ON HIGHER DIMENSIONAL SINGULARITIES FOR THE FRACTIONAL YAMABE PROBLEM: A NON-LOCAL MAZZEO-PACARD PROGRAM

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ABSTRACT. We consider the problem of constructing solutions to the fractional Yamabe problem that are singular at a given smooth sub-manifold, for which we establish the classical gluing method of Mazzeo and Pacard ([63]) for the scalar curvature in the fractional setting. This proof is based on the analysis of the model linearized operator, which amounts to the study of a fractional order ODE, and thus our main contribution here is the development of new methods coming from conformal geometry and scattering theory for the study of non-local ODEs. Note, however, that no traditional phase-plane analysis is available here. Instead, first, we provide a rigorous construction of radial fast-decaying solutions by a blow-up argument and a bifurcation method. Second, we use conformal geometry to rewrite this non-local ODE, giving a hint of what a non-local phase-plane analysis should be. Third, for the linear theory, we use complex analysis and some non-Euclidean harmonic analysis to examine a fractional Schrödinger equation with a Hardy type critical potential. We construct its Green's function, deduce Fredholm properties, and analyze its asymptotics at the singular points in the spirit of Frobenius method. Surprisingly enough, a fractional linear ODE may still have a two-dimensional kernel as in the second order case.

1. INTRODUCTION

We construct singular solutions to the following non-local semilinear problem

$$(1.1) \qquad (-\Delta_{\mathbb{R}^n})^\gamma u = u^p \text{ in } \mathbb{R}^n, \quad u > 0,$$

for $\gamma \in (0,1)$, $n \geq 2$, where the fractional Laplacian is defined by

$$(1.2) \qquad (-\Delta_{\mathbb{R}^n})^\gamma u(z) = k_{n,\gamma} P.V. \int_{\mathbb{R}^n} \frac{u(z) - u(\tilde{z})}{|z - \tilde{z}|^{n+2\gamma}} \, d\tilde{z}, \quad \text{for} \quad k_{n,\gamma} = \pi^{-n/2} 2^{2\gamma} \frac{\Gamma\left(\frac{n}{2} + \gamma\right)}{\Gamma(1 - \gamma)} \gamma.$$

Equation (1.1) for the critical power $p = \frac{n+2\gamma}{n-2\gamma}$ corresponds to the fractional Yamabe problem in conformal geometry, which asks to find a constant fractional curvature metric in a given conformal class (see [49, 50, 40, 58, 60]). In particular, for $\gamma = 1$ the fractional curvature coincides with the scalar curvature modulo a multiplicative constant, so (1.1) reduces to the classical Yamabe problem. However, classical methods for local equations do not generally work here and one needs to develop new ideas.

Non-local equations have attracted a great deal of interest in the community since they are of central importance in many fields, from the points of view of both pure analysis and applied modeling. By the substantial effort made in the past decade by many authors, we have learned that non-local elliptic equations do enjoy good PDE properties such as uniqueness, regularity and maximum principle. However, not so much is known when it comes to the study of an integro-differential equation such as (1.1) from an ODE perspective since most of the ODE theory relies on local properties and phase-plane analysis; our first achievement is the development of a suitable theory for the fractional order ODE (1.6), that arises when studying radial singular solutions to (1.1).

On the one hand, we construct singular radial solutions for (1.1) directly with a completely different argument. On the other hand, using ideas from conformal geometry and scattering theory we replace

phase-plane analysis by a global study to obtain that solutions of the nonlocal ODE (1.6) do have a behavior similar in spirit to a classical second-order autonomous ODE, and initiate the study of a non-local phase portrait. In particular, we show that a linear non-local ODE has a two-dimensional kernel. This is surprising since this non-local ODE has an infinite number of indicial roots at the origin and at infinity, which is very different from the local case where the solution to a homogeneous linear second order problem can be written as a linear combination of two particular solutions and thus, its asymptotic behavior is governed by two pairs of indicial roots.

Then, with these tools at hand, we arrive at our second accomplishment: to develop a Mazzeo-Pacard gluing program [63] for the construction of singular solutions to (1.1) in the non-local setting. This gluing method is indeed local by definition; so one needs to rethink the theory from a fresh perspective in order to adapt it for such non-local equation. More precisely, the program relies on the fact that the linearization to (1.1) has good properties. In the classical case, this linearization has been well studied applying microlocal analysis (see [61], for instance), and it reduces to the understanding of a second order ODE with two regular singular points. In the fractional case this is obviously not possible. Instead, we use conformal geometry, complex analysis and some non-Euclidean harmonic analysis coming from representation theory in order to provide a new proof.

Thus conformal geometry is the central core in this paper, but we provide an interdisciplinary approach in order to approach the following analytical problem:

Theorem 1.1. *Let $\Sigma = \bigcup_{i=1}^{K} \Sigma_i$ be a disjoint union of smooth, compact sub-manifolds Σ_i without boundary of dimensions k_i, $i = 1, \ldots, K$. Assume, in addition to $n - k_i \geq 2$, that*

$$\frac{n - k_i}{n - k_i - 2\gamma} < p < \frac{n - k_i + 2\gamma}{n - k_i - 2\gamma},$$

or equivalently,

$$n - \frac{2p\gamma + 2\gamma}{p - 1} < k_i < n - \frac{2p\gamma}{p - 1}$$

for all $i = 1, \ldots, K$. Then there exists a positive solution for the problem

$$(1.3) \qquad (-\Delta_{\mathbb{R}^n})^\gamma u = u^p \ in \ \mathbb{R}^n \setminus \Sigma$$

that blows up exactly at Σ.

As a consequence of the previous theorem we obtain:

Corollary 1.2. *Assume that the dimensions k_i satisfy*

$$(1.4) \qquad 0 < k_i < \frac{n - 2\gamma}{2}.$$

Then there exists a positive solution to the fractional Yamabe equation

$$(1.5) \qquad (-\Delta_{\mathbb{R}^n})^\gamma u = u^{\frac{n+2\gamma}{n-2\gamma}} \ in \ \mathbb{R}^n \setminus \Sigma$$

that blows up exactly at Σ.

The dimension estimate (1.4) is sharp in some sense. Indeed, it was proved by González, Mazzeo and Sire [48] that, if such u blows up at a smooth sub-manifold of dimension k and is polyhomogeneous, then k must satisfy the restriction

$$\Gamma\left(\frac{n}{4} - \frac{k}{2} + \frac{\gamma}{2}\right) \Big/ \Gamma\left(\frac{n}{4} - \frac{k}{2} - \frac{\gamma}{2}\right) \geq 0,$$

which in particular, includes (1.4). Here, and for the rest of the paper, Γ denotes the Gamma function. In addition, the asymptotic behavior of solutions to (1.5) when the singular set has fractional capacity zero has been considered in [57].

Let us describe our methods in detail. First, note that it is enough to let Σ be a single sub-manifold of dimension k, and we will restrict to this case for the rest of the paper. We denote $N = n - k$.

The first step is to construct the building block, i.e, a solution to (1.3) in $\mathbb{R}^n \setminus \mathbb{R}^k$ that blows up exactly at $\mathbb{R}^k$. For this, we write $\mathbb{R}^n \setminus \mathbb{R}^k = (\mathbb{R}^N \setminus \{0\}) \times \mathbb{R}^k$, parameterized with coordinates $z = (x, y)$, $x \in \mathbb{R}^N \setminus \{0\}$, $y \in \mathbb{R}^k$, and construct a solution u_1 that only depends on the radial variable $r = |x|$. Then u_1 is also a radial solution to

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \quad \text{in } \mathbb{R}^N \setminus \{0\}, \quad u > 0.$$

We write $u = r^{-\frac{2\gamma}{p-1}} v$, $r = e^{-t}$. Then, in the radially symmetric case, this equation can be written as the integro-differential ODE

$$(1.6) \qquad P.V. \int_{\mathbb{R}} K(t - t')[v(t) - v(t')] \, dt' + A_{N,p,\gamma} v(t) = A_{N,p,\gamma} v^p \quad \text{in } \mathbb{R}, \quad v > 0,$$

where the kernel K is given precisely in (4.5). However, in addition to having the right blow up rate at the origin, u_1 must decay fast as $r \to \infty$ in order to perform the Mazzeo-Pacard gluing argument later. The existence of such fast-decaying singular solutions in the case of $\gamma = 1$ is an easy consequence of phase-plane analysis as (1.6) is reduced to a second order autonomous ODE (see Proposition 1 of [63]). The analogue in the fractional case turns out to be quite non-trivial. To show the existence, we first use Kelvin transform to reduce our problem for entire solutions to a supercritical one (2.3). Then we consider an auxiliary non-local problem (2.4), for which we show that the minimal solution w_λ is unique using Schaaf's argument as in [38] and a fractional Pohožaev identity [73]. A blow up argument, together with a Crandall-Rabinowitz bifurcation scheme yields the existence of this u_1. This is the content of Section 2.

Then, in Section 3, we exploit the conformal properties of the equation to produce a geometric interpretation for (1.3) in terms of scattering theory and conformally covariant operators. Singular solutions for the standard fractional Laplacian in $\mathbb{R}^n \setminus \mathbb{R}^k$ can be better understood by considering the conformal metric g_k from (3.22), that is the product of a sphere $\mathbb{S}^{N-1}$ and a half-space $\mathbb{H}^{k+1}$. Inspired by the arguments by DelaTorre and González [29], our point of view is to rewrite the well known extension problem in $\mathbb{R}^{n+1}_+$ for the fractional Laplacian in $\mathbb{R}^n$ due to [19], as a different, but equivalent, extension problem and to consider the corresponding Dirichlet-to-Neumann operator $P_\gamma^{g_k}$, defined in $\mathbb{S}^{N-1} \times \mathbb{H}^{k+1}$. Here $\mathbb{R}^{n+1}_+$ is replaced by anti-de Sitter (AdS) space, but the arguments run in parallel.

This $P_\gamma^{g_k}$ turns out to be a conjugate operator for $(-\Delta_{\mathbb{R}^n})^\gamma$, (see (3.23)), and behaves well when the nonlinearity in (1.3) is the conformal power. However, the problem (1.3) is not conformal for a general p, so we need to perform a further conjugation (3.34) and to consider the new operator $\tilde{P}_\gamma^{g_k}$. Then the original equation (1.3) in $\mathbb{R}^n \setminus \mathbb{R}^k$ is equivalent to

$$(1.7) \qquad \tilde{P}_\gamma^{g_k}(v) = v^p \quad \text{in } \mathbb{S}^{N-1} \times \mathbb{H}^{k+1}, \quad v = r^{\frac{2\gamma}{p-1}} u, \quad v > 0 \text{ and smooth.}$$

Rather miraculously, both $P_\gamma^{g_k}$ and $\tilde{P}_\gamma^{g_k}$ diagonalize under the spherical harmonic decomposition of $\mathbb{S}^{N-1}$. In fact, they can be understood as pseudo-differential operators on hyperbolic space $\mathbb{H}^{k+1}$, and we calculate their symbols in Theorem 3.5 and Proposition 3.6, respectively, under the Fourier-Helgason transform (to be denoted by $\hat{\cdot}$) on the hyperbolic space understood as the symmetric space $M = G/K$ for $G = SO(1, k+1)$ and $K = SO(k+1)$ (see the Appendix for a short introduction to the subject). This is an original approach that yields new results even in the classical case $\gamma = 1$, simplifying some of the arguments in [63]. The precise knowledge of their symbols allows, as a consequence, for the development of the linear theory for our problem, as we will comment below.

Section 4 collects these ideas in order to develop new methods for the study of the non-local ODE (1.6), which is precisely the projection of equation (1.7) for $k = 0$, $n = N$, over the zero-eigenspace when projecting over spherical harmonics of $\mathbb{S}^{N-1}$. The advantage of shifting from u to v is that we

obtain a new equation that behaves very similarly to a second order autonomous ODE. This includes the existence of a Hamiltonian quantity along trajectories.

Moreover, one can take the spherical harmonic decomposition of $\mathbb{S}^{N-1}$ and consider all projections $m = 0, 1, \ldots$. In Proposition 4.3 we are able to write every projected equation as an integro-differential equation very similar to the $m = 0$ projection (1.6). This formulation immediately yields regularity and maximum principles for the solution of (1.7) following the arguments in [30].

Now, to continue with the proof of Theorem 1.1, one takes the fast decaying solution in $\mathbb{R}^n \setminus \mathbb{R}^k$ we have just constructed and, after some rescaling by ε, glues it to the background Euclidean space in order to have a global approximate solution $\bar{u}_\varepsilon$ in $\mathbb{R}^n \setminus \Sigma$. Even though the fractional Laplacian is a non-local operator, one is able to perform this gluing just by carefully estimating the tail terms that appear in the integrals after localizaton. This is done in Section 5.1 and, more precisely, Lemma 5.7, where we show that the error we generate when approximating a true solution by $\bar{u}_\varepsilon$, given by

$$f_\varepsilon := (-\Delta_{\mathbb{R}^n})^\gamma \bar{u}_\varepsilon - \bar{u}_\varepsilon^p,$$

is indeed small in suitable weighted Hölder spaces.

Once we have an approximate solution, we define the linearized operator around it,

$$L_\varepsilon \phi := (-\Delta_{\mathbb{R}^n})^\gamma \phi - p \bar{u}_\varepsilon^{p-1} \phi.$$

The general scheme of Mazzeo-Pacard's method is to set $u = \bar{u}_\varepsilon + \phi$ for an unknown perturbation ϕ and to rewrite equation (1.3) as

$$L_\varepsilon(\phi) + Q_\varepsilon(\phi) + f_\varepsilon = 0,$$

where Q_ε contains the remaining nonlinear terms. If L_ε is invertible, then we can write

$$\phi = (L_\varepsilon)^{-1}(-Q_\varepsilon(\phi) - f_\varepsilon),$$

and a standard fixed point argument for small ε will yield the existence of such ϕ, thus completing the proof of Theorem 1.1 (see Section 9).

Thus, a central argument here is the study of the linear theory for L_ε and, in particular, the analysis of its indicial roots, injectivity and Fredholm properties. However, while the behaviour of a second order ODE is governed by two boundary conditions (or behavior at the singular points using Frobenius method), this may not be true in general for a non-local operator.

We first consider the model operator $\mathcal{L}_1$ defined in (7.3) for an isolated singularity at the origin. Near the singularity $\mathcal{L}_1$ behaves like

$$(1.8) \qquad\qquad\qquad\qquad (-\Delta_{\mathbb{R}^N})^\gamma - \frac{\kappa}{r^{2\gamma}}$$

or, after conjugation, like $P_\gamma^{g_0} - \kappa$, which is a fractional Laplacian operator with a Hardy potential of critical type.

The central core of the linear theory deals with the operator (1.8). In Section 6 we perform a delicate study of the Green's function by inverting its Fourier symbol Θ_γ^m (see (3.15)). This requires a very careful analysis of the poles of the symbol, in both the stable and unstable cases. Contrary to the local case $\gamma = 1$, in which there are only two indicial roots for each projection m, here we find an infinite sequence for each m. But in any case, these are controlled. It is also interesting to observe that, even though we have a non-local operator, the first pair of indicial roots governs the asymptotic behavior of the operator and thus, its kernel is two-dimensional in some sense (see, for instance, Proposition 6.13 for a precise statement).

For the interested reader, we have prepared a separate survey paper [6] where we summarize the new ODE methods for non-local equations that we have developed here.

Then, in Section 7 we complete the calculation of the indicial roots (see Lemma 7.1). Next, we show the injectivity for $\mathcal{L}_1$ in weighted Hölder spaces, and an *a priori* estimate (Lemma 7.4) yields the injectivity for L_ε.

In addition, in Section 8 we work with weighted Hilbert spaces and we prove Fredholm properties for L_ε in the spirit of the results by Mazzeo [61, 62] for edge type operators by constructing a suitable parametrix with compact remainder. The difficulty lies precisely in the fact that we are working with a non-local operator, so the localization with a cut-off is the non-trivial step. However, by working with suitable weighted spaces we are able to localize the problem near the singularity; indeed, the tail terms are small. Then we conclude that L_ε must be surjective by purely functional analysis reasoning. Finally, we construct a right inverse for L_ε, with norm uniformly bounded independently of ε, and this concludes the proof of Theorem 1.1.

The Appendix contains some well known results on special functions and the Fourier-Helgason transform.

As a byproduct of the proof of Theorem 1.1, we will obtain the existence of solutions with isolated singularities in the subcritical regime (note the shift from n to N in the spatial dimension, which will fit better our purposes).

Theorem 1.3. *Let $\gamma \in (0,1)$, $N \geq 2$ and*

$$(1.9) \qquad \frac{N}{N-2\gamma} < p < \frac{N+2\gamma}{N-2\gamma}.$$

Let Σ be a finite number of points, $\Sigma = \{q_1, \ldots, q_K\}$. Then equation

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \ in \ \mathbb{R}^N \setminus \Sigma$$

has positive solutions that blow up exactly at Σ.

Remark 1.4. The constant $A_{N,p,\gamma}$ is chosen so that the model function $u_\gamma(x) = |x|^{-\frac{2\gamma}{p-1}}$ is a singular solution to (1.3) that blows up exactly at the origin. In particular,

$$(1.10) \qquad A_{N,p,\gamma} = \Lambda\big(\tfrac{N-2\gamma}{2} - \tfrac{2\gamma}{p-1}\big) \quad \text{for} \quad \Lambda(\alpha) = 2^{2\gamma} \frac{\Gamma(\frac{N+2\gamma+2\alpha}{4})\Gamma(\frac{N+2\gamma-2\alpha}{4})}{\Gamma(\frac{N-2\gamma-2\alpha}{4})\Gamma(\frac{N-2\gamma+2\alpha}{4})}.$$

Note that, for the critical exponent $p = \frac{N+2\gamma}{N-2\gamma}$, the constant $A_{N,p,\gamma}$ coincides with $\Lambda_{N,\gamma} = \Lambda(0)$, the sharp constant in the fractional Hardy inequality in $\mathbb{R}^N$. Its precise value is given in (3.21).

Let us make some comments on the bibliography. First, the Brezis-Nirenberg problem for the fractional Laplacian has been considered in [80] through variational techniques, which is one of the first papers on semi-linear equations with critical power non-linearity for the fractional Laplacian. In addition, the problem of uniqueness and non-degeneracy for some fractional ODE has been considered in [42, 43, 27], for instance.

The construction of singular solutions in the range of exponents for which the problem is stable, i.e., $\frac{N}{N-2\gamma} < p < p_1$ for $p_1 < \frac{N+2\gamma}{N-2\gamma}$ defined in (2.2), was studied in the previous paper by Ao, Chan, González and Wei [5]. In addition, for the critical case $p = \frac{N+2\gamma}{N-2\gamma}$, solutions with a finite number of isolated singularities were obtained in the article by Ao, DelaTorre, González and Wei [4] using a gluing method. The difficulty there was the presence of a non-trivial kernel for the linearized operator. With all these results, together with Theorem 1.1, we have successfully developed a complete fractional Mazzeo-Pacard program for the construction of singular solutions of the fractional Yamabe problem.

Gluing methods for fractional problems are starting to be developed. A finite dimensional reduction has been applied in [28] to construct standing-wave solutions to a fractional nonlinear Schrödinger equation and in [37] to construct layered solutions for a fractional inhomogeneous Allen-Cahn equation.

The next development came in [4] for the fractional Yamabe problem with isolated singularities, that we have just mentioned. There the model for an isolated singularity is a Delaunay-type metric (see also [65, 66, 79] for the construction of constant mean curvature surfaces with Delaunay ends and [64, 67] for

the scalar curvature case). However, in order to have enough freedom parameters at the perturbation step, for the non-local gluing in [4] the authors replace the Delaunay-type solution by a bubble tower (an infinite, but countable, sum of bubbles). As a consequence, the reduction method becomes infinite dimensional. Nevertheless, it can still be treated with the tools available in the finite dimensional case and one reduces the PDE to an infinite dimensional Toda type system. The most recent works related to gluing are [23, 24] for the construction of counterexamples to the fractional De Giorgi conjecture. This reduction is fully infinite dimensional.

For the fractional De Giorgi conjecture with $\gamma \in [\frac{1}{2}, 1)$ we refer to [16, 11, 77] and the most recent striking paper [41]. Related to this conjecture, in the case $\gamma \in (0, \frac{1}{2})$ there exists a notion of non-local mean curvature for hypersurfaces in $\mathbb{R}^n$, see [18] and the survey [85]. Much effort has been made regarding regularity [20, 12, 7] and various qualitative properties [35, 36]. More recent work on stability of non-local minimal surfaces can be found in [26]. Delaunay surfaces for this curvature have been constructed in [13, 14]. After the appearance of [33], Cabré has pointed out that this paper also constructs Delaunay surfaces with constant nonlocal mean curvature.

2. THE FAST DECAYING SOLUTION

We aim to construct a fast-decay singular solution to the fractional Lane–Emden equation

$$(2.1) \qquad (-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

for $\gamma \in (0, 1)$ and p in the range (1.9).

We consider the exponent $p_1 = p_1(N, \gamma) \in (\frac{N}{N-2\gamma}, \frac{N+2\gamma}{N-2\gamma})$ defined below by (2.2) such that the singular solution $u_\gamma(x) = |x|^{-\frac{2\gamma}{p-1}}$ is stable if and only if $\frac{N}{N-2\gamma} < p < p_1$. In the notation of Remark 1.4, p_1 as defined as the root of

$$(2.2) \qquad p A_{N,p,\gamma} = \Lambda(0).$$

The main result in this section is:

Proposition 2.1. *For any $\varepsilon \in (0, \infty)$ there exists a fast-decay entire singular solution u_ε of (2.1) such that*

$$u_\varepsilon(x) \sim \begin{cases} |x|^{-\frac{2\gamma}{p-1}} & \text{as } |x| \to 0, \\ \varepsilon |x|^{-(N-2\gamma)} & \text{as } |x| \to \infty. \end{cases}$$

The proof in the stable case $\frac{N}{N-2\gamma} < p < p_1 < \frac{N+2\gamma}{N-2\gamma}$ is already contained in the paper [5], so we will assume for the rest of the section that we are in the unstable regime

$$\frac{N}{N-2\gamma} < p_1 \le p < \frac{N+2\gamma}{N-2\gamma}.$$

We first prove uniqueness of minimal solutions for the non-local problem (2.4) using Schaaf's argument and a fractional Pohožaev identity obtained by Ros-Oton and Serra (Proposition 2.2 below). Then we perform a blow-up argument on an unbounded bifurcation branch. An application of Kelvin's transform yields an entire solution of the Lane–Emden equation with the desired asymptotics.

Set $A = A_{N,p,\gamma}$. Note that the Kelvin transform $w(x) = |x|^{-(N-2\gamma)} u\left(\frac{x}{|x|^2}\right)$ of u satisfies

$$(2.3) \qquad (-\Delta)^\gamma w(x) = A|x|^\beta w^p(x),$$

where $\beta =: p(N - 2\gamma) - (N + 2\gamma) \in (-2\gamma, 0)$.

Consider the following non-local Dirichlet problem in the unit ball $B_1 = B_1(0) \subset \mathbb{R}^N$,

$$(2.4) \qquad \begin{cases} (-\Delta)^\gamma w(x) = \lambda |x|^\beta A(1 + w(x))^p & \text{in } B_1, \\ w = 0 & \text{in } \mathbb{R}^N \setminus B_1. \end{cases}$$

Since $(-\Delta)^\gamma |x|^{\beta+2\gamma} = c_0 |x|^\beta$ and $(-\Delta)^\gamma (1 - |x|^2)_+^\gamma = c_1$ for some positive constants c_0 and c_1, we have that $|x|^{\beta+2\gamma} + (1 - |x|^2)_+^\gamma$ is a positive super-solution of (2.4) for small λ. Then one can follow classical arguments (we refer to [74], for instance), to show that there exists a minimal solution w_λ for small λ. Moreover, w_λ is non-decreasing in λ. Thus one can find a $\lambda^* > 0$ such that: (i) the minimal solution w_λ exists for each $\lambda \in (0, \lambda^*)$, and w_λ is radially symmetric and non-increasing in the radial variable; (ii) for $\lambda > \lambda^*$, (2.4) has no solutions.

We will show that w_λ is the unique solution of (2.4) for all small λ.

Proposition 2.2. *There exists a small $\lambda_0 > 0$ depending only on $N \geq 2$ and $\gamma \in (0,1)$ such that for any $0 \leq \lambda < \lambda_0$, w_λ is the unique solution to (2.4) among the class*

$$\tilde{\mathcal{C}}_0^2(B_1) = \left\{ w \in \mathcal{C}^2(B_1) \cap \mathcal{C}(\mathbb{R}^N) : w = 0 \text{ in } \mathbb{R}^N \setminus B_1 \right\}.$$

The idea of the proof of this Proposition follows from [38] and similar arguments can be found in [81], [54] and [55].

2.1. **Useful inequalities.** The first ingredient is the Pohožaev identity for the fractional Laplacian. Such identities for integro-differential operators have been recently studied in [73], [75] and [52].

Theorem 2.3 (Proposition 1.12 in [73]). *Let Ω be a bounded $\mathcal{C}^{1,1}$ domain, $f \in \mathcal{C}_{loc}^{0,1}(\overline{\Omega} \times \mathbb{R})$, u be a bounded solution of*

$$(2.5) \qquad \begin{cases} (-\Delta)^\gamma u = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

and $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. Then

$$u/\delta^\gamma \mid_\Omega \in \mathcal{C}^\alpha(\overline{\Omega}) \qquad \text{for some } \alpha \in (0, 1),$$

and there holds

$$\int_\Omega \left(F(x, u) + \frac{1}{N} x \cdot \nabla_x F(x, u) - \frac{N - 2\gamma}{2N} u f(x, u) \right) dx = \frac{\Gamma(1+\gamma)^2}{2N} \int_{\partial\Omega} \left(\frac{u}{\delta^\gamma} \right)^2 (x \cdot \nu) \, d\sigma$$

where $F(x, t) = \int_0^t f(x, \tau) \, d\tau$ and ν is the unit outward normal to $\partial\Omega$ at x.

Using integration by parts (see, for instance, equation (1.5) in [73]), it is clear that

$$\int_\Omega u f(x, u) \, dx = \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\gamma}{2}} u \right|^2 dx,$$

which yields our fundamental inequality:

Corollary 2.4. *Under the assumptions of Theorem 2.3, we have for any star-shaped domain Ω and any $\sigma \in \mathbb{R}$,*

$$(2.6) \qquad \int_\Omega \left(F(x, u) + \frac{1}{N} x \cdot \nabla_x F(x, u) - \sigma u f(x, u) \right) dx \geq \left(\frac{N - 2\gamma}{2N} - \sigma \right) \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\gamma}{2}} u \right|^2 dx$$

The second ingredient is the fractional Hardy–Sobolev inequality which, via Hölder inequality, is an interpolation of fractional Hardy inequality and fractional Sobolev inequality:

Theorem 2.5 (Lemma 2.1 in [46]). *Assume that $0 \leq \alpha < 2\gamma < \min\{2, N\}$. Then there exists a constant c such that*

$$(2.7) \qquad c \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\gamma}{2}} u \right|^2 dx \geq \left(\int_{\mathbb{R}^N} |x|^{-\alpha} |u|^{\frac{2(N-\alpha)}{N-2\gamma}} \right)^{\frac{N-2\gamma}{N-\alpha}}.$$

2.2. **Proof of Proposition 2.2.** We are now in a position to prove the uniqueness of solutions of (2.4) with small parameter.

Proof. Suppose w and w_λ are solutions to (2.4). Then $u = w - w_\lambda$ is a positive solution to the Dirichlet problem

$$\begin{cases} (-\Delta)^\gamma u = \lambda A |x|^\beta g_\lambda(x, u) & \text{in } B_1(0), \\ u = 0 & \text{in } \mathbb{R}^N \setminus B_1(0), \end{cases}$$

where $g_\lambda(x, u) = (1 + w_\lambda(x) + u)^p - (1 + w_\lambda(x))^p \geq 0$ for $u \geq 0$. Denoting

$$G_\lambda(x, u) = \int_0^u g_\lambda(x, t) \, dt,$$

we apply (2.6) with $f(x, u) = \lambda A |x|^\beta g_\lambda(x, u)$ over $\Omega = B_1$ to obtain

(2.8)
$$\begin{aligned} \left(\frac{N - 2\gamma}{2N} - \sigma \right) & \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\gamma}{2}} u \right|^2 dx \\ & \leq \lambda A \int_{B_1} \left(|x|^\beta G_\lambda(x, u) + \frac{1}{N} x \cdot \nabla_x \left(|x|^\beta G_\lambda(x, u) \right) - \sigma |x|^\beta u g_\lambda(x, u) \right) dx \\ & = \lambda A \int_{B_1} |x|^\beta \left(\left(1 + \frac{\beta}{N} \right) G_\lambda(x, u) + \frac{1}{N} x \cdot \nabla_x G_\lambda(x, u) - \sigma u g_\lambda(x, u) \right) dx. \end{aligned}$$

Note that

(2.9)
$$G_\lambda(x, u) = u^2 \int_0^1 \int_0^1 pt(1 + w_\lambda(x) + \tau t u)^{p-1} \, d\tau dt$$

and

$$\nabla_x G_\lambda(x, u) = u^2 \int_0^1 \int_0^1 p(p - 1)t(1 + w_\lambda(x) + \tau t u)^{p-2} \, d\tau dt \cdot \nabla w_\lambda(x).$$

Since w_λ is radially decreasing, $x \cdot \nabla w_\lambda(x) \leq 0$ and hence $x \cdot \nabla_x G_\lambda(x, u) \leq 0$. Then (2.8) becomes

(2.10)
$$\left(\frac{N - 2\gamma}{2N} - \sigma \right) \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{\gamma}{2}} u \right|^2 dx \leq \lambda A \int_{B_1} |x|^\beta \left(\left(1 + \frac{\beta}{N} \right) G_\lambda(x, u) - \sigma u g_\lambda(x, u) \right) dx.$$

Now, since for any $\lambda \in \left[0, \frac{\lambda^*}{2} \right]$ and any $x \in B_1$,

$$\lim_{t \to \infty} \frac{G_\lambda(x, t)}{t g_\lambda(x, t)} = \lim_{t \to \infty} \frac{\frac{1}{p+1} \left((1 + w_\lambda(x) + t)^{p+1} - (1 + w_\lambda(x))^{p+1} \right) - (1 + w_\lambda(x))^p t}{t \left((1 + w_\lambda(x) + t)^p - (1 + w_\lambda(x))^p \right)} = \frac{1}{p + 1},$$

we deduce that for any $\epsilon > 0$ there exists an $M = M(\epsilon) > 0$ such that

$$G_\lambda(x, t) \leq \frac{1 + \epsilon}{p + 1} u g_\lambda(x, t)$$

whenever $t \geq M$. From this we estimate the tail of the right hand side of (2.10) as

$$\begin{aligned} \int_{B_1 \cap \{u \geq M\}} & |x|^\beta \left(\left(1 + \frac{\beta}{N} \right) G_\lambda(x, u) - \sigma u g_\lambda(x, u) \right) dx \\ & \leq \int_{B_1 \cap \{u \geq M\}} |x|^\beta \left(\left(1 + \frac{\beta}{N} \right) \frac{1 + \epsilon}{p + 1} - \sigma \right) u g_\lambda(x, u) \, dx. \end{aligned}$$

We wish to choose ϵ and σ such that

$$\left(1 + \frac{\beta}{N} \right) \frac{1 + \epsilon}{p + 1} < \sigma < \frac{N - 2\gamma}{2N},$$

so that the above integral is non-positive. Indeed we observe that

$$\left(\frac{N+\beta}{N}\right)\frac{1}{p+1} - \frac{N-2\gamma}{2N} = \frac{2(p(N-2\gamma)-2\gamma)-(N-2\gamma)(p+1)}{2N(p+1)}$$
$$= \frac{(p-1)(N-2\gamma)-4\gamma}{2N(p+1)}$$
$$< 0$$

as $p - 1 \in \left(\frac{2\gamma}{N-2\gamma}, \frac{4\gamma}{N-2\gamma}\right)$. Then there exists a small $\epsilon > 0$ such that

$$\left(1+\frac{\beta}{N}\right)\frac{1+\epsilon}{p+1} < \frac{N-2\gamma}{2N},$$

from which the existence of such σ follows. With this choice of ϵ and σ, (2.10) gives

$$\left(\frac{N-2\gamma}{2N} - \sigma\right)\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{\gamma}{2}}u\right|^2 dx$$
$$\leq \lambda A \int_{B_1\cap\{u<M\}} |x|^\beta \left(\left(1+\frac{\beta}{N}\right)G_\lambda(x,u) - \sigma u g_\lambda(x,u)\right) dx$$
$$\leq \lambda A \left(1+\frac{\beta}{N}\right)\int_{B_1\cap\{u<M\}} |x|^\beta G_\lambda(x,u)\, dx.$$

Recalling the expression (2.9) for $G_\lambda(x,u)$, we have

$$\left(\frac{1}{2}-\frac{\sigma}{N}\right)\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{\gamma}{2}}u\right|^2 dx \leq \lambda A C_M \int_{B_1\cap\{u<M\}} |x|^\beta u^2\, dx,$$

where

(2.11) $$C_M = \frac{p}{2}\left(1+w_{\frac{\lambda^*}{2}}(0)+M\right)^{p-1}$$

by the monotonicity properties of w_λ.

