)} - G\,\frac{\partial u}{\partial \xi}\bigg|_{\xi=s_1(\tau)^+} \right\} d\tau = -\int_0^t \left\{ G|_{\xi=s(\tau)}\, M(\tau) + G\,\frac{\partial u}{\partial \xi}\bigg|_{\xi=s_1(\tau)^+} \right\} d\tau.$$

Here we have used that $\frac{\partial u}{\partial \xi}\Big|_{\xi=s(\tau)} = -M(\tau)$. For the third and fourth integrals we have

$$\begin{aligned} III &= -\int_0^t \left\{ \left(u\frac{\partial G}{\partial \xi}\right)\bigg|_{\xi=s(\tau)} - \left(u\frac{\partial G}{\partial \xi}\right)\bigg|_{\xi=-\infty} \right\} d\tau \\ &= -\int_0^t \left\{ (u(s(\tau),\tau)\frac{\partial G}{\partial \xi}\bigg|_{\xi=s(\tau)} - u(-\infty,\tau)\frac{\partial G}{\partial \xi}\bigg|_{\xi=-\infty} \right\} d\tau = 0, \end{aligned}$$

$$IV = \int_0^t \frac{\partial}{\partial \tau}\int_{-\infty}^{s(\tau)} Gu\,d\xi d\tau = \int_{-\infty}^{s(t)} Gu|_{\tau=t}\,d\xi - \int_{-\infty}^{s(0)} Gu|_{\tau=0}\,d\xi\,,$$

taking into account that $u(s(\tau),\tau) = u(-\infty,\tau) = 0$. Next, recalling $G(x,t,\xi,t) = \delta_{x=\xi}$, we get

$$IV = \int_{-\infty}^{s(t)} \delta_{\xi=x} u(\xi,t)\,d\xi - \int_{-\infty}^{s(0)} G(x,t,\xi,0)u_I(\xi)\,d\xi. \tag{3.3}$$

Combining (3.2)-(3.3), and part *ii)* of Lemma 2.3, yields that the solution u reads as

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{s(0)} G(x,t,\xi,0)u_I(\xi)\,d\xi + \int_0^t G(x,t,s_1(\tau))\,\frac{\partial u}{\partial \xi}\bigg|_{\xi=s_1(\tau)^-} d\tau \\ &\quad - \int_0^t M(\tau)G(x,t,s(\tau),\tau)\,d\tau - \int_0^t G(x,t,s_1(\tau))\frac{\partial u}{\partial \xi}\bigg|_{\xi=s_1(\tau)^+} d\tau \\ &= \int_{-\infty}^{s(0)} G(x,t,\xi,0)u_I(\xi)\,d\xi - \int_0^t M(\tau)G(x,t,s(\tau),\tau)\,d\tau + \int_0^t M(\tau)G(x,t,s_1(\tau),\tau)\,d\tau \\ &=:\ I_1 - I_2 + I_3. \end{aligned} \tag{3.4}$$

The term I_1 represents the solution to the homogeneous heat equation with initial data

$$u_0(\xi) = \begin{cases} u_I(\xi) & \xi \le s(0), \\ 0 & \xi > s(0). \end{cases}$$

All the calculations up to here are formal assuming that u is a solution of the equation (2.2) as in Definition 2.2. We now derive an equation for M which will be solved for short time using a fixed point argument. The first step is to obtain the space derivatives of the terms $I_i,\ i = 1,2,3$ and evaluate them at $x = s(t)^-$:

$$\frac{\partial I_1}{\partial x}\bigg|_{x=s(t)^-} = \int_{-\infty}^{s(0)} G_x(x,t,\xi,0)u_I(\xi)\,d\xi = -\int_{-\infty}^{s(0)} G(x,t,\xi,0)u_I'(\xi)\,d\xi.$$

To get the derivative of I_2, we use [9, Lemma 1, pag 217]. This lemma states that for any continuous function ρ the following limit holds:

$$\lim_{x\to s(t)^-} \frac{\partial}{\partial x}\int_0^t \rho(\tau)G(x,t,s(\tau),\tau)\,d\tau = \frac{1}{2}\rho(t) + \int_0^t \rho(\tau)\frac{\partial G}{\partial x}(s(t),t,s(\tau),\tau)\,d\tau. \tag{3.5}$$

As a consequence,

$$\left.\frac{\partial I_2}{\partial x}\right|_{x=s(t)^-} = \frac{1}{2}M(t) + \int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)\,d\tau.$$

For the derivative of I_3 note that problems can only occur if $t=\tau$ and $s(t)=s_1(\tau)$, but this is not possible by the definition of s_1. Thus,

$$\left.\frac{\partial I_3}{\partial x}\right|_{x=s(t)^-} = \int_0^t G_x(s(t),t;s_1(\tau),\tau)M(\tau)\,d\tau.$$

Substituting the estimates on I_1, I_2 and I_3 into (3.4) yields

$$\begin{aligned} M(t) = &-2\int_{-\infty}^{s(0)} G(s(t),t,\xi,0)u_I'(\xi)\,d\xi \\ &+2\int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)\,d\tau - 2\int_0^t M(\tau)G_x(s(t),t,s_1(\tau),\tau)\,d\tau. \end{aligned} \tag{3.6}$$

3.2 Local existence and uniqueness for M

Theorem 3.2. *Let $u_I(x)$ satisfy* **(H1)**. *There exists a unique solution $M(t)\in\mathcal{C}([0,T])$ to (3.6) and the maximal existence time T is estimated as*

$$T \le \left(\sup_{-\infty<x\le s_I}\left|\frac{\partial u_I}{\partial x}\right|\right)^{-1}.$$

Proof. The local in time existence of $M(t)$ is showed via a fixed point argument. For this, we modify the classical argument for the Stefan problem to account for the additional source term given by $M(t)\,\delta_{x=s_1(t)}$. For given constants $\sigma, m>0$ consider the set

$$C_{\sigma,m} := \{M\in\mathcal{C}([0,\sigma]) : \|M\| := \sup_{0\le t\le\sigma}|M(t)| < m\}.$$

Define

$$\begin{aligned} \mathbf{\Gamma}(M)(t) :=&-2\int_{-\infty}^{s(0)} G(s(t),t,\xi,0)u_I'(\xi)d\xi \\ &+2\int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)d\tau - 2\int_0^t M(\tau)G_x(s(t),t,s_1(\tau),\tau)d\tau \\ =:&\, J_1+J_2+J_3. \end{aligned} \tag{3.7}$$

In order to apply fixed point argument it is necessary to show that for sufficiently small σ the operator $\mathbf{\Gamma} : C_{\sigma,m}\to C_{\sigma,m}$ is a contraction.

