BOUNDARY CONNECTED SUM OF ESCOBAR MANIFOLDS

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ABSTRACT. Let $(X_1, \bar{g}_1)$ and $(X_2, \bar{g}_2)$ be two compact Riemannian manifolds with boundary (M_1, g_1) and (M_2, g_2) respectively. The Escobar problem consists in prescribing a conformal metric on a compact manifold with boundary with zero scalar curvature in the interior and constant mean curvature of the boundary. The present work is the construction of a connected sum $X = X_1 \# X_2$ by excising half ball near points on the boundary. The resulting metric on X has zero scalar curvature and a CMC boundary. We fully exploit the nonlocal aspect of the problem and use new tools developed in recent years to handle such kinds of issues. Our problem is of course a very well-known problem in geometric analysis and that is why we consider it but the results in the present paper can be extended to other more analytical problems involving connected sums of constant fractional curvatures.

CONTENTS

1. INTRODUCTION

Let $(X_i, \bar{g}_i)$, $i = 1, 2$, be two $(n+1)$-dimensional smooth compact Riemannian manifolds with boundaries M_i, $i = 1, 2$, for some $n \geq 2$, and set $g_i = \bar{g}_i|_{M_i}$. We assume that X_i are scalar-flat in the interior and have constant mean curvature H_0 on the boundary M_i.

We are interested in constructing connected sums on the boundary, producing a new scalar-flat manifold with CMC of the boundary. This is related

to the so-called Escobar problem (see [12]). Indeed, let u be a (positive) solution on X^{n+1} of the problem

$$\begin{cases} -\Delta_X u + \frac{n-1}{4n} R_X u & = & 0 & \text{in} & X, \\[2mm] \partial_\nu u + \frac{n-1}{2} H_M\, u & = & \frac{n-1}{2} H_0\, u^{\frac{n+1}{n-1}} & \text{on} & \partial X, \end{cases}$$

where R_X is the scalar curvature of X, H_M the mean curvature of $M = \partial X$, ν is the outer normal with respect to the metric g (on ∂X) and H_0 is a constant depending only on the conformal structure. Therefore, the new metric

$$\bar{g}' = u^{\frac{4}{n-1}} \bar{g}$$

has zero curvature and the boundary has constant mean curvature with respect to the metric $\bar{g}'$. Escobar in his seminal paper [12] solved the previous boundary problem in most of the cases (see also [5, 20, 21] for later developments on the problem).

The aim of the present work is to provide a different construction based on connected sums. While connected sums in the interior have been obtained in [18, 25], in this paper, we construct connected sums on the boundary. To this end, we fully use the analogy of the Escobar problem with the Dirichlet-to-Neumann operator approach as pointed out in [9, 17] (see also the survey [13]). Following the work of Graham and Zworski [15] and many others (see, for instance, [8, 16] and the references therein), one can see that the Dirichlet-to-Neumann operator associated to the Escobar problem is a conformally covariant pseudo-differential operator of order $1/2$.

We prove the following theorem:

THEOREM 1.1. *Let $(X_i, \bar{g}_i)$, $i = 1, 2$ be two smooth compact Riemannian manifolds with boundary (M_i, g_i), $g_i = \bar{g}_i|_{M_i}$, $i = 1, 2$. Assume that the manifolds X_i are non degenerate in the sense below, scalar-flat and with CMC boundary. Then there exists a connected sum $X_1 \# X_2$ obtained by excising an half-ball around a boundary point which is scalar-flat and has CMC boundary.*

The previous statement is well-known if one does connected sums in the interior as in the works of Joyce [18] and Mazzeo, Pollack and Uhlenbeck [25]. Our contribution is the development of new tools to handle connected sums on the boundary. In this case the problem becomes non-local since one has to deal with the Dirichlet-to-Neumann operator. While Escobar's problem is an old one, and gluing methods for constant positive scalar curvature are well known by now, when dealing with boundary gluing problems non-degeneracy is the main open question.

Due to the recent progress on non-local operators in conformal geometry, in particular [1], several major issues such as non-degeneracy can be by now resolved. Our point of view is to consider the boundary operator directly as a lower order perturbation in Hörmander classes of the half-Laplacian.

Other gluing problems for non-local equations have been considered in [1, 3, 4] with the objective of constructing singular metrics with prescribed non-local curvature. Another problem where non-degeneracy was a crucial point was in [2].

2. Formulation of the problem

As in the standard case (connected sums for Yamabe metrics), the idea is to use a perturbative argument, based on the non-degeneracy of each piece of the connected sum. In our context, this has to be done at a pseudo-differential level.

Let $(X, \bar{g})$ be an $(n+1)-$dimensional smooth compact Riemannian manifold with boundary $(M = \partial X, g = \bar{g}|_M)$ and consider the following boundary problem

$$(2.1) \qquad \begin{cases} L_{\bar{g}} u := -\Delta_{\bar{g}} u + \frac{n-1}{4n} R_{\bar{g}} u = 0 & \text{in} \quad X, \\ \\ \qquad\qquad\qquad\qquad\quad u = f & \text{on} \quad M, \end{cases}$$

where $R_{\bar{g}}$ is the scalar curvature of $(X, \bar{g})$. It has been proved in [17] and [9] (see the case 2, $\gamma = 1/2$, of Theorem 5.1) that a solution u of (2.1) exists and is unique. This allows to define the Dirichlet-to-Neumann type operator by

$$f \mapsto P_g f = \partial_\nu u|_{\partial X} + \frac{n-1}{2} H_g \, f$$

where H_g is the mean curvature of the boundary M with respect to the metric g. We also denote

$$Q_g := P_g(1).$$

REMARK 2.1. Notice that Q_g is different from $\frac{n-1}{2} H_g$ when $R_{\bar{g}}$ is not zero.