On the other hand, since $p > \frac{N}{N-2\gamma}$,

$$-\beta = -p(N-2\gamma)+(N+2\gamma) = 2\gamma-(N-2\gamma)\left(p-\frac{N}{N-2\gamma}\right) < 2\gamma,$$

and thus the fractional Hardy–Sobolev inequality (2.7) implies

$$c\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{\gamma}{2}}u\right|^2 dx \geq \left(\int_{\mathbb{R}^N}|x|^\beta u^{2\eta}\, dx\right)^{\frac{1}{\eta}} = \left(\int_{B_1}|x|^\beta u^{2\eta}\, dx\right)^{\frac{1}{\eta}},$$

where

$$\eta = \frac{N+\beta}{N-2\gamma} = \frac{p(N-2\gamma)-2\gamma}{N-2\gamma} = 1+\left(p-\frac{N}{N-2\gamma}\right) > 1.$$

Hence,

$$\left(\int_{B_1}|x|^\beta u^{2\gamma}\, dx\right)^{\frac{1}{\gamma}} \leq \frac{2N}{N-2\gamma}c\, C_M \lambda A \int_{B_1}|x|^\beta u^2\, dx.$$

However, by Hölder's inequality, we have

$$\int_{B_1}|x|^\beta u^2\, dx = \int_{B_1}|x|^{\frac{\beta}{\eta}}u^2 \cdot |x|^{\beta\left(1-\frac{1}{\eta}\right)}\, dx \leq \left(\int_{B_1}|x|^\beta u^{2\eta}\, dx\right)^{\frac{1}{\eta}}\left(\int_{B_1}|x|^\beta\, dx\right)^{1-\frac{1}{\eta}}$$
$$\leq (N+\beta)^{-\frac{N+2\gamma}{N+\beta}}\left(\int_{B_1}|x|^\beta u^{2\eta}\, dx\right)^{\frac{1}{\eta}}.$$

Therefore, we have

$$\left(\int_{B_1} |x|^\beta u^{2\eta}\, dx\right)^{\frac{1}{\eta}} \le \frac{2NcAC_M}{(N-2\gamma)(N+\beta)^{\frac{N+2\gamma}{N+\beta}}}\lambda \left(\int_{B_1} |x|^\beta u^{2\eta}\, dx\right)^{\frac{1}{\eta}},$$

which forces $u \equiv 0$ for any

$$(2.12) \qquad\qquad \lambda < \lambda_0 = \left(\frac{2NcAC_M}{(N-2\gamma)(N+\beta)^{\frac{N+2\gamma}{N+\beta}}}\right)^{-1}.$$

$\square$

2.3. **Existence of a fast-decay singular solution.** Consider the Banach space of bounded, continuous, non-negative and radially non-increasing functions supported in the unit ball,

$$E = \left\{w \in \mathcal{C}(\mathbb{R}^n) : w(x) = \tilde{w}(|x|),\ \tilde{w}(r_1) \le \tilde{w}(r_2) \text{ for } r_1 \ge r_2,\ w \ge 0 \text{ in } B_1 \text{ and } w \equiv 0 \text{ in } \mathbb{R}^N \setminus B_1\right\}.$$

We begin with an *a priori* estimate followed by a compactness result, from which a bifurcation argument follows.

In the following an equivalent integral formulation of (2.4) will be useful. Using the Green's function for the Dirichlet problem in the unit ball ([72, 10]), we see that (2.4) is equivalent to

$$(2.13) \qquad\qquad w(x) = T(\lambda, w)(x) := \int_{B_1} G(x,y)\lambda A|y|^\beta (1+w(y))^p\, dy, \quad x \in B_1,$$

where

$$G(x,y) = C(N,\gamma)\frac{1}{|x-y|^{N-2\gamma}}\int_0^{r_0(x,y)} \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}}\, dr$$

with

$$r_0(x,y) = \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}.$$

Here $C(N,\gamma)$ is some normalizing constant. T is a continuous operator from $\mathbb{R} \times E$ to E.

Lemma 2.6 (Uniform bound). *There exists a universal constant $C_0 = C_0(N,\gamma,p,\lambda^*)$ such that for any function $w \in E$, solving (2.4) and for any $x \in B_{1/2}(0) \setminus \{0\}$,*

$$w(x) \le C_0|x|^{-\frac{\beta+2\gamma}{p-1}} = C_0|x|^{-\frac{p(N-2\gamma)-N}{p-1}}.$$

Proof. The maximum principle implies that $w > 0$ in B_1. Let

$$y \in B_{\frac{|x|}{4}}\left(\frac{3x}{4}\right) \subset B_{\frac{|x|}{2}}(x) \cap B_{|x|}(0) \subset B_1(0).$$

From $y \in B_{\frac{|x|}{2}}(x)$, we have

$$|x-y| \le \frac{|x|}{2} \le \frac{1}{4} \quad \text{and} \quad |y| \le \frac{3|x|}{2} \le \frac{3}{4}$$

and so

$$r_0(x,y) \ge \frac{\left(1-\frac{1}{4}\right)\left(1-\frac{9}{16}\right)}{\frac{1}{16}} \ge \frac{21}{4} > 5.$$

On the other hand, since $y \in B_{|x|}(0)$ and w is radially non-increasing, we have

$$|y|^\beta \ge |x|^\beta \quad \text{and} \quad w(y) \ge w(x).$$

Therefore, we may conclude

$$G(x,y) \geq C(N,\gamma) \left(\frac{2}{|x|}\right)^{N-2\gamma} \int_0^5 \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}} \, dr$$

and

$$w(x) \geq A \int_{B_{\frac{|x|}{4}}\left(\frac{3x}{4}\right)} C(N,\gamma) \frac{2^{N-2\gamma}}{|x|^{N-2\gamma}} \left(\int_0^5 \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}} \, dr\right) \lambda_0 |x|^\beta w(x)^p \, dy$$

$$\geq C(N,\gamma) A 2^{N-2\gamma} \left(\int_0^5 \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}} \, dr\right) \lambda_0 \cdot \frac{|x|^\beta}{|x|^{N-2\gamma}} w(x)^p \cdot |B_1| \left(\frac{|x|}{4}\right)^N$$

$$\geq C_0^{-(p-1)} |x|^{\beta+2\gamma} w(x)^p,$$

where

$$C_0^{-(p-1)} = \frac{C(N,\gamma)|B_1|A\lambda_0}{2^{N+2\gamma}} \int_0^5 \frac{r^{\gamma-1}}{(r+1)^{\frac{N}{2}}} \, dr.$$

The inequality clearly rearranges to

$$w(x) \leq C_0 |x|^{-\frac{\beta+2\gamma}{p-1}},$$

as desired. The dependence of the constant C_0 follows from (2.12) and (2.11). $\qquad\square$

Lemma 2.7 (Compactness)**.** *The non-linear operator $T : \mathbb{R} \times E \to E$ defined in (2.13) is compact, i.e. it maps bounded sets to relatively compact sets.*

Proof. In a bounded set of $\mathbb{R} \times E$, it suffices to find a convergent subsequence via Arzelà-Ascoli theorem. Since $p > 1$, equi-boundedness and equi-continuity follow immediately once we have the bound

$$\int_{B_1} \frac{|y|^\beta}{|x-y|^{N-2\gamma}} \, dy \leq C,$$

for any $|x| < 1$. Indeed, it is true because $2\gamma + \beta > 0$. We prove it for the case where $N \geq 2$; the lower dimensional case is easier and omitted. Using polar coordinates with $r = |y|$ and $\theta_1 \in (0,\pi)$, and then changing variables to $r = |x|\rho$, $t = t_0 \sin \frac{\theta_1}{2}$ with $t_0 = \frac{2\sqrt{r|x|}}{|r-|x||} = \frac{2\sqrt{\rho}}{|\rho-1|}$, we compute

$$\int_{B_1} \frac{|y|^\beta}{|x-y|^{N-2\gamma}} \, dy = |\mathbb{S}^{N-2}| \int_0^1 r^{N-1+\beta} \int_0^\pi \frac{\sin^{N-2}\theta_1 \, d\theta_1}{((r-|x|)^2 + 2r|x|(1-\cos\theta_1))^{\frac{N-2\gamma}{2}}} \, dr$$

$$= |\mathbb{S}^{N-2}| \int_0^1 \frac{r^{N-1+\beta}}{|r-|x||^{N-2\gamma}} \int_0^{t_0} \frac{2^{N-2}\left(\frac{t}{t_0}\right)^{N-2}\left(1-\frac{t^2}{t_0^2}\right)^{\frac{N-3}{2}}\left(\frac{2}{t_0}dt\right)}{(1+t^2)^{\frac{N-2\gamma}{2}}} \, dr$$

$$= |\mathbb{S}^{N-2}| |x|^{2\gamma+\beta} \int_0^{\frac{1}{|x|}} \frac{\rho^{N-1+\beta}}{|\rho-1|^{N-2\gamma}} \left(\frac{2}{t_0}\right)^{N-1} \int_0^{t_0} \frac{t^{N-2}}{(1+t^2)^{\frac{N-2\gamma}{2}}} \left(1-\frac{t^2}{t_0^2}\right)^{\frac{N-3}{2}} dt d\rho$$

$$\leq C|x|^{2\gamma+\beta} \int_0^{\frac{1}{|x|}} \frac{\rho^{N-1+\beta}}{|\rho-1|^{N-2\gamma}} \min\left\{1, \left(\frac{\sqrt{\rho}}{|\rho-1|}\right)^{1-N} \int_1^{\frac{\sqrt{\rho}}{|\rho-1|}} t^{2\gamma-2} \, dt\right\} d\rho$$

$$\leq C|x|^{2\gamma+\beta} \left(\int_0^{\frac{1}{2}} \rho^{N-1+\beta} \, d\rho + \int_{\frac{1}{2}}^2 |\rho-1|^{2\gamma-1}\left[\int_1^{\frac{1}{|\rho-1|}} t^{2\gamma-2} \, dt\right] d\rho + \int_2^{\frac{1}{|x|}} \rho^{2\gamma-1+\beta} \, d\rho\right)$$

$$\leq C,$$

provided that $|x| \leq \frac{1}{2}$. When $\frac{1}{2} < |x| < 1$, we simply use

$$\frac{|y|^\beta}{|x-y|^{N-2\gamma}} \leq C\left(|y|^\beta + |x-y|^{-(N-2\gamma)} + 1\right).$$

This proves the claim.

Therefore, if (λ, u) and (μ, v) belong to a bounded set of $\mathbb{R} \times E$, then

$$\|T(\lambda, u) - T(\mu, v)\|_{L^\infty} \leq |\lambda - \mu| \int_{B_1} \frac{A|y|^\beta (1 + \|u\|_{L^\infty})^p}{|x-y|^{N-2\gamma}} \, dy$$

$$+ \|u - v\|_{L^\infty} \int_{B_1} |\mu| Ap \left(1 + \|u\|_{L^\infty} + \|v\|_{L^\infty}\right)^{p-1} \frac{|y|^\beta}{|x-y|^{N-2\gamma}} \, dy$$

$$\leq C\left(|\lambda - \mu| + \|u - v\|_{L^\infty}\right).$$

Hence, Arzelà-Ascoli theorem applies and the proof is complete. $\qquad\square$

Lemma 2.8 (Bifurcation). *There exists a sequence of solutions (λ_j, w_j) of (2.4) in $(0, \lambda^*] \times E$ such that*

$$\lim_{j\to\infty} \lambda_j = \lambda_\infty \in [\lambda_0, \lambda^*] \quad and \quad \lim_{t\to\infty} \|w_j\|_{L^\infty} = \infty,$$

where λ_0 is given in Proposition 2.2.

Proof. Consider the continuation

$$\mathscr{C} = \{(\lambda(t), w(t)) : t \geq 0\}$$

of the branch of minimal solutions $\{(\lambda, w_\lambda) : 0 \leq \lambda \leq \lambda_0\}$, where $(\lambda(0), w(0)) = (\lambda_0, w_{\lambda_0})$. By Proposition 2.2 and the fact that no solutions exist if $\lambda > \lambda^*$, we see that $\mathscr{C} \subset [\lambda_0, \lambda^*] \times E$. By the classical bifurcation theory [71, Theorem 6.2], $\mathscr{C}$ is unbounded in $[\lambda_0, \lambda^*] \times E$ and the existence of the desired sequence of pairs (λ_j, w_j) follows. $\qquad\square$

We are ready to establish the existence of a fast-decay singular solution.

Proof of Proposition 2.1. Let (λ_j, w_j) be as in Lemma 2.8. Define

$$m_j = \|w_j\|_{L^\infty(B_1)} = w_j(0) \quad \text{and} \quad R_j = m_j^{\frac{p-1}{\beta+2\gamma}} = m_j^{\frac{p-1}{p(N-2\gamma)-N}}$$

so that $m_j, R_j \to \infty$ as $j \to \infty$. Set also

$$W_j(x) = \lambda^{\frac{1}{p-1}} m_j^{-1} w_{\lambda_j}\left(\frac{x}{R_j}\right).$$

Then $0 \leq W_j \leq 1$ and W_j is a bounded solution to

$$\begin{cases} (-\Delta)^\gamma W_j = \lambda_j^{\frac{1}{p-1}+1} \lambda_j^{-\frac{p}{p-1}} m_j^{p-1} R_j^{-\beta-2\gamma} A|x|^\beta \left(\lambda_j^{\frac{1}{p-1}} m_j^{-1} + W_j\right)^p & \text{in } B_{R_j}(0), \\ W_j = 0 & \text{in } \mathbb{R}^N \setminus B_{R_j}(0), \end{cases}$$

that is,

$$\begin{cases} (-\Delta)^\gamma W_j = A|x|^\beta \left(\lambda_j^{\frac{1}{p-1}} m_j^{-1} + W_j\right)^p & \text{in } B_{R_j}(0), \\ W_j = 0 & \text{in } \mathbb{R}^N \setminus B_{R_j}(0). \end{cases}$$

Recall that by Lemma 2.6,

$$w_j(x) \leq C_0 |x|^{-\frac{\beta+2\gamma}{p-1}} \quad \text{for } x \in B_{\frac{1}{2}}(0).$$

Thus in $B_{\frac{R_j}{2}}(0)$, $W_j(x)$ has the upper bound

$$
(2.14) \quad
\begin{aligned}
W_j(x) &\leq \lambda^{\frac{1}{p-1}} m_j^{-1} \cdot C_0 \left(\frac{x}{R_j} \right)^{-\frac{\beta+2\gamma}{p-1}} \leq C_0 \left(\lambda_0^{\frac{1}{p-1}} + (\lambda^*)^{\frac{1}{p-1}} \right) |x|^{-\frac{\beta+2\gamma}{p-1}} \\
&= C_1 |x|^{-\frac{\beta+2\gamma}{p-1}} = C_1 |x|^{\frac{2\gamma}{p-1}-(N-2\gamma)}.
\end{aligned}
$$

Note that $|x|^\beta \in L^q(B_{R_j}(0))$ for any $\frac{N}{2\gamma} < q < \frac{N}{-\beta}$. Hence, for such q, by the regularity result in [74], $W_j \in \mathcal{C}^\eta_{loc}(\mathbb{R}^N)$ for $\eta = \min\left\{ \gamma, 2\gamma - \frac{N}{q} \right\} \in (0,1)$. Therefore, by passing to a subsequence, W_j converges uniformly on compact sets of $\mathbb{R}^N$ to a radially symmetric and non-increasing function w which satisfies

$$
\begin{cases}
(-\Delta)^\gamma w = A|x|^\beta w^p & \text{in } \mathbb{R}^N, \\
w(0) = 1, \\
w(x) \leq C_1 |x|^{\frac{2\gamma}{p-1}-(n-2\gamma)},
\end{cases}
$$

in view of (2.14).

Now the family of rescaled solutions $w_\varepsilon(x) = \varepsilon w \left(\varepsilon^{\frac{p-1}{\beta+2\gamma}} x \right)$ solves

$$
\begin{cases}
(-\Delta)^\gamma w_\varepsilon = A|x|^\beta w_\varepsilon^p & \text{in } \mathbb{R}^N, \\
w_\varepsilon(0) = \varepsilon, \\
w_\varepsilon(x) \leq C_1 |x|^{\frac{2\gamma}{p-1}-(N-2\gamma)} & \text{in } \mathbb{R}^N \setminus \{0\}.
\end{cases}
$$

Finally, its Kelvin transform $u_\varepsilon(x) = |x|^{-(N-2\gamma)} w_\varepsilon \left(\frac{x}{|x|^2} \right)$ satisfies

$$
\begin{cases}
(-\Delta)^\gamma u_\varepsilon = A u_\varepsilon^p & \text{in } \mathbb{R}^N \setminus \{0\}, \\
u_\varepsilon(x) \leq C_1 |x|^{-\frac{2\gamma}{p-1}} & \text{in } \mathbb{R}^N \setminus \{0\}, \\
u_\varepsilon(x) \sim \varepsilon |x|^{-(N-2\gamma)} & \text{as } |x| \to \infty,
\end{cases}
$$

as desired. □

3. The conformal fractional Laplacian in the presence of k-dimensional singularities

3.1. **A quick review on the conformal fractional Laplacian.** Here we review some basic facts on the conformal fractional Laplacian that will be needed in the next sections (see [25, 47] for the precise definitions and details).

If (X, g^+) is a $(n + 1)$-dimensional conformally compact Einstein manifold (which, in particular, includes the hyperbolic space), one can define a one-parameter family of operators P_γ of order 2γ on its conformal infinity $M^n = \partial_\infty X^{n+1}$. P_γ is known as the conformal fractional Laplacian and it can be understood as a Dirichlet-to-Neumann operator on M. In the particular case that X is the hyperbolic space $\mathbb{H}^{n+1}$, whose conformal infinity is $M = \mathbb{R}^n$ with the Euclidean metric, P_γ coincides with the standard fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\gamma$.

Let us explain this definition in detail. It is known that, having fixed a metric g_0 in the conformal infinity M, it is possible to write the metric g^+ in the normal form $g^+ = \rho^{-2}(d\rho^2 + g_\rho)$ in a tubular neighborhood $M \times (0, \delta]$. Here g_ρ is a one-parameter family of metrics on M satisfying $g_\rho|_{\rho=0} = g_0$ and ρ is a defining function in $\overline{X}$ for the boundary M (i.e., ρ is a non-degenerate function such that $\rho > 0$ in X and $\rho = 0$ on M).

Fix $\gamma \in (0, n/2)$ not an integer such that $n/2 + \gamma$ does not belong to the set of L^2-eigenvalues of $-\Delta_{g^+}$. Assume also that the first eigenvalue for $-\Delta_{g^+}$ satisfies $\lambda_1(-\Delta_{g^+}) > n^2/4 - \gamma^2$. It is well known

from scattering theory [51, 53] that, given $w \in \mathcal{C}^\infty(M)$, the eigenvalue problem

$$(3.1) \qquad -\Delta_{g^+}\mathcal{W} - \left(\tfrac{n^2}{2} - \gamma^2\right)\mathcal{W} = 0 \text{ in } X,$$

has a unique solution with the asymptotic expansion

$$(3.2) \qquad \mathcal{W} = \mathcal{W}_1 \rho^{\frac{n}{2}-\gamma} + \mathcal{W}_2 \rho^{\frac{n}{2}+\gamma}, \quad \mathcal{W}_1, \mathcal{W}_2 \in \mathcal{C}^\infty(\overline{X})$$

and Dirichlet condition on M

$$(3.3) \qquad \mathcal{W}_1|_{\rho=0} = w.$$

The conformal fractional Laplacian (or scattering operator, depending on the normalization constant) on (M, g_0) is defined taking the Neumann data

$$(3.4) \qquad P_\gamma^{g_0} w = d_\gamma \mathcal{W}_2|_{\rho=0}, \quad \text{where } d_\gamma = 2^{2\gamma} \frac{\Gamma(\gamma)}{\Gamma(-\gamma)},$$

and the fractional curvature as $Q_\gamma^{g_0} := P_\gamma^{g_0}(1)$.

$P_\gamma^{g_0}$ is a self-adjoint pseudodifferential operator of order 2γ on M with the same principal symbol as $(-\Delta_M)^\gamma$. In the case that the order of the operator is an even integer we recover the conformally invariant GJMS operators on M. In addition, for any $\gamma \in (0, \tfrac{n}{2})$, the operator is conformal. Indeed,

$$(3.5) \qquad P_\gamma^{g_w} f = w^{-\frac{n+2\gamma}{n-2\gamma}} P_\gamma^{g_0}(wf), \quad \forall f \in \mathcal{C}^\infty(M),$$

for a change of metric

$$g_w := w^{\frac{4}{n-2\gamma}} g_0, \ w > 0.$$

Moreover, (3.5) yields the Q_γ curvature equation

$$P_\gamma^{g_0}(w) = w^{\frac{n+2\gamma}{n-2\gamma}} Q_\gamma^{g_w}.$$

Explicit formulas for P_γ are not known in general. The formula for the cylinder will be given in Section 3.2, and it is one of the main ingredients for the linear theory arguments of Section 7.

The extension (3.1) takes a more familiar form under a conformal change of metric.

Proposition 3.1 ([25]). *Fix $\gamma \in (0, 1)$ and $\bar{g} = \rho^2 g^+$. Let $\mathcal{W}$ be the solution to the scattering problem (3.1)-(3.2) with Dirichlet data (3.3) set to w. Then $W = \rho^{-n/2+\gamma}\mathcal{W}$ is the unique solution to the extension problem*

$$(3.6) \qquad \begin{cases} -\operatorname{div}\left(\rho^{1-2\gamma}\nabla W\right) + E_{\bar{g}}(\rho)W = 0 & \text{in } (X, \bar{g}), \\ W|_{\rho=0} = w & \text{on } M, \end{cases}$$

where the derivatives are taken with respect to the metric $\bar{g}$, and the zero-th order term is given by

$$(3.7) \qquad \begin{aligned} E_{\bar{g}}(\rho) &= -\Delta_{\bar{g}}(\rho^{\frac{1-2\gamma}{2}})\rho^{\frac{1-2\gamma}{2}} + \left(\gamma^2 - \tfrac{1}{4}\right)\rho^{-(1+2\gamma)} + \tfrac{n-1}{4n}R_{\bar{g}}\rho^{1-2\gamma} \\ &= \rho^{-\frac{n}{2}-\gamma-1}\left\{-\Delta_{g^+} - \left(\tfrac{n^2}{4} - \gamma^2\right)\right\}\left(\rho^{\frac{n}{2}-\gamma}\right). \end{aligned}$$

Moreover, we recover the conformal fractional Laplacian as

$$P_\gamma^{g_0} w = -\tilde{d}_\gamma \lim_{\rho \to 0} \rho^{1-2\gamma}\partial_\rho W,$$

where

$$(3.8) \qquad \tilde{d}_\gamma = -\frac{d_\gamma}{2\gamma} = -\frac{2^{2\gamma-1}\Gamma(\gamma)}{\gamma\Gamma(-\gamma)}.$$

We also recall the following result, which allows us to rewrite (3.6) as a pure divergence equation with no zeroth order term. The more general statement can be found in Lemma 3.7, and it will be useful in the calculation of the Hamiltonian from Section 4.2.

Proposition 3.2 ([25, 21])**.** *Fix* $\gamma \in (0,1)$*. Let* $\mathcal{W}^0$ *be the solution to* (3.1)-(3.2) *with Dirichlet data* (3.3) *given by* $w \equiv 1$*, and set* $\rho^* = (\mathcal{W}^0)^{\frac{1}{n/2-\gamma}}$*. The function* ρ^* *is a defining function of* M *in* X *such that, if we define the metric* $\bar{g}^* = (\rho^*)^2 g^+$*, then* $E_{\bar{g}^*}(\rho^*) \equiv 0$*. Moreover,* ρ^* *has the asymptotic expansion near the conformal infinity*

$$\rho^*(\rho) = \rho\Big[1 + \frac{Q_\gamma^{g_0}}{(n/2-\gamma)d_\gamma}\rho^{2\gamma} + O(\rho^2)\Big].$$

By construction, if W^* *is the solution to*

$$\begin{cases} -\operatorname{div}\big((\rho^*)^{1-2\gamma}\nabla W^*\big) = 0 & in \ (X, \bar{g}^*), \\ \qquad\qquad\qquad\quad W^* = w & on \ (M, g_0), \end{cases}$$

with respect to the metric $\bar{g}^*$*, then*

$$P_\gamma^{g_0} w = -\tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^{1-2\gamma}\,\partial_{\rho^*}W^* + w\,Q_\gamma^{g_0}.$$

Remark 3.3. In the particular case that $X = \mathbb{R}_+^{n+1} = \{(x,\ell) : x \in \mathbb{R}^n, \ell > 0\}$ is hyperbolic space $\mathbb{H}^{n+1}$ with the metric $g^+ = \frac{d\ell^2 + |dx|^2}{\ell^2}$ and $M = \mathbb{R}^n$, this is just the construction for the fractional Laplacian $(-\Delta_{\mathbb{R}^n})^\gamma$ as a Dirichlet-to-Neumann operator for a degenerate elliptic extension problem from [19]. Indeed, let U be the solution to

$$(3.9) \qquad\qquad \begin{cases} \partial_{\ell\ell}U + \dfrac{1-2\gamma}{\ell}\partial_\ell U + \Delta_{\mathbb{R}^n}U = 0 & in \ \mathbb{R}_+^{n+1}, \\ \qquad\qquad\qquad\qquad U|_{\ell=0} = u & on \ \mathbb{R}^n, \end{cases}$$

then

$$(3.10) \qquad\qquad\qquad (-\Delta_{\mathbb{R}^n})^\gamma u = -\tilde{d}_\gamma \lim_{\ell \to 0} \ell^{1-2\gamma}\partial_\ell U.$$

From now on, (X, g^+) will be fixed to be hyperbolic space with its standard metric. Our point of view in this paper is to rewrite this extension problem (3.9)-(3.10) using different coordinates for the hyperbolic metric in X, such as (3.16).

3.2. **An isolated singularity.** Before we go to the general problem, let us look at positive solutions to

$$(3.11) \qquad\qquad (-\Delta_{\mathbb{R}^N})^\gamma u = \Lambda_{N,\gamma} u^{\frac{N+2\gamma}{N-2\gamma}} \quad in \ \ \mathbb{R}^N \setminus \{0\}$$

that have an isolated singularity at the origin. It is known ([17]) that such solutions have the asymptotic behavior near the origin like $r^{-\frac{N-2\gamma}{2}}$, for $r = |x|$. Thus it is natural to write

$$(3.12) \qquad\qquad\qquad u = r^{-\frac{N-2\gamma}{2}} w.$$

Note that the power of the nonlinearity in the right hand side of (3.11) is chosen so that the equation has good conformal properties. Indeed, let $r = e^{-t}$ and $\theta \in \mathbb{S}^{N-1}$ and write the Euclidean metric in $\mathbb{R}^N$ as

$$|dx|^2 = dr^2 + r^2 g_{\mathbb{S}^{N-1}}$$

in polar coordinates. We use conformal geometry to rewrite equation (3.11). For this, consider the conformal change

$$g_0 := \frac{1}{r^2}|dx|^2 = dt^2 + g_{\mathbb{S}^{N-1}},$$

which is a complete metric defined on the cylinder $M_0 := \mathbb{R} \times \mathbb{S}^{N-1}$. The advantage of using g_0 as a background metric instead of the Euclidean one on $\mathbb{R}^N$ is the following: since the two metrics are conformally related, any conformal change may be rewritten as

$$\tilde{g} = u^{\frac{4}{N-2\gamma}}|dx|^2 = w^{\frac{4}{N-2\gamma}} g_0,$$

where we have used relation (3.12). Then, looking at the conformal transformation property (3.5) for the conformal fractional Laplacian P_γ, it is clear that

$$(3.13) \qquad P_\gamma^{g_0}(w) = r^{\frac{N+2\gamma}{2}} P_\gamma^{|dx|^2}\left(r^{-\frac{N-2\gamma}{2}} w\right) = r^{\frac{N+2\gamma}{2}} (-\Delta_{\mathbb{R}^N})^\gamma u,$$

and thus equation (3.11) is equivalent to

$$P_\gamma^{g_0}(w) = \Lambda_{N,\gamma} w^{\frac{N+2\gamma}{N-2\gamma}} \quad \text{in} \quad \mathbb{R} \times \mathbb{S}^{N-1}.$$

The operator $P_\gamma^{g_0}$ on $\mathbb{R} \times \mathbb{S}^{N-1}$ is explicit. Indeed, in [29] the authors calculate its principal symbol using the spherical harmonic decomposition for $\mathbb{S}^{N-1}$. With some abuse of notation, let μ_m, $m = 0, 1, 2, \ldots$ be the eigenvalues of $\Delta_{\mathbb{S}^{N-1}}$, repeated according to multiplicity (this is, $\mu_0 = 0$, $\mu_1, \ldots, \mu_N = N-1, \ldots$). Then any function on $\mathbb{R} \times \mathbb{S}^{N-1}$ may be decomposed as $\sum_m w_m(t) E_m$, where $\{E_m(\theta)\}$ is the corresponding basis of eigenfunctions. The operator $P_\gamma^{g_0}$ diagonalizes under such eigenspace decomposition, and moreover, it is possible to calculate the Fourier symbol of each projection. Let

$$(3.14) \qquad \hat{w}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi \cdot t} w(t)\, dt$$

be our normalization for the one-dimensional Fourier transform.