Step 1. We fist show that for σ sufficiently small $\mathbf{\Gamma}(C_{\sigma,m})\subseteq C_{\sigma,m}$. For simplicity, we focus on the proof in the case $b<0$. At the end we make the necessary changes for $b>0$. Choose σ sufficiently small so that

i. $\alpha^{-1}(t)\le 2,\ \forall\, t\le\sigma$,

ii. $\frac{m(|b_0|+2m|b|)}{\sqrt{\pi}}\sigma^{1/2}\le 1/2$,

iii. $|v_R|-|b_0|\sigma>0$,

iv. $\dfrac{2m}{\sqrt{\pi}}\displaystyle\int_{\frac{|v_R|-|b_0|\sigma}{\sqrt{8\sigma}}}^{\infty} z^{-1}\exp\{-z^2\}dz\le 1/2$, and define

$$m := 1 + 2 \sup_{-\infty < x \le s(0)} \left| \frac{\partial u_I}{\partial x} \right| . \tag{3.8}$$

We obtain first an auxiliary estimate. Since σ has been chosen so small that condition *i.* holds and $\alpha^{-1}\sqrt{1+2t}$ is a 1-Lipschitz function for $t \ge 0$, if $M \in C_{\sigma,m}$ we have

$$|s(t) - s(\tau)| \le |b_0||t-\tau| + |b| \int_\tau^t M(s)\alpha^{-1}(s)\, ds \le (|b_0| + 2|b|m)\, |t-\tau|, \tag{3.9}$$

i.e., $s(t)$ is a Lipschitz continuous function of time.

To estimate the image of the operator $\mathbf{\Gamma}(M)$ as defined in (3.7) for $M \in C_{\sigma,m}$ we find separately a bound for each $J_i,\ i = 1,2,3$. It is straightforward to check

$$|J_1| \le 2 \left\{ \sup_{-\infty < x \le s(0)} \left| \frac{\partial u_I}{\partial x} \right| \right\} \int_{-\infty}^{s(0)} G(x,t,\xi,0)\, d\xi \le 2 \sup_{-\infty < x \le s(0)} \left| \frac{\partial u_I}{\partial x} \right| .$$

On the other hand, the Lipschitz bound (3.9) for s yields

$$|G_x(s(t),t,s(\tau),\tau)| \le \frac{1}{2\sqrt{4\pi}} \frac{|s(t)-s(\tau)|}{(t-\tau)^{3/2}} \le \frac{(|b_0| + 2m\,|b|)}{2\sqrt{4\pi}} \frac{1}{(t-\tau)^{1/2}},$$

and thus, we bound the following integral as

$$\int_0^t |G_x(s(t),t,s(\tau),\tau)|\, d\tau \le \frac{(|b_0| + 2m|b|)}{\sqrt{4\pi}} t^{1/2} \le \frac{(|b_0| + 2m|b|)}{\sqrt{4\pi}} \sigma^{1/2} \le \frac{1}{4m}, \tag{3.10}$$

taking into account the choice of σ given by *ii.*. The above estimates imply $|J_2| \le \frac{1}{2}$. Next, we estimate J_3. The inequality $y \exp\{-y^2\} \le \exp\{-y^2/2\}$ yields

$$|G_x(x,t,\xi,\tau)| \le \frac{1}{\sqrt{4\pi}(t-\tau)} \exp\left\{ -\frac{|x-\xi|^2}{8(t-\tau)} \right\}. \tag{3.11}$$

The definitions of $s(t)$ and $s_1(\tau)$, using that $b < 0$ and condition *iii.*, yield

$$|s(t) - s_1(\tau)| \ge |v_R| - |b_0|\sigma > 0. \tag{3.12}$$

If we integrate (3.11) we get

$$\begin{aligned} \int_0^t |G_x(s(t),t,s_1(\tau),\tau)|\, d\tau &\le \frac{1}{\sqrt{4\pi}} \int_0^t \frac{1}{t-\tau} \exp\left\{ -\frac{|s(t)-s_1(\tau)|^2}{8(t-\tau)} \right\} d\tau \\ &\le \frac{1}{\sqrt{4\pi}} \int_0^t \frac{1}{t-\tau} \exp\left\{ -\frac{(|v_R| - |b_0|\sigma)^2}{8(t-\tau)} \right\} d\tau \\ &= \frac{1}{\sqrt{\pi}} \int_{\frac{|v_R|-|b_0|\sigma}{\sqrt{8t}}}^\infty \frac{1}{z} e^{-z^2}\, dz \le \frac{1}{\sqrt{\pi}} \int_{\frac{|v_R|-|b_0|\sigma}{\sqrt{8\sigma}}}^\infty \frac{1}{z} e^{-z^2}\, dz, \end{aligned} \tag{3.13}$$

where we used the change of variables $z = \frac{|v_R|-|b_0|\sigma}{\sqrt{8(t-\tau)}}$. By the last estimate and by condition *iv.*, it follows

$$|J_3| \le 2m \int_0^t |G_x(s(t),t,s_1(\tau),\tau)|\, d\tau \le \frac{2m}{\sqrt{\pi}} \int_{\frac{|v_R|-|b_0|\sigma}{\sqrt{8\sigma}}}^\infty \frac{1}{z} e^{-z^2} dz \le \frac{1}{2}. \tag{3.14}$$

The estimates for $J_i, i = 1,2,3$ establish that $\mathbf{\Gamma}(M) \in C_{\sigma,m}$ since $\|\mathbf{\Gamma}(M)\| \le J_1 + J_2 + J_3 \le m$, for all $M \in C_{\sigma,m}$, by the choice of m in (3.8).