The operator pair $(L_{\bar{g}}, P_g)$ is conformally convariant, indeed, under the change of metric $\bar{g}_u = u^{\frac{4}{n-1}} \bar{g}$, and consequently, $g_f = f^{\frac{4}{n-1}} g$ for $f = u|_M$, we have that

$$(2.2) \qquad \begin{cases} L_{\bar{g}_u}(\cdot) = u^{-\frac{n+1}{n-1}} L_{\bar{g}}(u \, \cdot) & \text{in} \quad X, \\ \\ P_{g_f}(\cdot) = f^{-\frac{n+1}{n-1}} P_g(f \, \cdot) & \text{on} \quad M. \end{cases}$$

By evaluating at the function identically one, one obtains the equation pair:

$$\begin{cases} L_{\bar{g}} u = \frac{n-1}{4n} R_{\bar{g}} u^{\frac{n+1}{n-1}} & \text{in} \quad X, \\ \\ P_g f = Q_{g_f} f^{\frac{n+1}{n-1}} & \text{on} \quad M, \end{cases}$$

where $u|_M = f$.

As a consequence, the problem under consideration is written as

$$(2.3) \qquad \begin{cases} L_{\bar{g}} u = 0 & \text{in} \quad X, \\ \\ P_g f = c f^{\frac{n+1}{n-1}} & \text{on} \quad M, \end{cases}$$

for a constant c. In order to simplify our notation we will simply write

$$(2.4) \qquad\qquad\qquad \mathcal{Q}_g(f) := f^{-\frac{n+1}{n-1}} P_g f = c$$

for problem (2.3) and the extension equation will be implicitly understood.

In this paper we will only consider the case c positive. The Yamabe problem in the non-positive case is much easier to handle and follows directly from the variational methods in [12].

By construction, the operator P_g is a pseudo-differential operator of order 1 (recall the extension formulation for the half-Laplacian [7, 9]). Furthermore,

$$P_g = (-\Delta_g)^{1/2} + \Psi_g,$$

where Ψ_g is a pseudo-differential operator of lower order. Then the analysis for (2.3) is inspired by equation

$$(-\Delta_g)^{1/2} f = c f^{\frac{n+1}{n-1}}.$$

We could think now of the problem

$$(-\Delta_g)^\gamma f = c f^{\frac{n+2\gamma}{n-2\gamma}}, \quad \gamma \in (0,1).$$

The gluing construction for other powers $\gamma \in (0,1)$ in order to have connected sums of constant non-local γ-curvature should still hold, but it is more delicate since when building an approximate metric $\bar{g}_\varepsilon$ in Section 3, we move outside of the conformal class of $\bar{g}$. The technicalities involved should be the same ones as in Section 6.8 of [13].

We conclude this section by noting that a key operator is the linearization of P_g around a constant $f \equiv 1$ defined on M. Since P_g is linear we see that

$$(2.5) \qquad \mathcal{L}_g v := D\mathcal{Q}_g(1)v = \frac{d}{dt}\Big|_{t=0} \mathcal{Q}_g(1+tv) = P_g v - \frac{n+1}{n-1} \mathcal{Q}_g v.$$

3. Construction of the approximate metric

To construct the approximate solution, we follow the approach of Mazzeo, Pollack and Uhlenbeck [25] and, in particular, the one of Mazzeo and Pacard [24]. There the gluing procedure to obtain $X_1 \# X_2$ is along an half-ring around a point at infinity. Here we consider the same type of construction at a point on the boundary.

Let $(X_i, \bar{g}_i)$, $i = 1, 2$ be two manifolds with boundary $(M_i = \partial X_i, g_i = \bar{g}_i|_{M_i})$, $i = 1, 2$ as in the hypothesis of Theorem 1.1. Consider $p_1 \in M_1 = \partial X_1$ and let $\{y_\ell\}$ be a system of normal coordinates on M_1 centered at p_1. We set Fermi coordinates $z = (x, y) \in X_1$ and the annulus

$$A_\varepsilon = \{\varepsilon\delta < |z| < \delta\},$$

where $|z|^2 = x^2 + |y|^2$. We will take $\delta = \delta(\varepsilon)$ satisfying $\delta \to 0$ but this will not make a difference in the following as long as $\varepsilon^{1/2} << \delta$ as $\varepsilon \to 0$.

In Fermi coordinates, the metric is written as

$$\bar{g}_1 = (dx^2 + dy^2) + O(\delta^2).$$

Thanks to the dilation

$$R_\delta : A_1 \ni z \longrightarrow \delta z \in A_\delta,$$

one can work on A_1 (the dependence of δ should still be taken into account, though). The metric on A_1 becomes

$$\delta^2(dx^2 + dy^2) + O(\delta^4).$$

Multiplying by δ^{-2} (the metric stays scalar-flat with CMC on the boundary), one can work with the new metric

$$\bar{g}_{1,\delta} = (dx^2 + dy^2) + O(\delta^2).$$

Then one reproduces the same construction on X_2 in an annulus $A_1' = \{\varepsilon < |z'| < 1\}$, parameterized with Fermi coordinates $z' = (x', y')$ near a point $p_2 \in M_2$. Thanks to an inversion $z' = I(z) = \frac{z}{|z|^2}\varepsilon$, one can identify both annulus A_1 and A_1', and we can define a smooth manifold with boundary

$$X := X_1 \# X_2.$$

Denote also $M = \partial X$. Note that, often, we will just refer to the manifold M, even though the manifold X is present in the background at all times.

Note that the annulus $A_1 \sim A_1'$ is naturally embedded in X. Let us now write the metric on this connected sum.

Set $z = (x, y) = (r\cos\phi, r\sin\phi\,\theta)$ with $r \in \mathbb{R}^+$, $\phi \in [0, \pi/2]$, $\theta \in \mathbb{S}^{n-1}$ be the coordinates in A_1. We would like to glue the background metric $\bar{g}_{1,\delta}$ to the cylinder

$$\bar{g}_0 := \frac{dr^2 + r^2 d\phi^2 + r^2\sin^2\phi d\theta^2}{r^2} = ds^2 + d\phi^2 + \sin^2\phi\,d\theta^2,$$

where we have defined the variable $s = -\log r$. For this, consider the approximate cylindrical metric

$$\tilde{g}_{1,\delta} = \left(\chi(r) + (1 - \chi(r))\frac{1}{r^2}\right)\bar{g}_{1,\delta},$$

for χ is a cutoff satisfying $\chi \equiv 1$ if $r \geq 2$, $\chi \equiv 0$ if $r \leq 1$. In the smaller ball $B_1(p_1)$, this metric is isometric to a cylinder plus a small error term. Repeat for g_2.