Proposition 3.4 ([29]). *Fix $\gamma \in (0, \frac{N}{2})$ and let P_γ^m be the projection of the operator $P_\gamma^{g_0}$ over each eigenspace $\langle E_m \rangle$. Then*

$$\widehat{P_\gamma^m(w_m)} = \Theta_\gamma^m(\xi)\, \widehat{w_m},$$

and this Fourier symbol is given by

$$(3.15) \qquad \Theta_\gamma^m(\xi) = 2^{2\gamma} \frac{\left| \Gamma\left(\frac{1}{2} + \frac{\gamma}{2} + \frac{1}{2}\sqrt{(\frac{N}{2}-1)^2 + \mu_m} + \frac{\xi}{2}i \right) \right|^2}{\left| \Gamma\left(\frac{1}{2} - \frac{\gamma}{2} + \frac{1}{2}\sqrt{(\frac{N}{2}-1)^2 + \mu_m} + \frac{\xi}{2}i \right) \right|^2}.$$

Proof. Let us give some ideas in the proof because they will be needed in the next subsections. It is inspired in the calculation of the Fourier symbol for the conformal fractional Laplacian on the sphere $\mathbb{S}^n$ (see the survey [47], for instance). The method is, using spherical harmonics, to reduce the scattering equation (3.1) to an ODE. For this, we go back to the scattering theory definition for the fractional Laplacian and use different coordinates for the hyperbolic metric g^+. More precisely,

$$(3.16) \qquad g^+ = \rho^{-2}\left\{ \rho^2 + \left(1 + \frac{\rho^2}{4}\right)^2 dt^2 + \left(1 - \frac{\rho^2}{4}\right)^2 g_{\mathbb{S}^{N-1}} \right\}, \quad \bar{g} = \rho^2 g^+,$$

where $\rho \in (0, 2)$, $t \in \mathbb{R}$. The conformal infinity $\{\rho = 0\}$ is precisely the cylinder $(\mathbb{R} \times \mathbb{S}^{N-1}, g_0)$. Actually, for the particular calculation here it is better to use the new variable $\sigma = -\log(\rho/2)$, and write

$$(3.17) \qquad g^+ = d\sigma^2 + (\cosh\sigma)^2 dt^2 + (\sinh\sigma)^2 g_{\mathbb{S}^{N-1}}.$$

Using this metric, the scattering equation (3.1) is

$$(3.18) \qquad \partial_{\sigma\sigma}\mathcal{W} + R(\sigma)\partial_\sigma\mathcal{W} + (\cosh\sigma)^{-2}\partial_{tt}\mathcal{W} + (\sinh\sigma)^{-2}\Delta_{\mathbb{S}^{N-1}}\mathcal{W} + \left(\frac{N^2}{4} - \gamma^2\right)\mathcal{W} = 0,$$

where $\mathcal{W} = \mathcal{W}(\sigma, t, \theta)$, $\sigma \in (0, \infty)$, $t \in \mathbb{R}$, $\theta \in \mathbb{S}^{N-1}$, and

$$R(\sigma) = \frac{\partial_\sigma\left(\cosh\sigma \sinh^{N-1}\sigma\right)}{\cosh\sigma \sinh^{N-1}\sigma}.$$

After projection over spherical harmonics, and Fourier transform in t, the solution to equation (3.18) maybe written as

$$\widehat{\mathcal{W}_m} = \widehat{w_m}\, \varphi(\tau),$$

where we have used the change of variable $\tau = \tanh(\sigma)$ and $\varphi := \varphi^{(m)}$ is a solution to the boundary value problem

$$\begin{cases} (1-\tau^2)\partial_{\tau\tau}\varphi + \left(\frac{N-1}{\tau} - \tau\right)\partial_\tau\varphi + \left[-\mu_m\frac{1}{\tau^2} + \left(\frac{n^2}{4} - \gamma^2\right)\frac{1}{1-\tau^2} - \xi^2\right]\varphi = 0, \\[2mm] \text{has the expansion (3.2) with } w \equiv 1 \text{ near the conformal infinity } \{\tau = 1\}, \\[2mm] \varphi \text{ is regular at } \tau = 0. \end{cases}$$

This is an ODE that can be explicitly solved in terms of hypergeometric functions, and indeed,

$$
\begin{aligned}
(3.19) \quad \varphi(\tau) = &(1+\tau)^{\frac{N}{4}-\frac{\gamma}{2}}(1-\tau)^{\frac{N}{4}-\frac{\gamma}{2}}\tau^{1-\frac{N}{2}+\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}} \cdot {}_2F_1(a,b;a+b-c+1;1-\tau^2) \\
&+ S(1+\tau)^{\frac{N}{4}+\frac{\gamma}{2}}(1-\tau)^{\frac{N}{4}+\frac{\gamma}{2}}\tau^{1-\frac{N}{2}+\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}} \cdot {}_2F_1(c-a,c-b;c-a-b+1;1-\tau^2),
\end{aligned}
$$

where

$$
S(\xi) = \frac{\Gamma(-\gamma)}{\Gamma(\gamma)} \frac{\left|\Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{1}{2}\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}+\frac{\xi}{2}i\right)\right|^2}{\left|\Gamma\left(\frac{1}{2}-\frac{\gamma}{2}+\frac{1}{2}\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}+\frac{\xi}{2}i\right)\right|^2},
$$

and

$$a = \frac{-\gamma}{2}+\frac{1}{2}+\frac{1}{2}\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}+i\frac{\xi}{2}, \ b = \frac{-\gamma}{2}+\frac{1}{2}+\frac{1}{2}\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}-i\frac{\xi}{2}, \ c = 1+\sqrt{\left(\frac{N}{2}-1\right)^2+\mu_m}.$$

The Proposition follows by looking at the Neumann condition in the expansion (3.2). $\qquad\square$

The interest of this proposition will become clear in Section 7, where we calculate the indicial roots for the linearized problem. It is also the crucial ingredient in the calculation of the Green's function for the fractional Laplacian with Hardy potential in Section 6.

We finally recall the fractional Hardy's inequality in $\mathbb{R}^N$ ([59, 86, 8, 44])

$$(3.20) \qquad \int_{\mathbb{R}^N} u(-\Delta_{\mathbb{R}^N})^\gamma u\,dx \geq \Lambda_{N,\gamma}\int_{\mathbb{R}^N}\frac{u^2}{r^{2\gamma}}\,dx,$$

where $\Lambda_{N,\gamma}$ is the Hardy constant given by

$$(3.21) \qquad \Lambda_{N,\gamma} = 2^{2\gamma}\frac{\Gamma^2(\frac{N+2\gamma}{4})}{\Gamma^2(\frac{N-2\gamma}{4})} = \Theta_\gamma^0(0).$$

Under the conjugation (3.12), inequality (3.20) is written as

$$\int_{\mathbb{R}\times\mathbb{S}^{N-1}} wP_\gamma^{g_0}w\,dtd\theta \geq \Lambda_{N,\gamma}\int_{\mathbb{R}\times\mathbb{S}^{N-1}} w^2\,dtd\theta.$$

3.3. **The full symbol.** Now we consider the singular Yamabe problem (1.5) in $\mathbb{R}^n\setminus\mathbb{R}^k$. This particular case is important because it is the model for a general higher dimensional singularity (see [57]).

As in the introduction, set $N := n-k$. We define the coordinates $z = (x,y)$, $x \in \mathbb{R}^N$, $y \in \mathbb{R}^k$ in the product space $\mathbb{R}^n\setminus\mathbb{R}^k = (\mathbb{R}^{n-k}\setminus\{0\})\times\mathbb{R}^k$. Sometimes we will consider polar coordinates for x, which are

$$r = |x| = \text{dist}(\cdot,\mathbb{R}^k) \in \mathbb{R}_+, \ \theta \in \mathbb{S}^{N-1}.$$

We write the Euclidean metric in $\mathbb{R}^n$ as

$$|dz|^2 = |dx|^2 + |dy|^2 = dr^2 + r^2 g_{\mathbb{S}^{N-1}} + |dy|^2.$$

Our model manifold M is going to be given by the conformal change

$$(3.22) \qquad g_k := \frac{1}{r^2}|dz|^2 = g_{\mathbb{S}^{N-1}} + \frac{dr^2 + |dy|^2}{r^2} = g_{\mathbb{S}^{N-1}} + g_{\mathbb{H}^{k+1}},$$

which is a complete metric, singular along $\mathbb{R}^k$. In particular, $M := \mathbb{S}^{N-1} \times \mathbb{H}^{k+1}$. As in the previous case, any conformal change may be rewritten as

$$\tilde{g} = u^{\frac{4}{n-2\gamma}}|dz|^2 = w^{\frac{4}{n-2\gamma}}g_k,$$

where we have used relation

$$u = r^{-\frac{n-2\gamma}{2}}w,$$

so we may just use g_k as our background metric. As a consequence, arguing as in the previous subsection, the conformal transformation property (3.5) for the conformal fractional Laplacian yields that

$$(3.23) \qquad P_\gamma^{g_k}(w) = r^{\frac{n+2\gamma}{2}}P_\gamma^{|dz|^2}(r^{-\frac{n-2\gamma}{2}}w) = r^{\frac{n+2\gamma}{2}}(-\Delta_{\mathbb{R}^n})^\gamma u,$$

and thus the original Yamabe problem (1.5) is equivalent to the following:

$$P_\gamma^{g_k}(w) = \Lambda_{n,\gamma}w^{\frac{n+2\gamma}{n-2\gamma}} \quad \text{on} \quad M.$$

Moreover, the expression for $P_\gamma^{g_k}$ in the metric g_k (with respect to the standard extension to hyperbolic space $X = \mathbb{H}^{n+1}$) is explicit, and this is the statement of the following theorem. For our purposes, it will be more convenient to write the standard hyperbolic metric as

$$(3.24) \qquad g^+ = \rho^{-2}\left\{d\rho^2 + \left(1 + \tfrac{\rho^2}{4}\right)^2 g_{\mathbb{H}^{k+1}} + \left(1 - \tfrac{\rho^2}{4}\right)^2 g_{\mathbb{S}^{N-1}}\right\},$$

for $\rho \in (0, 2)$, so its conformal infinity $\{\rho = 0\}$ is precisely (M, g_k).

Consider the spherical harmonic decomposition for $\mathbb{S}^{N-1}$ as in Section 3.2. Then any function w on M may be decomposed as $w = \sum_m w_m E_m$, where $w_m = w_m(\zeta)$ for $\zeta \in \mathbb{H}^{k+1}$. We show that the operator $P_\gamma^{g_k}$ diagonalizes under such eigenspace decomposition, and moreover, it is possible to calculate the Fourier symbol for each projection. Let $\widehat{\cdot}$ denote the Fourier-Helgason transform on $\mathbb{H}^{k+1}$, as described in the Appendix (section 10.2).

Theorem 3.5. *Fix $\gamma \in (0, \frac{n}{2})$ and let P_γ^m be the projection of the operator $P_\gamma^{g_k}$ over each eigenspace $\langle E_m \rangle$. Then*

$$\widehat{P_\gamma^m(w_m)} = \Theta_\gamma^m(\lambda)\,\widehat{w_m},$$

and this Fourier symbol is given by

$$(3.25) \qquad \Theta_\gamma^m(\lambda) = 2^{2\gamma}\frac{\left|\Gamma\left(\frac{1}{2} + \frac{\gamma}{2} + \frac{1}{2}\sqrt{\left(\frac{N}{2} - 1\right)^2 + \mu_m} + \frac{\lambda}{2}i\right)\right|^2}{\left|\Gamma\left(\frac{1}{2} - \frac{\gamma}{2} + \frac{1}{2}\sqrt{\left(\frac{N}{2} - 1\right)^2 + \mu_m} + \frac{\lambda}{2}i\right)\right|^2}.$$

Proof. We follow the arguments in Proposition 3.4, however, the additional ingredient here is to use Fourier-Helgason transform to handle the extra term $\Delta_{\mathbb{H}^{k+1}}$ that will appear.

For the calculations below it is better to use the new variable

$$\sigma = -\log(\rho/2), \quad \rho \in (0, 2),$$

and to rewrite the hyperbolic metric in $\mathbb{H}^{n+1}$ from (3.24) as

$$g^+ = d\sigma^2 + (\cosh\sigma)^2 g_{\mathbb{H}^{k+1}} + (\sinh\sigma)^2 g_{\mathbb{S}^{N-1}},$$

for the variables $\sigma \in (0, \infty)$, $\zeta \in \mathbb{H}^{k+1}$ and $\theta \in \mathbb{S}^{N-1}$. The conformal infinity is now $\{\sigma = +\infty\}$ and the scattering equation (3.1) is written as

$$(3.26) \qquad \partial_{\sigma\sigma}\mathcal{W} + R(\sigma)\partial_\sigma\mathcal{W} + (\cosh\sigma)^{-2}\Delta_{\mathbb{H}^{k+1}}\mathcal{W} + (\sinh\sigma)^{-2}\Delta_{\mathbb{S}^{N-1}}\mathcal{W} + \left(\tfrac{n^2}{4} - \gamma^2\right)\mathcal{W} = 0,$$

where $\mathcal{W} = \mathcal{W}(\sigma, \zeta, \theta)$, and

$$R(\sigma) = \frac{\partial_\sigma\left((\cosh\sigma)^{k+1}(\sinh\sigma)^{N-1}\right)}{(\cosh\sigma)^{k+1}(\sinh\sigma)^{N-1}}.$$

The change of variable

(3.27)
$$\tau = \tanh(\sigma),$$

transforms equation (3.26) into

$$(1-\tau^2)^2\partial_{\tau\tau}\mathcal{W} + \left(\tfrac{n-k-1}{\tau} + (k-1)\tau\right)(1-\tau^2)\partial_\tau\mathcal{W} + (1-\tau^2)\Delta_{\mathbb{H}^{k+1}}\mathcal{W}$$
$$+ \left(\tfrac{1}{\tau^2} - 1\right)\Delta_{\mathbb{S}^{N-1}}\mathcal{W} + \left(\tfrac{n^2}{4} - \gamma^2\right)\mathcal{W} = 0.$$

Now we project onto spherical harmonics. This is, let $\mathcal{W}_m(\tau,\zeta)$ be the projection of $\mathcal{W}$ over the eigenspace $\langle E_m\rangle$. Then each $\mathcal{W}_m$ satisfies

(3.28) $$(1-\tau^2)\partial_{\tau\tau}\mathcal{W}_m + \left(\tfrac{n-k-1}{\tau} + (k-1)\tau\right)\partial_\tau\mathcal{W}_m + \Delta_{\mathbb{H}^{k+1}}\mathcal{W}_m - \mu_m\tfrac{1}{\tau^2}\mathcal{W}_m + \tfrac{\frac{n^2}{4}-\gamma^2}{1-\tau^2}\mathcal{W}_m = 0.$$

Taking the Fourier-Helgason transform in $\mathbb{H}^{k+1}$ we obtain

$$(1-\tau^2)\partial_{\tau\tau}\widehat{\mathcal{W}_m} + \left(\tfrac{n-k-1}{\tau} + (k-1)\tau\right)\partial_\tau\widehat{\mathcal{W}_m} + \left[-\mu_m\tfrac{1}{\tau^2} + (\tfrac{n^2}{4}-\gamma^2)\tfrac{1}{1-\tau^2} - (\lambda^2 + \tfrac{k^2}{4})\right]\widehat{\mathcal{W}_m} = 0$$

for $\widehat{\mathcal{W}_m} = \widehat{\mathcal{W}_m}(\tau,\lambda,\omega)$. Fixed $m = 0,1,\ldots,\lambda \in \mathbb{R}$ and $\omega \in \mathbb{S}^k$, we know that

$$\widehat{\mathcal{W}_m} = \widehat{w_m}\,\varphi_k^\lambda(\tau),$$

where $\varphi := \varphi_k^\lambda(\tau)$ is the solution to the following boundary value problem:

(3.29)
$$\begin{cases} (1-\tau^2)\partial_{\tau\tau}\varphi + \left(\tfrac{n-k-1}{\tau} + (k-1)\tau\right)\partial_\tau\varphi + \left[-\mu_m\tfrac{1}{\tau^2} + (\tfrac{n^2}{4}-\gamma^2)\tfrac{1}{1-\tau^2} - (\lambda^2 + \tfrac{k^2}{4})\right]\varphi = 0, \\ \text{has the expansion (3.2) with } w \equiv 1 \text{ near the conformal infinity } \{\tau = 1\}, \\ \varphi \text{ is regular at } \tau = 0. \end{cases}$$

This is an ODE in τ that has only regular singular points, and can be explicitly solved. Indeed, from the first equation in (3.29) we obtain

(3.30)
$$\varphi(\tau) = A(1-\tau^2)^{\frac{n}{4}-\frac{\gamma}{2}}\tau^{1-\frac{n}{2}+\frac{k}{2}+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}}\,{}_2F_1(a,b;c;\tau^2)$$
$$+ B(1-\tau^2)^{\frac{n}{4}-\frac{\gamma}{2}}\tau^{1-\frac{n}{2}-\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}}\,{}_2F_1(\tilde{a},\tilde{b};\tilde{c};,\tau^2),$$

for any real constants A, B, where

$$a = \tfrac{-\gamma}{2} + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m} + i\tfrac{\lambda}{2}, \qquad \tilde{a} = \tfrac{-\gamma}{2} + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m} + i\tfrac{\lambda}{2},$$

$$b = \tfrac{-\gamma}{2} + \tfrac{1}{2} + \tfrac{1}{2}\sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m} - i\tfrac{\lambda}{2}, \qquad \tilde{b} = \tfrac{-\gamma}{2} + \tfrac{1}{2} - \tfrac{1}{2}\sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m} - i\tfrac{\lambda}{2},$$

$$c = 1 + \sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m}, \qquad\qquad \tilde{c} = 1 - \sqrt{(\tfrac{n-k}{2}-1)^2 + \mu_m},$$

and $_2F_1$ denotes the standard hypergeometric function described in Lemma 10.1. Note that we can write λ instead of $|\lambda|$ in the arguments of the hypergeometric functions because $a = \bar{b}$, $\tilde{a} = \bar{\tilde{b}}$ and property (10.3).

The regularity at the origin $\tau = 0$ implies $B = 0$ in (3.30). Moreover, using (10.2) we can write

$$\varphi(\tau) = A\left[\alpha(1-\tau^2)^{\frac{n}{4}-\frac{\gamma}{2}}\tau^{1-\frac{n}{2}+\frac{k}{2}+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}}\,{}_2F_1(a,b;a+b-c+1;1-\tau^2)\right.$$

$$\left. + \beta(1-\tau^2)^{\frac{n}{4}+\frac{\gamma}{2}}\tau^{1-\frac{n}{2}+\frac{k}{2}+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}}\,{}_2F_1(c-a,c-b;c-a-b+1;1-\tau^2)\right],$$

where

$$\alpha = \frac{\Gamma\left(1+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}\right)\Gamma(\gamma)}{\Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}-i\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}+i\frac{\lambda}{2}\right)},$$

$$\beta = \frac{\Gamma\left(1+\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}\right)\Gamma(-\gamma)}{\Gamma\left(\frac{1}{2}-\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}+i\frac{\lambda}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\gamma}{2}+\frac{1}{2}\sqrt{(\frac{n-k}{2}-1)^2+\mu_m}-i\frac{\lambda}{2}\right)}.$$

Note that our changes of variable give

$$(3.31) \qquad\qquad \tau = \tanh(\sigma) = \frac{4-\rho^2}{4+\rho^2} = 1 - \frac{1}{2}\rho^2 + \cdots,$$

which yields, as $\rho \to 0$,

$$\varphi(\rho) \sim A\left[\alpha\rho^{\frac{n}{2}-\gamma} + \beta\rho^{\frac{n}{2}+\gamma} + \ldots\right].$$

Here we have used (10.1) for the hypergeometric function.

Looking at the expansion for the scattering solution (3.2) and the definition of the conformal fractional Laplacian (3.4), we must have

$$(3.32) \qquad\qquad A = \alpha^{-1}, \quad \text{and} \quad \Theta_\gamma^m(\lambda) = d_\gamma\beta\alpha^{-1}.$$

Property (10.5) yields (3.25) and completes the proof of Theorem 3.5. $\qquad\qquad\square$

3.4. **Conjugation.** We now go back to the discussion in Section 3.2 for an isolated singularity but we allow any subcritical power $p \in (\frac{N}{N-2\gamma}, \frac{N+2\gamma}{N-2\gamma})$ in the right hand side of (3.11); this is,

$$(3.33) \qquad\qquad (-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma}u^p \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

This equation does not have good conformal properties. But, given $u \in \mathcal{C}^\infty(\mathbb{R}^N \setminus \{0\})$, we can consider

$$u = r^{-\frac{N-2\gamma}{2}}w = r^{-\frac{2\gamma}{p-1}}v, \quad r = e^{-t},$$

and define the conjugate operator

$$(3.34) \qquad \tilde{P}_\gamma^{g_0}(v) := r^{-\frac{N-2\gamma}{2}+\frac{2\gamma}{p-1}}P_\gamma^{g_0}\left(r^{\frac{N-2\gamma}{2}-\frac{2\gamma}{p-1}}v\right) = r^{\frac{2\gamma}{p-1}p}(-\Delta_{\mathbb{R}^N})^\gamma u.$$

Then problem (3.33) is equivalent to

$$\tilde{P}_\gamma^{g_0}(v) = A_{N,p,\gamma}v^p \quad \text{in } \mathbb{R} \times \mathbb{S}^{N-1},$$

for some $v = v(t,\theta)$ smooth, $t \in \mathbb{R}$, $\theta \in \mathbb{S}^{N-1}$.

This $\tilde{P}_\gamma^{g_0}$ can then be seen from the perspective of scattering theory, and thus be characterized as a Dirichlet-to-Neumann operator for a special extension problem in Proposition 3.9, as inspired by the paper of Chang and González [25]. Note the Neumann condition (3.53), which differs from the one of the standard fractional Laplacian.

In the notation of Section 3.2, we set $X = \mathbb{H}^{N+1}$ with the metric given by (3.16). Its conformal infinity is $M = \mathbb{R} \times \mathbb{S}^{N-1}$ with the metric g_0. We would like to repeat the arguments of Section 3 for the conjugate operator $\tilde{P}_\gamma^{g_0}$. But this operator does not have good conformal properties. In any case, we are able to define a new eigenvalue problem that replaces (3.1)-(3.2).

More precisely, let $\mathcal{W}$ be the unique solution to the scattering problem (3.1)-(3.2) with Dirichlet data (3.3) set to w. We define the function $\mathcal{V}$ by the following relation

$$(3.35) \qquad\qquad r^{Q_0}\mathcal{W} = \mathcal{V}, \quad Q_0 := -\frac{N-2\gamma}{2} + \frac{2\gamma}{p-1},$$

Substituting into (3.1), the new scattering problem is

$$(3.36) \qquad -\Delta_{g^+}\mathcal{V} + \left(\tfrac{4+\rho^2}{4\rho}\right)^{-2}\left[-2Q_0\,\partial_t V - Q_0^2\mathcal{V}\right] - \left(\tfrac{N^4}{2} - \gamma^2\right)\mathcal{V} = 0 \text{ in } X,$$

Moreover, if we set

$$(3.37) \qquad \mathcal{V} = \rho^{\frac{N}{2}-\gamma}\mathcal{V}_1 + \rho^{\frac{N}{2}+\gamma}\mathcal{V}_2,$$

the Dirichlet condition (3.3) will turn into

$$(3.38) \qquad \mathcal{V}_1|_{\rho=0} = v,$$

and the Neumann one (3.4) into

$$(3.39) \qquad d_\gamma\mathcal{V}_2|_{\rho=0} = \tilde{P}_\gamma^{g_0}(v).$$

The following proposition is the analogous to Proposition 3.4 for $\tilde{P}_\gamma^{g_0}$:

Proposition 3.6. *Fix $\gamma \in (0, \tfrac{n}{2})$ and let $\tilde{P}_\gamma^m$ be the projection of the operator $\tilde{P}_\gamma^{g_0}$ over each eigenspace $\langle E_m \rangle$. Then*

$$\widehat{\tilde{P}_\gamma^m(v_m)} = \tilde{\Theta}_\gamma^m(\xi)\,\widehat{v_m},$$

and this Fourier symbol is given by

$$(3.40) \quad \tilde{\Theta}_\gamma^m(\xi) = 2^{2\gamma}\frac{\Gamma\left(\tfrac{1}{2} + \tfrac{\gamma}{2} + \frac{\sqrt{(\frac{N}{2}-1)^2+\mu_m}}{2} + \tfrac{1}{2}(Q_0 + \xi i)\right)\Gamma\left(\tfrac{1}{2} + \tfrac{\gamma}{2} + \frac{\sqrt{(\frac{N}{2}-1)^2+\mu_m}}{2} - \tfrac{1}{2}(Q_0 + \xi i)\right)}{\Gamma\left(\tfrac{1}{2} - \tfrac{\gamma}{2} + \frac{\sqrt{(\frac{N}{2}-1)^2+\mu_m}}{2} + \tfrac{1}{2}(Q_0 + \xi i)\right)\Gamma\left(\tfrac{1}{2} - \tfrac{\gamma}{2} + \frac{\sqrt{(\frac{N}{2}-1)^2+\mu_m}}{2} - \tfrac{1}{2}(Q_0 + \xi i)\right)}.$$

Proof. We write the hyperbolic metric as (3.17) using the change of variable $\sigma = -\log(\rho/2)$. The scattering equation for $\mathcal{W}$ is (3.26) in the particular case $k = 0$, $n = N$, and thus, we follow the arguments in the proof of Theorem 3.5. Set $r = e^{-t}$ and project over spherical harmonics as in (3.28), which yields

$$(3.41) \qquad \partial_{\sigma\sigma}\mathcal{W}_m + R(\sigma)\partial_\sigma\mathcal{W}_m + (\cosh\sigma)^{-2}\partial_{tt}\mathcal{W}_m - (\sinh\sigma)^{-2}\mu_m\mathcal{W}_m + \left(\tfrac{N^2}{4} - \gamma^2\right)\mathcal{W}_m = 0$$

for

$$R(\sigma) = \frac{\partial_\sigma(\cosh\sigma\,\sinh^{N-1}\sigma)}{\cosh\sigma\,\sinh^{N-1}\sigma}.$$

Recall the relation (3.35) and rewrite the extension equation (3.41) in terms of each projection $\mathcal{V}_m$ of $\mathcal{V}$. This gives

$$(3.42) \qquad \begin{aligned} \partial_{\sigma\sigma}\mathcal{V}_m + R(\sigma)\partial_\sigma\mathcal{V}_m &+ (\cosh\sigma)^{-2}\left\{\partial_{tt}\mathcal{V}_m + 2Q_0\partial_t\mathcal{V}_m + Q_0^2\mathcal{V}_m\right\} \\ &- (\sinh\sigma)^{-2}\mu_m\mathcal{V}_m + \left(\tfrac{N^2}{4} - \gamma^2\right)\mathcal{V}_m = 0. \end{aligned}$$

Now we use the change of variable (3.27), and take Fourier transform (3.14) with respect to the variable t. Then

$$(3.43) \qquad (1-\tau^2)\partial_{\tau\tau}\widehat{\mathcal{V}_m} + \left(\tfrac{N-1}{\tau} - \tau\right)\partial_\tau\widehat{\mathcal{V}_m} + \left[-\mu_m\tfrac{1}{\tau^2} + \left(\tfrac{N^2}{4} - \gamma^2\right)\tfrac{1}{1-\tau^2} - (\xi - iQ_0)^2\right]\widehat{\mathcal{V}_m} = 0.$$

The Fourier symbol (3.40) is obtained following the same steps as in the proof of Theorem 3.5. Note that the only difference is the coefficient of $\widehat{\mathcal{V}}_m$ in (3.43).

We note here than an alternative way to calculate the symbol is by taking Fourier transform in relation $\tilde{P}_\gamma^{g_0}(v) = e^{-Q_0 t}P_\gamma^{g_0}(w)$, as follows:

$$\widehat{\tilde{P}_\gamma^m v_m}(t) = \widehat{P_\gamma^m w_m}(\xi - iQ_0) = \Theta_\gamma^m(\xi - iQ_0)\hat{w}_m(\xi - iQ_0) = \Theta_\gamma^m(\xi - iQ_0)\hat{v}_m(\xi).$$

Thus $\tilde{\Theta}_\gamma^m(\xi) = \Theta_\gamma^m(\xi - iQ_0)$, as desired. $\qquad\qquad\square$

Now we turn to Proposition 3.2, and we show that there exists a very special defining function adapted to $\mathcal{V}$.

Lemma 3.7. *Let $\gamma \in (0,1)$. There exists a new defining function ρ^* such that, if we define the metric $\bar{g}^* = (\rho^*)^2 g^+$, then*

$$E_{\bar{g}^*}(\rho^*) = (\rho^*)^{-(1+2\gamma)} \left(\tfrac{4\rho}{4+\rho^2}\right)^2 Q_0^2,$$

where $E_{\bar{g}^*}(\rho^*)$ is defined in (3.7). The precise expression for ρ^* is

$$(3.44) \qquad \rho^*(\rho) = \left[\alpha^{-1} \left(\tfrac{4\rho}{4+\rho^2}\right)^{\frac{N-2\gamma}{2}} \; {}_2F_1\left(\tfrac{\gamma}{p-1}, \tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}; \tfrac{N}{2}; \left(\tfrac{4-\rho^2}{4+\rho^2}\right)^2\right) \right]^{2/(N-2\gamma)}, \quad \rho \in (0,2),$$

where

$$\alpha = \frac{\Gamma(\tfrac{N}{2})\Gamma(\gamma)}{\Gamma\left(\gamma + \tfrac{\gamma}{p-1}\right)\Gamma\left(\tfrac{N}{2} - \tfrac{\gamma}{p-1}\right)}.$$

The function ρ^* is strictly monotone with respect to ρ, and in particular, $\rho^* \in (0, \rho_0^*)$ for

$$(3.45) \qquad \rho_0^* := \rho^*(2) = \alpha^{\frac{-2}{N-2\gamma}}.$$

Moreover, it has the asymptotic expansion near the conformal infinity

$$(3.46) \qquad \rho^*(\rho) = \rho \left[1 + O(\rho^{2\gamma}) + O(\rho^2) \right].$$

Proof. The proof follows Lemma 4.5 in [25]. The scattering equation (3.1) for $\mathcal{W}$ is modified to (3.36) when we substitute (3.35), but the additional terms do not affect the overall result. Then we know that, given $v \equiv 1$ on M, (3.36) has a unique solution $\mathcal{V}^0$ with the asymptotic expansion

$$\mathcal{V}^0 = \mathcal{V}_1^0 \rho^{\frac{N}{2}-\gamma} + \mathcal{V}_2^0 \rho^{\frac{N}{2}+\gamma}, \quad \mathcal{V}_1^0, \mathcal{V}_2^0 \in \mathcal{C}^\infty(\overline{X})$$

and Dirichlet condition on $M = \mathbb{R} \times \mathbb{S}^{N-1}$

$$(3.47) \qquad \mathcal{V}_1^0|_{\rho=0} = 1.$$

Actually, from the proof of Proposition 3.6 and the modifications of Proposition 3.4 we do obtain an explicit formula for such $\mathcal{V}^0$. Indeed, from (3.30) and (3.32) for $k = 0$, $n = N$, $m = 0$, replacing $i\lambda$ by Q_0, we arrive at

$$\mathcal{V}^0(\tau) = \varphi(\tau) = \alpha^{-1}(1 - \tau^2)^{\frac{N}{4} - \frac{\gamma}{2}} \; {}_2F_1\left(\tfrac{\gamma}{p-1}, \tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}; \tfrac{N}{2}; \tau^2\right).$$

Finally, substitute in the relation between τ and ρ from (3.31) and set

$$(3.48) \qquad \rho^*(\rho) = (\mathcal{V}^0)^{\frac{1}{N/2-\gamma}}(\rho).$$

Then, recalling (3.7), for this ρ^* we have

$$E_{\bar{g}^*}(\rho^*) = (\rho^*)^{-\frac{N}{2}-\gamma-1} \left\{ -\Delta_{g^+} - \left(\tfrac{N^2}{4} - \gamma^2\right) \right\} (\mathcal{V}^0) = (\rho^*)^{-(1+2\gamma)} \left(\tfrac{4\rho}{4+\rho^2}\right)^2 Q_0^2,$$

as desired. Here we have used the scattering equation for $\mathcal{V}^0$ from (3.36) and the fact that $\mathcal{V}^0$ does not depend on the variable t.