It remains to consider the case $b > 0$. It is clear that the only modification needed is in (3.12). For this use

$$|s(t)-s_1(\tau)| = |s(t)-s(\tau)-v_R\alpha^{-1}(\tau)| \geq \big||v_R|\alpha^{-1}(\tau) - |s(t)-s(\tau)|\big| \geq |v_R|-(|b_0|+m)\sigma, \quad (3.15)$$

which may be estimated from below by a positive constant for some σ small enough. The main difference between the cases $b \leq 0$ and $b > 0$ may be found by comparing (3.12) with (3.15): for $b > 0$ the bound on the distance of $s(t)$ to $s_1(\tau)$ for $0 \leq \tau \leq t$ now depends on the initial data (3.8).

Step 2. The map $\mathbf{\Gamma} : C_{\sigma,m} \to C_{\sigma,m}$ defined in (3.7) is a contraction for σ small enough. In the sequel, constants C are arbitrary and may change from line to line. Let $M, \tilde{M} \in C_{\sigma,m}$,

$$s(t) = s_I - b_0\left(\sqrt{1+2t}-1\right) - b\int_0^t M(\tau)\alpha^{-1}(\tau)\,d\tau\,, \quad (3.16)$$

and analogously for $\tilde{s}(t)$ and $\tilde{M}(t)$. The following auxiliary estimate holds:

$$|s(t)-\tilde{s}(t)| \leq |b|\int_0^t |M(\tau)-\tilde{M}(\tau)|\alpha^{-1}(\tau)\,d\tau \leq \frac{|b|}{3}\|M-\tilde{M}\|\left[(2t+1)^{3/2}-1\right]. \quad (3.17)$$

It is straightforward from (3.16) that

$$|\dot{s}(t)-\dot{\tilde{s}}(t)| \leq 2\,|b|\,\|M-\tilde{M}\|, \quad 0 < t \leq \sigma < 1. \quad (3.18)$$

From condition *i.* on σ and (3.9), it follows that

$$\max\{|s(t)-s(\tau)|, |\tilde{s}(t)-\tilde{s}(\tau)|\} \leq (|b_0|+2m\,|b|)|t-\tau| \leq (|b_0|+2\,|b|)m|t-\tau|\,. \quad (3.19)$$

To show that $\mathbf{\Gamma}$ is a contraction we proceed as follows.

$$\begin{aligned}|\mathbf{\Gamma}(M)-\mathbf{\Gamma}(\tilde{M})| \leq\,& 2\left[\int_{-\infty}^{s(0)} |u_I'(\xi)||G(s(t),t,\xi,0)-G(\tilde{s}(t),t,\xi,0)|\,d\xi\right]\\ &+2\left|\int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)-\tilde{M}(\tau)G_x(\tilde{s}(t),t,\tilde{s}(\tau),\tau)\,d\tau\right|\\ &+2\left|\int_0^t M(\tau)G_x(s(t),t,s_1(\tau),\tau)-\tilde{M}(\tau)G_x(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)\,d\tau\right|\\ =:\,& \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3.\end{aligned}$$

Without loss of generality assume that $\tilde{s}(t) > s(t)$. The mean value theorem applied to the kernel $G(x,t,\xi,0)$ implies the following inequality for some $\bar{s} \in [s(t), \tilde{s}(t)]$:

$$|G(s(t),t,\xi,0)-G(\tilde{s}(t),t,\xi,0)| \leq |G_x(\bar{s},t,\xi,0)|\,|s(t)-\tilde{s}(t)|. \quad (3.20)$$

Recall that

$$|G_x(\bar{s},t,\xi,0)| = \frac{|\bar{s}-\xi|}{2t}\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{|\bar{s}-\xi|^2}{4t}\right\} \leq \frac{1}{\sqrt{t}}\frac{1}{\sqrt{4\pi t}}\exp\left\{-\frac{|\bar{s}-\xi|^2}{8t}\right\},$$

where we have used the relation $ye^{-y^2} \leq e^{-y^2/2}$. Hence (3.20) simply reduces to

$$|G(s(t),t,\xi,0)-G(\tilde{s}(t),t,\xi,0)| \leq \frac{C}{\sqrt{t}}G(\bar{s}(t),2t,\xi,0))|s(t)-\tilde{s}(t)|.$$

Integrating in ξ, together with (3.17) yields

$$\mathcal{A}_1 \leq C|b|\|u_I'\|\|M-\tilde{M}\|\left\{\frac{(1+2t)^{3/2}-1}{t^{1/2}}\right\}.$$

Since $\lim\limits_{t\to 0} t^{-1/2}((1+2t)^{3/2}-1)=0$, for σ small we have $\mathcal{A}_1 \le \frac{1}{6}\|M-\tilde{M}\|$. To estimate $\mathcal{A}_2$ we consider first

$$\begin{aligned}|\mathcal{A}_2| \le & 2\left|\int_0^t M(\tau)G_x(s(t),t,s(\tau),\tau)-\tilde{M}(\tau)G_x(s(t),t,s(\tau),\tau)\,d\tau\right| \\ &+2\left|\int_0^t \tilde{M}(\tau)G_x(s(t),t,s(\tau),\tau)-\tilde{M}(\tau)G_x(\tilde{s}(t),t,\tilde{s}(\tau),\tau)\,d\tau\right| \\ =&:\mathcal{A}_{21}+\mathcal{A}_{22}.\end{aligned}$$

Using (3.10), we get

$$|\mathcal{A}_{21}| \le \frac{(|b_0|+2m|b|)}{\sqrt{4\pi}}\sigma^{1/2}\,\|M-\tilde{M}\| \le \frac{1}{12}\|M-\tilde{M}\|,$$

for σ small enough. To estimate $\mathcal{A}_{22}$ proceed as follows:

$$\begin{aligned}|G_x(s(t),&t,s(\tau),\tau)-G_x(\tilde{s}(t),t,\tilde{s}(\tau),\tau)| \\ &= C\left|\frac{s(t)-s(\tau)}{t-\tau}G(s(t),t,s(\tau),\tau)-\frac{\tilde{s}(t)-\tilde{s}(\tau)}{t-\tau}G(\tilde{s}(t),t,\tilde{s}(\tau),\tau)\right| \\ &\le C\left|\frac{s(t)-s(\tau)}{t-\tau}-\frac{\tilde{s}(t)-\tilde{s}(\tau)}{t-\tau}\right|G(s(t),t,s(\tau),\tau) \\ &\quad + C\frac{\tilde{s}(t)-\tilde{s}(\tau)}{t-\tau}\,|G(s(t),t,s(\tau),\tau)-G(\tilde{s}(t),t,\tilde{s}(\tau),\tau)| \\ &=: \mathcal{B}_1+\mathcal{B}_2.\end{aligned}$$