Now we glue both metrics $\tilde{g}_{1,\delta}$, $\tilde{g}_{2,\delta}$ by a cutoff. Let $S_\varepsilon := -\log\varepsilon$ and consider, as above,

$$A_1 = \{\varepsilon < r < 1\} = \{0 < s < S_\varepsilon\}.$$

Let $\tilde{\chi}(s)$ be a cutoff function on A_1, satisfying $\tilde{\chi} \equiv 1$ for $s \leq S_\varepsilon/2 - 1$, $\tilde{\chi} \equiv 0$ for $s \geq S_\varepsilon/2 + 1$.

Now repeat with A_1', and recall that $s' = -s + S_\varepsilon$. The approximate solution metric $\bar{g}_\varepsilon$ is constructed as

$$\bar{g}_\varepsilon = \tilde{\chi}(s)\tilde{g}_{1,\delta} + \tilde{\chi}'(s')\tilde{g}_{2,\delta}.$$

On $M := \partial X$ we consider the metric $g_\varepsilon := \bar{g}_\varepsilon|_M$. Note that $\bar{g}_\varepsilon = \bar{g}_{i,\delta}$ on $X_i \backslash B_{C\sqrt{\varepsilon}}(p_i)$, $i = 1, 2$.

It will be convenient to define a new variable $\bar{s} = s - S_\varepsilon/2$, and to write the neck $A_1 \sim A_1'$ as $\mathcal{T}_\varepsilon = \{-S_\varepsilon/2 < s < S_\varepsilon/2\}$. We let also $\mathcal{M}_\varepsilon := \mathcal{T}_\varepsilon|_{\phi=\frac{\pi}{2}}$. Then, as $\varepsilon \to 0$, this neck converges to $\mathcal{T}_0 = \mathbb{R} \times \mathbb{S}_+^n$ with the metric

$$\bar{g}_0 = d\bar{s}^2 + d\phi^2 + \sin^2\phi\,d\theta^2 = d\bar{s}^2 + g_{\mathbb{S}^n},$$

where $g_{\mathbb{S}^n}$ is the standard metric on $\mathbb{S}_+^n$. Hence $(\mathcal{T}_0, \bar{g}_0)$ is half a solid cylinder, with boundary given by a lower dimensional cylinder $\mathcal{M}_0 = \{\phi = \pi/2\} = \mathbb{R} \times \mathbb{S}^{n-1}$ and metric $g_0 = d\bar{s}^2 + d\theta^2$.

In Lemma 6.1 we will show that we indeed have constructed a good approximate solution.

4. The model cylinder

From now on, we will take $\bar{s}$ as variable, and drop the bar in the notation if there is no risk of confusion.

Here we calculate precisely the Fourier symbol of the operator P_{g_0} in the model cylinder. As in the previous section, let $\mathcal{T}_0 = \mathbb{R} \times \mathbb{S}_+^n$ be the half solid cylinder with the metric

$$\bar{g}_0 = ds^2 + d\phi^2 + \sin^2 \phi \, d\theta^2,$$

where $s \in \mathbb{R}$, $\phi \in [0, \pi/2]$ and $\theta \in \mathbb{S}^{n-1}$. Its boundary (or, more precisely, its conformal infinity) is the model cylinder $\mathcal{M}_0 = \{\phi = \pi/2\} = \mathbb{R} \times \mathbb{S}^{n-1}$ with metric

$$g_0 = ds^2 + d\theta^2.$$

Consider the spherical harmonic decomposition for $\mathbb{S}^{n-1}$. With some abuse of notation, let $\mu_m = m(m+n-2)$, $m = 0, 1, 2, \ldots$ be the eigenvalues of $-\Delta_{\mathbb{S}^{n-1}}$, repeated according to multiplicity. Then any function on $\mathbb{R} \times \mathbb{S}^{n-1}$ may be decomposed as $u(s, \theta) = \sum_m u_m(s) E_m(\theta)$, where $\{E_m(\theta)\}_m$ is the corresponding basis of eigenfunctions.

Following the construction of [11], let us write the hyperbolic AdS metric as

$$g_0^+ = \frac{1}{1 + R^2} dR^2 + (1 + R^2) ds^2 + R^2 d\theta^2,$$

and make the change of variables

$$R = \tan \phi.$$

This yields

$$(4.1) \qquad g_0^+ = \frac{1}{\cos^2 \phi} \left[d\phi^2 + ds^2 + \sin \phi \, d\theta^2 \right] = \frac{\bar{g}_0}{\cos^2 \phi}.$$

The function $\varrho = \cos \phi$ is a defining function for $\mathcal{M}_0$, which is the conformal infinity for this metric.

The conformal fractional Laplacian is calculated from the scattering operator of the Einstein metric g_0^+. The operator P_{g_0} is defined as the Dirichlet-to-Neumann operator for the extension problem (2.1) in the metric $\bar{g}_0$. As it is explained in [9] (see also [17]), P_{g_0} is precisely the conformal version of the half-Laplacian coming from scattering theory in the metric g_0^+. Here note that (4.1) is not written in normal form, but for the half-Laplacian case this is not an issue since the defining function does not appear explicitly in (2.1).

The calculation of the scattering operator and the conformal fractional Laplacian in this setting can be found in [10, 11] and, in particular:

PROPOSITION 4.1 ([11]). *Fix* $\gamma \in (0, \frac{n}{2})$ *and let* $P_\gamma^{(m)}$ *be the projection of the operator* $P_\gamma^{g_0}$ *over each eigenspace* $\langle E_m \rangle$. *Then*

$$\widehat{P_\gamma^{(m)}(f_m)} = \Theta_m(\xi) \, \widehat{f_m}.$$

Here $\widehat{}$ *denotes the usual Fourier transform*

$$\widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(s) e^{-i\xi s} \, ds.$$

The Fourier symbol Θ_m is given by

$$\Theta_m(\xi) = 2^{2\gamma} \frac{\left| \Gamma\left(\frac{1}{2} + \frac{\gamma}{2} + \frac{1}{2} \left(\frac{n}{2} + m - 1 \right) + \frac{\xi}{2} i \right) \right|^2}{\left| \Gamma\left(\frac{1}{2} - \frac{\gamma}{2} + \frac{1}{2} \left(\frac{n}{2} + m - 1 \right) + \frac{\xi}{2} i \right) \right|^2}.$$

In this paper we are only interested in the case $\gamma = 1/2$, so for the Fourier symbol of $P_{g_0} := P_{1/2}^{g_0}$ simplifies to

$$\Theta_m(\xi) = 2 \frac{\left| \Gamma\left(\frac{1}{4} + \frac{1}{2} \left(\frac{n}{2} + m \right) + \frac{\xi}{2} i \right) \right|^2}{\left| \Gamma\left(-\frac{1}{4} + \frac{1}{2} \left(\frac{n}{2} + m \right) + \frac{\xi}{2} i \right) \right|^2}.$$

In addition, one can check that

$$(4.2) \qquad Q_{g_0} = P_{g_0}(1) = \Theta_0(0) = \frac{2\,\Gamma\left(\frac{n+1}{4} \right)^2}{\Gamma\left(\frac{n-1}{4} \right)^2} =: c.$$

Note that this constant, in the paper [11], is denoted by $c_{n,1/2}$.