To show monotonicity, denote $\eta := \left(\tfrac{4-\rho^2}{4+\rho^2}\right)^2$ for $\eta \in (0,1)$. It is enough to check that

$$f(\eta) := (1-\eta)^{\frac{N-2\gamma}{4}} \; {}_2F_1\left(\tfrac{\gamma}{p-1}, \tfrac{N-2\gamma}{2} - \tfrac{\gamma}{p-1}; \tfrac{N}{2}; \eta\right)$$

is monotone with respect to η. From properties (10.3) and (10.4) of the Hypergeometric function and the possible values for p in (1.9) we can assert that

$$
\begin{aligned}
\frac{d}{d\eta} f(\eta) &= \frac{d}{d\eta} \left((1-\eta)^{-\frac{N-2\gamma}{4} + \frac{\gamma}{p-1}} (1-\eta)^{\frac{N-2\gamma}{2} - \frac{\gamma}{p-1}} \, _2F_1 \left(\frac{N-2\gamma}{2} - \frac{\gamma}{p-1}, \frac{\gamma}{p-1}; \frac{N}{2}; \eta \right) \right) \\
&= \left(\frac{N-2\gamma}{4} - \frac{\gamma}{p-1} \right) (1-\eta)^{-\frac{N-2\gamma}{4} + \frac{\gamma}{p-1} - 1} (1-\eta)^{\frac{N-2\gamma}{2} - \frac{\gamma}{p-1}} \, _2F_1 \left(\frac{N-2\gamma}{2} - \frac{\gamma}{p-1}, \frac{\gamma}{p-1}; \frac{N}{2}; \eta \right) \\
&\quad - \frac{2}{N} \left(\frac{N-2\gamma}{2} - \frac{\gamma}{p-1} \right) \left(\frac{N}{2} - \frac{\gamma}{p-1} \right) (1-\eta)^{\frac{N-2\gamma}{2} - \frac{\gamma}{p-1} - 1} \, _2F_1 \left(\frac{N-2\gamma}{2} - \frac{\gamma}{p-1} + 1, \frac{\gamma}{p-1}; \frac{N}{2} + 1; \eta \right) \\
&< 0.
\end{aligned}
$$

$\square$

Remark 3.8. For the Neumann condition, note that, by construction,

$$
\tag{3.49} \tilde{P}_\gamma^{g_0}(1) = d_\gamma \mathcal{V}_2^0|_{\rho=0},
$$

while from (3.34) and the definition of $A_{N,p,\gamma}$ from (1.10),

$$
\tilde{P}_\gamma^{g_0}(1) = r^{\frac{2\gamma}{p-1}p}(-\Delta_{\mathbb{R}^N})^\gamma (r^{-\frac{2\gamma}{p-1}}) = A_{N,p,\gamma}.
$$

The last result in this section shows that the scattering problem for $\mathcal{V}$ (3.36) can be transformed into a new extension problem as in Proposition 3.2, and whose Dirichlet-to-Neumann operator is precisely $\tilde{P}_\gamma^{g_0}$. For this we will introduce the new metric on $\mathbb{R}^N \setminus \{0\}$

$$
\tag{3.50} \bar{g}^* = (\rho^*)^2 g^+,
$$

where ρ^* is the defining function defined in (3.44), and let us denote

$$
\tag{3.51} V^* = (\rho^*)^{-(N/2-\gamma)} \mathcal{V}.
$$

Proposition 3.9. *Let v be a smooth function on $M = \mathbb{R} \times \mathbb{S}^{N-1}$. The extension problem*

$$
\tag{3.52}
\begin{cases}
- \operatorname{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma} \nabla_{\bar{g}^*} V^*) - (\rho^*)^{-(1+2\gamma)} \left(\frac{4\rho}{4+\rho^2} \right)^2 2Q_0 \, \partial_t V^* = 0 & \text{in } (X, \bar{g}^*), \\
V^*|_{\rho=0} = v & \text{on } (M, g_0),
\end{cases}
$$

has a unique solution V^*. Moreover, for its Neumann data,

$$
\tag{3.53} \tilde{P}_\gamma^{g_0}(v) = -\tilde{d}_\gamma \lim_{\rho^* \to 0} (\rho^*)^{1-2\gamma} \partial_{\rho^*}(V^*) + A_{N,p,\gamma} v.
$$

Proof. The original scattering equation (3.1)-(3.2) was rewritten in terms of $\mathcal{V}$ (recall (3.35)) as (3.36)-(3.37) with Dirichlet condition $\mathcal{V}_1|_{\rho=0} = v$. Let us rewrite this equation into the more familiar form of Proposition 3.2. We follow the arguments in [25]; the difference comes from some additional terms that appear when changing to $\mathcal{V}$.

First use the definition of the classical conformal Laplacian for g^+ (that has constant scalar curvature $R_{g^+} = -N(N+1)$),

$$
P_1^{g^+} = -\Delta_{g^+} - \frac{N^2-1}{4},
$$

and the conformal property of this operator (3.5) to assure that

$$
P_1^{g^+}(\mathcal{V}) = (\rho^*)^{\frac{N+3}{2}} P_1^{\bar{g}}((\rho^*)^{-\frac{N-1}{2}} \mathcal{V}).
$$

Using (3.51) we can rewrite equation (3.36) in terms of V^* as

$$
P_1^{\bar{g}}((\rho^*)^{\frac{1-2\gamma}{2}} V^*) + (\rho^*)^{\frac{-3-2\gamma}{2}} \left\{ \left(\frac{4+\rho^2}{4\rho} \right)^{-2} \left(-2Q_0 \, \partial_t V^* - Q_0^2 V^* \right) + \left(\gamma^2 - \frac{1}{4} \right) V^* \right\} = 0
$$

or, equivalently, using that for $\varrho := \rho^{\frac{1-2\gamma}{2}}$,

$$\varrho \Delta_{\bar{g}^*}(\varrho V) = \mathrm{div}_{\bar{g}^*}(\varrho^2 \nabla_{\bar{g}^*} V) + \varrho V \Delta_{\bar{g}^*}(\varrho),$$

we have

$$-\mathrm{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma}\nabla_{\bar{g}^*}V^*) + E_{\bar{g}^*}(\rho^*)V^* + (\rho^*)^{-(1+2\gamma)}\left(\frac{4+\rho^2}{2\rho}\right)^{-2}\left(-2Q_0\,\partial_t V^* - Q_0^2\,V^*\right) = 0,$$

with $E_{\bar{g}^*}(\rho^*)$ defined as in (3.7). Finally, note that the defining function ρ^* was chosen as in Lemma 3.7. This yields (3.52).

For the boundary conditions let us recall the asymptotics (3.46). The Dirichlet condition follows directly from (3.3) and the asymptotics. For the Neumann condition, we recall the definition of ρ^* from (3.48), so

$$V^* = (\rho^*)^{-\frac{N}{2}+\gamma}\mathcal{V} = \frac{\mathcal{V}}{\mathcal{V}^0} = \frac{\mathcal{V}_1 + \rho^{2\gamma}\mathcal{V}_2}{\mathcal{V}_1^0 + \rho^{2\gamma}\mathcal{V}_2^0},$$

and thus

$$-\tilde{d}_\gamma \lim_{\rho \to 0} \rho^{1-2\gamma}\partial_\rho V^* = d_\gamma \left(\mathcal{V}_2\mathcal{V}_1^0 - \mathcal{V}_1\mathcal{V}_2^0\right)\big|_{\rho=0} = \tilde{P}_\gamma^{g_0}v - A_{N,p,\gamma}v,$$

where we have used (3.38) and (3.39) for $\mathcal{V}$, and (3.47) and (3.49) for $\mathcal{V}^0$. This completes the proof of the Proposition. □

4. New ODE methods for non-local equations

In this section we use the conformal properties developed in the previous section to study positive singular solutions to equation

$$(4.1) \qquad\qquad (-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma}u^p \text{ in } \mathbb{R}^N \setminus \{0\}.$$

The first idea is, in the notation of Section 3.4, to set $v = r^{\frac{2\gamma}{p-1}}u$ and rewrite this equation as

$$(4.2) \qquad\qquad \tilde{P}_\gamma^{g_0}(v) = A_{N,p,\gamma}v^p, \quad \text{in } \mathbb{R} \times \mathbb{S}^{N-1},$$

and to consider the projection over spherical harmonics in $\mathbb{S}^{N-1}$,

$$\tilde{P}_\gamma^m(v_m) = A_{N,p,\gamma}(v_m)^p, \quad \text{for } v = v(t),$$

While in Proposition 3.6 we calculated the Fourier symbol for $\tilde{P}_\gamma^m$, now we will write it as an integro-differential operator for a well behaved convolution kernel. The advantage of this formulation is that immediately yields regularity for v_m as in [30].

Now we look at the $m = 0$ projection, which corresponds to finding radially symmetric singular solutions to (4.1). This is a non-local ODE for $u = u(r)$. In the second part of the section we define a suitable Hamiltonian quantity in conformal coordinates in the spirit a classical second order ODE.

4.1. **The kernel.** We consider first the projection $m = 0$. Following the argument in [30], one can use polar coordinates to rewrite $\tilde{P}_\gamma^0$ as an integro-differential operator with a new convolution kernel. Indeed, polar coordinates $x = (r, \theta)$ and $\bar{x} = (\bar{r}, \bar{\theta})$ in the definition of the fractional Laplacian (1.2) give

$$(-\Delta_{\mathbb{R}^N})^\gamma u(x) = k_{N,\gamma}P.V.\int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{r^{-\frac{2\gamma}{p-1}}v(r) - \bar{r}^{-\frac{2\gamma}{p-1}}v(\bar{r})}{|r^2 + \bar{r}^2 + 2r\bar{r}\langle\theta,\bar{\theta}\rangle|^{\frac{N+2\gamma}{2}}}\,\bar{r}^{N-1}\,d\bar{r}\,d\bar{\theta}.$$

After the substitutions $\bar{r} = rs$ and $v(r) = (1 - s^{-\frac{2\gamma}{p-1}})v(r) + s^{-\frac{2\gamma}{p-1}}v(r)$, and recalling the definition for $\tilde{P}_\gamma^0$ from (3.34) we have

$$\tilde{P}_\gamma^0(v) = k_{N,\gamma}P.V.\int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{s^{-\frac{2\gamma}{p-1}+N-1}(v(r) - v(rs))}{|1 + s^2 - 2s\langle\theta,\bar{\theta}\rangle|^{\frac{N+2\gamma}{2}}}\,ds\,d\bar{\theta} + Cv(r),$$

where

$$C = k_{N,\gamma} P.V. \int_0^\infty \int_{\mathbb{S}^{N-1}} \frac{(1 - s^{-\frac{2\gamma}{p-1}})s^{N-1}}{|1 + s^2 - 2s\langle \theta, \bar{\theta}\rangle|^{\frac{N+2\gamma}{2}}} \, ds \, d\bar{\theta}.$$

Using the fact that $v \equiv 1$ is a solution, one gets that $C = A_{N,p,\gamma}$. Finally, the change of variables $r = e^{-t}$, $\bar{r} = e^{-t'}$ yields

$$(4.3) \qquad \tilde{P}_\gamma^0(v)(t) = P.V. \int_{\mathbb{R}} \tilde{\mathcal{K}}_0(t - t')[v(t) - v(t')] \, dt' + A_{N,p,\gamma} v(t)$$

for the convolution kernel

$$(4.4) \quad \tilde{\mathcal{K}}_0(t) = \int_{\mathbb{S}^{N-1}} \frac{k_{N,\gamma} e^{-(\frac{2\gamma}{p-1} - N)t}}{|1 + e^{2t} - 2e^t \langle \theta, \bar{\theta}\rangle|^{\frac{N+2\gamma}{2}}} \, d\bar{\theta} = c\, e^{-(\frac{2\gamma}{p-1} - \frac{N-2\gamma}{2})t} \int_0^\pi \frac{(\sin \phi_1)^{N-2}}{(\cosh t - \cos \phi_1)^{\frac{N+2\gamma}{2}}} \, d\phi_1,$$

where ϕ_1 is the angle between θ and $\bar{\theta}$ in spherical coordinates, and c is a positive constant that only depends on N and γ. From here we have the explicit expression

$$(4.5) \qquad \tilde{\mathcal{K}}_0(t) = c\, e^{-(\frac{2\gamma}{p-1} - \frac{N-2\gamma}{2})t} e^{-\frac{N+2\gamma}{2}|t|} \, {}_2F_1\left(\frac{N+2\gamma}{2}, 1 + \gamma; \frac{N}{2}; e^{-2|t|}\right),$$

for a different constant c. As in [30], one can calculate its asymptotic behavior, and we refer to this paper for details:

Lemma 4.1. *The kernel $\tilde{\mathcal{K}}_0(t)$ is decaying as $t \to \pm\infty$. More precisely,*

$$\tilde{\mathcal{K}}_0(t) \sim \begin{cases} |t|^{-1-2\gamma} & \text{as } |t| \to 0, \\ e^{-(N - \frac{2\gamma}{p-1})|t|} & \text{as } t \to -\infty, \\ e^{-\frac{2p\gamma}{p-1}|t|} & \text{as } t \to +\infty. \end{cases}$$

Remark 4.2. Note that (4.4) is a special case of the Funk-Hecke formula for the $m = 0$ spherical harmonic (see [69], pages 29-30).

We would like to obtain analogous results for any projection $m = 1, 2, \ldots$. The main result in this section is that one also has (4.3) for any projection $\tilde{P}_\gamma^m$. However, we have not been able to use the previous argument (in the spirit of the Funk-Hecke formula) and, instead, we develop a new approach using conformal geometry and the special defining function ρ^* from Proposition 3.2.

Set $Q_0 = \frac{2\gamma}{p-1} - \frac{N-2\gamma}{2}$. In the notation of Proposition 3.6 we have:

Proposition 4.3. *For the m-th projection of the operator $\tilde{P}_\gamma^{g_0}$ we have the expression*

$$\tilde{P}_\gamma^m(v_m)(t) = \int_{\mathbb{R}} \tilde{\mathcal{K}}_m(t - t')[v_m(t) - v_m(t')] \, dt' + A_{N,p,\gamma} v_m(t),$$

for a convolution kernel $\tilde{\mathcal{K}}_m$ on $\mathbb{R}$ with the asymptotic behavior

$$\tilde{\mathcal{K}}_m(t) \sim \begin{cases} |t|^{-1-2\gamma} & \text{as } |t| \to 0, \\ e^{-\left(1 + \gamma + \sqrt{(\frac{N-2}{2})^2 + \mu_m} + Q_0\right)t} & \text{as } t \to +\infty, \\ e^{\left(1 + \gamma + \sqrt{(\frac{N-2}{2})^2 + \mu_m} - Q_0\right)t} & \text{as } t \to -\infty. \end{cases}$$

Proof. We first consider the case that $p = \frac{N+2\gamma}{N-2\gamma}$ so that $Q_0 = 0$, and look at the operator $P_\gamma^{g_0}(w)$ from Proposition 3.4. Let ρ^* be the new defining function from Proposition 3.2 and write a new extension problem for w in the corresponding metric $\bar{g}^*$. In this particular case, we can use (3.44) to write

$$\rho^*(\rho) = \left[\alpha^{-1}\left(\frac{4\rho}{4+\rho^2}\right)^{\frac{N-2\gamma}{2}} \, {}_2F_1\left(\frac{N-2\gamma}{4}, \frac{N-2\gamma}{4}, \frac{N}{2}, \left(\frac{4-\rho^2}{4+\rho^2}\right)^2\right)\right]^{\frac{2}{N-2\gamma}}, \qquad \alpha = \frac{\Gamma(\frac{N}{2})\Gamma(\gamma)}{\Gamma(\frac{N}{4} + \frac{\gamma}{2})^2}.$$

The extension problem for $\bar{g}^*$ is

$$\begin{cases} -\operatorname{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma}\nabla_{\bar{g}^*}W^*) = 0 & \text{in } (X, \bar{g}^*), \\ \hspace{3cm} W^*|_{\rho=0} = w & \text{on } (M, g_0); \end{cases}$$

notice that it does not have a zero-th order term. Moreover, for the Neumann data,

$$P_\gamma^{g_0}(w) = -\tilde{d}_\gamma \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\partial_{\rho^*}(W^*) + \Lambda_{N,\gamma}w.$$

From the proof of Proposition 3.2 we know that $W^* = (\rho^*)^{-(N/2-\gamma)}\mathcal{W}$, where $\mathcal{W}$ is the solution to (3.18). Taking the projection over spherical harmonics, and arguing as in the proof of Proposition 3.4, we have that $\widehat{\mathcal{W}_m}(\tau,\xi) = \widehat{w_m}(\xi)\,\varphi(\tau)$, and $\varphi = \varphi_\xi^{(m)}$ is given in (3.19). Let us undo all the changes of variable, but let us keep the notation $\varphi(\rho^*) = \varphi_\xi^{(m)}(\tau)$.

Taking the inverse Fourier transform, we obtain a Poisson formula

$$W_m^*(\rho^*, t) = \int_{\mathbb{R}} \mathcal{P}_m(\rho^*, t-t')w_m(t')\,dt',$$

where

$$\mathcal{P}_m(\rho^*, t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\rho^*)^{-(N/2-\gamma)}\varphi(\rho^*)e^{i\xi t}\,d\xi.$$

Note that, by construction, $\int_{\mathbb{R}}\mathcal{P}_m(\rho^*, t)\,dt = 1$ for all ρ^*. Now we calculate

$$\lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\partial_{\rho^*}(W_m^*) = \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\frac{W_m^*(\rho^*, t) - W_m^*(0, t)}{\rho^*}$$

$$= \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\int_{\mathbb{R}}\frac{\mathcal{P}_m(\rho^*, t-t')}{\rho^*}[w_m(t') - w_m(t)]\,dt'.$$

This implies that

$$(4.6) \hspace{2cm} P_\gamma^m(w_m)(t) = \int_{\mathbb{R}}\mathcal{K}_m(t-t')[w_m(t) - w_m(t')]\,dt' + \Lambda_{N,\gamma}w_m(t),$$

where the convolution kernel is defined as

$$\mathcal{K}_m(t) = \tilde{d}_\gamma \lim_{\rho^*\to 0}(\rho^*)^{1-2\gamma}\frac{\mathcal{P}_m(\rho^*, t)}{\rho^*}.$$

If we calculate this limit, the precise expression for φ from (3.19) yields that

$$\mathcal{K}_m(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}(\Theta_\gamma^m(\xi) - \Lambda_{N,\gamma})e^{i\xi t}\,d\xi, \quad \mathcal{K}_m(-t) = \mathcal{K}_m(t).$$

which, of course, agrees with Proposition 3.4.

The asymptotic behavior for the kernel follows from the arguments in Section 6, for instance. In particular, the limit as $t \to 0$ is an easy calculation since Stirling's formula implies that $\Theta_\gamma^m(\xi) \sim |\xi|^{2\gamma}$ as $\xi \to \infty$. For the limit as $|t| \to \infty$ we use that the first pole of the symbol happens at $\pm i(1 + \gamma + \sqrt{(\frac{N-2}{2})^2 + \mu_m})$ so it extends analytically to a strip that contains the real axis. We have:

$$\mathcal{K}_m(t) \sim \begin{cases} |t|^{-1-2\gamma} & \text{as } |t| \to 0, \\ e^{-\left(1+\gamma+\sqrt{(\frac{N-2}{2})^2+\mu_m}\right)|t|} & \text{as } t \to \pm\infty. \end{cases}$$

Now we move on to $\tilde{P}_\gamma^{g_0}(v)$, whose symbol is calculated in Proposition 3.6. Recall that under the change $w(t) = r^{-Q_0}v(t)$, we have

$$\tilde{P}_\gamma^{g_0}(v) = e^{-Q_0 t}P_\gamma^{g_0}(e^{Q_0 t}v).$$

From (4.6), if we split $e^{Q_0 t} v_m(t) = (e^{Q_0 t} - e^{Q_0 t'}) v_m(t) + e^{Q_0 t'} v_m(t)$, then

$$\tilde{P}_\gamma^{g_0}(v)(t) = Cv(t) + \int_{\mathbb{R}} \tilde{\mathcal{K}}_m(t - t')(v_m(t) - v_m(t')) \, dt'$$

for the kernel

$$\tilde{\mathcal{K}}_m(t) = \mathcal{K}_m(t) e^{-Q_0 t} = \frac{1}{\sqrt{2\pi}} e^{-Q_0 t} \int_{\mathbb{R}} (\Theta_\gamma^m(\xi) - \Lambda_{N,\gamma}) e^{i\xi t} \, d\xi,$$

and the constant

$$C = \Lambda_{N,\gamma} + \int_{\mathbb{R}} \mathcal{K}_m(t - t')(1 - e^{Q_0(t' - t)}) \, dt'.$$

We have not attempted a direct calculation for the constant C. Instead, by noting that $v \equiv 1$ is an exact solution to the equation $\tilde{P}_\gamma^{g_0}(v) = A_{N,p,\gamma} v^p$, we have that $C = A_{N,p,\gamma}$, and this completes the proof of the proposition. $\qquad\square$

4.2. The Hamiltonian along trajectories.

Now we concentrate on positive radial solutions to (4.2). These satisfy

(4.7) $$\tilde{P}_\gamma^0(v) = A_{N,p,\gamma} v^p, \quad v = v(t).$$

We prove the existence of a Hamiltonian type quantity for (4.7), decreasing along trajectories when p is in the subcritical range, while this Hamiltonian remains constant in t for critical p. Monotonicity formulas for non-local equations in the form of a Hamiltonian have been known for some time ([16, 15, 43]). Our main innovation is that our formula (4.8) gives a precise analogue of the ODE local case (see Proposition 1 in [63], and the notes [78]), and hints what the phase portrait for v should be in the non-local setting. We hope to return to this problem elsewhere.

Theorem 4.4. *Fix $\gamma \in (0,1)$ and $p \in (\frac{N}{N-2\gamma}, \frac{N+2\gamma}{N-2\gamma})$. Let $v = v(t)$ be a solution to (4.7) and set V^* its extension from Proposition 3.9. Then, the Hamiltonian quantity*

(4.8)
$$H_\gamma^*(t) = \frac{A_{N,p,\gamma}}{\tilde{d}_\gamma} \left(-\frac{1}{2} v^2 + \frac{1}{p+1} v^{p+1} \right) + \frac{1}{2} \int_0^{\rho_0^*} (\rho^*)^{1-2\gamma} \left\{ -e_1^*(\partial_{\rho^*} V^*)^2 + e_2^*(\partial_t V^*)^2 \right\} d\rho^*$$
$$=: H_1(t) + H_2(t)$$

is decreasing with respect to t. In addition, if $p = \frac{N+2\gamma}{N-2\gamma}$, then $H_\gamma^*(t)$ is constant along trajectories. Here we write, using Lemma 3.7, ρ as a function of ρ^*, and

(4.9)
$$e^* = \left(\tfrac{\rho^*}{\rho} \right)^2 \left(1 + \tfrac{\rho^2}{4} \right) \left(1 - \tfrac{\rho^2}{4} \right)^{N-1},$$
$$e_1^* = \left(\tfrac{\rho^*}{\rho} \right)^{-2} e^*,$$
$$e_2^* = \left(\tfrac{\rho^*}{\rho} \right)^{-2} \left(1 + \tfrac{\rho^2}{4} \right)^{-2} e^*.$$

The constants $A_{N,p,\gamma}$ and $\tilde{d}_\gamma$ are given in (1.10) and (3.8), respectively.

Proof. In the notation of Proposition 3.9, let v be a function on $M = \mathbb{R} \times \mathbb{S}^{N-1}$ only depending on the variable $t \in \mathbb{R}$, and let V^* be the corresponding solution to the extension problem (3.52). Then $V^* = V^*(\rho, t)$. Use that

$$\text{div}_{\bar{g}^*}((\rho^*)^{1-2\gamma} \nabla_{\bar{g}^*} V^*) = \frac{1}{e^*} \partial_{\rho^*} \left(e^*(\rho^*)^{-(1+2\gamma)} \rho^2 \partial_{\rho^*} V^* \right) + (\rho^*)^{1-2\gamma} \left(\tfrac{\rho^*}{\rho} \right)^{-2} \left(1 + \tfrac{\rho^2}{4} \right)^{-2} \partial_{tt} V^*,$$

where $e^* = |\sqrt{g^*}|$ is given in (4.9), so equation (3.52) reads

$$-\partial_{\rho^*}\left(e^*\rho^2(\rho^*)^{-(1+2\gamma)}\partial_{\rho^*}V^*\right) - (\rho^*)^{1-2\gamma}e^*\left(\tfrac{\rho^*}{\rho}\right)^{-2}\left(1+\tfrac{\rho^2}{4}\right)^{-2}\partial_{tt}V^*$$
$$-(\rho^*)^{-(1+2\gamma)}e^*\left(\tfrac{4+\rho^2}{4\rho}\right)^{-2}2\left(-\tfrac{N-2\gamma}{2}+\tfrac{2\gamma}{p-1}\right)\partial_t V^* = 0.$$

We follow the same steps as in [29]: multiply this equation by $\partial_t V^*$ and integrate with respect to $\rho^* \in (0,\rho_0^*)$, where ρ_0^* is given in (3.45). Using integration by parts in the first term, the regularity of the function V^* at ρ_0^*, and the fact that $\frac{1}{2}\partial_t\left[(\partial_t V^*)^2\right] = \partial_{tt}V^*\partial_t V^*$ and $\frac{1}{2}\partial_t\left[(\partial_{\rho^*}V^*)^2\right] = \partial_{t\rho^*}(V^*)\partial_{\rho^*}V^*$, it holds

$$\lim_{\rho^*\to 0}\left(\partial_t(V^*)e^*(\rho^*)^{-(1+2\gamma)}\rho^2\partial_{\rho^*}V^*\right)$$
$$+\int_0^{\rho_0^*}\left[\tfrac{1}{2}e^*(\rho^*)^{-(1+2\gamma)}\rho^2\partial_t\left[(\partial_{\rho^*}V^*)^2\right]\right]d\rho^* - \int_0^{\rho_0^*}\left[\tfrac{1}{2}(\rho^*)^{1-2\gamma}e^*\left(\tfrac{\rho}{\rho^*}\right)^2\left(1+\tfrac{\rho^2}{4}\right)^{-2}\partial_t\left[(\partial_t V^*)^2\right]\right]d\rho^*$$
$$-\int_0^{\rho_0^*}\left[(\rho^*)^{-(1+2\gamma)}e^*\left(\tfrac{4\rho}{4+\rho^2}\right)^2 2\left(-\tfrac{N-2\gamma}{2}+\tfrac{2\gamma}{p-1}\right)[\partial_t V^*]^2\right]d\rho^*$$
$$= 0.$$

But, for the limit as $\rho^* \to 0$, we use (3.53) and equation (4.7),

$$\tilde{d}_\gamma\lim_{\rho^*\to 0}\left((\rho^*)^{-(1+2\gamma)}\rho^2 e^*\partial_t V^*\partial_{\rho^*}V^*\right) = \left[-\tilde{P}_\gamma^{g_0}v + A_{N,p,\gamma}v\right]\partial_t v = A_{N,p,\gamma}(v-v^p)\partial_t v$$
$$= A_{N,p,\gamma}\partial_t\left(\tfrac{1}{2}v^2 - \tfrac{1}{p+1}v^{p+1}\right).$$

Then, for $H(t)$ defined as in (4.8), we have

$$\partial_t[H(t)] = -2\int_0^{\rho_0^*}\left[(\rho^*)^{1-2\gamma}e^*\left(\tfrac{\rho}{\rho^*}\right)^{-2}\left(1+\tfrac{\rho^2}{4}\right)^{-2}\left(-\tfrac{N-2\gamma}{2}+\tfrac{2\gamma}{p-1}\right)[\partial_t V^*]^2\right]d\rho^* \le 0,$$

which proves the result. $\square$

5. The approximate solution

5.1. **Function spaces.** In this section we define the weighted Hölder space $\mathcal{C}^{2,\alpha}_{\mu,\nu}(\mathbb{R}^n\setminus\Sigma)$ tailored for this problem, following the notations and definitions in Section 3 of [64]. Intuitively, these spaces consist of functions which are products of powers of the distance to Σ with functions whose Hölder norms are invariant under homothetic transformations centered at an arbitrary point on Σ.

Despite the non-local setting, the local Fermi coordinates are still in use around each component Σ_i of Σ. When Σ_i is a point, these are simply polar coordinates around it. In case Σ_i is a higher dimensional sub-manifold, let $\mathcal{T}^i_\sigma$ be the tubular neighbourhood of radius σ around Σ_i. It is well known that $\mathcal{T}^i_\sigma$ is a disk bundle over Σ_i; more precisely, it is diffeomorphic to the bundle of radius σ in the normal bundle $\mathcal{N}\Sigma_i$. The Fermi coordinates will be constructed as coordinates in the normal bundle transferred to $\mathcal{T}^i_\sigma$ via such diffeomorphism. Let r be the distance to Σ_i, which is well defined and smooth away from Σ_i for small σ. Let also y be a local coordinate system on Σ_i and θ the angular variable on the sphere in each normal space $\mathcal{N}_y\Sigma_i$. We denote by $B_\sigma^{\mathcal{N}}$ the ball of radius σ in $\mathcal{N}_y\Sigma_i$. Finally we let x denote the rectangular coordinate in these normal spaces, so that $r = |x|$, $\theta = \frac{x}{|x|}$.

Let u be a function in this tubular neighbourhood and define

$$\|u\|^{\mathcal{T}^i_\sigma}_{0,\alpha,0} = \sup_{z\in\mathcal{T}^i_\sigma}|u| + \sup_{z,\tilde{z}\in\mathcal{T}^i_\sigma}\frac{(r+\tilde{r})^\alpha|u(z)-u(\tilde{z})|}{|r-\tilde{r}|^\alpha + |y-\tilde{y}|^\alpha + (r+\tilde{r})^\alpha|\theta-\tilde{\theta}|^\alpha},$$

where $z,\tilde{z}$ are two points in $\mathcal{T}^i_\sigma$ and $(r,\theta,y),(\tilde{r},\tilde{\theta},\tilde{y})$ are their Fermi coordinates.

We fix a $R > 0$ be large enough such that $\Sigma \subset B_{\frac{R}{2}}(0)$ in $\mathbb{R}^n$. Hereafter the letter z is reserved to denote a point in $\mathbb{R}^n \setminus \Sigma$. For notational convenience let us also fix a positive function $\varrho \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \Sigma)$ that is equal to the polar distance r in each $\mathcal{T}_\sigma^i$, and to $|z|$ in $\mathbb{R}^n \setminus B_R(0)$.

Definition 5.1. *The space $\mathcal{C}_0^{l,\alpha}(\mathbb{R}^n \setminus \Sigma)$ is defined to be the set of all $u \in \mathcal{C}^{l,\alpha}(\mathbb{R}^n \setminus \Sigma)$ for which the norm*

$$\|u\|_{l,\alpha,0} = \|u\|_{\mathcal{C}^{l,\alpha}(\Sigma_{\sigma/2}^c)} + \sum_{i=1}^{K} \sum_{j=0}^{l} \|\nabla^j u\|_{\mathcal{C}^{0,\alpha}(\mathcal{T}_\sigma^i)}$$

is finite. Here $\Sigma_{\sigma/2}^c = \mathbb{R}^n \setminus \bigcup_{i=1}^{K} \mathcal{T}_{\sigma/2}^i$.

Let us define a weighted Hölder space for functions having different behaviors near Σ and at ∞. With $R > 0$ fixed, for any $\mu, \nu \in \mathbb{R}$ we set

$$\mathcal{C}_\mu^{l,\alpha}(B_R \setminus \Sigma) = \{u = \varrho^\mu \bar{u} : \bar{u} \in \mathcal{C}_0^{l,\alpha}(B_R \setminus \Sigma)\},$$
$$\mathcal{C}_\nu^{l,\alpha}(\mathbb{R}^n \setminus B_R) = \{u = \varrho^\nu \bar{u} : \bar{u} \in \mathcal{C}_0^{l,\alpha}(\mathbb{R}^n \setminus B_R)\},$$

and thus we can define:

Definition 5.2. *The space $\mathcal{C}_{\mu,\nu}^{l,\alpha}(\mathbb{R}^n \setminus \Sigma)$ consists of all functions u for which the norm*

$$\|u\|_{\mathcal{C}_{\mu,\nu}^{l,\alpha}} = \sup_{B_R \setminus \Sigma} \|\varrho^{-\mu} u\|_{l,\alpha,0} + \sup_{\mathbb{R}^n \setminus B_R} \|\varrho^{-\nu} u\|_{l,\alpha,0}$$

is finite. The spaces $\mathcal{C}_{\mu,\nu}^{l,\alpha}(\mathbb{R}^N \setminus \{0\})$ and $\mathcal{C}_{\mu,\nu}^{l,\alpha}(\mathbb{R}^n \setminus \mathbb{R}^k)$ are defined similarly, in terms of the (global) Fermi coordinates (r, θ) or (r, θ, y) and the weights r^μ, r^ν.