For $\mathcal{B}_1$ we use the mean value theorem

$$\frac{[s(t)-\tilde{s}(t)]-[s(\tau)-\tilde{s}(\tau)]}{t-\tau} = \dot{s}(\bar{\tau})-\dot{\tilde{s}}(\bar{\tau}) \tag{3.21}$$

for some $0<\bar{\tau}<t$. By the previous equality and (3.18) we have

$$\mathcal{B}_1 \le C(t-\tau)^{-1/2}|\dot{s}(\bar{\tau})-\dot{\tilde{s}}(\bar{\tau})| \le C(t-\tau)^{-1/2}\|M-\tilde{M}\|.$$

On the other hand, to handle the term $\mathcal{B}_2$, we first note that

$$\begin{aligned}|G(s(t),t,s(\tau),\tau)-&G(\tilde{s}(t),t,\tilde{s}(\tau),\tau)| \\ &\le G(s(t),t,s(\tau),\tau)\left|1-\exp\left\{\tfrac{-(\tilde{s}(t)-\tilde{s}(\tau))^2+(s(t)-s(\tau))^2}{4(t-\tau)}\right\}\right|.\end{aligned} \tag{3.22}$$

Define now

$$S := (s(t)-s(\tau))^2-(\tilde{s}(t)-\tilde{s}(\tau))^2 = [s(t)-s(\tau)+\tilde{s}(t)-\tilde{s}(\tau)]\,[s(t)-\tilde{s}(t)-(s(\tau)-\tilde{s}(\tau))].$$

The mean value theorem (3.21) and the estimate (3.18) lead to

$$|[s(t)-\tilde{s}(t)]-[s(\tau)-\tilde{s}(\tau)]| = \left|\dot{s}(\bar{\tau})-\dot{\tilde{s}}(\bar{\tau})\right|(t-\tau) \le C\|M-\tilde{M}\|(t-\tau). \tag{3.23}$$

On the other hand, we recall again the Lipschitz estimate (3.19), i.e.,

$$\max\{|s(t)-s(\tau)|,|\tilde{s}(t)-\tilde{s}(\tau)|\} \le Cm(t-\tau), \tag{3.24}$$

for a constant C depending on $|b|$, $|b_0|$, which yields $|S| \le C(t-\tau)m\sigma\|M-\tilde{M}\|$. The combination of the last inequality with (3.22) together with the mean value theorem shows that

$$|G(s(t),t,s(\tau),\tau)-G(\tilde{s}(t),t,\tilde{s}(\tau),\tau)| \le G(s(t),t,s(\tau),\tau)Cm\sigma\|M-\tilde{M}\|,$$

and thus the term $\mathcal{B}_2$ is estimated using (3.24) as

$$\mathcal{B}_2 \le C(t-\tau)^{-1/2} m^2 \|M-\tilde{M}\|\sigma.$$

Multiplying $\mathcal{B}_1+\mathcal{B}_2$ by $\tilde{M}(\tau)$ and integrating over the interval $[0,t]$ yields

$$\begin{aligned}\mathcal{A}_{22} &\le Cm\int_0^t |G_x(s(t),t,s(\tau),\tau) - G_x(\tilde{s}(t),t,\tilde{s}(\tau),\tau)|\,d\tau \\ &\le Cm\int_0^t (\mathcal{B}_1+\mathcal{B}_2)\,d\tau \le Cm^3\|M-\tilde{M}\|\sigma^{1/2} < \frac{1}{12}\|M-\tilde{M}\|,\end{aligned}$$

for σ small enough. The next step is to estimate $\mathcal{A}_3$. Split the integral into two terms

$$\begin{aligned}|\mathcal{A}_3| &\le 2\left|\int_0^t M(\tau)G_x(s(t),t,s_1(\tau),\tau) - \tilde{M}(\tau)G_x(s(t),t,s_1(\tau),\tau)d\tau\right| \\ &\quad + 2\left|\int_0^t \tilde{M}(\tau)G_x(s(t),t,s_1(\tau),\tau) - \tilde{M}(\tau)G_x(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)d\tau\right| \\ &=: \mathcal{A}_{31}+\mathcal{A}_{32}.\end{aligned}$$

The estimate for $\mathcal{A}_{31}$ is very similar to that of J_3 from (3.14). Indeed,

$$\begin{aligned}|\mathcal{A}_{31}| &\le 2\|M-\tilde{M}\|\int_0^t |G_x(s(t),t,s_1(\tau),\tau)|d\tau \\ &\le C\|M-\tilde{M}\|\int_{\frac{\Lambda}{\sqrt{8\sigma}}}^\infty \frac{1}{z}e^{-z^2}dz < \frac{1}{12}\|M-\tilde{M}\|,\end{aligned}$$

where we used that $\tilde{s}(t)-\tilde{s}_1(\tau) \ge \Lambda > 0$ for σ sufficiently small with

$$\Lambda := \begin{cases} |v_R| - |b_0|\sigma & \text{for } b<0 \\ |v_R| - (|b_0|+m)\sigma & \text{for } b>0. \end{cases}$$

To bound $\mathcal{A}_{32}$ we split

$$\begin{aligned}|G_x(s(t),&t,s_1(\tau),\tau) - G_x(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)| \\ &= C\left|\frac{s(t)-s_1(\tau)}{t-\tau}G(s(t),t,s_1(\tau),\tau) - \frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}G(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)\right| \\ &\le C\left|\frac{s(t)-s_1(\tau)}{t-\tau} - \frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}\right| G(s(t),t,s_1(\tau),\tau) \\ &\quad + C\frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}\,|G(s(t),t,s_1(\tau),\tau) - G(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)| \\ &=: \mathcal{B}_1'+\mathcal{B}_2'.\end{aligned}$$