5. Linear study

We endow $X = X_1 \# X_2$ with the approximate metric $\bar{g}_\varepsilon$ constructed in Section 3. Let $\mathcal{L}_\varepsilon$ be the linearized operator (2.5) around the approximate metric $g_\varepsilon = \bar{g}_\varepsilon|_M$. The main step in the proof Theorem 1.1 is to show that $\mathcal{L}_\varepsilon$ is invertible, provided that it is invertible on each X_i, $i = 1, 2$. Furthermore, we need to have uniform bounds for the inverse with respect to ε.

Before we study the operator $\mathcal{L}_\varepsilon$, let us look at the model cylinder g_0, for which the linearized operator has the simple formula

$$\mathcal{L}_0 v = P_{g_0} v - \kappa v,$$

where we have defined the constant $\kappa := \frac{n+1}{n-1} c > 0$. Taking into account the conformal transformation law for the conformal fractional Laplacian, if we write our metric as

$$g_0 = \rho^{-2}(d\rho^2 + \rho^2 d\theta^2) = \rho^{-2} g_{\mathbb{R}^n},$$

for $\rho = e^{-s}$ and make the change

$$(5.1) \qquad \tilde{v} = \rho^{-\frac{n-1}{2}} v,$$

then we can see that $\mathcal{L}_0$ is a conjugate of a fractional Laplacian operator on $\mathbb{R}^n$ with critical Hardy potential, this is,

$$(5.2) \qquad \mathcal{L}_0 v = \rho^{\frac{n+1}{n-1}} \left[(-\Delta_{\mathbb{R}^n})^{1/2} \tilde{v} - \frac{\kappa}{\rho} \tilde{v} \right] =: \rho^{\frac{n+1}{n-1}} \tilde{\mathcal{L}}_0 \tilde{v}.$$

One should keep in mind this conjugation in the analysis below, specially the shift in powers from (5.1).

5.1. **Elliptic regularity in weighted spaces.** Consider the weight ω_ε defined as a smooth version of the function defined by

$$\tilde{\omega}_\varepsilon = \begin{cases} \cosh(s)/\cosh(S_\varepsilon), & \text{on } \mathcal{T}_\varepsilon, \\ 1, & \text{on } X\backslash\mathcal{T}_\varepsilon. \end{cases}$$

We require that ω_ε is smooth, and agrees with ω_ε except on a small neighborhood of the neck, where it remains bounded between $\frac{1}{2}$ and 2. The previous weight has been introduced by Mazzeo and Pacard in [24, Section 4]. Denote

$$\overline{\omega}_\varepsilon = \omega_\varepsilon|_{\phi=\pi/2},$$

its restriction to the boundary. We note for comparison that when s is fixed, and $\varepsilon \to 0$,

$$\overline{\omega}_\varepsilon \sim 2\varepsilon\cosh(s).$$

Now introduce the weighted Hölder spaces $\omega_\varepsilon^\mu \mathcal{C}^{k,\alpha}$ endowed with their natural norm

$$\|v\|_{\mu,k,\alpha} = \|(\overline{\omega}_\varepsilon)^{-\mu}v\|_{\mathcal{C}^{k,\alpha}}.$$

We have the following lemma.

LEMMA 5.1. *For each $\varepsilon > 0$, the map*

$$\mathcal{L}_\varepsilon : \overline{\omega}_\varepsilon^\mu \mathcal{C}^{2,\alpha}(M) \to \overline{\omega}_\varepsilon^\mu \mathcal{C}^{1,\alpha}(M)$$

is bounded.

Proof. Far from the point p_i in M_i, $\mathcal{L}_\varepsilon$ is a lower order perturbation of $(-\Delta_{g_i})^{1/2}$, $i = 1, 2$. There the weights do not play any role and thus $\mathcal{L}_\varepsilon$ is bounded in $\mathcal{C}^{2,\alpha}$.

On the other hand, near the point p_1, for instance, after conjugation, $\mathcal{L}_\varepsilon$ is also a lower order perturbation of $(-\Delta_g)^{1/2}$. Thus the mapping properties of $\mathcal{L}_\varepsilon$ can be deduced from the ones of $(-\Delta_g)^{1/2}$, and the Lemma follows by standard pseudodifferential calculus in weighted spaces (see the references [22] and [26] on regularity and boundary regularity for edge operators). $\square$

5.2. **A Liouville theorem for the model metric.** One of the main results in [1] implies that the behavior of a singular solution to $\mathcal{L}_0 v = 0$ is given in terms of the inditial roots of the problem. This yields a Liouville theorem for $\mathcal{L}_0$ (see Theorem 5.4 below). For this, in the notation of Proposition 4.1 for the spherical harmonic decomposition, set

$$(5.3) \qquad\qquad \mathcal{L}_0^{(m)}v := P_{1/2}^m v - \kappa v, \quad m = 0, 1, \dots$$

to be the m-th projection of the operator $\mathcal{L}_0$.

First, the indicial roots for $\mathcal{L}_0^{(m)}$ are given in the following Lemma from [1], and we refer to this paper for a more detailed asymptotic behavior. The crucial point of the later proofs is that, even though there exists an infinite sequence of inditial roots for each mode $m = 0, 1, \dots$, the behavior of (5.3) is governed by the first inditial root pair; its real part will be denoted by $\gamma_m^\pm$.