Remark 5.3. From the definition of $\mathcal{C}_{\mu,\nu}^{l,\alpha}$, functions in this space are allowed to blow up like ϱ^μ near Σ_i and decay like ϱ^ν at ∞. Moreover, near Σ_i, their derivatives with respect to up to l-fold products of the vector fields $r\partial_r, r\partial_y, \partial_\theta$ blow up no faster than ϱ^μ while at ∞, their derivatives with respect to up to l-fold products of the vector fields $|z|\partial_i$ decay at least like ϱ^ν.

Remark 5.4. As it is customary in the analysis of fractional order operators, we write many times, with some abuse of notation, $\mathcal{C}_{\mu,\nu}^{2\gamma+\alpha}$.

5.2. Approximate solution with isolated singularities.
Let $\Sigma = \{q_1, \cdots, q_K\}$ be a prescribed set of singular points. In the next paragraphs we construct an approximate solution to

$$(-\Delta_{\mathbb{R}^N})^\gamma u = A_{N,p,\gamma} u^p \text{ in } \mathbb{R}^N \setminus \Sigma,$$

and check that it is indeed a good approximation in certain weighted spaces.

Let u_1 be the fast decaying solution to (4.1) that we constructed in Proposition 2.1. Now consider the following rescaling

$$(5.1) \qquad u_\varepsilon(x) = \varepsilon^{-\frac{2\gamma}{p-1}} u_1\left(\frac{x}{\varepsilon}\right) \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.$$

Choose χ_d to be a smooth cut-off function such that $\chi_d = 1$ if $|x| \le d$ and $\chi_d(x) = 0$ for $|x| \ge 2d$, where $d > 0$ is a positive constant such that $d < d_0 = \inf_{i \ne j}\{\text{dist}(q_i, q_j)/2\}$. Let $\bar{\varepsilon} = \{\varepsilon_1, \cdots, \varepsilon_K\}$ be a K-tuple of dilation parameters satisfying $c\varepsilon \le \varepsilon_i \le \varepsilon < 1$ for $i = 1, \ldots, K$. Now define our approximate solution by

$$\bar{u}_\varepsilon(x) = \sum_{i=1}^{K} \chi_d(x - q_i) u_{\varepsilon_i}(x - q_i).$$

Set also

$$(5.2) \qquad f_\varepsilon := (-\Delta_x)^\gamma \bar{u}_\varepsilon - A_{N,p,\gamma} \bar{u}_\varepsilon^p.$$

For the rest of the section, we consider the spaces $\mathcal{C}^{0,\alpha}_{\tilde{\mu},\tilde{\nu}}$, where

$$(5.3) \qquad -\frac{2\gamma}{p-1} < \tilde{\mu} < 2\gamma \quad \text{and} \quad -(n-2\gamma) < \tilde{\nu}.$$

Lemma 5.5. *There exists a constant C, depending on $d, \tilde{\mu}, \tilde{\nu}$ only, such that*

$$(5.4) \qquad \|f_\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}} \le C\varepsilon^{N-\frac{2p\gamma}{p-1}}.$$

Proof. Using the definition of $(-\Delta)^\gamma$ in $\mathbb{R}^N$, one has

$$(-\Delta_x)^\gamma(\chi_i u_{\varepsilon_i})(x - q_i) = k_{N,\gamma} P.V. \int_{\mathbb{R}^N} \frac{\chi_i(x-q_i)u_{\varepsilon_i}(x-q_i) - \chi_i(\tilde{x}-q_i)u_{\varepsilon_i}(\tilde{x}-q_i)}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x}$$
$$= \chi_i(x-q_i)(-\Delta_x)^\gamma u_{\varepsilon_i}(x-q_i)$$
$$+ k_{N,\gamma} P.V. \int_{\mathbb{R}^N} \frac{(\chi_i(x-q_i)-\chi_i(\tilde{x}-q_i))u_{\varepsilon_i}(\tilde{x}-q_i)}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x}$$

for each $i = 1, \dots, K$. Using the equation (2.1) satisfied by u_{ε_i} we have

$$f_\varepsilon(x) = A_{N,p,\gamma}\sum_{i=1}^K (\chi_i - \chi_i^p)u_{\varepsilon_i}^p(x-q_i) + k_{N,\gamma}\sum_{i=1}^K P.V. \int_{\mathbb{R}^N} \frac{(\chi_i(x-q_i)-\chi_i(\tilde{x}-q_i))u_{\varepsilon_i}(\tilde{x}-q_i)}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x}$$
$$=: I_1 + k_{N,\gamma}I_2.$$

Let us look first at the term I_1. It vanishes unless $|x - q_i| \in [d, 2d]$ for some $i = 1, \dots, K$. But then, one knows from the asymptotic behaviour of u_{ε_i} that

$$u_{\varepsilon_i}(x) = O\left(\varepsilon_i^{-\frac{2\gamma}{p-1}}\left|\frac{x-q_i}{\varepsilon_i}\right|^{-(N-2\gamma)}\right) = O(\varepsilon^{N-2\gamma-\frac{2\gamma}{p-1}})|x-q_i|^{-(N-2\gamma)},$$

so one has

$$I_1(x) \le C\varepsilon^{N-2\gamma-\frac{2\gamma}{p-1}} \quad \text{if } |x-q_i| \in [d, 2d].$$

For the second term $I_2 = I_2(x)$, we fix $i = 1, \dots, K$, and divide it into three cases: $x \in B_{d/2}(q_i)$, $x \in B_{2d}(q_i) \setminus B_{d/2}(q_i)$ and $x \in \mathbb{R}^N \setminus B_{2d}(q_i)$. In the first case, $x \in B_{d/2}(q_i)$, without loss of generality, assume that $q_i = 0$, so

$$I_2(x) = P.V. \int_{\mathbb{R}^N} \frac{(\chi_i(x)-\chi_i(\tilde{x}))u_{\varepsilon_i}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x}$$
$$= P.V\left[\int_{B_d(0)} \cdots + \int_{B_{2d}\setminus B_d(0)} \cdots + \int_{\mathbb{R}^N\setminus B_{2d}(0)} \cdots\right]$$
$$\lesssim \int_{\{d<|\tilde{x}|<2d\}} \frac{u_{\varepsilon_i}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma-2}}\,d\tilde{x} + \int_{\{|\tilde{x}|>2d\}} \frac{u_{\varepsilon_i}(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\,d\tilde{x}.$$

Hereafter "$\cdots$" carries its obvious meaning, replacing the previously written integrand. Using that $|x - \tilde{x}| \ge \frac{1}{2}|\tilde{x}|$ for $|\tilde{x}| > 2d$ when $|x| < \frac{d}{2}$ we easily estimate

$$I_2(x) \le O(\varepsilon^{N-\frac{2p\gamma}{p-1}}) + \int_{2d}^{\infty} \frac{u_{\varepsilon_i}(r)}{r^{1+2\gamma}}\,dr = O(\varepsilon^{N-\frac{2p\gamma}{p-1}}).$$

Next, if $x \in B_{2d}(q_i) \setminus B_{d/2}(q_i)$,

$$I_2(x) = P.V.\Big[\int_{B_{d/4}(q_i)} \cdots + \int_{B_{2d}(q_i)\setminus B_{d/4}(q_i)} \cdots + \int_{\mathbb{R}^N \setminus B_{2d}(q_i)} \cdots \Big]$$

$$= O\Big(\int_0^{\frac{d}{4\varepsilon}} \varepsilon^{N-\frac{2p\gamma}{p-1}} u_1(\tilde{x})\, d\tilde{x}\Big) + O\Big(\int_{B_{2d}(q_i)\setminus B_{d/4}(q_i)} \frac{\varepsilon^{N-\frac{2p\gamma}{p-1}}}{|x-\tilde{x}|^{N+2\gamma-2}}\, d\tilde{x}\Big)$$

$$+ O\Big(\int_{\mathbb{R}^N \setminus B_{2d}(q_i)} \frac{u_{\varepsilon_i}(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x}\Big)$$

$$= O(\varepsilon^{N-\frac{2p\gamma}{p-1}}).$$

Finally, if $x \in \mathbb{R}^N \setminus B_{2d}(q_i)$,

$$I_2(x) = P.V.\Big[\int_{B_d(q_i)} \cdots + \int_{B_{2d}\setminus B_d(q_i)} \cdots + \int_{\mathbb{R}^N \setminus B_{2d}(q_i)} \cdots \Big]$$

$$= O(\varepsilon^{N-\frac{2p\gamma}{p-1}}|x|^{-(N+2\gamma)}).$$

Combining all the estimates above we get a $\mathcal{C}^0_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}$ bound for a pair of weights satisfying (5.3). But passing to $\mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}$ is analogous and thus we obtain (5.4). $\qquad\square$

5.3. **Approximate solution in general case.** First note that our ODE argument for u_1 also yields a fast decaying positive solution to the general problem

$$(5.5) \qquad\qquad (-\Delta_{\mathbb{R}^n})^\gamma u = A_{N,p,\gamma} u^p \text{ in } \mathbb{R}^n \setminus \mathbb{R}^k.$$

that is singular along $\mathbb{R}^k$. Recall that we have set $N = n - k$.

Indeed, define $\tilde{u}_1(x,y) := u_1(x)$, where $z = (x,y) \in \mathbb{R}^{n-k} \times \mathbb{R}^k$, and use the Lemma below. For this reason, many times we will use indistinctly the notations $u_1(z)$ and $u_1(x)$. Moreover, after a straightforward rescaling, the constant $A_{N,p,\gamma}$ may be taken to be one.

Lemma 5.6. *If u is defined on $\mathbb{R}^N$ and we set $\tilde{u}(z) := u(x)$ in $\mathbb{R}^n$ in the notation above, then*

$$(-\Delta_{\mathbb{R}^n})^\gamma \tilde{u} = (-\Delta_{\mathbb{R}^N})^\gamma u.$$

Proof. We compute, first evaluating the y-integral,

$$(-\Delta_{\mathbb{R}^n})^\gamma \tilde{u}(z) = k_{n,\gamma} P.V. \int_{\mathbb{R}^n} \frac{\tilde{u}(z) - \tilde{u}(\tilde{z})}{|z-\tilde{z}|^{n+2\gamma}}\, d\tilde{z} = k_{n,\gamma} P.V. \int_{\mathbb{R}^k}\int_{\mathbb{R}^N} \frac{u(x)-u(\tilde{x})}{\big[|x-\tilde{x}|^2 + |y-\tilde{y}|^2\big]^{\frac{n+2\gamma}{2}}}\, d\tilde{x}\, d\tilde{y}$$

$$= k_{n,\gamma} P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\, d\tilde{x} \int_{\mathbb{R}^k} \frac{1}{(1+|\tilde{y}|^2)^{\frac{n+2\gamma}{2}}}\, d\tilde{y} = (-\Delta_{\mathbb{R}^N})^\gamma u(x).$$

Here we have used

$$(5.6) \qquad\qquad k_{n,\gamma}\int_{\mathbb{R}^k} \frac{1}{(1+|\tilde{y}|^2)^{\frac{n+2\gamma}{2}}}\, d\tilde{y} = k_{N,\gamma}.$$

(See Lemma A.1 and Corollary A.1 in [22]). $\qquad\square$

Now we turn to the construction of an approximate solution for (1.3). Let Σ be a k-dimensional compact sub-manifold in $\mathbb{R}^n$. We shall use local Fermi coordinates around Σ, as defined in Section 5.1. Let $\mathcal{T}_\sigma$ be the tubular neighbourhood of radius σ around Σ. For a point $z \in \mathcal{T}_\sigma$, denote it by $z = (x,y) \in \mathcal{N}\Sigma \times \Sigma$ where $\mathcal{N}\Sigma$ is the normal bundle of Σ. Let B a ball in $\mathcal{N}\Sigma$. We identify $\mathcal{T}_\sigma$ with $B \times \Sigma$. In these coordinates, the Euclidean metric is written as (see, for instance, [68])

$$|dz|^2 = \begin{pmatrix} |dx|^2 & O(r) \\ O(r) & g_\Sigma + O(r) \end{pmatrix},$$

where $|dx|^2$ is the standard flat metric in B and g_Σ the metric in Σ. The volume form reduces to

$$dz = dx\sqrt{\det g_\Sigma} + O(r).$$

In the ball B we use standard polar coordinates $r > 0$, $\theta \in \mathbb{S}^{N-1}$. In addition, near each $q \in \Sigma$, we will consider normal coordinates for g_Σ centered at q. A neighborhood of $\Sigma \ni q$ is then identified with a neighborhood of $\mathbb{R}^k \ni 0$ with the metric

$$g_\Sigma = |dy|^2 + O(|y|^2),$$

which yields the volume form

(5.7) $$dz = dx\,dy\,(1 + O(r) + O(|y|^2)).$$

Note that Σ is compact, so we can cover it by a finite number of small balls B.

As in the isolated singularity case, we define an approximate solution as follows:

$$\bar{u}_\varepsilon(x, y) = \chi_d(x)u_\varepsilon(x)$$

where χ_d is a cut-off function such that $\chi_d = 1$ if $|x| \leq d$ and $\chi_d(x) = 0$ for $|x| \geq 2d$. In the following we always assume $d < \frac{\sigma}{2}$. Let

$$f_\varepsilon := (-\Delta_{\mathbb{R}^n})^\gamma \bar{u}_\varepsilon - \bar{u}_\varepsilon^p.$$

Lemma 5.7. *Assume, in addition to (5.3), that $-\frac{2\gamma}{p-1} < \tilde{\mu} < \min\{\gamma - \frac{2\gamma}{p-1}, \frac{1}{2} - \frac{2\gamma}{p-1}\}$. Then there exists a positive constant C depending only on $d, \tilde{\mu}, \tilde{\nu}$ but independent of ε such that for $\varepsilon \ll 1$,*

(5.8) $$\|f_\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}} \leq C\varepsilon^q,$$

where $q = \min\{\frac{(p-3)\gamma}{p-1} - \tilde{\mu}, \frac{1}{2} - \gamma + \frac{(p-3)\gamma}{p-1} - \tilde{\mu}, N - \frac{2p\gamma}{p-1}\} > 0$.

Proof. Let us fix a point $z = (x, y) \in \mathcal{T}_\sigma$, i.e. $|x| < \sigma$. By the definition of the fractional Laplacian,

$$(-\Delta_z)^\gamma \bar{u}_\varepsilon(z) = k_{n,\gamma} P.V. \int_{\mathbb{R}^n} \frac{\bar{u}_\varepsilon(z) - \bar{u}_\varepsilon(\tilde{z})}{|z - \tilde{z}|^{n+2\gamma}}\,d\tilde{z}$$

$$= k_{n,\gamma}\left[P.V. \int_{\mathcal{T}_\sigma} \cdots + \int_{\mathcal{T}_\sigma^c} \cdots\right] =: I_1 + I_2.$$

Note that in this neighborhood we can write $\bar{u}_\varepsilon(z) := \bar{u}_\varepsilon(x)$.

For I_2, since $\bar{u}_\varepsilon(\tilde{x}) = 0$ when $\tilde{z} = (\tilde{x}, \tilde{y}) \in \mathcal{T}_\sigma^c$, one has

$$I_2 = k_{n,\gamma} \int_{\mathcal{T}_\sigma^c} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|z - \tilde{z}|^{n+2\gamma}}\,d\tilde{z} = \bar{u}_\varepsilon(x)\,k_{n,\gamma} \int_{\mathcal{T}_\sigma^c} \frac{1}{|z - \tilde{z}|^{n+2\gamma}}\,d\tilde{z} \leq C\bar{u}_\varepsilon(x),$$

so $I_2 = O(1)\bar{u}_\varepsilon(x)$ (the precise constant depends on σ). Next, for I_1, use normal coordinates $\tilde{y}$ in Σ centered at y in a neighborhood $\{|y - \tilde{y}| < \sigma_1\}$ for some σ_1 small but fixed. The constants will also depend on this σ_1. We have

$$I_1 = k_{n,\gamma}P.V. \int_{\mathcal{T}_\sigma} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|z - \tilde{z}|^{n+2\gamma}}\,d\tilde{z}$$

$$= k_{n,\gamma}P.V.\left[\int_{\{|y-\tilde{y}|\leq|x|^\beta\}\cap\mathcal{T}_\sigma} \cdots + \int_{\{\sigma_1>|y-\tilde{y}|>|x|^\beta\}\cap\mathcal{T}_\sigma} \cdots + \int_{\{|y-\tilde{y}|>\sigma_1\}\cap\mathcal{T}_\sigma} \cdots\right]$$

$$=: k_{n,\gamma}[I_{11} + I_{12} + I_{13}],$$

where $\beta \in (0, 1)$ is to be determined later. The main term will be I_{11}; let us calculate the other two. First, for I_{12} we recall the expansion of the volume form (5.7), and approximate $|z-\tilde{z}|^2 = |x-\tilde{x}|^2 + |y-\tilde{y}|^2$

and $d\tilde{z} = d\tilde{x}\,d\tilde{y}$ modulo lower order perturbations. Then

$$
\begin{aligned}
I_{12} &= \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\} \cap \mathcal{T}_\sigma} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|z - \tilde{z}|^{n+2\gamma}}\,d\tilde{z} \\
&= \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \int_{\{|x-\tilde{x}| \le |x|^\beta\} \cap \mathcal{T}_\sigma} \cdots + \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \int_{\{|x-\tilde{x}| > |x|^\beta\} \cap \mathcal{T}_\sigma} \cdots \\
&\lesssim \int_{\{|x-\tilde{x}| \le |x|^\beta\}} (\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})) \Big(\int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \frac{1}{|z-\tilde{z}|^{n+2\gamma}}\,d\tilde{y} \Big) d\tilde{x} \\
&\quad + \int_{\{|x-\tilde{x}| > |x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \Big(\int_{\{|\hat{y}| > \frac{|x|^\beta}{|x-\tilde{x}|}\}} \frac{1}{(1 + |\hat{y}|^2)^{\frac{n+2\gamma}{2}}}\,d\hat{y} \Big) d\tilde{x}.
\end{aligned}
$$

We estimate the above integrals in $d\tilde{y}$. For instance, for the first term, we have used that

$$
\begin{aligned}
\int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \frac{1}{|z-\tilde{z}|^{n+2\gamma}}\,d\tilde{y} &\le \int_{\{\sigma_1 > |y-\tilde{y}| > |x|^\beta\}} \frac{1}{|y-\tilde{y}|^{n+2\gamma}}\,d\tilde{y} \lesssim \int_{\{|y| \ge |x|^\beta\}} \frac{1}{|y|^{n+2\gamma}}\,dy \\
&\lesssim \int_{|x|^\beta}^{\infty} \frac{r^{k-1}}{r^{n+2\gamma}}\,dr \lesssim |x|^{-\beta(N+2\gamma)},
\end{aligned}
$$

which yields,

$$
I_{12} \lesssim \int_{\{|x-\tilde{x}| \le |x|^\beta\}} (\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x}))|x|^{-\beta(N+2\gamma)}\,d\tilde{x} + \int_{\{|x-\tilde{x}| > |x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}}\,d\tilde{x}.
$$

Now, since $|x - \tilde{x}| > |x|^\beta$ implies that $|\tilde{x}| > c_0|x|^\beta$ and $|x - \tilde{x}| \sim |\tilde{x}|$ for some $c_0 > 0$ independent of $|x|$ small,

$$
\begin{aligned}
I_{12} &\lesssim |x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) + |x|^{-\beta(N+2\gamma)} \int_{\{|x-\tilde{x}| \le |x|^\beta\}} \bar{u}_\varepsilon(\tilde{x})\,d\tilde{x} + \int_{\{|\tilde{x}| > c_0|x|^\beta\}} \frac{\bar{u}_\varepsilon(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\,d\tilde{x} \\
&\quad + \int_{\{|\tilde{x}| \ge c_0|x|^\beta\}} \frac{\bar{u}_\varepsilon(x)}{|\tilde{x}|^{N+2\gamma}}.
\end{aligned}
$$

We conclude, using the definition of $\bar{u}_\varepsilon$ and the rescaling (5.1), that

$$
\begin{aligned}
I_{12} &\lesssim |x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) + \varepsilon^{N - \frac{2\gamma}{p-1}}|x|^{-\beta(N+2\gamma)} \int_{\{|\tilde{x}| \le \frac{|x|^\beta}{\varepsilon}\}} u_1(\tilde{x})\,d\tilde{x} + \varepsilon^{-\frac{2p\gamma}{p-1}} \int_{\{|\tilde{x}| > \frac{|x|^\beta}{\varepsilon}\}} \frac{u_1(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\,d\tilde{x} \\
&\quad + |x|^{-2\gamma\beta}\bar{u}_\varepsilon(x).
\end{aligned}
$$

For I_{13}, one has

$$
I_{13} \le C \int_{\mathcal{T}_\sigma} |\bar{u}_\varepsilon(x)| + |\bar{u}_\varepsilon(\tilde{x})|\,d\tilde{x} \le C(\bar{u}_\varepsilon(x) + \varepsilon^{N - \frac{2p\gamma}{p-1}}(1 + |x|)^{-(N-2\gamma)}).
$$

We look now into the main term I_{11}, for which we need to be more precise,

$$
\begin{aligned}
I_{11} &= P.V. \int_{\{|y-\tilde{y}| \le |x|^\beta\}} \int_{\{|\tilde{x}| < \sigma\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|z - \tilde{z}|^{n+2\gamma}}\,d\tilde{z} \\
&= P.V. \Big[\int_{\{|y-\tilde{y}| \le |x|^\beta\}} \int_{\{|x-\tilde{x}| \le |x|^\beta\} \cap \{|\tilde{x}| < \sigma\}} \cdots + \int_{\{|y-\tilde{y}| \le |x|^\beta\}} \int_{\{|x-\tilde{x}| > |x|^\beta\} \cap \{|\tilde{x}| < \sigma\}} \cdots \Big] \\
&=: I_{111} + I_{112}.
\end{aligned}
$$

Let us estimate these two integrals. First, since for $|x|$ small, $|x - \tilde{x}| < |x|^\beta$ implies that $|\tilde{x}| < \sigma$, we have

$$k_{n,\gamma} I_{111} = k_{n,\gamma}\, P.V. \int_{\{|y-\tilde{y}|\leq|x|^\beta\}} \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|z - \tilde{z}|^{n+2\gamma}}\, d\tilde{z}$$

$$= k_{n,\gamma}\, P.V. \int_{\{|y-\tilde{y}|\leq|x|^\beta\}} \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{\left[|x - \tilde{x}|^2 + |y - \tilde{y}|^2\right]^{\frac{n+2\gamma}{2}}} (1 + O(|\tilde{x}|) + O(|y - \tilde{y}|))\, d\tilde{x}\, d\tilde{y}$$

$$= (1 + O(|x|^\beta)) k_{n,\gamma}\, P.V. \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \int_{\{|y|\leq\frac{|x|^\beta}{|x-\tilde{x}|}\}} \frac{1}{(1 + |y|^2)^{\frac{n+2\gamma}{2}}}\, dy\, d\tilde{x}$$

$$= (1 + O(|x|^\beta)) k_{n,\gamma}\, P.V. \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}}$$

$$\cdot \left[\int_{\mathbb{R}^k} \frac{1}{(1 + |y|^2)^{\frac{n+2\gamma}{2}}}\, dy - \int_{\{|y|>\frac{|x|^\beta}{|x-\tilde{x}|}\}} \frac{1}{(1 + |y|^2)^{\frac{n+2\gamma}{2}}}\, dy \right] d\tilde{x}.$$

Recall relation (5.6), then

$$k_{n,\gamma} I_{111} = (1 + O(|x|^\beta)) k_{N,\gamma} P.V. \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}}\, d\tilde{x}$$

$$+ O(1) \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \left(\frac{|x|^\beta}{|x - \tilde{x}|}\right)^{-(N+2\gamma)} d\tilde{x}$$

$$= (1 + O(|x|^\beta)) k_{N,\gamma} P.V. \int_{\mathbb{R}^N} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}}\, d\tilde{x}$$

$$+ O(1) \int_{\{|x-\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}}\, d\tilde{x}$$

$$+ O(1) \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \left(\frac{|x|^\beta}{|x - \tilde{x}|}\right)^{-(N+2\gamma)} d\tilde{x}.$$

Using the definition of the fractional Laplacian in $\mathbb{R}^N$,

$$k_{n,\gamma} I_{111} = (1 + O(|x|^\beta))(-\Delta_x)^\gamma \bar{u}_\varepsilon(x)$$

$$+ O(1) \int_{\{|x-\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}}\, d\tilde{x} + O(1)|x|^{-\beta(N+2\gamma)} \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} (\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x}))\, d\tilde{x}.$$

Now we use a similar argument to that of I_{12}, which yields

$$\int_{\{|x-\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}}\, d\tilde{x} \lesssim \int_{\{|\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(x)}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x} + \int_{\{|\tilde{x}|>|x|^\beta\}} \frac{\bar{u}_\varepsilon(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x}$$

$$\lesssim |x|^{-2\beta\gamma} \bar{u}_\varepsilon(x) + \varepsilon^{-\frac{2p\gamma}{p-1}} \int_{\{\frac{|x|^\beta}{\varepsilon} < |\tilde{x}| < \frac{\sigma}{\varepsilon}\}} \frac{u_1(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\, d\tilde{x}$$

and also,

$$|x|^{-\beta(N+2\gamma)} \int_{\{|x-\tilde{x}|\leq|x|^\beta\}} (\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x}))\, d\tilde{x}$$

$$\lesssim |x|^{-\beta(N+2\gamma)} \left[|x|^{\beta N} \bar{u}_\varepsilon(x) + \varepsilon^{N-\frac{2\gamma}{p-1}} \int_{\{|\tilde{x}|\leq\frac{|x|^\beta}{\varepsilon}\}} u_1(\tilde{x})\, d\tilde{x} \right]$$

$$\lesssim |x|^{-2\beta\gamma} \bar{u}_\varepsilon(x) + \varepsilon^{N-\frac{2\gamma}{p-1}} |x|^{-\beta(N+2\gamma)} \int_{\{|\tilde{x}|\leq\frac{|x|^\beta}{\varepsilon}\}} u_1(\tilde{x})\, d\tilde{x}.$$

In conclusion, one has

$$k_{n,\gamma}I_{111} = (1+O(|x|^\beta))(-\Delta_x)^\gamma \bar{u}_\varepsilon(x) + O(1)\Big[|x|^{-2\beta\gamma}\bar{u}_\varepsilon(x)$$

$$+ \varepsilon^{-\frac{2p\gamma}{p-1}}\int_{\{\frac{|x|^\beta}{\varepsilon}\leq|\tilde{x}|<\frac{\sigma}{\varepsilon}\}}\frac{u_1(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}\,d\tilde{x} + \varepsilon^{N-\frac{2\gamma}{p-1}}|x|^{-\beta(N+2\gamma)}\int_{\{|\tilde{x}|\leq\frac{|x|^\beta}{\varepsilon}\}}u_1(\tilde{x})\,d\tilde{x}\Big].$$

Next, for I_{112} we calculate similarly

$$I_{112} \lesssim \int_{\{|y-\tilde{y}|\leq|x|^\beta\}}\int_{\{|x-\tilde{x}|>|x|^\beta\}}\frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{\big[|x-\tilde{x}|^2 + |y-\tilde{y}|^2\big]^{\frac{n+2\gamma}{2}}}\,d\tilde{x}\,d\tilde{y}$$

$$= \int_{\{|x-\tilde{x}|>|x|^\beta\}}\frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\int_{\{|y|\leq\frac{|x|^\beta}{|x-\tilde{x}|}\}}\frac{1}{(1+|y|^2)^{\frac{n+2\gamma}{2}}}\,dy\,d\tilde{x}$$

$$= \int_{\{|x-\tilde{x}|>|x|^\beta\}}\frac{\bar{u}_\varepsilon(x) - \bar{u}_\varepsilon(\tilde{x})}{|x-\tilde{x}|^{N+2\gamma}}\Big(\frac{|x|^\beta}{|x-\tilde{x}|}\Big)^k\,d\tilde{x}$$

$$= |x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) + \varepsilon^{-k-\frac{2p\gamma}{p-1}}|x|^{\beta k}\int_{\{|\tilde{x}|>\frac{|x|^\beta}{\varepsilon}\}}\frac{u_1(\tilde{x})}{|\tilde{x}|^{n+2\gamma}}\,d\tilde{x}.$$

Combining the estimates for I_{111}, I_{112} and I_{12}, I_{13} we obtain

$$(-\Delta_z)^\gamma \bar{u}_\varepsilon(x) = (1 + O(|x|^\beta))(-\Delta_x)^\gamma \bar{u}_\varepsilon(x) + O(1)\Big[|x|^{-2\beta\gamma}\bar{u}_\varepsilon(x)$$

$$+ \varepsilon^{-\frac{2p\gamma}{p-1}}\int_{\{\frac{|x|^\beta}{\varepsilon}<|\tilde{x}|<\frac{\sigma}{\varepsilon}\}}\frac{u_1(\tilde{x})}{|\tilde{x}|^{N+2\gamma}}d\tilde{x} + \varepsilon^{N-\frac{2\gamma}{p-1}}|x|^{-\beta(N+2\gamma)}\int_{\{|\tilde{x}|<\frac{|x|^\beta}{\varepsilon}\}}u_1(\tilde{x})\,d\tilde{x}$$

$$+ \varepsilon^{-k-\frac{2p\gamma}{p-1}}|x|^{\beta k}\int_{\{|\tilde{x}|>\frac{|x|^\beta}{\varepsilon}\}}\frac{u_1(\tilde{x})}{|\tilde{x}|^{n+2\gamma}}\,d\tilde{x} + O(\varepsilon^{N-\frac{2p\gamma}{p-1}}(1+|x|)^{-(N-2\gamma)})\Big]$$

$$= (1 + O(|x|^\beta))(-\Delta_x)^\gamma \bar{u}_\varepsilon(x) + O(1)|x|^{-2\beta\gamma}\bar{u}_\varepsilon(x) + \mathcal{R}_1.$$

In order to estimate $\mathcal{R}_1$ we use the asymptotic behavior of $u_1(x)$ at 0 and ∞. By direct computation one sees that

$$\mathcal{R}_1(x) = \begin{cases} |x|^{-\beta\frac{2p\gamma}{p-1}}, & \text{if } |x|^\beta < \varepsilon, \\ \varepsilon^{N-\frac{2p\gamma}{p-1}}|x|^{-\beta N}, & \text{if } |x|^\beta > \varepsilon. \end{cases}$$

The choice $\beta = \frac{1}{2}$ yields that $\mathcal{R}_1 = O(|x|^{-\gamma})\bar{u}_\varepsilon(x)$, and thus

$$(-\Delta_z)^\gamma \bar{u}_\varepsilon(z) = (1 + O(|x|^\beta))(-\Delta_x)^\gamma \bar{u}_\varepsilon(x) + O(1)|x|^{-\gamma}\bar{u}_\varepsilon(x).$$

Finally, recall that $\bar{u}_\varepsilon(x) = \chi_d(x)u_\varepsilon(x)$, then by the estimates in the previous subsection (5.4), one has

(5.9) $$|f_\varepsilon(z)| \lesssim |x|^{\frac{1}{2}}|(-\Delta_x)^\gamma \bar{u}_\varepsilon(x)| + |x|^{-\gamma}\bar{u}_\varepsilon(x) + \mathcal{E},$$

where the weighted norm of $\mathcal{E}$ can be bounded by $\varepsilon^{N-\frac{2p\gamma}{p-1}}$.