We observe that $\mathcal{B}_1'$ is estimated exactly in the same way as $\mathcal{B}_1$. This is a consequence of $[s(t)-\tilde{s}(t)]-[s_1(\tau)-\tilde{s}_1(\tau)] = [s(t)-\tilde{s}(t)]-[s(\tau)-\tilde{s}(\tau)]$. We can continue from (3.21) as before to obtain

$$|B_1'| \le C(t-\tau)^{-1/2}\|M-\tilde{M}\|. \tag{3.25}$$

The estimate for $\mathcal{B}_2'$ is slightly more involved. We write

$$\mathcal{B}_2' = C(\tilde{s}(t)-\tilde{s}_1(\tau))(t-\tau)^{-1}|G(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)|\left(1-\exp\left\{\tfrac{S'}{4(t-\tau)}\right\}\right) \tag{3.26}$$

for $S' := -(s(t)-s_1(\tau))^2 + (\tilde{s}(t)-\tilde{s}_1(\tau))^2 = [\tilde{s}(t)-\tilde{s}_1(\tau)+s(t)-s_1(\tau)]\,[\tilde{s}(t)-\tilde{s}_1(\tau)-s(t)+s_1(\tau)]$. By the definitions of s_1 and $\tilde{s}_1$ (see (2.2)) we have that

$$|\tilde{s}(t)-\tilde{s}_1(\tau)-s(t)+s_1(\tau)| = |\tilde{s}(t)-\tilde{s}(\tau)-s(t)+s(\tau)| \le C\|M-\tilde{M}\|(t-\tau), \tag{3.27}$$

where in the last inequality we have used estimate (3.23). On the other hand,

$$|[\tilde{s}(t)-\tilde{s}_1(\tau)+s(t)-s_1(\tau)]| \le |s(t)-s(\tau)|+|\tilde{s}(t)-\tilde{s}(\tau)|+2|v_R|\sqrt{2\tau+1} \le Cm\sigma, \quad (3.28)$$

by the Lipschitz estimate (3.24). Hence combining (3.27) with (3.28) we get again that $|S'| \le C(t-\tau)m\sigma\|M-\tilde{M}\|$. Consequently (3.26) reduces to

$$|\mathcal{B}_2'| \le Cm\frac{\tilde{s}(t)-\tilde{s}_1(\tau)}{t-\tau}G(\tilde{s}(t),t,\tilde{s}_1(\tau),\tau)\sigma\|M-\tilde{M}\|.$$

Integrating the previous expression, using again the inequality $y\exp\{-y^2\} \le \exp\{-y^2/2\}$, and noting that $\tilde{s}(t)-\tilde{s}_1(\tau) \ge \Lambda > 0$, we can give a very rough estimate that is enough to our purposes:

$$\int_0^t |\mathcal{B}_2'|\,d\tau \le Cm\sigma\|M-\tilde{M}\|. \quad (3.29)$$

Thus, from the estimates for $\mathcal{B}_1'$ and $\mathcal{B}_2'$ from (3.25) and (3.29) respectively,

$$|\mathcal{A}_{32}| \le Cm\int_0^t(\mathcal{B}_1'+\mathcal{B}_2')\,d\tau \le C\|M-\tilde{M}\|\left(m\sigma^{1/2}+m^2\sigma\right) < \frac{1}{12}\|M-\tilde{M}\|,$$

for some suitable σ small enough. Then, adding the estimates obtained for $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$ yields that $\mathbf{\Gamma}$ is a contraction satisfying for some σ small enough inversely proportional to m:

$$\mathbf{\Gamma}(M)-\mathbf{\Gamma}(\tilde{M})\| \le \frac{1}{2}\|M-\tilde{M}\|.$$

This concludes the proof of Theorem 3.2 as desired. □

3.3 Recovery of u

Theorem 3.2 shows that we have short time existence of a mild solution for problem (2.2) (i.e., a solution in the integral sense). However, one can easily show that:

Corollary 3.3. *There exists a unique solution of problem* (2.2) *in the sense of Definition* 2.2 *for* $t \in [0,T]$.

Proof. Once M is known, one can construct u from Duhamel's formula (3.4). The smoothness and decay of u follow immediately from here. One needs to check also that u has well defined side derivatives at s_1. But this follows from formula (3.5) applied to $s_1(t)$ and from the estimate for $|G_x(s_1(t),t,s_1(\tau),\tau)|$ similarly as the calculation in (3.13). □

This completes the proof of Theorem 3.1.

4 Proofs of the Main Results

From the previous arguments, and in particular from (3.8), it is clear that the obstacle for long time existence in this case is the possible blow up in time of $\|u_x(\cdot,t)\|_\infty$ particularly at the free boundary, i.e. the blow up of $M(t)$. We now formalize this idea by showing that we can extend the solution as long as the firing rate $M(t)$ is bounded.

Proposition 4.1. *Let* (u,s) *be a classical solution to* (2.2) *in the time interval* $[0,T]$*, as proven in Theorem* 3.1*. Assume, in addition, that*

$$U_0 := \sup_{x\in(-\infty,s(t_0-\varepsilon)]}|u_x(x,t_0-\varepsilon)| < \infty \qquad \textit{and that} \qquad M^* = \sup_{t\in(t_0-\varepsilon,t_0)} M(t) < \infty\,,$$

for some $0<\varepsilon<t_0\le T$*. Then*

$$\sup\{|u_x(x,t)| \ \textit{with}\ x\in(-\infty,s(t)]\,,\, t\in[t_0-\varepsilon,t_0)\} < \infty\,,$$

with a bound depending only on the quantities M^**,* U_0*, and* t_0*.*

Proof. Differentiating (3.4) with respect to x yields

$$
\begin{aligned}
u_x(x,t) =& \int_{-\infty}^{s(t_0-\varepsilon)} G(x,t,\xi,t_0-\varepsilon)u_x(\xi,t_0-\varepsilon)d\xi \\
&- \int_{t_0-\varepsilon}^{t} M(\tau)G_x(x,t,s(\tau),\tau)d\tau + \int_{t_0-\varepsilon}^{t} M(\tau)G_x(x,t,s_1(\tau),\tau)d\tau \\
=:& I_1 - I_2 + I_3.
\end{aligned}
$$

The estimate for I_1 is straightforward from heat kernel properties and depends only on U_0. Consider now I_2: since M is uniformly bounded in the whole interval $t_0-\varepsilon<t<t_0$, we get

$$
|I_2| \le C\int_{t_0-\varepsilon}^{t} |G_x(x,t,s(\tau),\tau)|d\tau. \tag{4.1}
$$

Next, it is shown in [9, Eq. (1.16), pag. 219] that for any Lipschitz continuous function $s(t)$ there exists a constant C depending on t_0, ε and on the Lipschitz constant of s such that

$$
\int_{t-\varepsilon}^{t} \frac{|x-s(\tau)|}{(t-\tau)} G(x,t,s(\tau),\tau)d\tau \le C, \quad t\in(t_0-\varepsilon,t_0).
$$

The previous estimate allows to bound (4.1). However such bound for I_2 may depend on t_0 and M^* since the Lipschitz constant of s does, see (3.16).