LEMMA 5.2. *Fix $m = 0, 1, \ldots$ Both at $t = +\infty$ and $t = -\infty$, there exist two sequences of indicial roots for $\mathcal{L}_0^{(m)}$, with no accumulation points:*

$$\{\sigma_j^{(m)} \pm i\tau_j^{(m)}\}_{j=0}^{\infty} \quad and \quad \{-\sigma_j^{(m)} \pm i\tau_j^{(m)}\}_{j=0}^{\infty}.$$

We denote $\gamma_m^{\pm} := \pm\sigma_0^{(m)}$. Then

a) *For the mode $m = 0$, $\gamma_0^{\pm} = \sigma_0^{(0)} = 0$, so we have a pair of inditial roots that are complex conjugates with real part zero.*

b) *For the remaining modes $m \geq 1$, $\tau_0^{(m)} = 0$, so the first inditial root $\gamma_m^{\pm}$ is real. Actually, for the mode $m = 1$, $\gamma_1^- := -\sigma_0^{(1)} = -1$.*

c) *$\{\gamma_m^+\}$ is an increasing sequence in m.*

d) *In addition, for all $j \geq 1$, $m \geq 0$, $\sigma_j^{(m)} > \frac{n-1}{2}$.*

Proof. This follows from Lemma 7.1 in [1] for the special case $\gamma = \frac{1}{2}$ and $p = \frac{n+2\gamma}{n-2\gamma}$. There the indicial roots are written for the operator $\tilde{\mathcal{L}}_0$ from (5.2), but they only differ by a translation of $\frac{n-1}{2}$. Moreover, the indicial roots at $t = +\infty$ and at $t = -\infty$ are the same in our case.

Note, however, that in [1] the lemma is stated only for $p < \frac{n+2\gamma}{n-2\gamma}$, but one can easily check that it also holds for $p = \frac{n+2\gamma}{n-2\gamma}$. $\quad\square$

Now we adapt Theorem 6.1, Remark 6.2. and Proposition 6.3 in [1] on the behavior of the Green's function for $\mathcal{L}_0^{(m)}$ to our setting (note that it corresponds to the unstable case):

THEOREM 5.3. *Consider the equation*

(5.4)
$$\mathcal{L}_0^{(m)} v = h.$$

Assume that the right hand side h in (5.4) satisfies

$$h(s) = \begin{cases} O(e^{-\delta s}) & as \ s \to +\infty, \\ O(e^{\delta_0 s}) & as \ s \to -\infty, \end{cases}$$

for some real constants δ, δ_0. It holds:

i. *If $\delta > 0$ and $\delta_0 \geq 0$, then a particular solution of (5.4) can be written as*

$$v_m(s) = \int_{\mathbb{R}} h(\tilde{s})\mathcal{G}_m(s - \tilde{s}) \, d\tilde{s},$$

where

$$\mathcal{G}_m(s) = d_0 \sin(\tau_0 s)\chi_{(-\infty,0)}(s) + \sum_{j=1}^{\infty} d_j e^{-\sigma_j|s|} \cos(\tau_j|s|)$$

for some constants d_j, $j = 0, 1, \ldots$. Moreover, $\mathcal{G}_m$ is an even $\mathcal{C}^{\infty}$ function when $s \neq 0$ and

(5.5) $\quad v_m(s) = O(e^{-\delta s})$ *as $s \to +\infty$,* $\quad v_m(s) = O(e^{\delta_0 s})$ *as $s \to -\infty$.*

ii. *Now assume only that $\delta + \delta_0 \geq 0$. If $\sigma_J < \delta < \sigma_{J+1}$ (and thus $\delta_0 > -\sigma_{J+1}$), then a particular solution is*

$$v_m(s) = \int_{\mathbb{R}} h(\tilde{s})\tilde{\mathcal{G}}_m(s - \tilde{s}) \, d\tilde{s},$$

where

$$\tilde{\mathcal{G}}_m(s) = \sum_{j=J+1}^{\infty} d_j e^{-\sigma_j|s|} \cos(\tau_j|s|).$$

Moreover, $\tilde{\mathcal{G}}_m$ is an even $\mathcal{C}^\infty$ function when $s \neq 0$ and the same conclusion as in (5.5) holds.

iii. All solutions of the homogeneous problem $\mathcal{L}_0^{(0)}v = 0$ are of the form

(5.6)
$$v(s) = C_0^+ \sin(\tau_0^{(0)}s) + C_0^- \cos(\tau_0^{(0)}s)$$
$$+ \sum_{j=1}^{\infty} C_j^- e^{-\sigma_j^{(0)}s} \cos(\tau_j^{(0)}s) + \sum_{j=1}^{\infty} C_j^+ e^{+\sigma_j^{(0)}s} \cos(\tau_j^{(0)}s)$$

for some real constants C_j^-, C_j^+, $j = 0, 1, \ldots$, and all solutions to $\mathcal{L}_0^{(m)}v = 0$ for $m \geq 1$ are of the form

(5.7) $$v(s) = \sum_{j=0}^{\infty} C_j^- e^{-\sigma_j^{(m)}s} \cos(\tau_j^{(m)}s) + \sum_{j=0}^{\infty} C_j^+ e^{+\sigma_j^{(m)}s} \cos(\tau_j^{(m)}s)$$

iv. The only solution to (5.4), in both the cases i. *and* ii., *with decay as in (5.5) is precisely v_m. Moreover we have unique continuation: any solution that decays faster than any exponential at $t \to \infty$ or $t \to -\infty$ must vanish identically.*

Then one obtains, as a consequence of parts *iii.-iv.* above, the following Liouville theorem by choosing the weight μ accordingly:

THEOREM 5.4. *Assume that $\mu \in (-\frac{n-1}{2}, 0)$ and $\mu \neq -\sigma_0^{(m)}$ for $m \geq 0$. Then any solution to $\mathcal{L}_0 v = 0$ satisfying $|v(s, \theta)| \leq ce^{\mu s}$ or $|v(s, \theta)| \leq ce^{-\mu s}$ must vanish everywhere.*