For $z \in \mathbb{R}^n \setminus \mathcal{T}_{\frac{\sigma}{2}}$, the estimate is similar to the isolated singularity case, we omit the details here. Then we may conclude

$$\|f_\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}} \leq C\varepsilon^q,$$

where $q = \min\big\{\frac{(p-3)\gamma}{p-1} - \tilde{\mu}, \frac{1}{2} - \gamma + \frac{(p-3)\gamma}{p-1} - \tilde{\mu}, N - \frac{2p\gamma}{p-1}\big\}$, and it is positive if $-\frac{2\gamma}{p-1} < \tilde{\mu} < \min\big\{\gamma - \frac{2\gamma}{p-1}, \frac{1}{2} - \frac{2\gamma}{p-1}\big\}$. $\qquad\square$

Remark 5.8. In general, in terms of the local Fermi coordinates (x, y) around a fixed $z_0 = (0, 0) \in \Sigma$, for $u \in \mathcal{C}^{\alpha+2\gamma}_{\tilde{\mu},\tilde{\nu}}(\mathbb{R}^n \setminus \Sigma)$, one has the following estimate:

$$(-\Delta)^\gamma u = (-\Delta_{\mathbb{R}^n \setminus \mathbb{R}^k})^\gamma u(x, y) + |x|^\tau \|u\|_*$$

for $|x| \ll 1, |y| \ll 1$, and some $\tau > \tilde{\mu} - 2\gamma$. Indeed, similar to the estimates in Lemma 5.7, except the main term in I_{111}, in the estimates, it suffices to control the terms $u(\tilde{x})$ by $\|u\|_* |x|^{\tilde{\mu}}$.

6. Hardy type operators with fractional Laplacian

Here we give a formula for the Green's function for the Hardy type operator in $\mathbb{R}^N$,

$$(6.1) \qquad L\phi := (-\Delta_{\mathbb{R}^N})^\gamma \phi - \frac{\kappa}{r^{2\gamma}} \phi,$$

where $\kappa \in \mathbb{R}$. In the notation of Section 3.2, after the conjugation (3.13) we may study the equivalent operator

$$\tilde{\mathcal{L}}w := e^{-\frac{N+2\gamma}{2}t} \mathcal{L}(e^{\frac{N-2\gamma}{2}t} w) = P_\gamma^{g_0} w - \kappa w \quad \text{on } \mathbb{R} \times \mathbb{S}^{N-1}$$

for $\phi = e^{\frac{N-2\gamma}{2}} w$. Consider the projections over spherical harmonics: for $m = 0, 1 \ldots$, let w_m be a solution to

$$(6.2) \qquad \tilde{\mathcal{L}}_m w := P_\gamma^m w_m - \kappa w_m = h_m \quad \text{on } \mathbb{R}.$$

Recall Proposition 3.4. Then, in Fourier variables, equation (6.2) simply becomes

$$(\Theta_\gamma^m(\xi) - \kappa)\hat{w}_m = \hat{h}_m.$$

The behavior of this equation depends on the zeroes of the symbol $\Theta_\gamma^m(\xi) - \kappa$. In any case, we can formally write

$$(6.3) \qquad w_m(t) = \int_{\mathbb{R}} \frac{1}{\Theta_\gamma^m(\xi) - \kappa} \hat{h}_m(\xi) e^{i\xi t} \, d\xi = \int_{\mathbb{R}} h_m(t') \mathcal{G}_m(t - t') \, dt',$$

where the Green's function for the problem is formally given by

$$(6.4) \qquad \mathcal{G}_m(t) = \int_{\mathbb{R}} e^{i\xi t} \frac{1}{\Theta_\gamma^m(\xi) - \kappa} \, d\xi.$$

Let us make this statement rigorous in the stable case (this is, below the Hardy constant (3.21)), by taking into account the poles that may occur in the above integral. For this, let us define

$$(6.5) \qquad A_m = \tfrac{1}{2} + \tfrac{\gamma}{2} + \tfrac{1}{2}\sqrt{\left(\tfrac{N}{2} - 1\right)^2 + \mu_m}, \quad B_m = \tfrac{1}{2} - \tfrac{\gamma}{2} + \tfrac{1}{2}\sqrt{\left(\tfrac{N}{2} - 1\right)^2 + \mu_m},$$

and observe that the symbol

$$\Theta_\gamma^m(\xi) = 2^{2\gamma} \frac{\left|\Gamma\left(A_m + \tfrac{\xi}{2}i\right)\right|^2}{\left|\Gamma\left(B_m + \tfrac{\xi}{2}i\right)\right|^2} = 2^{2\gamma} \frac{\Gamma\left(A_m + \tfrac{\xi}{2}i\right)\Gamma\left(A_m - \tfrac{\xi}{2}i\right)}{\Gamma\left(B_m + \tfrac{\xi}{2}i\right)\Gamma\left(B_m - \tfrac{\xi}{2}i\right)}$$

can be extended meromorphically to the complex plane; this extension will be denoted simply by

$$\Theta_m(z) := 2^{2\gamma} \frac{\Gamma\left(A_m + \tfrac{z}{2}i\right)\Gamma\left(A_m - \tfrac{z}{2}i\right)}{\Gamma\left(B_m + \tfrac{z}{2}i\right)\Gamma\left(B_m - \tfrac{z}{2}i\right)},$$

for $z \in \mathbb{C}$. We have:

Theorem 6.1. *Let $0 \le \kappa < \Lambda_{N,\gamma}$ and fix $m = 0, 1, \ldots$. Assume that the right hand side h_m in (6.2) satisfies*

$$(6.6) \qquad h_m(t) = \begin{cases} O(e^{-\delta t}) & \text{as } t \to +\infty, \\ O(e^{\delta_0 t}) & \text{as } t \to -\infty, \end{cases}$$

for some real constants δ, δ_0. It holds:

i. The function $\frac{1}{\Theta_m(z)-\kappa}$ is meromorphic in $z \in \mathbb{C}$. Its poles are located at points of the form $\tau_j \pm i\sigma_j$ and $-\tau_j \pm i\sigma_j$, where $\sigma_j > \sigma_0 > 0$ for $j = 1,\ldots,$ and $\tau_j \geq 0$ for $j = 0, 1, \ldots$. In addition, $\tau_0 = 0$, and $\tau_j = 0$ for j large enough. For such j, $\{\sigma_j\}$ is an increasing sequence with no accumulation points.

ii. If $\delta > 0$, $\delta_0 \geq 0$, then a particular solution of (6.2) can be written as

$$(6.7) \qquad w_m(t) = \int_{\mathbb{R}} h_m(t') \mathcal{G}_m(t - t')\, dt',$$

where

$$(6.8) \qquad \mathcal{G}_m(t) = d_0 e^{-\sigma_0 |t|} + \sum_{j=1}^{\infty} e^{-\sigma_j |t|}[d_j \cos(\tau_j |t|) + d_j' \sin(\tau_j |t|)],$$

for some constants d_j, d_j' given precisely in the proof. Moreover, $\mathcal{G}_m$ is an even $\mathcal{C}^\infty$ function when $t \neq 0$.

iii. Now assume only that $\delta + \delta_0 \geq 0$. If $\sigma_J < \delta < \sigma_{J+1}$ (and thus $\delta_0 > -\sigma_{J+1}$), then a particular solution is

$$(6.9) \qquad w_m(t) = \int_{\mathbb{R}} h_m(t') \widetilde{\mathcal{G}}_m(t - t')\, dt'$$

where

$$\widetilde{\mathcal{G}}_m(t) = \sum_{j=J+1}^{\infty} e^{-\sigma_j t}[d_j \cos(\tau_j t) + d_j' \sin(\tau_j t)], \quad \text{for} \quad t > 0.$$

$\widetilde{\mathcal{G}}_m$ is a $\mathcal{C}^\infty$ function when $t \neq 0$. In addition, w_m satisfies the asymptotic behavior

$$(6.10) \qquad w_m(t) = O(e^{-\delta t}) \quad as \quad t \to +\infty, \quad w_m(t) \sim \sum_{j=0}^{J} D_j(t) e^{-\sigma_j t} + O(e^{\delta_0 t}) \quad as \quad t \to -\infty,$$

for some functions $D_0(t), \ldots, D_J(t)$. In the case $\tau_j = 0$, we have

$$D_j(t) \to \tilde{D}_j := d_j \int_{\mathbb{R}} e^{\sigma_j t'} h_m(t')\, dt', \quad as \quad t \to -\infty,$$

while when $\tau_j \neq 0$,

$$D_j(t) = d_j \left[\cos(\tau_j t) \int_t^{+\infty} e^{\sigma_j t'} \cos(\tau_j t') h(t')\, dt' + \sin(\tau_j t) \int_t^{+\infty} e^{\sigma_j t'} \sin(\tau_j t') h(t')\, dt' \right]$$

$$+ d_j' \left[\sin(\tau_j t) \int_t^{+\infty} e^{\sigma_j t'} \cos(\tau_j t') h(t')\, dt' - \cos(\tau_j t) \int_t^{+\infty} e^{\sigma_j t'} \sin(\tau_j t') h(t')\, dt' \right].$$

Remark 6.2. All solutions of (6.2) are obtained from the particular solution w_m above by adding those of the homogeneous problem $\tilde{\mathcal{L}}_m w = 0$, which are of the form

$$w_{m,h}(t) = C_0^- e^{-\sigma_0 t} + C_0^+ e^{\sigma_0 t} + \sum_{j=1}^{\infty} e^{-\sigma_j t}[C_j^- \cos(\tau_j t) + C_j'^- \sin(\tau_j t)]$$

$$+ \sum_{j=1}^{\infty} e^{+\sigma_j t}[C_j^+ \cos(\tau_j t) + C_j'^+ \sin(\tau_j t)]$$

for some real constants $C_j^-, C_j^+, C_j'^-, C_j'^+$, $j = 0, 1, \ldots$.

We also look at the case when κ leaves the stability regime. In order to simplify the presentation, we only consider the projection $m = 0$ and the equation

$$\tilde{\mathcal{L}}_0 w = h. \tag{6.11}$$

In addition, we assume that only the first pole leaves the stability regime, which happens if $\Lambda_{N,\gamma} < \kappa < \Lambda'_{N,\gamma}$ for some $\Lambda'_{N,\gamma}$. Then, in addition to the poles above, we will have two real poles τ_0 and $-\tau_0$. Some study regarding $\Lambda'_{N,\gamma}$ will be given in the next section but we are not interested in its explicit formula.

Proposition 6.3. *Let $\Lambda_{N,\gamma} < \kappa < \Lambda'_{N,\gamma}$. Assume that h decays like $O(e^{-\delta t})$ as $t \to \infty$, and $O(e^{\delta_0 t})$ as $t \to -\infty$ for some real constants δ, δ_0. It holds:*

    i. The function $\frac{1}{\Theta_m(z)-\kappa}$ is meromorphic in $z \in \mathbb{C}$. Its poles are located at points of the form $\tau_j \pm i\sigma_j$ and $-\tau_j \pm i\sigma_j$, where $\tau_j, \sigma_j \geq 0$, for $j = 0, 1, \ldots$. In addition, $\sigma_0 = 0$, and $\tau_j = 0$ for j large enough. For such j, $\{\sigma_j\}$ is an increasing sequence with no accumulation points.

    ii. If $\delta > 0$, $\delta_0 > 0$, then a particular solution of (6.11) can be written as

$$w_0(t) = \int_{\mathbb{R}} h(t')\mathcal{G}_0(t - t') \, dt', \tag{6.12}$$

      where

$$\mathcal{G}_0(t) = d_0 \sin(\tau_0 t)\chi_{(-\infty,0)}(t) + \sum_{j=1}^{\infty} e^{-\sigma_j|t|}[d_j \cos(\tau_j t) + d'_j \sin(\tau_j|t|)]$$

      for some constants d_0, d_j, d'_j, $j = 1, \ldots$. Moreover, $\mathcal{G}_0$ is an even $\mathcal{C}^\infty$ function when $t \neq 0$.

    iii. The analogous statements to Theorem 6.1, iii., and Remark 6.2 hold.

Further study of fractional non-linear equations with critical Hardy potential has been done in [2, 34], for instance.

Before we start the proof of Theorem 6.1, let us give some preliminaries regarding the symbol the symbol Θ_m.

Remark 6.4. It is interesting to observe that

$$\Theta_m(z) = \Theta_m(-z).$$

Moreover, thanks to Stirling formula (expression 6.1.37 in [3])

$$\Gamma(z) \sim e^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}}, \quad \text{as} \quad |z| \to \infty \text{ in } |\arg z| < \pi, \tag{6.13}$$

one may check that for $\xi \in \mathbb{R}$,

$$\Theta_m(\xi) \sim |m + \xi i|^{2\gamma}, \quad \text{as} \quad |\xi| \to \infty, \tag{6.14}$$

and this limit is uniform in m. Here the symbol $\sim$ means that one can bound one quantity, above and below, by constant times the other. This also shows that, for fixed m, the behavior at infinity is the same as the one for the standard fractional Laplacian $(-\Delta)^\gamma$.

The following proposition uses this idea to study the behavior as $|t| \to 0$. Recall that the Green's function for the fractional Laplacian $(-\Delta_{\mathbb{R}})^\gamma$ in one space dimension is precisely

$$G(t) = |t|^{-(1-2\gamma)}.$$

We will prove that $\mathcal{G}_m$ has a similar behavior.

Proposition 6.5. *Let* $\gamma \in (0, 1/2)$. *Then*

$$\lim_{|t| \to 0} \frac{\mathcal{G}_m(t)}{|t|^{-(1-2\gamma)}} = c$$

for some positive constant c.

Proof. Indeed, recalling (6.14), we have

$$\lim_{|t| \to 0} \frac{\int_{\mathbb{R}} \frac{1}{\Theta_m(\xi) - \lambda} e^{i\xi t} d\xi}{|t|^{-(1-2\gamma)}} = \lim_{|t| \to 0} \int_{\mathbb{R}} \frac{1}{|t|^{2\gamma} \left[\Theta_m(\frac{\zeta}{t}) - \lambda\right]} e^{i\zeta} d\zeta$$

$$= \lim_{t \to 0} \left[\int_{\{|\zeta| > t^\delta\}} \cdots + \int_{\{|\zeta| \leq t^\delta\}} \cdots \right] =: \lim_{t \to 0} [I_1 + I_2]$$

for some $2\gamma < \delta < 1$.

For I_1, we use Stirling's formula (6.13) to estimate

$$I_1 \sim \int_{\{|\zeta| > t^\delta\}} \frac{\cos(\zeta)}{|\zeta|^{2\gamma}} d\zeta \to c \quad \text{as} \quad t \to 0,$$

while for I_2,

$$|I_2| \leq \int_{\{|\zeta| \leq t^\delta\}} \frac{1}{|t|^{2\gamma} \left[\Theta_m(0) - \lambda\right]} d\zeta \to 0 \quad \text{as} \quad t \to 0,$$

as desired. $\qquad\square$

Lemma 6.6. *Define the function* $\Phi(x, \xi) = 2^{2\gamma} \frac{\left|\Gamma\left(A_m + x + \frac{\xi}{2}i\right)\right|^2}{\left|\Gamma\left(B_m + x + \frac{\xi}{2}i\right)\right|^2}$. *Then:*

   i. *Fixed* $x > B_m$, $\Psi(x, \xi)$ *is a (strictly) increasing function of* $\xi > 0$.
   ii. $\Psi(x, 0)$ *is a (strictly) increasing function of* $x > 0$.

Proof. As in [29], section 7, one calculates using (10.9),

$$\partial_\xi (\log \Theta_m(\xi)) = \text{Im} \left\{ \psi\left(B_m + x + \tfrac{\xi}{2}i\right) - \psi\left(A_m + x + \tfrac{\xi}{2}i\right) \right\}$$

$$= c \, \text{Im} \sum_{l=0}^{\infty} \left(\frac{1}{l + A_m + x + \frac{\xi}{2}i} - \frac{1}{l + B_m + x + \frac{\xi}{2}i} \right) > 0,$$

as claimed. A similar argument yields the monotonicity in x. $\qquad\square$

Now we give the proof of Theorem 6.1. Before we consider the general case, let us study first when $\kappa = 0$, for which $\mathcal{G}_m$ can be computed almost explicitly. Fix $m = 0, 1, \ldots$. The poles of the function $\frac{1}{\Theta_m(z)}$ happen at points $z \in \mathbb{C}$ such that

$$\pm \frac{z}{2}i + B_m = -j, \quad \text{for } j \in \mathbb{N} \cup \{0\},$$

i.e, at points $\{\pm i\sigma_j\}$ for

(6.15) $$\sigma_j := 2(B_m + j), \quad j = 0, 1, \ldots.$$

Then the integral in (6.4) can be computed in terms of the usual residue formula. Define the region in the complex plane

(6.16) $$\Omega_+ = \{z \in \mathbb{C} : |z| < R, \text{Im } z > 0\}.$$

A standard contour integration along $\partial\Omega_+$ gives, as $R \to \infty$, that

$$\text{(6.17)} \qquad \mathcal{G}_m(t) = 2\pi i \sum_{j=0}^{\infty} \text{Res}\left(e^{izt}\frac{1}{\Theta_m(z)}, i\sigma_j\right) = 2\pi i \sum_{j=0}^{\infty} e^{-\sigma_j t} c_j,$$

where $c_j = c_j(m)$ is the residue of the function $\frac{1}{\Theta_m(z)}$ at the pole $i\sigma_j$. This argument is valid as long as the integral in the upper semicircle tends to zero as $R \to \infty$. This happens when $t > 0$ since $|e^{izt}| = e^{-t\,\text{Im}\,z}$. For $t < 0$, we need to modify the contour of integration to $\Omega_- = \{z \in \mathbb{C} : |z| < R, \text{Im}\,z < 0\}$, and we have that, for $t < 0$,

$$\mathcal{G}_m(t) = 2\pi i \sum_{j=0}^{\infty} c_j e^{\sigma_j t},$$

which of course gives that $\mathcal{G}_m$ is an even function in t. In any case $\mathcal{G}_m$ is exponentially decaying as $|t| \to \infty$ with speed given by the first pole $|\sigma_0| = 2B_m$.

In addition, recalling the formula for the residues of the Gamma function from (10.8), we have that

$$c_j = \frac{1}{2^{2\gamma}} \frac{\Gamma(2B_m + j)}{\Gamma(A_m - B_m - j)\Gamma(A_m + B_m + j)} \lim_{z \to i\sigma_j} \Gamma(B_m + \tfrac{z}{2}i)(z - i\sigma_j)$$

$$= \frac{2}{2^{2\gamma}} \frac{\Gamma\left(1 - \gamma + \sqrt{(\tfrac{n}{2} - 1)^2 + \mu_m} + j\right)}{\Gamma(\gamma - j)\Gamma\left(1 + \sqrt{(\tfrac{n}{2} - 1)^2 + \mu_m} + j\right)} \frac{-i(-1)^j}{j!}$$

for $j \geq 1$, which yields the (uniform) convergence of the series (6.8) by Stirling's formula (6.13).

Next, take a general $0 < \kappa < \Lambda_{n,\gamma}$. The function $e^{izt}\frac{1}{\Theta_m(z) - \kappa}$ is meromorphic in the complex plane $\mathbb{C}$. Moreover, if z is a root of $\Theta_m(z) = \kappa$, so are $-z$, $\bar{z}$ and $-\bar{z}$.

Let us check then that there are no poles on the real line. Indeed, the first statement in Lemma 6.6 implies that is enough to show that

$$\Theta_m(0) - \kappa > 0.$$

But again, from the second statement of the lemma, $\Theta_m(0) > \Theta_0(0)$, so we only need to look at the case $m = 0$. Finally, just note that $\Theta_0(0) = \Lambda_{N,\gamma} > \kappa$.

Next, we look for poles on the imaginary axis. For $\sigma > 0$, $\Theta_m(i\sigma) = \Psi(-\sigma, 0)$ and this function is (strictly) decreasing in σ. Moreover, $\Psi(0,0) = \Theta_m(0) = \Lambda_{N,\gamma} > \kappa$. Let $\sigma_0 \in (0, +\infty]$ be the first point where $\Theta_m(i\sigma_0) = \kappa$. Then $\pm i\sigma_0$ are poles on the imaginary axis. Moreover, the first statement of Lemma 6.6 shows that there are no other poles in the strip $\{z : |\text{Im}(z)| \leq \sigma_0\}$.

Denote the rest of the poles by $z_j := \tau_j + i\sigma_j$, $\tau_j - i\sigma_j$, $-\tau_j + i\sigma_j$ and $-\tau_j - i\sigma_j$, $j = 1, 2, \ldots$. Here we take $\sigma_j > \sigma_0 > 0$, $\tau_j \geq 0$. A detailed study of the poles is given in the Section 6.4. In particular, for large j, all poles lie there on the imaginary axis, and their asymptotic behavior is similar to that of (6.15).

Now we can complete the proof of Theorem 6.1. Since we have shown that there is a spectral gap σ_0 from the real line, it is possible to modify the contour of integration in (6.17) to prove a similar residue formula: for $t > 0$,

$$\mathcal{G}_m(t) = 2\pi i\, \text{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, i\sigma_0\right)$$

$$\text{(6.18)} \qquad + 2\pi i \sum_{j=1}^{\infty} \left[\text{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, \tau_j + i\sigma_j\right) + \text{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, -\tau_j + i\sigma_j\right)\right]$$

$$= 2\pi i c_0 e^{-\sigma_0 t} + 2\pi i \sum_{j=1}^{\infty} e^{-\sigma_j t}[c_j \cos(\tau_j t) + c_j' \sin(\tau_j t)].$$

Here $-i(c'_j + ic_j)/2$ is the residue of the function $\frac{1}{\Theta_m(z)-\kappa}$ at the point $\tau_j + i\sigma_j$ if $\tau_j \neq 0$ (if $\tau_j = 0$ then we write instead ic_j for this residue).

By the monotonicity properties of Θ_m, we see that

$$d_0 = 2\pi i c_0 = 2\pi i \operatorname{Res}\left(\frac{1}{\Theta_m(z)-\kappa}, i\sigma_0\right)$$

$$= 2\pi i \lim_{z\to i\sigma_0} \frac{z - i\sigma_0}{\Theta_m(z)-\kappa} = -2\pi \lim_{\zeta\to\sigma_0} \frac{\zeta - \sigma_0}{\Psi(-\frac{\zeta}{2},0)-\kappa} > 0.$$

Using the symmetries of the function Θ_m, it can be easily shown that c_j, c'_j are purely imaginary and that the residues at the points $\pm\tau_j + i\sigma_j$ (denoted by $r_\pm$) are related by $r_+ = -\overline{r_-}$. Moreover, the asymptotic behavior for this residue is calculated in (6.34); indeed, $c_j \sim Cj^{-2\gamma}$. The convergence of the series is guaranteed, and we have proved the desired expression (6.8) (for $t < 0$, $\mathcal{G}$ it is defined evenly).

We have produced a particular solution to (6.2), given explicitly by (6.7). Due to the presence of a non-trivial kernel, one has an infinite dimensional family of solutions. The main idea in the proof of *iii.* is to modify (6.7) by adding elements in the kernel in order to obtain a particular solution that decays as $O(e^{-\delta t})$ as $t \to +\infty$ (at the expense of worsening the behavior as $t \to -\infty$). We obtain what is known as a multipole expansion. We start with a simple lemma:

Lemma 6.7. *If $f_1(t) = O(e^{-a|t|})$ as $t \to \infty$, $f_2(t) = O(e^{-a_+ t})$ as $t \to +\infty$ and $f_2(t) = O(e^{a_- t})$ as $t \to -\infty$ for some $a, a_+ > 0$, $a_- > -a$, then*

$$f_1 * f_2(t) = O(e^{-\min\{a,a_+\}t}) \quad as \quad t \to +\infty.$$

Proof. Indeed, for $t > 0$,

$$|f_1 * f_2(t)| = \left|\int_{\mathbb{R}} f_1(t-t')f_2(t')\,dt'\right|$$

$$\lesssim \int_{-\infty}^{0} e^{-a(t-t')}e^{a_- t'}\,dt' + \int_{0}^{t} e^{-a(t-t')}e^{-a_+ t'}\,ds + \int_{t}^{+\infty} e^{a(t-t')}e^{-a_+ t'}\,dt'.$$

The lemma follows by straightforward computations. $\square$

Remark 6.8. It is interesting to observe that a_- is not involved in the decay as $t \to +\infty$. Moreover, by reversing the role of t and $-t$, it is possible to obtain the analogous statement for $t \to -\infty$ with the obvious modifications.

Let us assume the bounds (6.6). Suppose first that $0 < \delta \leq \sigma_0$. Then (6.7) is already our solution, since

$$|w_m(t)| = \left|\int_{\mathbb{R}} \mathcal{G}_m(t-t')h(t')\,dt'\right| \leq \int_{\mathbb{R}} O(e^{-\sigma_0|t-t'|})|h(t')|\,dt' = O(e^{-\delta t}).$$

Similarly as $t \to -\infty$, $w(t) = O(e^{\min\{\delta_0, \sigma_0\}t})$.

Fix $m = 0, 1, \ldots$, and drop the subindex m for simplicity. Now we take $\sigma_J < \delta \leq \sigma_{J+1}$ for some $J \geq 0$ and $\delta + \delta_0 > 0$ (and thus, $\delta_0 > -\sigma_{J+1}$). The idea is to change our Fourier transform in order to integrate on a different horizontal line $\zeta \in \mathbb{R} + i\vartheta$ for some $\vartheta > 0$. This is, for $w = w(t)$, set

$$\widetilde{w}(\zeta) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\zeta t}\,w(t)\,dt = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i(\xi+i\vartheta)t}\,w(t)\,dt = \hat{w}(\xi + i\vartheta),$$

whose inverse Fourier transform is

$$w(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}+i\vartheta} e^{i\zeta t}\,\widetilde{w}(\zeta)\,d\zeta,$$

Moreover, in the new variable $\zeta = \xi + i\vartheta$ we have that

$$\widetilde{h}(\zeta) = \widetilde{P^{(m)}(w)}(\zeta) = (\Theta_m(\zeta) - \kappa)\widetilde{w}(\zeta).$$

Inverting this symbol we obtain a particular solution

$$w(t) = \int_{\mathbb{R}+i\vartheta} \frac{1}{\Theta_m(\zeta) - \kappa}\widetilde{h}(\zeta)e^{i\zeta t}\,d\zeta = \int_{\mathbb{R}} h(t')\widetilde{\mathcal{G}}(t - t')\,dt',$$

where

$$\widetilde{\mathcal{G}}(t) = \int_{\mathbb{R}+i\vartheta} e^{i\zeta t}\frac{1}{\Theta_m(\zeta) - \kappa}\,d\zeta.$$

In order to evaluate this integral, for $t > 0$ we need to replace the contour of integration from (6.16) to $\partial\Omega_{\vartheta,+}$ for

$$\Omega_{\vartheta,+} = \{z \in \mathbb{C} : |z| < R,\, \operatorname{Im} z > \vartheta\},$$

in the case $\kappa = 0$, and with the obvious modifications, for $\kappa \neq 0$. We will take $\vartheta \in (\sigma_J, \delta)$. Letting $R \to \infty$, this yields, similarly to (6.18), that

$$\widetilde{\mathcal{G}}(t) = 2\pi i \sum_{j=J+1}^{\infty} \operatorname{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, \pm\tau_j + i\sigma_j\right) = 2\pi i \sum_{j=J+1}^{\infty} e^{-\sigma_j t}c_j, \quad \text{for} \quad t > 0,$$

where, as above, c_j is the residue of the function $\frac{1}{\Theta_m(z)-\kappa}$ at the pole $\tau_j + i\sigma_j$.

On the other hand, for $t < 0$, we need to change the contour of integration to $\partial\Omega_{\vartheta,-}$, where

$$\Omega_{\vartheta,-} = \{z \in \mathbb{C} : -R < \operatorname{Re}(z) < R,\, -R < \operatorname{Im}(z) < \vartheta\}.$$

We see again that, for $t < 0$, $|e^{izt}| = e^{-t\operatorname{Im} z}$, so the contour integral is well defined in the lower half-space $\{\operatorname{Im}(z) < 0\}$. However, for the region $\{\operatorname{Im}(z) > 0\} \cap \partial\Omega_{\vartheta,-}$ we need the additional estimate $\frac{1}{\Theta_m(z)-\kappa} = O(R^{-2\gamma})$, which follows directly from (6.14). Using the residue formula we obtain

$$\widetilde{\mathcal{G}}(t) = 2\pi i \left[\sum_{j=1}^{\infty}\operatorname{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, \pm\tau_j - i\sigma_j\right) + \sum_{j=0}^{J}\operatorname{Res}\left(e^{izt}\frac{1}{\Theta_m(z) - \kappa}, \pm\tau_j + i\sigma_j\right)\right]$$

$$= 2\pi i \sum_{j=1}^{\infty} e^{\sigma_j t}[c_j \cos(\tau_j t) + c_j' \sin(\tau_j t)] + 2\pi i \sum_{j=1}^{J} e^{-\sigma_j t}[c_j \cos(\tau_j t) + c_j' \sin(\tau_j t)] + 2\pi i c_0 e^{-\sigma_0 t},$$

for $t < 0$. In order to check the asymptotic behavior of w_m we write

$$w_m(t) = \left[\int_{-\infty}^{t} + \int_{t}^{+\infty}\right]\widetilde{\mathcal{G}}(t - t')h(t')\,dt' = W_1(t) + W_2(t),$$

where

$$W_1(t) = \sum_{j=J+1}^{\infty} e^{-\sigma_j t}\int_{-\infty}^{t} e^{\sigma_j t'}[d_j \cos(\tau_j(t - t')) + d_j' \sin(\tau_j(t - t'))]h(t')\,dt',$$

$$W_2(t) = \sum_{j=1}^{\infty} e^{\sigma_j t}\int_{t}^{+\infty} e^{-\sigma_j t'}[d_j \cos(\tau_j(t - t')) + d_j' \sin(\tau_j(t - t'))]h(t')\,dt'$$

$$+ \sum_{j=0}^{J} e^{-\sigma_j t}\int_{t}^{+\infty} e^{\sigma_j t'}[d_j \cos(\tau_j(t - t')) + d_j' \sin(\tau_j(t - t'))]h(t')\,dt'.$$

Let us check its asymptotic behavior. The growth hypothesis on h from (6.6) imply that

$$\left|\int_{\mathbb{R}} h(t')\widetilde{\mathcal{G}}(t - t')\,dt'\right| = O(e^{-\delta t}),$$

as $t \to +\infty$. Thus we have found a different particular solution w_m with the desired decay. On the other hand, when $t \to -\infty$, the worst term above is the second one in W_2, as expected. Let

$$D_j(t) = \int_t^{+\infty} e^{\sigma_j t'} [d_j \cos(\tau_j(t - t')) + d_j' \sin(\tau_j(t - t'))] h(t') \, dt', \quad j = 1, \ldots, J.$$

Let us assume for a moment, that $\tau_j = 0$. Two cases can happen as $t \to -\infty$:

- either $D_j(t) \to \tilde{D}_j \neq 0$, so $w_m(t) \sim e^{-\sigma_j t}$, or
- $D_j(-\infty) = 0$, so $w_m(t) = O(e^{\delta_0 t})$, by L'Hôpital's rule,

and this completes the proof of (6.10). When $\tau_j \neq 0$ we need to write

(6.19)
$$\begin{aligned}
D_j(t) = d_j &\left[\cos(\tau_j t) \int_t^{+\infty} e^{\sigma_j t'} \cos(\tau_j t') h(t') \, dt' + \sin(\tau_j t) \int_t^{+\infty} e^{\sigma_j t'} \sin(\tau_j t') h(t') \, dt' \right] \\
+ d_j' &\left[\sin(\tau_j t) \int_t^{+\infty} e^{\sigma_j t'} \cos(\tau_j t') h(t') \, dt' - \cos(\tau_j t) \int_t^{+\infty} e^{\sigma_j t'} \sin(\tau_j t') h(t') \, dt' \right],
\end{aligned}$$

but we have an analogous result.