Finally, the same argument works for I_3, replacing s by s_1 in the previous calculations. □

With this result in hand, our solutions can be extended to a maximal time of existence. The maximal time can be characterized, as shown in the following theorem. Note that the result does not depend on the sign of b.

Theorem 4.2. *Let (u,s) be a classical solution to* (2.2)*, as proven in Theorem* 3.1*. Then the solution u can be extended up to a maximal time $0<T^*\le\infty$ given by*

$$
T^* = \sup\{t>0 \,:\, M(t)<\infty\}\,.
$$

Proof. Assume that the maximal time of existence of a classical solution $(u(t),s(t))$ to (2.2) in the sense of Definition 2.2 is $T^*<\infty$. If $T^*=\infty$ there is nothing to show. By definition we have $T^*\le\sup\{t>0 \,:\, M(t)<\infty\}$. Let us show the equality by contradiction. Let us assume that $T^*<\sup\{t>0 \,:\, M(t)<\infty\}$ and then, there exists $0<\varepsilon<T^*$ such that

$$
M^* = \sup_{t\in(T^*-\varepsilon,T^*)} M(t) < \infty\,.
$$

Let U_0 be defined as in Proposition 4.1 with $t_0=T^*$. Applying Proposition 4.1, we deduce that $u_x(x,t)$ is also uniformly bounded for $x\in(-\infty,s(t)]$ and $t\in[T^*-\varepsilon,T^*)$ by a constant, denoted U^*. The same proposition tells us that U^* only depends on M^* and on U_0, i.e. on the uniform bound of $u_x(x,T^*-\varepsilon)$ for $x\in(-\infty,s(T^*-\varepsilon)]$. We may now use Theorem 3.1 to show that problem (2.2) has a classical solution in the time interval $[t_0,t_0+\delta]$, with $t_0\in[T^*-\varepsilon,T^*)$ and δ depending only on U^*. Thus, we can extend the solution $(u(t),s(t))$ to (2.2) after T^* and find a continuous extension of $M(t)$ past T^*. We have then reached a contradiction and the conclusion of the Theorem follows. □

We now show, following Friedman's ideas [9], that it is possible to extend the solution for a short (but uniform) time ε for $b<0$.

Proposition 4.3. *For $b<0$, let (u,s), $t\in[0,t_0)$, be a classical solution to* (2.2) *as proven in Theorem* 3.1*. There exists $\varepsilon>0$ small enough such that, if*

$$
\mathcal{M}_0 := \sup_{x\in(-\infty,s(t_0-\varepsilon)]} |u_x(x,t_0-\varepsilon)| < \infty,
$$

for $0 < \varepsilon < t_0$ *then*
$$\sup_{t_0-\varepsilon<t<t_0} M(t) \le C < \infty.$$
The constant ε *does not depend on* t_0*, and the constant* C *above only depends on* $\mathcal{M}_0$*.*

Proof. We use the integral formulation (3.6) for M, this time with initial condition at time $t_0-\varepsilon$ for some fixed ε chosen below, and $t \in (t_0-\varepsilon, t_0)$. It holds

$$\begin{aligned} M(t) =& -2\int_{-\infty}^{s(t_0-\varepsilon)} G(s(t),t,\xi,t_0-\varepsilon)u_x(\xi,t_0-\varepsilon)d\xi \\ &+2\int_{t_0-\varepsilon}^{t} M(\tau)G_x(s(t),t,s(\tau),\tau)d\tau - 2\int_{t_0-\varepsilon}^{t} M(\tau)G_x(s(t),t,s_1(\tau),\tau)d\tau \\ =:&\ K_1+K_2+K_3. \end{aligned} \tag{4.2}$$

Since $s(t) \ge s(\tau)$, it follows that $G_x(s(t),t,s(\tau),\tau) \le 0$; hence $K_2 \le 0$ by taking into account that $M \ge 0$. To estimate K_3 let
$$\Phi(t) := \sup_{t_0-\varepsilon<\tau<t} M(\tau).$$
Note that
$$|K_3| \le \Phi(t)\int_{t_0-\varepsilon}^{t} |G_x(s(t),t,s_1(\tau),\tau)|\,d\tau. \tag{4.3}$$
To estimate the derivative $|G_x(s(t),t,s_1(\tau),\tau)|$ we use the fact that the nonlinear part of s is an increasing function in the case $b<0$ as in (3.12). Thus, for ε small enough, we conclude that
$$s(t)-s_1(\tau) = s(t)-s(\tau)-v_R\alpha^{-1}(\tau) \ge |v_R| - |b_0|\varepsilon > 0\,. \tag{4.4}$$
Hence, repeating the computations in (3.13) to estimate
$$\int_{t_0-\varepsilon}^{t} |G_x(s(t),t,s_1(\tau),\tau)|\,d\tau \le C\int_{\frac{|v_R|-|b_0|\varepsilon}{\sqrt{8(t-t_0+\varepsilon)}}}^{\infty} \frac{1}{z}e^{-z^2}\,dz \le C\int_{\frac{|v_R|-|b_0|\varepsilon}{\sqrt{8\varepsilon}}}^{\infty} \frac{1}{z}e^{-z^2}\,dz.$$
It is clear that this last integral can be made less than $1/2$ for some ε small enough independently of t_0. Substituting the above inequality into (4.3) gives the estimate $|K_3| \le \frac{1}{2}\Phi(t)$. Finally, note that $|K_1| \le C$ depending on $\sup|u_x(x,t_0-\varepsilon)|$. Combining the estimates for K_1, K_2, K_3 with (4.2) yields
$$M(t) \le \tfrac{1}{2}\Phi(t) + \mathcal{M}_0.$$
Since our solution is classical in $[0,t_0)$, we have that $\Phi(t) < \infty$ for $t \in (t_0-\varepsilon, t_0)$. Note that $\Phi(t)$ is an increasing function. For any $\delta < \varepsilon$, take $t \in (t_0-\varepsilon, t_0-\delta)$, then
$$M(t) \le \tfrac{1}{2}\Phi(t-\delta) + \mathcal{M}_0.$$
Taking the supremum on the left hand side, we get that $\Phi(t_0-\delta) \le 2\mathcal{M}_0$ for all $t \in (t_0-\varepsilon, t_0-\delta)$. Now let $\delta \to 0$ and the conclusion of the Proposition follows. □

Remark. Let us point out that the key estimate (4.4) comes from the fact that the nonlinear part of the free boundary $s(t)$ is monotone increasing. For the case $b>0$, instead of (4.4) we got (3.15), which makes impossible to get a uniform estimate with respect to t_0 since m will depend on it.