Proof. From statements *iii.* of Theorem 5.3, all the solutions to the homogeneous problem $\mathcal{L}_0^{(m)}v = 0$ are given by (5.6) and (5.7). If the weight $\mu \in (-\frac{n-1}{2}, 0)$, then by *d)* of Lemma 5.2, $e^{\sigma_j^{(m)}s} > e^{\mu s}$ for $s > 0$ and $e^{-\sigma_j^{(m)}s} > e^{\mu s}$ for $s < 0$ when $j \geq 1$. So all $C_j^\pm = 0$ for $j \geq 1$, $m \geq 0$. Moreover, if $\mu \neq -\sigma_0^{(m)}$ for all m, using *c.* of Lemma 5.2, there exists $J \geq 0$ such that $-\sigma_0^{J+1} < \mu < \sigma_0^J$. Similar to the previous argument again, $e^{\mu s} < e^{-\sigma_0^i s}$ when $s > 0$ for $i \leq J$ (here we agree that $-\sigma_0^{(i)} = \sigma_0^{(-i)}$ if $i < 0$) and $e^{\mu s} < e^{-\sigma_0^i s}$ when $s < 0$ for $i \geq J + 1$, so $C_0^\pm = 0$ for all m. Combining all the above results, one can elliminate all the nontrivial solutions to $\mathcal{L}_0 v = 0$. Then in this weighted space, only the trivial solution exists. $\qquad\square$

5.3. **Removability of singularities.** We present here a standard result concerning removability of isolated singularities for an elliptic problem if the solution grows slower than its Green's function.

PROPOSITION 5.5. *Let $(X, \bar{g})$ be a compact manifold with boundary $(M, g := \bar{g}|_M)$. Assume $\mathcal{L}_g v = 0$ on $M \backslash \{p\}$ and*

(5.8) $$|v(\cdot)| < C \operatorname{dist}(p, \cdot)^\mu, \quad \mu \in (1 - n, 0)$$

then u extends on all of M so that $\mathcal{L}_g v = 0$ on M.

Proof. This is a standard result but we have not been able to find an exact reference for it. In normal coordinates near p, $\mathcal{L}_g$ is a lower order perturbation of $(-\Delta_g)^{1/2}$ plus a zero-th order term. A classical Bôcher theorem for the half-Laplacian on a punctured ball (see, for instance, [19]) yields that v should be the sum of a smooth function plus a constant times the fundamental solution of the half-Laplacian. But our hypothesis (5.8) prevents this second case and thus the constant must be zero. Then v can be extended smoothly all the way across the singularity. $\qquad\square$

REMARK 5.6. This proposition has to be compared to the last statement in Theorem 1.1 in [14].

5.4. **Non-degeneracy of the linearized operator.** We now prove the main result of this section.

PROPOSITION 5.7. *In the setting of Theorem 1.1, assume that the maps $\mathcal{L}_{g_i} : \mathcal{C}^{1,\alpha}(M_i) \to \mathcal{C}^\alpha(M_i)$ on M_i, $i = 1, 2$, have trivial kernel. Then for*

$$(5.9) \qquad \mu \in \left(-\frac{n-1}{2}, 0 \right) \quad and \quad \mu \neq -\sigma_0^{(m)} \text{ for } m \geq 0,$$

the linearized operator $\mathcal{L}_\varepsilon$ is invertible on $M = \partial X$. Moreover, its inverse can be bounded independently of ε (small enough).

Proof. We only have to show that there exist a constant C such that for all $v \in \overline{\omega}_\varepsilon^\mu \mathcal{C}^{2,\alpha}$,

$$\|v\|_{\mu,1,\alpha} \leq C \|\mathcal{L}_\varepsilon v\|_{\mu,0,\alpha}$$

for some C independent of ε small enough. We follow the proof of proposition 11 of [24]. By contradiction, there would exist sequences $\varepsilon_j \to 0$ and $\{v_j\}$ for which

$$\|v_j\|_{\mu,1,\alpha} = 1,$$

while

$$\|\mathcal{L}_{\varepsilon_j} v_j\|_{\mu,0,\alpha} \to 0.$$

In particular this gives the estimates

$$|v_j(y)| \leq \overline{\omega}_{\varepsilon_j}^\mu(y),$$

and

$$|\mathcal{L}_{\varepsilon_j} v_j(y)| \leq \eta_j \overline{\omega}_{\varepsilon_j}^\mu(y)$$

for all $y \in M$, where $\eta_j \to 0$. Let q_j a point where $|v_j|(\overline{\omega}_{\varepsilon_j})^{-\mu}$ attains its maximum. Possibly passing to a subsequence, different cases can occur:

   (i) $\{q_j\}$ converges to a point q in $M_1 \backslash \{p_1\}$ or $M_2 \backslash \{p_2\}$,

   (ii) $\{q_j\}$ lies in $\mathcal{M}_\varepsilon$, and $|s_j| \leq C$.

   (iii) $\{q_j\}$ lies in $\mathcal{M}_\varepsilon$, and $|s_j| \to \infty$.

We start by ruling out the case *(i)*. We may assume that $q_j \to q \in M_1 \backslash \{p_1\}$. Then $\{v_j\}$ converges in $\mathcal{C}^{2,\alpha}$ on any compact set of $M_1 \backslash \{p_1\}$ to v such that $\mathcal{L}_{g_1} v = 0$. Now recall that on $B_C(p_1) \backslash B_\varepsilon(p_1)$ we have $\overline{\omega}_\varepsilon$ varying from $\frac{2}{\varepsilon + \varepsilon^{-1}} \sim 2\varepsilon = 2d(p_1, \cdot)$ to 1, so using that $|v_j(y)| \leq \overline{\omega}_{\varepsilon_j}(y)^\mu$, we obtain as $\varepsilon \to 0$, that $|v| \leq Cd(p_1, \cdot)^\mu$. Hence by Proposition 5.5 and the hypothesis

that $\mathcal{L}_{g_1}$ has trivial kernel, we have $v \equiv 0$. However, $v(q) \neq 0$ since $\|v\|_{\mu,1,\alpha} = 1$. This yields a contradiction.