This completes the proof of Theorem 6.1.

$\square$

Remark 6.9. We now look at the proof of Theorem 6.1, *iii.*, in terms of the variation of constants method and Fredholm theory. Starting from equations (6.7)-(6.8), we estimate, for $t > 0$,

(6.20)
$$\left| \int_{\mathbb{R}} \left[\mathcal{G}_m(t - t') - \sum_{j=0}^{J} e^{-\sigma_j|t-t'|} (d_j \cos(\tau_j|t - t'|) + d_j' \sin(\tau_j|t - t'|)) \right] h(t') \, dt' \right|$$
$$\leq \int_{\mathbb{R}} O(e^{-\sigma_{J+1}|t-t'|}) |h(t')| \, dt' = O(e^{-\delta t}).$$

Let

$$\varphi_j(t) := \int_{\mathbb{R}} e^{-\sigma_j|t-t'|} (d_j \cos(\tau_j|t - t'|) + d_j' \sin(\tau_j|t - t'|)) h(t') \, dt', \quad j = 0, \ldots, J,$$

so

(6.21)
$$w_m(t) = \sum_{j=0}^{J} \varphi_j(t) + O(e^{-\delta t}) \quad \text{as} \quad t \to +\infty.$$

The underlying idea is that, if one wishes to improve the decay of (6.21) as $t \to +\infty$, one needs to add to this particular solution elements in the kernel, say, $C_j e^{-\sigma_j}$ for a precise constant C_j, $j = 0, \ldots, J$. But this worsens the decay at $t \to -\infty$. Let us make this computation more precise in the case that $\tau_j = 0$. The general case is similar, but lengthy.

(6.22)
$$\begin{aligned}
\varphi_j(t) &= d_j \int_{\mathbb{R}} e^{-\sigma_j|t-t'|} h(t') \, dt' \\
&= d_j \int_{-\infty}^{t} e^{-\sigma_j(t-t')} h(t') \, dt' + d_j \int_t^{+\infty} e^{\sigma_j(t-t')} h(t') \, dt'.
\end{aligned}$$

The first integral in the right hand side above can be rewritten using that, by Fredholm theory, the following compatibility condition must be satisfied:

(6.23)
$$0 = \int_{\mathbb{R}} e^{\sigma_j t'} h(t') \, dt' = \int_{-\infty}^{t} \ldots + \int_t^{+\infty} \ldots.$$

Thus we have

$$\varphi_j(t) = d_j \int_t^{+\infty} \left[-e^{-\sigma_j(t-t')} + e^{\sigma_j(t-t')} \right] h(t') \, dt'.$$

Then, except for the factor d_j in front, this is the standard variation of constants formula to produce a particular solution for the second order ODE

$$\varphi_j''(t) = \sigma_j^2 \varphi_j(t) - 2\sigma_j h(t),$$

and in particular shows that $\varphi_j(t)$ decays like $h(t)$ as $t \to +\infty$, which is $O(e^{-\delta t})$. One could also check behavior as $t \to -\infty$, similarly to the above.

6.1. **Beyond the stability regime.** Now we look at the proof of Proposition 6.3. As we have mentioned, in order to simplify the presentation, we only consider the projection $m = 0$. Let $\Lambda_{N,\gamma} < \kappa < \Lambda'_{N,\gamma}$ be the region where we have exactly two real poles at τ_0 and $-\tau_0$, for $\tau > 0$. For this, just note that, for real $\xi > 0$, Lemma 6.6 shows that $\Theta_0(\xi)$ is an increasing function in ξ, and it is even. Denote the rest of the poles as in the previous subsection, for $j = 1, 2, \ldots$.

We proceed as in the proof of Theorem 6.1 and write

$$(6.24) \qquad w_0(t) = \int_{\mathbb{R}} \frac{1}{\Theta_0(\xi) - \kappa} \hat{h}(\xi) e^{i\xi t}\, d\xi = \int_{\mathbb{R}} h(t') \mathcal{G}_0(t - t')\, dt'.$$

There are two real poles now. So in order to give sense to the integral above we can integrate on a different horizontal line, just below the horizontal axis, or to regularize the integral by some $\varepsilon > 0$ small. For this, calculate

$$\mathcal{G}_0^\varepsilon(t) = \int_{\mathbb{R}} e^{i\xi t} \frac{1}{\Theta_0(\xi - \varepsilon i) - \kappa}\, d\xi.$$

The poles are now $\tau_0 + \varepsilon i$ and $\tau_0 - \varepsilon i$. Define the region $\Omega_+ = \{z \in \mathbb{C} : |z - (\tau + \varepsilon i)| < R, \operatorname{Re}(z) > 0\}$. A standard contour integration along $\partial \Omega_+$ gives, as $R \to \infty$, that for $t > 0$,

$$\mathcal{G}_0^\varepsilon(t) = 2\pi i c_0^\varepsilon e^{i(\tau_0 + \varepsilon i)t} + 2\pi i \sum_{j=1}^\infty e^{-\sigma_j^\varepsilon t}[c_j^\varepsilon \cos(\tau_j^\varepsilon t) + c_j'^\varepsilon \sin(\tau_j^\varepsilon t)]$$

where

$$c_0^\varepsilon = \operatorname{Res}\left(\frac{1}{\Theta_0(z - \varepsilon i) - \kappa}, \tau_0 + \varepsilon i\right).$$

Taking the limit $\varepsilon \to 0$,

$$\mathcal{G}_0^\varepsilon(t) \to \mathcal{G}_0(t) = 2\pi i[c_0 \cos(\tau_0 t) + c_0' \sin(\tau_0 t)] + 2\pi i \sum_{j=1}^\infty e^{-\sigma_j t}[c_j \cos(\tau_j t) + c_j' \sin(\tau_j t)],$$

for $t > 0$, and extended evenly to the real line.

Let us simplify this formula for the $j = 0$ case. Using Fredholm theory, to have a decaying solution of equation (6.11), h must satisfy the compatibility condition

$$0 = e^{-i\tau_0 t} \int_{\mathbb{R}} h(t') e^{i\tau_0 t'}\, dt' = \int_{\mathbb{R}} h(t - t') e^{-i\tau_0 t'}\, dt'$$

$$= \int_0^{+\infty} h(t - t') e^{-i\tau_0 t'}\, dt' + \int_{-\infty}^0 h(t - t') e^{-i\tau_0 t'}\, dt'.$$

Substitute this expression into the formula below (and take imaginary part)

$$\int_0^{+\infty} h(t - t') e^{-i\tau_0 t'}\, dt' + \int_{-\infty}^0 h(t - t') e^{i\tau_0 t'}\, dt'$$

$$= \int_{-\infty}^0 h(t - t') e^{-i\tau_0 t'} - \int_{-\infty}^0 h(t - t') e^{i\tau_0 t'}\, dt' = -2i \int_t^{+\infty} h(t') \sin(\tau_0(t - t'))\, dt'.$$

Arguing as in the proof of Theorem 6.1 we obtain *ii.* The only difference with the stable case is that the $j = 0$ term in the summation in formula (6.20) needs to be replaced by

$$\int_t^{+\infty} \sin(\tau_0(t - t'))h(t') \, dt'$$

(note that $c_0 = 0$). A similar argument yields *iii.* too.

6.2. **A-priori estimates in weighted Sobolev spaces.** For $s > 0$, we define the norm in $\mathbb{R} \times \mathbb{S}^{N-1}$ given by

$$(6.25) \qquad \|w\|_s^2 = \sum_{m=0}^{\infty} \int_{\mathbb{R}} (1 + \xi^2 + m^2)^{2s} |\hat{w}_m(\xi)|^2 \, d\xi.$$

These are homogeneous norms in the variable $r = e^{-t}$, and formulate the Sobolev counterpart to the Hölder norms in $\mathbb{R}^N \setminus \{0\}$ from Section 5.1. That is, for $w^*(r) := w(t)$ and s integer we have

$$\|w\|_0^2 = \sum_{m=0}^{\infty} \int_0^{\infty} |\tilde{w}_m^*|^2 r^{-1} \, dr,$$

$$\|w\|_1^2 = \sum_{m=0}^{\infty} \int_0^{\infty} (|\tilde{w}_m^*|^2 + |\partial_r \tilde{w}_m^*|^2 r^2) \, r^{-1} \, dr,$$

$$\|w\|_2^2 = \sum_{m=0}^{\infty} \int_0^{\infty} (|\tilde{w}_m^*|^2 + |\partial_r \tilde{w}_m^*|^2 r^2 + |\partial_{rr} \tilde{w}_m^*|^2 r^4) \, r^{-1} \, dr.$$

One may also give the corresponding weighted norms, for a weight of the type $r^{-\vartheta} = e^{\vartheta t}$. Indeed, one just needs to modify the norm (6.25) to

$$\|w\|_{s,\vartheta}^2 = \sum_{m=0}^{\infty} \int_{\mathbb{R}+i\vartheta} (1 + \xi^2 + m^2)^{2s} |\hat{w}_m(\xi)|^2 \, d\xi.$$

For instance, in the particular case $s = 1$, this is

$$\|w\|_{1,\vartheta}^2 = \sum_{m=0}^{\infty} \int_0^{\infty} (|w_m^*|^2 r^{-2\vartheta} + |\partial_r w_m^*|^2 r^{2-2\vartheta}) \, r^{-1} \, dr.$$

Proposition 6.10. *Let $s \geq 2\gamma$, and fix $\vartheta \in \mathbb{R}$ such that the horizontal line $\mathbb{R} + i\vartheta$ does not cross any pole $\tau_j^{(m)} \pm i\sigma_j^{(m)}$, $j = 0, 1, \ldots$, $m = 0, 1, \ldots$. If w is a solution to*

$$\tilde{\mathcal{L}}w = h \quad in \ \mathbb{R} \times \mathbb{S}^{N-1}$$

of the form (6.7), then

$$\|w\|_{s,\vartheta} \leq C\|h\|_{s-2\gamma,\vartheta}$$

for some constant $C > 0$.

Proof. We project over spherical harmonics $w = \sum_m w_m E_m$, where w_m is a solution to $\tilde{\mathcal{L}}_m w_m = h_m$. Assume, without loss of generality, that $\vartheta = 0$, otherwise replace the Fourier transform $\hat{\cdot}$ by $\tilde{\cdot}$ on a different horizontal line. In particular, $\hat{w}_m(\xi) = (\Theta_m(\xi) - \kappa)^{-1} \hat{h}_m(\xi)$, and we simply estimate

$$\|w\|_s^2 = \sum_{m=0}^{\infty} \int_{\mathbb{R}} \frac{(1 + |\xi|^2 + m^2)^{2s}}{|\Theta_m(\xi) - \kappa|^2} |\hat{h}_m(\xi)|^2 \, d\xi \leq C \sum_{m=0}^{\infty} \int_{\mathbb{R}} (1 + |\xi|^2 + m^2)^{2s-4\gamma} |\hat{h}_m(\xi)|^2 \, d\xi$$

$$= C\|h\|_{s-2\gamma}^2,$$

where we have used that

$$\frac{(1 + |\xi|^2 + m^2)^{2s}}{|\Theta_m(\xi) - \kappa|^2} \le C(1 + |\xi|^2 + m^2)^{2s - 4\gamma},$$

which follows from (6.14). $\qquad\qquad\square$

6.3. **An application to a non-local ODE.** For simplicity, denote by u_1 be the fast decaying solution from Proposition 2.1 for $\epsilon = 1$. We would like to study the kernel for the linearization of (2.1) in the space of radial functions, this is,

$$(6.26) \qquad\qquad L_1\varphi := (-\Delta)^\gamma \varphi - p(u_1)^{p-1}\varphi = 0, \quad \varphi = \varphi(r).$$

In terms of the t variable we can write as follows: set $r = e^{-t}$, setting $u_1 = r^{-\frac{2\gamma}{p-1}} v_1$, $w = r^{\frac{N-2\gamma}{2}}\phi$, we can write this equation as

$$\mathcal{L}_1 w := r^{\frac{n+2\gamma}{2}} L_1\varphi = P_\gamma w - V(t)w, \quad w = w(t),$$

for the potential

$$(6.27) \qquad\qquad V(t) = p(v_1)^{p-1} = \begin{cases} \kappa + O(e^{-qt}), & \text{if } t \to +\infty, \\ O(e^{q_1 t}), & \text{if } t \to -\infty, \end{cases}$$

where we have set $\kappa = pA_{N,p,\gamma}^{p-1}$, and for some $q, q_1 > 0$.

The following Hamiltonian identity will be useful (the proof is exactly the same as in Theorem 4.4):

Lemma 6.11. *Let $w_i = w_i(t)$, $i = 1, 2$, be two solutions to $\mathcal{L}_* w = 0$ and set W_i, $i = 1, 2$, the corresponding extensions from Proposition 3.2 (for simplicity, the special defining function ρ^* will just be denoted by ρ). Then, the Hamiltonian quantity*

$$\tilde{H}_\gamma(t) := \int_0^{\rho_0} \rho^{1-2\gamma} e(\rho)[W_1 \partial_t W_2 - W_2 \partial_t W_1]\, d\rho$$

is constant along trajectories. Here $e(\rho)$ is a smooth function satisfying $e(\rho) \to 1$ as $\rho \to 0$.

First, let us look at non-singular solutions of (2.1). Since this equation is invariant under rescaling, it is well known that

$$\varphi_* := r\partial_r u_1 + \tfrac{p-1}{2\gamma} u_1$$

belongs to the kernel of L_*. If we set

$$w_* = r^{\frac{N-2\gamma}{2}} \varphi_*,$$

then w_* is a solution to $\mathcal{L}_1 w = 0$. Moreover, in the stable case,

$$w_*(t) \sim \begin{cases} e^{-\sigma_0 t} & \text{as } t \to +\infty, \\ e^{-\frac{N-2\gamma}{2} t} & \text{as } t \to -\infty. \end{cases}$$

In the unstable case, the picture is more complicated but has a similar behavior (see [6]).

From Lemma 6.11 above we have a non-degeneracy result, which will be used in the proof of Proposition 7.2:

Corollary 6.12. *Any other solution to $\mathcal{L}_* w = 0$ that decays both at $\pm\infty$ must be a multiple of w_*.*

We conclude the section which a statement that is not needed in the proof of the main theorem, but we have decided to include it here because it showcases a classical ODE type behavior for a non-local equation, and it motivates the arguments in Section 7.

Assume that we are in the unstable case, i.e., the setting of Proposition 6.3.

Proposition 6.13. *Let $q > 0$ and fix a potential on $\mathbb{R}$ as in (6.27), with the asymptotic behavior*

$$V(t) = \begin{cases} \kappa + O(e^{-qt}) & \text{as } t \to +\infty, \\ O(1) & \text{as } t \to -\infty, \end{cases}$$

for $r = e^{-t}$. Then the space of radial solutions to equation

(6.28)
$$(-\Delta)^\gamma u - \frac{V}{r^{2\gamma}} u = 0 \quad \text{in } \mathbb{R}^N$$

that have a bound of the form $|u(r)| \leq C r^{-\frac{N-2\gamma}{2}}$ is at most two-dimensional.

Proof. Let u be one of such solutions, and write $w = u r^{\frac{N-2\gamma}{2}}$, $w = w(t)$. By assumption, w is bounded on $\mathbb{R}$. Moreover, w satisfies the equation $P_\gamma^{(0)} w - Vw = 0$.

First we look at the indicial roots at $t \to -\infty$, these come from studying the problem $P_\gamma^{(0)} w = 0$, this is, for $\kappa = 0$. We have

$$P_\gamma^{(0)} w = \mathcal{V}_*(t) w =: h_0 = \begin{cases} O(1) & \text{as } t \to +\infty \\ O(e^{-q_1|t|}) & \text{as } t \to -\infty. \end{cases}$$

Thus we get a particular solution decaying as $t \to -\infty$. Using this piece of information, we now look at the indicial roots as $t \to +\infty$, which come from studying the equation

$$\tilde{\mathcal{L}}_0 w = h, \quad \text{for} \quad h := (V - \kappa) w.$$

Then we have the bounds for h

$$h(t) = \begin{cases} O(e^{-qt}) & \text{as } t \to +\infty, \\ O(e^{-q_1|t|}) & \text{as } t \to -\infty, \end{cases}$$

so we take $\delta = q > 0$, $\delta_0 = 0$, and apply Proposition 6.3. Then w must be of the form

$$w(t) = w_0(t) + C_0^1 \sin(\tau_0 t) + C_0^2 \cos(\tau_0 t)$$

$$+ \sum_{j=1}^\infty e^{-\sigma_j t} \left[C_j^1 \sin(\tau_j t) + C_j^2 \cos(\tau_j t) \right] + \sum_{j=1}^\infty e^{\sigma_j t} \left[D_j^1 \sin(\tau_j t) + D_j^2 \cos(\tau_j t) \right]$$

for some real constants $C_0^1, C_0^2, C_j^1, C_j^2, D_j^1, D_j^2$, $j = 1, 2, \ldots$, and w_0 is given by (6.12). The same proposition yields that w_0 is decaying as $O(e^{-\delta t})$ when $t \to +\infty$, so we must have $D_j^1, D_j^2 = 0$ for $j = 1, 2, \ldots$. Moreover, as $t \to -\infty$, if we want a bounded solution, this fixes the values of C_j^1, C_j^2 for $j = 1, 2, \ldots$, which implies that only C_0^1 and C_0^2 are free, and determined from w.

The Hamiltonian from Lemma 6.11 shows that there are no decaying solutions on both sides $t \to \pm\infty$ except for the trivial one ϕ_*, which is controlled; indeed, its asymptotic behavior is given by the indicial roots (if it decays exponentially as $O(e^{-\delta t})$ for $t \to +\infty$, we can iterate statement *iii.* with $\delta = lq$, $l = 2, 3, \ldots$ and $\delta_0 = 0$, to show that it decays faster than any $O(e^{-\delta t})$, $\delta > 0$, as $t \to +\infty$). Next, we use a unique continuation result for equation (6.28) to show that it must vanish. In the stable case, unique continuation was proved in [39] using a monotonicity formula, while in the unstable case it follows from [76], where Carleman estimates were the crucial ingredient.

We remark that if, in addition, the potential satisfies a monotonicity condition, one can give a direct proof of unique continuation using Theorem 1 from [43]. Note that, however, in [43] the potential is assumed to be smooth at the origin. But one can check that the lack of regularity of the potential at the origin can be handled by the higher order of vanishing of u.

$\square$

6.4. **Technical results.** Here we give a more precise calculation of the poles of the function $\frac{1}{\Theta_m(z)-\kappa}$. For this, given $\kappa \in \mathbb{R}$, we aim to solve the equation

$$(6.29) \qquad \frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta + iz)\Gamma(\beta - iz)} - \kappa = 0$$

with $|\alpha - \beta| < 1$ and $\beta < \alpha$.

Lemma 6.14. *Let*

$$z = iR + \zeta$$

with $|z| > R_0$ and R_0 sufficiently large. Then the solutions to (6.29) are contained in balls of radius $\frac{C\kappa \sin((\alpha-\beta)\pi)}{\mathcal{N}^{2(\alpha-\beta)}}$ around the points $z = (\mathcal{N} + \beta)i$, with $\mathcal{N} = [R]$ and C depending solely on α and β.

Proof. First we note, by using the identity $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, that

$$\frac{\Gamma(\alpha - R + i\zeta)}{\Gamma(\beta - R + i\zeta)} = \frac{\Gamma(1 - \beta + R - i\zeta)}{\Gamma(1 - \alpha + R - i\zeta)} \frac{\sin(\pi(\beta - R + i\zeta))}{\sin(\pi(\alpha - R + i\zeta))}$$

$$= \frac{\Gamma(1 - \beta + R - i\zeta)}{\Gamma(1 - \alpha + R - i\zeta)} \frac{\sin(\pi(\beta - \delta + i\zeta))}{\sin(\pi(\alpha - \delta + i\zeta))},$$

where we have denoted

$$\delta = R - [R].$$

Then, Stirling's formula (6.13) yields

$$|\Gamma(1 + z)| \sim |z|^{\operatorname{Re} z} e^{-(\operatorname{Im} z)\arg(z)} e^{-\operatorname{Re} z} \sqrt{2\pi} |z|^{\frac{1}{2}},$$

which implies

$$\left| \frac{\Gamma(1 - \beta + R - i\zeta)\Gamma(\alpha + R - i\zeta)}{\Gamma(1 - \alpha + R - i\zeta)\Gamma(\beta + R - i\zeta)} \right|$$

$$\sim (R^2 + \zeta^2)^{\alpha-\beta} e^{\zeta\left(\arctan\frac{\zeta}{1-\beta+R} + \arctan\frac{\zeta}{\alpha+R} - \arctan\frac{\zeta}{1-\alpha+R} - \arctan\frac{\zeta}{\beta+R}\right)} e^{-2(\alpha-\beta)}.$$

Since

$$\arctan\frac{\zeta}{1 - \beta + R} + \arctan\frac{\zeta}{\alpha + R} - \arctan\frac{\zeta}{1 - \alpha + R} - \arctan\frac{\zeta}{\beta + R}$$

$$= \arctan\frac{\frac{\zeta}{1-\beta+R} - \frac{\zeta}{1-\alpha+R}}{1 + \frac{\zeta}{1-\beta+R}\frac{\zeta}{1-\alpha+R}} + \arctan\frac{\frac{\zeta}{\alpha+R} - \frac{\zeta}{\beta+R}}{1 + \frac{\zeta}{\alpha+R}\frac{\zeta}{\beta+R}}$$

$$\sim -2\arctan\frac{(\alpha - \beta)\zeta}{R^2 + \zeta^2} \sim -2\frac{(\alpha - \beta)\zeta}{R^2 + \zeta^2},$$

we can estimate, for $R^2 + \zeta^2$ sufficiently large,

$$\left| \frac{\Gamma(1 - \beta + R - i\zeta)\Gamma(\alpha + R - i\zeta)}{\Gamma(1 - \alpha + R - i\zeta)\Gamma(\beta + R - i\zeta)} \right| \sim (R^2 + \zeta^2)^{\alpha-\beta} e^{-2\frac{(\alpha-\beta)\zeta^2}{R^2+\zeta^2}} e^{-2(\alpha-\beta)}.$$

Therefore, for $R^2 + \zeta^2 > R_0^2$ with R_0 sufficiently large, we have the bound

$$C^{-1}(R^2 + \zeta^2)^{\alpha-\beta} \leq \left| \frac{\Gamma(1 - \beta + R - i\zeta)\Gamma(\alpha + R - i\zeta)}{\Gamma(1 - \alpha + R - i\zeta)\Gamma(\beta + R - i\zeta)} \right| \leq C(R^2 + \zeta^2)^{\alpha-\beta},$$

where C depends only on α and β. Hence,

$$\frac{\kappa}{C(R^2 + \zeta^2)^{\alpha-\beta}} \leq \left| \frac{\sin(\pi(\beta - \delta + i\zeta))}{\sin(\pi(\alpha - \delta + i\zeta))} \right| \leq \frac{\kappa}{C^{-1}(R^2 + \zeta^2)^{\alpha-\beta}},$$

and by writing

$$\delta - i\zeta = \beta + \tilde{z},$$

we conclude that necessarily

$$|\tilde{z}| \leq \frac{C\kappa \sin\left((\alpha - \beta)\pi\right)}{R^{2(\alpha-\beta)}},$$

which implies that solutions to (6.29) lie at

$$z = iR + \zeta = i\,[R] + i\beta + i\tilde{z} = [R] + \beta + O\left(\frac{C\kappa \sin\left((\alpha - \beta)\pi\right)}{[R]^{2\,(\alpha-\beta)}}\right),$$

and this proves the Lemma.

$\square$

Next we write

$$z = i(\beta + \mathcal{N}) + \tilde{z}$$

with $\mathcal{N}$ sufficiently large (according to the previous lemma) natural number, and equation (6.29) reads

(6.30)
$$\frac{\Gamma(\alpha - \beta - \mathcal{N} + i\tilde{z})\Gamma(\alpha + \beta + \mathcal{N} - i\tilde{z})}{\Gamma(-\mathcal{N} + i\tilde{z})\Gamma(2\beta + \mathcal{N} - i\tilde{z})} - \kappa = 0.$$

Since

$$\Gamma(-\mathcal{N} + i\tilde{z}) = (-1)^{\mathcal{N}-1}\frac{\Gamma(-i\tilde{z})\Gamma(1 + i\tilde{z})}{\Gamma(\mathcal{N} + 1 - i\tilde{z})}$$

and

$$\Gamma(\alpha - \beta - \mathcal{N} + i\tilde{z}) = (-1)^{\mathcal{N}-1}\frac{\Gamma(\beta - \alpha - i\tilde{z})\Gamma(1 + \alpha - \beta + i\tilde{z})}{\Gamma(\mathcal{N} + 1 - \alpha + \beta - i\tilde{z})},$$

we can write

$$\frac{\Gamma(\alpha - \beta - \mathcal{N} + i\tilde{z})\Gamma(\alpha + \beta + \mathcal{N} - i\tilde{z})}{\Gamma(-\mathcal{N} + i\tilde{z})\Gamma(2\beta + \mathcal{N} - i\tilde{z})}$$
$$= \frac{\Gamma(\mathcal{N} + 1 - i\tilde{z})\Gamma(\alpha + \beta + \mathcal{N} - i\tilde{z})}{\Gamma(\mathcal{N} + 1 - \alpha + \beta - i\tilde{z})\Gamma(2\beta + \mathcal{N} - i\tilde{z})}\frac{\Gamma(\beta - \alpha - i\tilde{z})\Gamma(1 + \alpha - \beta + i\tilde{z})}{\Gamma(1 + i\tilde{z})\Gamma(-i\tilde{z})}.$$

Using that

$$\frac{\Gamma(\beta - \alpha - i\tilde{z})\Gamma(1 + \alpha - \beta + i\tilde{z})}{\Gamma(1 + i\tilde{z})\Gamma(-i\tilde{z})} = \frac{\sin(-\pi i\tilde{z})}{\sin\left(\pi(\beta - \alpha - i\tilde{z})\right)},$$

as well as Stirling's formula (6.13) to estimate

$$\frac{\Gamma(\mathcal{N} + 1 - i\tilde{z})\Gamma(\alpha + \beta + \mathcal{N} - i\tilde{z})}{\Gamma(\mathcal{N} + 1 - \alpha + \beta - i\tilde{z})\Gamma(2\beta + \mathcal{N} - i\tilde{z})}$$
$$\sim \frac{(\mathcal{N} - i\tilde{z})^{\mathcal{N}-i\tilde{z}}(\alpha + \beta + \mathcal{N} - 1 - i\tilde{z})^{\alpha+\beta+\mathcal{N}-1-i\tilde{z}}}{(\mathcal{N} - \alpha + \beta - i\tilde{z})^{\mathcal{N}-\alpha+\beta-i\tilde{z}}(2\beta - 1 + \mathcal{N} - i\tilde{z})^{2\beta-1+\mathcal{N}-i\tilde{z}}}$$
$$\cdot e^{-2(\alpha-\beta)}\sqrt{\frac{(\mathcal{N} - i\tilde{z})(\alpha + \beta + \mathcal{N} - 1 - i\tilde{z})}{(\mathcal{N} - \alpha + \beta - i\tilde{z})(2\beta - 1 + \mathcal{N} - i\tilde{z})}}$$
$$\sim \mathcal{N}^{2(\alpha-\beta)}e^{-2i(\alpha-\beta)\tilde{z}}e^{-2(\alpha-\beta)},$$

we arrive at the relation

$$\frac{\sin(-\pi i\tilde{z})}{\sin\left(\pi(\beta - \alpha - i\tilde{z})\right)}e^{-2i(\alpha-\beta)\tilde{z}} \sim \frac{\kappa}{\mathcal{N}^{2(\alpha-\beta)}e^{-2(\alpha-\beta)}},$$

which implies

$$\tilde{z} \sim \frac{i}{\pi}\frac{\kappa \sin\left(\pi(\beta - \alpha)\right)}{\mathcal{N}^{2(\alpha-\beta)}e^{-2(\alpha-\beta)}}.$$

In fact, it is easy to see from (6.30) and the estimates above that a purely imaginary solution $\tilde{z}$ does exist and a standard fixed point argument in each of the balls in the previous lemma would show that it is unique.

Finally, the half-ball of radius R_0 around the origin in the upper half-plane is a compact set. Since the function at the left hand side of (6.29) is meromorphic, there cannot exist accumulation points of zeros and this necessarily implies that the number of zeros in that half-ball is finite.

We conclude then that the set of solutions to (6.29) consists of a finite number of solutions in a half ball of radius R_0 around the origin in the upper half-plane together with an infinite sequence of roots at the imaginary axis located at

$$(6.31) \qquad z_{\mathcal{N}} = i(\beta + \mathcal{N}) + O\left(\frac{\kappa \sin(\pi(\beta - \alpha))}{\mathcal{N}^{2(\alpha-\beta)}e^{-2(\alpha-\beta)}}\right) \quad \text{for} \quad \mathcal{N} > R_0,$$

as desired.