The combination of Proposition 4.3 with Theorem 4.2 and Theorem 4.1 gives global existence for $b<0$, as summarized in the following result:

Theorem 4.4. *Let* $u_I(x)$ *satisfy* **(H1)***. For* $b<0$ *problem* (2.2) *has a unique global classical solution* (u,s)*. Furthermore, the function* $s(t)$ *is a monotone increasing function of time* t *if both* b *and* b_0 *are negative.*

The main Theorem 1.1 is now complete. We emphasize that Theorem 4.2 characterizes the possible blow-up of classical solutions in finite time as the time of divergence of the firing rate $N(t)$.

5 Study of the spectrum

In this section we study the spectrum of the linear version $\mu = 0$ of (1.4):

$$p_t - \partial_v(vp) - \partial_{vv}p = N(t)\delta_{v=v_R} \quad \text{on } (-\infty, 0),$$

where $N(t) = -p_v(0,t)$ and $p(0,t) = 0$. The objective is to solve the eigenvalue problem

$$\begin{cases} \partial_{vv}p + \partial_v(vp) - p_v(0)\delta_{v=v_R} = \lambda\, p, \quad v \in (-\infty, 0), \\ p(0) = 0, \end{cases} \tag{5.1}$$

with eigenfunctions $p(v)$ in the space $L^2_{exp}(\mathbb{R})$ defined as

$$L^2_{exp}(\mathbb{R}) := \left\{ p \in L^2(\mathbb{R}) \,:\, \|p\|_{L^2_{exp}(\mathbb{R})} < \infty \right\}, \qquad \text{with } \|p\|^2_{L^2_{exp}(\mathbb{R})} := \int_{\mathbb{R}} \left(e^{v^2/2}|p(v)| \right)^2 dv.$$

Note that although problem (5.1) is only defined in $(-\infty, 0)$, it can be easily extended to $\mathbb{R}$ by odd reflection. Following an idea developed in [13, 14], we consider the equivalent problem to (5.1) defined as

$$\mathcal{L}(p_\lambda) \;:=\; \partial_{vv}p_\lambda + \partial_v(vp_\lambda) = \lambda\, p_\lambda \quad \text{in } (-\infty, v_R) \cup (v_R, 0), \tag{5.2}$$

with p_λ satisfying the following properties:

(F1) $p_\lambda \in L^2_{exp}(\mathbb{R})$ **(F2)** $p_\lambda(0) = 0$,
(F3) Matching: $p_\lambda(v_R^+) = p_\lambda(v_R^-)$, **(F4)** Jump: $\partial_v p_\lambda(v_R^+) = \partial_v p_\lambda(v_R^-) + \partial_v p_\lambda(0)$.

The main result of this section is:

Theorem 5.1. *Consider the eigenvalue problem* (5.1) *subject to conditions* **(F1)** - **(F4)**.

1. *There is no continuous spectrum.*
2. *The value $\lambda = 0$ is an eigenvalue with a one-dimensional eigenspace spanned by the function*

$$p_\infty(v) = \begin{cases} e^{-v^2/2} & v \in (-\infty, v_R), \\ \alpha_0 e^{-v^2/2} \displaystyle\int_v^0 e^{v^2/2}\, dv & v \in (v_R, 0], \end{cases}$$

for $\alpha_0 := \left(\displaystyle\int_{v_R}^0 e^{v^2/2}\, dv \right)^{-1}$.

3. *There exists a countable set $S \subset R$ such that for all $v_R \notin S$, there are no other eigenvalues.*
4. *If n and v_R happen to satisfy the compatibility condition* (5.9)*, then $\lambda = -2n$ is an eigenvalue with eigenspace of finite dimension spanned by $p_{2n}(v)$ defined in* (5.8).

We are going to write the solution for (5.1) in the form:

$$p_\lambda(v) = \chi_{(-\infty, v_R)} p^1(v) + \chi_{(v_R, 0)} p^2(v), \tag{5.3}$$

where each $p^i(v)$, $i = 1, 2$, is a linear combination of the two linearly independent solutions of (5.2) in $\mathbb{R}$, and such that the combination (5.3) satisfies **(F1)-(F4)**.

The functions $p^1(v)$ and $p^2(v)$ will be calculated by a standard classical method used to compute the spectrum for the classical Fokker-Planck equation given by $\mathcal{L}(p) = \lambda p$ in $\mathbb{R}$. Define first the function space

$$L^2_m(\mathbb{R}) = \left\{ p \in L^2(\mathbb{R}) \,:\, \|p\|_{L^2_m(\mathbb{R})} < \infty \right\}, \qquad \text{with } \|p\|^2_{L^2_m(\mathbb{R})} := \int_{\mathbb{R}} \left(1 + v^2\right)^m |p(v)|^2\, dv.$$

For completeness we recall a well known result on the spectrum for the classical operator $\mathcal{L}$, see for instance [11, 26]:

Lemma 5.2. *For any $m \geq 0$, the spectrum of the operator $\mathcal{L}$ on $L^2_m(\mathbb{R})$ is given by*

$$\sigma(\mathcal{L}) = \left\{\lambda \in \mathbb{C} \,:\, \Re(\lambda) \leq \tfrac{1}{2} - m\right\} \cup \{-n \,:\, n \in \mathbb{N} \cup \{0\}\}.$$

Moreover, if $m > \frac{1}{2}$ and if $n \in \mathbb{N} \cup \{0\}$ satisfies $n + \frac{1}{2} < m$, then $\lambda_n = -n$ is an isolated eigenvalue of $\mathcal{L}$, with multiplicity one, and eigenfunction given by the n-th Hermite polynomial

$$H_n(v) = (-1)^n e^{v^2/2} \frac{d^n}{dv^n} e^{-v^2/2}.$$

In particular, the spectrum of the Fokker-Planck operator $\mathcal{L}$ in the space $L^2_{exp}(\mathbb{R})$ reduces to the eigenvalues $\lambda = -n, \quad n \in \mathbb{N} \cup \{0\}$.