We now study cases (ii) and (iii). For the case (ii), as $\varepsilon \to 0$, $\mathcal{M}_\varepsilon \to \mathcal{M}_0$, and by the assumption that $\{q_j\}$ stays in a compact set of $\mathcal{M}_0$, we may take $q_j \to q \in \mathcal{M}_0$. But when $y \in \mathcal{M}_0$,

$$\frac{\overline{\omega}_{\varepsilon_j}(y)}{\overline{\omega}_{\varepsilon_j}(q_j)} = \frac{\cosh s}{\cosh s_j} \to c' \cosh s,$$

where $c' > 0$ and the convergence is $\mathcal{C}^\infty$ on any compact set. We define

$$u_j := \overline{\omega}_{\varepsilon_j}(q_j)^{-\mu} v_j,$$

so $u_j \to u$ with $u(q) \neq 0$, $\mathcal{L}_0 u = 0$ (where the last operator is with respect to the metric g_0) and $|u(y)| \leq c \cosh^\mu(s) \leq c e^{\mu|s|}$. By Theorem 5.4, one has $u \equiv 0$, which contradicts to $u(q) \neq 0$.

Now consider the case (iii) and define

$$u_j(s,\theta) := \overline{\omega}_{\varepsilon_j}(q_j)^{-\mu} v_j(s+s_j,\theta).$$

As $|v_j(s+s_j,\theta)| \leq c\overline{\omega}_{\varepsilon_j}(s+s_j,\theta)^\mu$ and

$$\frac{\overline{\omega}_{\varepsilon_j}(s+s_j)}{\overline{\omega}_{\varepsilon_j}(s_j)} = \frac{\cosh(s+s_j)}{\cosh S_{\varepsilon_j}} \frac{\cosh S_{\varepsilon_j}}{\cosh s_j} \to \begin{cases} e^s, & \text{if } s_j \to +\infty, \\ e^{-s}, & \text{if } s_j \to -\infty, \end{cases}$$

then $u_j \to u$ with $u(0,\theta) \neq 0$ and u is a solution to $\mathcal{L}_0 u = 0$. Since $|u(y)| \leq c(e^s)^\mu$ everywhere or $|u(y)| \leq c(e^{-s})^\mu$ everywhere, again by Theorem 5.4, we arrive at a contradiction. □

REMARK 5.8. In Proposition 5.7 it is enough to assume that M_1, M_2 are non-degenerate (in the sense that $\mathcal{L}_\varepsilon$ is well behaved only after adding a *deficiency* space). This is the case if some M_i, $i = 1, 2$, is a Delaunay metric, for instance. However, this generalization only adds technicalities in the analysis and we are not interested in those here.

5.5. **Surjectivity.** To find an inverse for the linearized operator $\mathcal{L}_\varepsilon$ we set up the problem in the Hilbert space $L^2(X)$. For this functional analysis part we do not need to introduce weights since the manifold $X = X_1 \# X_2$ is compact; in the noncompact case, one would need to consider weighted Lebesgue spaces and to work with deficiency spaces (see [25] and later references).

First note that $\overline{\omega}_\varepsilon^\mu \mathcal{C}^\alpha \hookrightarrow L^2$ if $\mu < 0$. We claim that, in weighted Lebesgue spaces, $\mathcal{L}_\varepsilon : L^2(M) \to L^2(M)$ is injective. To see this, let $v \in L^2$ such that $\mathcal{L}_\varepsilon v = 0$. By Proposition 5.7, it is enough to show that $v \in \overline{\omega}_\varepsilon^\mu \mathcal{C}^{1,\alpha}$. But this is a consequence of Theorem 5.3 and elliptic regularity again.

Next, since $\mathcal{L}_\varepsilon$ is a self-adjoint operator in L^2, surjectivity follows immediately. By elliptic regularity, one can obviously produce a right inverse G_ε in the weighted space $\overline{\omega}_\varepsilon^\mu \mathcal{C}^\alpha$. Finally, Proposition 5.7 shows that this inverse has norm uniformly bounded in ε.

6. THE CONSTRUCTION: PROOF OF THEOREM 1.1

Let g_ε be the metric constructed in Section 3 for the connected sum $M = \partial X$, $X = X_1 \# X_2$. The proof of Theorem 1.1 follows from solving

$$\mathcal{Q}_{g_\varepsilon}(f) := f^{-\frac{n+1}{n-1}} P_{g_\varepsilon} f = c \ \text{ constant}.$$

Recalling (2.3), this equation should always be coupled with

$$(6.1) \qquad\qquad L_{\bar{g}_\varepsilon}(f) = 0 \quad \text{in } X,$$

even if not always explicitly written. Without loss of generality, let us normalize the constant c as in (4.2).

The procedure is by now standard, using a fixed point argument (see, for instance, Section 9 in [23]). Define $f = 1 + v$. Expanding in Taylor series the operator $\mathcal{Q}$, one gets

$$\mathcal{Q}_{g_\varepsilon}(1 + v) = \mathcal{Q}_{g_\varepsilon}(1) - c + \mathcal{L}_\varepsilon v + \mathcal{N}_\varepsilon(v),$$

where $\mathcal{N}_\varepsilon(v)$ is quadratic in v. Denote

$$\mathcal{A}_\varepsilon v := -G_\varepsilon(\mathcal{Q}_{g_\varepsilon}(1) - c + \mathcal{N}_\varepsilon(v)),$$

where $G_\varepsilon : \overline{w}_\varepsilon^\mu \mathcal{C}^{0,\alpha} \to \overline{w}_\varepsilon^\mu \mathcal{C}^{1,\alpha}$ is the right inverse constructed in the previous section, for μ satisfying (5.9).

Then equation (2.4) is reduced to $v = \mathcal{A}_\varepsilon v$, and we just need to show that $\mathcal{A}_\varepsilon$ is contractive for ε small enough.

First we check that g_ε is indeed a good approximate metric:

LEMMA 6.1. *Assume that $\mu < 0$. Then*

$$\|Q_{g_\varepsilon} - c\|_{\mu,0,\alpha} \leq C_\varepsilon,$$

where $C_\varepsilon \to 0$ as $\varepsilon \to 0$.

Proof. Again, we will prove the result for the weighted L^∞ norm $\|\cdot\|_{\mu,0,0}$; to pass to the weighted Hölder norm is standard.

In order to calculate Q_{g_ε}, we set up the problem (2.3) with Dirichlet condition identically one, this is

$$(6.2) \qquad\qquad L_{\bar{g}_\varepsilon} u = 0 \quad \text{on } X.$$

Our second remark is that we are dealing with edge operators on manifolds with boundary in the sense in the sense of [22, 26], so their mapping properties are well known.

Let us relate the metric $\bar{g}_\varepsilon$ to $\bar{g}_0$. Assume first that we are in a neighborhood where $\tilde{g}_{1,\delta}$ lives, this is, $X_1 \backslash B_1(p_1)$. If we write $\tilde{g}_{1,\delta} = \varphi^{\frac{4}{n-1}} \bar{g}_1$, recalling the conformal transformation law for the conformal Laplacian (2.2) and the fact that $\bar{g}_1$ is scalar flat, then equation (6.2) is simply

$$-\Delta_{\bar{g}_1}(\varphi u) = 0.$$

But $\varphi = 1$ outside a neighborhood $B_2(p_1)$, so we recover the original problem for $\bar{g}_1$.

Inside the neighborhood $B_2(p_1)$ we use normal coordinates $\bar{g}_2 = |dz|^2(1 + o(1))$, so up to a small error, we can approximate (6.2) by

$$(6.3) \qquad\qquad -\Delta_{|dz|^2}(\varphi u) = 0.$$

On the other hand, in $B_1(p_1)$ but before gluing to $\bar{g}_2$, $\varphi = r^{-\frac{n-1}{2}}$, so (6.3) can be written as a problem in the cylinder, with metric $\bar{g}_0$, and it can be included in the theory of edge operators, which are well behaved with respect to weights. In the intermediate neighborhood $B_2(p_1) \backslash B_1(p_1)$ we have error terms depending on $\nabla \varphi$ and $\Delta \varphi$, but these are of lower order.

Let us look closer to the neighborhood $B_1(p_1)$. In this region, equation (6.2) is written as

$$-\Delta_{\bar{g}_0} u + \tfrac{n-1}{4n} R_{\bar{g}_0} u = 0.$$

The gluing $\tilde{g}_{1,\delta}$ to $\tilde{g}_{2,\delta}$ using the cutoff $\tilde{\chi}$ happens in a region $\{C^{-1}\varepsilon^{1/2} \leq r \leq C\varepsilon^{1/2}\}$. Since $\Delta(\tilde{\chi}u) - \tilde{\chi}\Delta u = 2\nabla\tilde{\chi} \cdot \nabla u + u\Delta\tilde{\chi} =: \tilde{h}$, we are creating two new error terms (depending on ε). Let us translate everything in the notation s (for which our weights are adapted). Taking into account the conformal transformation law for $\mathcal{L}$, our equation is simply

$$\mathcal{L}_0 u = h$$

for $h := \tilde{h}r^{-\frac{n+1}{2}}$. This f has support only on $\Sigma := \{S_\varepsilon/2 - 1 \leq s \leq S_\varepsilon/2 + 1\}$.

Adding the Dirichlet condition $u \equiv 1$ to the extension problem (6.2), we recover Q_{g_ε} as the Dirichlet-to-Neumann operator

(6.4) $$Q_{g_\varepsilon} = -\partial_\nu u + \tfrac{n-1}{2} H_{g_\varepsilon} u|_M.$$

When we couple the boundary operator with (6.2), we are in the framework of studying edge operators on manifolds with boundary from [26].

We will compare (6.2)-(6.4) to the model (zero scalar curvature in the bulk, constant mean curvature of the boundary). Standard regularity estimates yield that the H^2 norm of u is bounded, for instance, in terms of the L^2 norm in the right hand side. It is standard to add a suitable weight to these estimates. So, as we are comparing two linear problems, to have good estimates for the Neumann boundary term (6.4) it is enough to estimate, in a weighted $L^2(dt)$ norm, the error terms produced in the gluing process above, for the weight ω_ε^μ. Since the weight has a factor $\varepsilon^{-\mu}$ for $\mu < 0$, we only need to take into account the neck, which is the noncompact part. But here the error term only lives in the neighborhood Σ, so

$$\int_\Sigma (\omega_\varepsilon)^{-2\mu} h^2 \, dt d\phi d\theta \sim \left(S_\varepsilon^n \varepsilon^{-\mu}\right)^{1/2} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

This completes the proof of the Lemma. $\qquad\square$

We finally check that $\mathcal{A}_\varepsilon$ is a contraction in a suitable space. Define

$$\mathcal{B}_\mu(r_\varepsilon) = \left\{v \in (\overline{\omega}_\varepsilon)^\mu \mathcal{C}^{1,\alpha} : \ \|v\|_{1,\alpha,\mu} \leq r_\varepsilon\right\}.$$

LEMMA 6.2. *For sufficiently small $\varepsilon > 0$, there exists a radius r_ε such that*

$$\mathcal{A}_\varepsilon(\mathcal{B}_\mu(r_\varepsilon)) \subset \mathcal{B}_\mu(r_\varepsilon)$$

and

$$\|\mathcal{A}_\varepsilon v_1 - \mathcal{A}_\varepsilon v_2\|_{1,\alpha,\mu} \leq \frac{1}{2}\|v_1 - v_2\|_{1,\alpha,\mu}$$

for all $v_1, v_2 \in \mathcal{B}_\mu(r_\varepsilon)$.

Proof. For simplicity, we will show the result for the weighted $\mathcal{C}^0$ norm, and not $\mathcal{C}^\alpha$. Modifications in the general case are straightforward.

Recalling the definition of the operator from (2.4), we have $\mathcal{Q}_{g_\varepsilon}(1) = Q_{g_\varepsilon}$, so Lemma 6.1 yields an estimate for the term $\mathcal{Q}_{g_\varepsilon}(1) - c$. Then, taking into account the discussion in Section 5.5, to complete the proof of the Lemma we just need to bound

$$\|\mathcal{N}_\varepsilon(v_1) - \mathcal{N}_\varepsilon(v_2)\|_{0,0,\mu} \leq o(1)\|v_1 - v_2\|_{1,0,\mu}.$$

The proof of this inequality is standard and therefore we omit the argument.

□

From the previous lemma, we have constructed a solution $v \in (\overline{\omega}_\varepsilon)^\mu$ such that the metric $g' = (1+v)^{\frac{4}{n-1}} g_\varepsilon$ has constant Q-curvature. The usual maximum principle for Dirichlet-to-Neumann operators (see [6], for instance) implies that $1 + v$ is a positive function. Moreover, by equation (6.1), the metric $\bar{g}' = (1+v)^{\frac{4}{n-1}} \bar{g}_\varepsilon$, has zero scalar curvature. This completes the proof of Theorem 1.1.

□

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