We consider now the original problem (5.1) and seek for solutions $p(v)$ of the form (5.3). Our first observation is that the values for λ are determined only by the decay of p as $v \to -\infty$. Consequently, if we impose that the function p^1 belongs to $L^2_{exp}(\mathbb{R})$, then this fixes the possible values of the eigenvalues λ as in Lemma 5.2. In particular, there is no continuous spectrum. Moreover, for each $\lambda_n = -n$, $n \in \mathbb{N}$, we must have for some $\alpha \in \mathbb{R}$

$$p^1(v) = \alpha H_n(v) e^{-v^2/2}. \tag{5.4}$$

The difference between our problem and the classical Fokker-Planck operator lies in the fact in the interval $(v_R, 0)$ all solutions to the ODE $\mathcal{L}(p) = \lambda\, p$ for $\lambda = -n$ are admissible since the behavior at infinity does not play any role. One of the solutions is given by (5.4) and the other one can be easily found by making the following ansatz: $p^2(v) = e^{-v^2/2} H_n(v) g(v)$.

By imposing that $p^2(v)$ satisfies (5.2) one can obtain an equation for $g(v)$ that reads: $2H_n' g' - v g' H_n + H_n g'' = 0$. This equation has the following general solution:

$$g(v) = \beta_1 \int_{v_0}^{v} \frac{e^{s^2/2}}{H_n^2(s)}\, ds + \beta_2,$$

for some constants $\beta_1, \beta_2 \in \mathbb{R}$, and where we have fixed any $v_0 \in (v_R, 0)$ such that $H_n(v_0) \neq 0$ for the integral to be well defined. Note that g is well defined for all v even where the denominator vanishes because the Hermite polynomials only have single roots. Consequently we define

$$p^2(v) := \beta_1 e^{-v^2/2} H_n(v) \int_{v_0}^{v} \frac{e^{s^2/2}}{H_n^2(s)}\, ds + \beta_2 e^{-v^2/2} H_n(v),$$

and the eigenfunction corresponding to $\lambda = -n$ is simply

$$p_n(v) = \begin{cases} \alpha e^{-v^2/2} H_n(v), & v \in (-\infty, v_R), \\ \beta_1 e^{-v^2/2} H_n(v) \displaystyle\int_{v_0}^{v} \frac{e^{s^2/2}}{H_n^2(s)}\, ds + \beta_2 e^{-v^2/2} H_n(v), & v \in (v_R, 0]. \end{cases} \tag{5.5}$$

for some real constants α, β. For simplicity define

$$\theta_n(v) := H_n(v) \int_{v_0}^{v} \frac{e^{s^2/2}}{H_n^2(s)}\, ds.$$

It is clear, by doing a careful Taylor expansion, that if v_1 is a root of H_n, then there exists a finite limit for $\Delta_{v_1,n} := \lim_{v \to v_1} \theta_n(v) \neq 0$. Now we are ready to check if (5.5) is an admissible eigenfunction. In the case n is odd integer, the Hermite polynomial H_{2n+1} vanishes at zero, but as we have mentioned, $\theta_{2n+1}(v) \to \Delta_{0,2n+1} \neq 0$, as $v \to 0$ for any $n \in \mathbb{N}$. Then in this case condition **(F2)** is satisfied only when $\beta_1 = 0$. Then, if we wish p_{2n+1} to be a continuous function as stated in condition **(F3)**, we must have $\alpha = \beta_2$ unless $H_{2n+1}(v_R) = 0$ that will be considered afterwards. The solution constructed this way does not satisfy condition **(F4)**, so we conclude that $2n + 1$ is not an admissible eigenvalue.

On the other hand, let us check if p_{2n} is an admissible eigenvalue. For even integers it holds that $H_{2n}(0) \neq 0$. Thus we can simply take $v_0 = 0$. Consequently condition **(F2)** is satisfied if and only if $\beta_2 = 0$. The matching condition **(F3)** implies

$$\alpha H_{2n}(v_R) = \beta_1 H_{2n}(v_R) \int_0^{v_R} \frac{e^{s^2/2}}{H_n^2(s)}\, ds. \tag{5.6}$$

Here we distinguish two cases: if v_R is not a root for any H_{2n}, then the above equality implies

$$\alpha = \beta_1 \int_0^{v_R} \frac{e^{s^2/2}}{H_n^2(s)}\, ds. \tag{5.7}$$

If instead $H_{2n}(v_R) = 0$ (note that Hermite polynomials only have single roots), one can repeat a Taylor expansion around v_R for $\theta_{2n}(v)$ and see that $\theta_{2n}(v) \to \Delta_{v_R,2n} \neq 0$, as $v \to v_R$. Consequently (5.6) cannot be satisfied for these n such that $H_{2n}(v_R) = 0$. Using conditions $\beta_2 = 0$ and (5.7) for p_n we get

$$p_{2n}(v) = \beta_1 e^{-v^2/2} H_{2n}(v) \cdot \begin{cases} \displaystyle\int_0^{v_R} \frac{e^{s^2/2}}{H_{2n}^2(s)}\, ds, & v \in (-\infty, v_R), \\ \displaystyle\int_0^{v} \frac{e^{s^2/2}}{H_{2n}^2(s)}\, ds, & v \in (v_R, 0]. \end{cases} \tag{5.8}$$

One can easily check that the jump condition **(F4)** is satisfied if and only if

$$H_{2n}(0) = H_{2n}(v_R). \tag{5.9}$$

Remark. We remark that the steady state $p_\infty(v)$ was previously obtained in [13, 3]. In this last paper, it was also shown exponential decay towards equilibrium p_∞. However, the speed of convergence is unknown and the spectral analysis does not seem to give any insight.

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