Now we consider the asymptotics for the residues. We define

$$g(z) := \frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta + iz)\Gamma(\beta - iz)} - \kappa.$$

We will estimate the residue of the function $\frac{1}{g(z)}$ at the poles $z_{\mathcal{N}}$ when $\mathcal{N}$ is sufficiently large. Given the fact that the poles are simple and the function $1/g(z)$ is analytic outside its poles, we have

$$\text{Res}\left(\frac{1}{g(z)}, z_{\mathcal{N}}\right) = \lim_{z \to z_{\mathcal{N}}} \left((z - z_{\mathcal{N}})\frac{1}{g(z)}\right) = \frac{1}{g'(z_{\mathcal{N}})}.$$

Hence

$$\begin{aligned}
g'(z) &= \frac{d}{dz}\left(\frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta + iz)\Gamma(\beta - iz)}\right) \\
&= -i\frac{\Gamma'(\beta + iz)}{\Gamma^2(\beta + iz)}\left(\frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta - iz)}\right) + \frac{1}{\Gamma(\beta + iz)}\left(\frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta - iz)}\right)' \\
&=: S_1 + S_2.
\end{aligned}$$

Notice that $g(z_{\mathcal{N}}) = 0$ implies that

$$\Gamma(\beta + iz_{\mathcal{N}}) = \frac{\Gamma(\alpha + iz_{\mathcal{N}})\Gamma(\alpha - iz_{\mathcal{N}})}{\kappa\Gamma(\beta - iz_{\mathcal{N}})}$$

and therefore,

$$S_1 = -i\frac{\Gamma'(\beta + iz_{\mathcal{N}})}{\Gamma^2(\beta + iz_{\mathcal{N}})}\left(\frac{\Gamma(\alpha + iz_{\mathcal{N}})\Gamma(\alpha - iz_{\mathcal{N}})}{\Gamma(\beta - iz_{\mathcal{N}})}\right) = -i\kappa\frac{\Gamma'(\beta + iz_{\mathcal{N}})}{\Gamma(\beta + iz_{\mathcal{N}})} = -i\kappa\psi(\beta + iz_{\mathcal{N}}),$$

where $\psi(z)$ is the digamma function. We recall the expansion (10.9),

$$\psi(z) = -\gamma + \sum_{l=0}^{\infty}\left(\frac{1}{l+1} - \frac{1}{l+z}\right).$$

In this section, γ denotes the Euler constant. Then

$$(6.32) \qquad S_1 = i\kappa\left(\gamma + \sum_{l=0}^{\infty}\left(\frac{1}{l+\beta+iz_{\mathcal{N}}} - \frac{1}{l+1}\right)\right) = \frac{\pi e^{-2(\alpha-\beta)}}{i\sin(\pi(\alpha - \beta))}\mathcal{N}^{2(\alpha-\beta)} + O(1),$$

where we have used the asymptotics of $l + \beta + iz_{\mathcal{N}}$ when $l = \mathcal{N}$ from (6.31). Next, using again (6.31) we estimate

$$S_2 = \frac{1}{\Gamma(\beta + iz)} \left(\frac{\Gamma(\alpha + iz)\Gamma(\alpha - iz)}{\Gamma(\beta - iz)} \right)' \Bigg|_{z=z_{\mathcal{N}}} = \kappa i \left(\frac{\Gamma'(\alpha + iz)}{\Gamma(\alpha + iz)} - \frac{\Gamma'(\alpha - iz)}{\Gamma(\alpha - iz)} + \frac{\Gamma'(\beta - iz)}{\Gamma(\beta - iz)} \right)$$

$$= \kappa i \left(\psi(\alpha - \beta - \mathcal{N} + O(\mathcal{N}^{-2(\alpha - \beta)})) - \psi(\alpha + \beta + \mathcal{N} + O(\mathcal{N}^{-2(\alpha - \beta)})) \right.$$

$$\left. + \psi(2\beta + \mathcal{N} + O(\mathcal{N}^{-2(\alpha - \beta)})) \right).$$

By using the relations

$$\psi(1 - z) - \psi(z) = \pi \cot(\pi z)$$
$$\psi(z) \sim \ln(z - \gamma) + 2\gamma, \text{ as } |z| \to \infty, \text{ Re } z > 0,$$

we conclude, as $\mathcal{N} \to \infty$,

$$\psi(\alpha - \beta - \mathcal{N} + O(\mathcal{N}^{-2(\alpha - \beta)}))$$
$$= \psi(1 - \alpha + \beta + \mathcal{N} + O(\mathcal{N}^{-2(\alpha - \beta)})) + \pi \cot(\pi(1 - \alpha + \beta + \mathcal{N} + O(\mathcal{N}^{-2(\alpha - \beta)})))$$
$$= \ln(\mathcal{N}) + O(1),$$

and hence

(6.33) $$S_2 = i\kappa \ln \mathcal{N} + O(1).$$

Putting together (6.32) and (6.33) we find

$$S_1 + S_2 = \frac{\pi e^{-2(\alpha - \beta)}}{i \sin(\pi(\alpha - \beta))} \mathcal{N}^{2(\alpha - \beta)} + i\kappa \ln \mathcal{N} + O(1),$$

and hence

(6.34) $$\mathrm{Res}\left(\frac{1}{g(z)}, z_{\mathcal{N}} \right) = \frac{i}{\frac{\pi e^{-2(\alpha - \beta)}}{\sin(\pi(\alpha - \beta))} \mathcal{N}^{2(\alpha - \beta)} - \kappa \ln \mathcal{N} + O(1)}$$

$$= i \frac{\sin(\pi(\alpha - \beta)) e^{2(\alpha - \beta)}}{\pi} \mathcal{N}^{-2(\alpha - \beta)} + O\left(\frac{\ln \mathcal{N}}{\mathcal{N}^{4(\alpha - \beta)}} \right)$$

as $\mathcal{N} \to \infty$.

7. LINEAR THEORY - INJECTIVITY

Let $\bar{u}_\varepsilon$ be the approximate solution from the Section 5.1. In this section we consider the linearized operator

(7.1) $$L_\varepsilon \phi := (-\Delta_{\mathbb{R}^n})^\gamma \phi - p \bar{u}_\varepsilon^{p-1} \phi, \quad \text{in } \mathbb{R}^n \setminus \Sigma,$$

where Σ is a sub-manifold of dimension k (or a disjoint union of smooth k-dimensional manifolds), and

(7.2) $$L_\varepsilon \phi := (-\Delta_{\mathbb{R}^N})^\gamma \phi - p A_{N,p,\gamma} \bar{u}_\varepsilon^{p-1} \phi, \quad \text{in } \mathbb{R}^N \setminus \{q_1, \ldots, q_K\}.$$

For this, we first need to study the model linearization

(7.3) $$\mathcal{L}_1 \phi := (-\Delta_{\mathbb{R}^N})^\gamma \phi - p A_{N,p,\gamma} u_1^{p-1} \phi = 0 \quad \text{in} \quad \mathbb{R}^N \setminus \{0\}.$$

We will show that any solution (in suitable weighted spaces) to this equation must vanish everywhere (from which injectivity in $\mathbb{R}^n \setminus \mathbb{R}^k$ follows easily), and then we will prove injectivity for the operator L_ε.

Let us rewrite (7.3) using conformal properties and the conjugation (3.13). If we define

(7.4) $$w = r^{\frac{N-2\gamma}{2}} \phi,$$

then this equation is equivalent to

(7.5)
$$P_\gamma^{g_0}(w) - Vw = 0,$$

for the radial potential

(7.6)
$$V = V(r) = r^{2\gamma} p A_{N,p,\gamma} u_1^{p-1}.$$

The asymptotic behavior of this potential is easily calculated using Proposition 2.1 and, indeed, for $r = e^{-t}$,

(7.7)
$$V(t) = \begin{cases} pA_{N,p,\gamma} + O(e^{-q_1 t}) & \text{as } t \to +\infty, \\ O(e^{q_0 t}) & \text{as } t \to -\infty, \end{cases}$$

for $q_0 = (N - 2\gamma)(p - 1) - 2\gamma > 0$ and $q_1 > 0$.

Let $\gamma \in (0, 1)$. By the well known extension theorem for the fractional Laplacian (3.9)-(3.10), equation (7.3) is equivalent to the boundary reaction problem

$$\begin{cases} \partial_{\ell\ell}\Phi + \dfrac{1 - 2\gamma}{\ell}\partial_\ell\Phi + \Delta_{\mathbb{R}^N}\Phi = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -\tilde{d}_\gamma \lim\limits_{\ell\to 0} \ell^{1-2\gamma}\partial_\ell\Phi = pA_{N,p,\gamma} u_1^{p-1}\Phi & \text{on } \mathbb{R}^N \setminus \{0\}, \end{cases}$$

where $\tilde{d}_\gamma$ is defined in (3.8) and $\Phi|_{\ell=0} = \phi$.

Keeping the notations of Section 3.2 for the spherical harmonic decomposition of $\mathbb{S}^{N-1}$, by μ_m we denote the m-th eigenvalue for $-\Delta_{\mathbb{S}^{N-1}}$, repeated according to multiplicity, and by $E_m(\theta)$ the corresponding eigenfunction. Then we can write $\Phi = \sum_{m=0}^\infty \Phi_m(r, \ell)E_m(\theta)$, where Φ_m satisfies the following:

(7.8)
$$\begin{cases} \partial_{\ell\ell}\Phi_m + \dfrac{1 - 2\gamma}{\ell}\partial_\ell\Phi_m + \Delta_{\mathbb{R}^N}\Phi_m - \dfrac{\mu_m}{r^2}\Phi_m = 0 & \text{in } \mathbb{R}_+^{N+1}, \\ -\tilde{d}_\gamma \lim\limits_{\ell\to 0} \ell^{1-2\gamma}\partial_\ell\Phi_m = pA_{N,p,\gamma} u_1^{p-1}\Phi_m & \text{on } \mathbb{R}^N \setminus \{0\}, \end{cases}$$

or equivalently, from (7.5),

(7.9)
$$P_\gamma^m(w) - Vw = 0,$$

for $w = w_m = r^{\frac{N-2\gamma}{2}}\phi_m$, $\phi_m = \Phi_m(\cdot, 0)$.

7.1. **Indicial roots.** Let us calculate the indicial roots for the model linearized operator defined in (7.3) as $r \to 0$ and as $r \to \infty$. Recalling (7.7), $\mathcal{L}_1$ behaves like the Hardy operator (6.1) with $\kappa = pA_{N,p,\gamma}$ as $r \to 0$ and $\kappa = 0$ as $r \to \infty$. Moreover, we can characterize very precisely the location of the poles in Theorem 6.1 and Proposition (6.3).

Here we find a crucial difference from the local case $\gamma = 1$, where the Fourier symbol for the m-th projection $\Theta_m(\xi) - \kappa$ is quadratic in ξ, implying that there are only two poles. In contrast, in the non-local case, we have just seen that there exist *infinitely many* poles. Surprisingly, even though $\mathcal{L}_1$ is a non-local operator, its behavior is controlled by just four indicial roots, so we obtain results analogous to the local case.

For the statement of the next result, recall the shift (7.4).

Lemma 7.1. *For the operator $\mathcal{L}_1$ we have that, for each fixed mode $m = 0, 1, \ldots$,*

i. *At $r = \infty$, there exist two sequences of indicial roots*
$$\{\tilde{\sigma}_j^{(m)} \pm i\tilde{\tau}_j^{(m)} - \tfrac{N-2\gamma}{2}\}_{j=0}^\infty \quad and \quad \{-\tilde{\sigma}_j^{(m)} \pm i\tilde{\tau}_j^{(m)} - \tfrac{N-2\gamma}{2}\}_{j=0}^\infty.$$

Moreover,

$$\tilde{\gamma}_m^\pm := \pm\tilde{\sigma}_0^{(m)} - \tfrac{N-2\gamma}{2} = -\tfrac{N-2\gamma}{2} \pm \left[1 - \gamma + \sqrt{(\tfrac{N-2}{2})^2 + \mu_m}\right], \quad m = 0, 1, \ldots,$$

and $\tilde{\gamma}_m^+$ is an increasing sequence (except for multiplicity repetitions).

ii. At $r = 0$, there exist two sequences of indicial roots

$$\{\sigma_j^{(m)} \pm i\tau_j^{(m)} - \tfrac{N-2\gamma}{2}\}_{j=0}^\infty \quad and \quad \{-\sigma_j^{(m)} \pm i\tau_j^{(m)} - \tfrac{N-2\gamma}{2}\}_{j=0}^\infty.$$

Moreover,

a) For the mode $m = 0$, there exists p_1 with $\tfrac{N}{N-2\gamma} < p_1 < \tfrac{N+2\gamma}{N-2\gamma}$ (and it is given by (2.2)), such that for $\tfrac{N}{N-2\gamma} < p < p_1$ (the stable case), the indicial roots $\gamma_0^\pm := \pm\sigma_0^{(0)} - \tfrac{N-2\gamma}{2}$ are real with

$$-\tfrac{2\gamma}{p-1} < \gamma_0^- < -\tfrac{N-2\gamma}{2} < \gamma_0^+,$$

while if $p_1 < p < \tfrac{N+2\gamma}{N-2\gamma}$ (the unstable case), then $\gamma_0^\pm$ are a pair of complex conjugates with real part $-\tfrac{N-2\gamma}{2}$ and imaginary part $\pm\tau_0^{(0)}$.

b) In addition, for all $j \geq 1$,

$$\sigma_j^{(0)} > \tfrac{N-2\gamma}{2}.$$

c) For the mode $m = 1$,

$$\gamma_1^- := -\sigma_0^{(1)} - \tfrac{N-2\gamma}{2} = -\tfrac{2\gamma}{p-1} - 1.$$

Proof. First we consider statement *ii.* and calculate the indicial roots at $r = 0$. Recalling the shift (7.4), let $\mathcal{L}_1$ act on the function $r^{-\frac{N-2\gamma}{2}+\delta}$, and consider instead the operator in (7.5). Because of Proposition 3.4, for each $m = 0, 1, \ldots$, the indicial root $\gamma_m := -\frac{N-2\gamma}{2} + \delta$ satisfies

$$(7.10) \qquad 2^{2\gamma}\frac{\Gamma(A_m + \tfrac{\delta}{2})\Gamma(A_m - \tfrac{\delta}{2})}{\Gamma(B_m + \tfrac{\delta}{2})\Gamma(B_m - \tfrac{\delta}{2})} = pA_{N,p,\gamma},$$

where A_m, B_m are defined in (6.5).

Note that if $\delta \in \mathbb{C}$ is a solution, then $-\delta$ and $\pm\bar{\delta}$ are also solutions. Let us write $\frac{\delta}{2} = \alpha + i\beta$, and denote

$$\Phi_m(\alpha, \beta) = 2^{2\gamma}\frac{\Gamma(A_m + \tfrac{\delta}{2})\Gamma(A_m - \tfrac{\delta}{2})}{\Gamma(B_m + \tfrac{\delta}{2})\Gamma(B_m - \tfrac{\delta}{2})}.$$

From the expression, one can see that $\Phi_m(\alpha, 0)$ and $\Phi_m(0, \beta)$ are real functions.

We first claim that on the $\alpha\beta$-plane, provided that $|\alpha| \leq B_m$, any solution of (7.10) must satisfy $\alpha = 0$ or $\beta = 0$, i.e., δ must be real or purely imaginary. Observing that the right hand side of (7.10) is real and so is $\Phi_m(0, \beta)$ for $\beta \neq 0$, the claim follows from the strict monotonicity of the imaginary part with respect to α, namely

$$\frac{\partial}{\partial\alpha}\text{Im}(\Phi_m(\alpha, \beta)) = -\frac{i}{2}\frac{\partial}{\partial\alpha}\left[\Phi_m(\alpha, \beta) - \Phi_m(\alpha, -\beta)\right]$$

$$(7.11) \qquad = \sum_{j=0}^\infty \text{Im}\left[\frac{1}{j+A_m+\alpha-i\beta} + \frac{1}{j+A_m-\alpha-i\beta} + \frac{1}{j+B_m+\alpha+i\beta} + \frac{1}{j+B_m-\alpha+i\beta}\right]$$

$$= \beta\sum_{j=0}^\infty \left[\frac{1}{(j+A_m+\alpha)^2+\beta^2} + \frac{1}{(j+A_m-\alpha)^2+\beta^2} - \frac{1}{(j+B_m+\alpha)^2+\beta^2} - \frac{1}{(j+B_m-\alpha)^2+\beta^2}\right],$$

the summands being strictly negative since $A_m > B_m$. If $\beta \neq 0$ and $|\alpha| \leq B_m$, it is easy to see that the above expression is not zero. This yields the proof of the claim.

Moreover, $\Phi_m(\alpha, 0)$ and $\Phi_m(0, \beta)$ are even functions in α, β, respectively. Using the properties of the digamma function again, one can check that

$$(7.12) \qquad \frac{\partial \Phi_m(\alpha, 0)}{\partial \alpha} < 0 \text{ for } \alpha > 0$$

and

$$(7.13) \qquad \frac{\partial \Phi_m(0, \beta)}{\partial \beta} > 0 \text{ for } \beta > 0.$$

Let us consider now the case $m = 0$. Using the explicit expression for $A_{N,p,\gamma}$ from (1.10), then δ must be a solution of

$$(7.14) \qquad \frac{\Gamma\left(\frac{N}{4} + \frac{\gamma}{2} + \frac{\delta}{2}\right)\Gamma\left(\frac{N}{4} + \frac{\gamma}{2} - \frac{\delta}{2}\right)}{\Gamma\left(\frac{N}{4} - \frac{\gamma}{2} + \frac{\delta}{2}\right)\Gamma\left(\frac{N}{4} - \frac{\gamma}{2} - \frac{\delta}{2}\right)} = p \frac{\Gamma\left(\frac{N}{2} - \frac{\gamma}{p-1}\right)\Gamma\left(\frac{\gamma}{p-1} + \gamma\right)}{\Gamma\left(\frac{\gamma}{p-1}\right)\Gamma\left(\frac{N}{2} - \gamma - \frac{\gamma}{p-1}\right)} =: \lambda(p).$$

From the arguments in [5] (see also the definition of p_1 in (2.2)), there exists a unique p_1 satisfying $\frac{N}{N-2\gamma} < p_1 < \frac{N+2\gamma}{N-2\gamma}$ such that $\Phi_0(0,0) = \lambda(p_1)$, and $\Phi_0(0,0) > \lambda(p)$ when $\frac{N}{N-2\gamma} < p < p_1$, and $\Phi_0(0,0) < \lambda(p)$ when $p_1 < p < \frac{N+2\gamma}{N-2\gamma}$.

Assume first that $\frac{N}{N-2\gamma} < p < p_1$ (the stable case). Then from (7.13), we know that there are no indicial roots on the imaginary axis. Next we consider the real axis. Since $\Phi_0(B_0, 0) = 0$, by (7.12), there exists an unique root $\alpha^* \in (0, B_0)$ such that $\Phi_0(\pm \alpha^*, 0) = \lambda(p)$. We now show that $\alpha^* \in (0, \frac{2\gamma}{p-1} - \frac{N-2\gamma}{2})$. Note that

$$\Phi_0\left(\frac{2\gamma}{p-1} - \frac{N-2\gamma}{2}, 0\right) - pA_{N,p,\gamma} = (1-p)\frac{\Gamma\left(\frac{N}{2} - \frac{\gamma}{p-1}\right)\Gamma\left(\frac{\gamma}{p-1} + \gamma\right)}{\Gamma\left(\frac{\gamma}{p-1}\right)\Gamma\left(\frac{N}{2} - \gamma - \frac{\gamma}{p-1}\right)} < 0.$$

We conclude using the monotonicity of $\Phi_0(\alpha, 0)$ in α.

Now we consider the unstable case, i.e., for $p > p_1$. First by (7.12), there are no indicial roots on the real axis. Then by (7.11), in the region $|\alpha| \leq B_0$, if a solution exists, then δ must stay in the imaginary axis. Since $\Phi_0(0, \beta)$ is increasing in β and $\lim_{\beta \to \infty} \Phi_0(0, \beta) = +\infty$, we get an unique $\beta^* > 0$ such that $\Phi_0(0, \pm\beta^*) = \lambda(p)$.

In the notation of Section 6, we denote all the solutions to (7.14) to be $\sigma_j^{(0)} \pm i\tau_j^{(0)}$ and $-\sigma_j^{(0)} \pm i\tau_j^{(0)}$, such that σ_j is increasing sequence, then from the above argument, one has the following properties:

$$\begin{cases} \sigma_0^{(0)} \in \left(0, \frac{2\gamma}{p-1} - \frac{N-2\gamma}{2}\right), & \tau_0^{(0)} = 0, & \text{for } \frac{N}{N-2\gamma} < p < p_1, \\ \sigma_0^{(0)} = 0, & \tau_0^{(0)} \in (0, \infty), & \text{for } p_1 \leq p < \frac{N+2\gamma}{N-2\gamma}, \end{cases}$$

and

$$\sigma_j^{(0)} > 2B_0 = \frac{N-2\gamma}{2} \quad \text{for } j \geq 1.$$

For the next mode $m = 1$, one can check by direct calculation that $\alpha_1 = \frac{2\gamma}{p-1} + 1 - \frac{N-2\gamma}{2}$ is a solution to (7.10). By the monotonicity (7.12), there are no other real solutions in $(0, \alpha_1)$. This also implies that $\Phi_1(0,0) > \lambda(p)$, by (7.13), there are no solutions in the imaginary axis.

Moreover, using the fact that $\Phi_m(\alpha, 0)$ is increasing in m, and $\Phi_m(\pm B_m, 0) = 0$, we can get a sequence of real solutions $\alpha_m \in (0, B_m)$ for $m \geq 1$ that is increasing. Moreover, from (7.11), one also has that in the region $|\alpha| \leq B_m$, all the solutions to (7.10) are real.

Then, denoting the solutions to (7.10) by $\sigma_j^{(m)} \pm i\tau_j^{(m)}$ and $-\sigma_j^{(m)} \pm i\tau_j^{(m)}$ for $m \geq 1$, we conclude that:

$$\sigma_0^{(1)} = \frac{2\gamma}{p-1} + 1 - \frac{N-2\gamma}{2}, \quad \{\sigma_0^{(m)}\} \text{ is increasing}, \quad \tau_0^{(m)} = 0.$$

We finally consider statement $i.$ in the Lemma and look for the indicial roots of $\mathcal{L}_1$ at $r = +\infty$. In this case, δ will satisfy the following equation:

$$2^{2\gamma} \frac{\left| \Gamma\left(\frac{1}{2} + \frac{\gamma}{2} + \frac{1}{2}\sqrt{\left(\frac{N}{2} - 1\right)^2 + \mu_m} + \frac{\delta}{2} \right) \right|^2}{\left| \Gamma\left(\frac{1}{2} - \frac{\gamma}{2} + \frac{1}{2}\sqrt{\left(\frac{N}{2} - 1\right)^2 + \mu_m} + \frac{\delta}{2} \right) \right|^2} = 0.$$

For each fixed $m = 0, 1, \ldots$, the indicial roots occur when

$$\frac{1}{2} - \frac{\gamma}{2} + \frac{1}{2}\sqrt{\left(\frac{N}{2} - 1\right)^2 + \mu_m} \pm \frac{\delta}{2} = j, \quad \text{for } j = 0, -1, -2, \ldots, -\infty,$$

or

$$\pm\delta = (1 - \gamma) + \sqrt{(\tfrac{N}{2} - 1)^2 + \mu_m} + 2j, \quad j = 0, 1, 2, \ldots.$$

Thus, the indicial roots for $\mathcal{L}_1$ at $r = +\infty$ are given by

$$-\frac{N - 2\gamma}{2} \pm \left\{ (1 - \gamma) + \sqrt{(\tfrac{N}{2} - 1)^2 + \mu_m} \right\} \pm 2j, \quad j = 0, 1, \ldots.$$

This finishes the proof of the Lemma. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7.2. Injectivity of $\mathcal{L}_1$ in the weighted space $\mathcal{C}^{2\gamma+\alpha}_{\mu,\nu_1}$.
The arguments in this section rely heavily on the results from Section 6. We fix

$$(7.15) \qquad\qquad \mu > \operatorname{Re}(\gamma_0^+) \geq -\frac{N - 2\gamma}{2}, \quad \nu_1 \leq \min\{0, \mu\}.$$

Such μ is chosen to exclude the trivial solution $\phi_* = r\partial_r u_1 + \frac{p-1}{2\gamma} u_1$.

Proposition 7.2. *Under the hypothesis* (7.15), *the only solution $\phi \in \mathcal{C}^{2\gamma+\alpha}_{\mu,\nu_1}(\mathbb{R}^N \setminus \{0\})$ of the equation $\mathcal{L}_1\phi = 0$ is the trivial solution $\phi \equiv 0$.*

Proof. We would like to classify solutions to the following equation:

$$(-\Delta_{\mathbb{R}^N})^\gamma \phi = p A_{N,p,\gamma} u_1^{p-1} \phi \text{ in } \mathbb{R}^N \setminus \{0\},$$

or equivalently, (7.8) or (7.9) for each $m = 0, 1, \ldots$.

Step 1: the mode $m = 0$. Define the constant $\tau = p A_{N,p,\gamma}$ and rewrite equation (7.5), i.e. equation (7.9) for $m = 0$, as

$$(7.16) \qquad\qquad P^0_\gamma(w) - \tau w = (V - \tau)w =: h,$$

for $w = w(t)$, the conjugation of ϕ_0 according to (7.4), and the right hand side $h = h(t)$, both written in the variable $t = -\log r \in \mathbb{R}$. By definition,

$$(7.17) \qquad\qquad w(t) = \begin{cases} O(e^{-(\mu + \frac{N-2\gamma}{2})t}) & \text{as } t \to +\infty, \\ O(e^{-(\nu_1 + \frac{N-2\gamma}{2})t}) & \text{as } t \to -\infty. \end{cases}$$

We use also (7.7) to estimate the right hand side,

$$h(t) = \begin{cases} O(e^{-(q_1 + \mu + \frac{N-2\gamma}{2})t}) & \text{as } t \to +\infty, \\ O(e^{-(\nu_1 + \frac{N-2\gamma}{2})t}) & \text{as } t \to -\infty. \end{cases}$$

By our choice of weights μ, ν_1 from (7.15), we have that $\mu + \frac{N-2\gamma}{2} > \sigma_0^{(0)}$ and $\nu_1 + \frac{N-2\gamma}{2} < \sigma_1^{(0)}$ (for this, recall statements *a)* and *b)* in Lemma 7.1).

We use Theorem 6.1 and Proposition 6.3 with $\delta = q_1 + \mu + \frac{N-2\gamma}{2} > \operatorname{Re}(\gamma_0^+ + \frac{N-2\gamma}{2}) = \sigma_0^{(0)}$ and $\delta_0 = -(\nu_1 + \frac{N-2\gamma}{2}) > -\sigma_1^{(0)}$. Obviously, $\delta + \delta_0 = q_1 + (\mu - \nu_1) > 0$. Let $J \geq 0$ be such that $\sigma_J^{(0)} < \delta < \sigma_{J+1}^{(0)}$. Then we can find a particular solution w_p (depending on J) such that

$$w_p(t) = O(e^{-\delta t}), \quad \text{as} \quad t \to +\infty.$$

Our solution will be obtained from this particular solution by adding elements in the kernel. By our choice of μ (hence δ), we could only add exponentials $e^{-\sigma_j^{(0)} t}$, $j > J$. But these grow fast at $-\infty$ and thus are not allowed by the choice of ν_1, and we must have $w = w_p$. But again, by the theorem, either w_p grows as $t \to -\infty$ like $O(e^{\delta_0 t})$, or it has the asymptotic behavior $w_p \sim e^{-\sigma_j^{(0)} t}$ for some $j = 0, \ldots, J$ (maybe after passing to a subsequence in t, in case $\tau_j^{(0)} \neq 0$). But by (7.17) this second scenario is not allowed unless $j = 0$, so we must conclude that $w = w_p$ satisfies

$$w(t) = \begin{cases} O(e^{-\delta t}) & \text{as} \quad t \to +\infty, \\ O(e^{\min\{-\sigma_0^{(0)}, \delta_0\} t}) & \text{as} \quad t \to -\infty. \end{cases}$$

With this and (7.17) we see that we have improved the decay of $w = w_p$ at $+\infty$ by $e^{-q_1 t}$ (at the expense of slightly worsening the behavior at $-\infty$).

Now, by the definition of h in (7.16), we can iterate this process with $\delta' = lq_1 + \mu + \frac{N-2\gamma}{2}$, $l \geq 2$, and $\delta_0' = \min\{-\sigma_0, \delta_0\}$, to obtain better decay when $t \to +\infty$. Indeed, with these choices of the parameters we still have $\delta' + \delta_0' = (l-1)q_1 + \delta + \min\{-\sigma_0, \delta_0\} > 0$, so that the results in Section 6 remain applicable. The Hamiltonian argument from Corollary 6.12 shows that there are no decaying solutions on both sides $t \to \pm\infty$ except for ϕ_*. As a consequence, we have that w decays faster than any $e^{-\delta t}$ as $t \to +\infty$, which translated to ϕ means that $\phi = o(r^a)$ as $r \to 0$ for every $a \in \mathbb{N}$. Note the strong unique continuation result of [39] (stable case) and [76] (unstable case) for the operator $P_\gamma^0 - V$ implies that ϕ must vanish everywhere.

Step 2: the modes $m = 1, \ldots, N$. Differentiating the equation (2.1) we get

$$\mathcal{L}_1 \frac{\partial u_1}{\partial x_m} = 0.$$

Since u_1 only depends on r, we have $\frac{\partial u_1}{\partial x_m} = u_1'(r) E_m$, where $E_m = \frac{x_m}{|x|}$. Using the fact that $-\Delta_{\mathbb{S}^{N-1}} E_m = \mu_m E_m$, the extension for $u_1'(r)$ to $\mathbb{R}_+^{N+1}$ solves (7.8) with eigenvalue $N-1$, and $w_1 := r^{\frac{N-2\gamma}{2}} u_1'$ satisfies $P_\gamma^m w - V w = 0$. Note that u_1' decays like $r^{-(N+1-2\gamma)}$ as $r \to \infty$ and blows up like $r^{-\frac{2\gamma}{p-1}-1}$ as $r \to 0$.

We know that also ϕ_m solves (7.9). Assume it decays like $r^{-(N+1-2\gamma)}$ as $r \to \infty$ and blows up like $r^{\gamma_m^+}$ as $r \to 0$. Then we can find a non-trivial combination of u_1' and ϕ_m that decays faster than $r^{-(N+1-2\gamma)}$ at infinity. Since their singularities at 0 cannot cancel, this combination is non-trivial.

Now we claim that no solution to (7.9) can decay faster than $r^{-(N+1-2\gamma)}$ at ∞, which is a contradiction and yields that $\phi_m = 0$ for $m = 1, \ldots, N$.

To show this claim we argue as in Step 1, using the indicial roots at infinity (namely $-(N+1-2\gamma)$ and 1) and interchanging the role of $+\infty$ and $-\infty$ in the decay estimate. Using the facts that the solution decays like r^σ for some $\sigma < -(N-2\gamma+1)$, i.e. $\sigma + \frac{N-2\gamma}{2} < -\frac{N-2\gamma}{2} - 1 = -\sigma_0^{(1)}$ and $\operatorname{Re}(\gamma_m^+) + \frac{N-2\gamma}{2} < \sigma_1^{(1)}$, one can show that the solution is identically zero.

Step 3: the remaining modes $m \geq N+1$. We use an integral estimate involving the first mode which has a sign, as in [32, 31]. We note that, in particular, $\phi_1(r) = -u_1'(r) > 0$, which also implies

that its extension Φ_1 is positive. In general, the γ-harmonic extension Φ_m of ϕ_m satisfies

$$
\begin{cases}
\operatorname{div}(\ell^{1-2\gamma}\nabla\Phi_m) = \mu_m \dfrac{\ell^{1-2\gamma}}{r^2}\Phi_m & \text{in } \mathbb{R}_+^{N+1}, \\
-\tilde{d}_\gamma \lim\limits_{\ell\to 0} \ell^{1-2\gamma}\partial_\ell\Phi_m = pu_1^{p-1}\phi_m & \text{on } \mathbb{R}_+^{N+1}.
\end{cases}
$$

We multiply this equation by Φ_1 and the one with $m=1$ by Φ_m. Their difference gives the equality

$$
(\mu_m - \mu_1)\frac{\ell^{1-2\gamma}}{r^2}\Phi_m\Phi_1 = \Phi_1 \operatorname{div}(\ell^{1-2\gamma}\nabla\Phi_m) - \Phi_m \operatorname{div}(\ell^{1-2\gamma}\nabla\Phi_1)
$$
$$
= \operatorname{div}(\ell^{1-2\gamma}(\Phi_1\nabla\Phi_m - \Phi_m\nabla\Phi_1)).
$$

Let us integrate over the region where $\Phi_m > 0$. The functions are regular enough near $x = 0$ by the restriction (7.15). The boundary $\partial\{\Phi_m > 0\}$ is decomposed into a disjoint union of $\partial^0\{\Phi_m > 0\}$ and $\partial^+\{\Phi_m > 0\}$, on which the extension variable $\ell = 0$ and $\ell > 0$, respectively. Hence

$$
0 \le \tilde{d}_\gamma(\mu_m - \mu_1)\int_{\{\Phi_m>0\}} \frac{\Phi_m\Phi_1}{r^2}\, dxd\ell
$$
$$
= \int_{\partial^0\{\Phi_m>0\}}\left(\phi_1 \lim_{\ell\to 0}\ell^{1-2\gamma}\frac{\partial\Phi_m}{\partial\nu} - \phi_m \lim_{\ell\to 0}\ell^{1-2\gamma}\frac{\partial\Phi_1}{\partial\nu}\right) dx
$$
$$
+ \int_{\partial^+\{\Phi_m>0\}} \ell^{1-2\gamma}\left(\Phi_1\frac{\partial\Phi_m}{\partial\nu} - \Phi_m\frac{\partial\Phi_1}{\partial\nu}\right) dxd\ell.
$$

The first integral on the right hand side vanishes due to the equations Φ_1 and Φ_m satisfy. Then we observe that on $\partial^+\{\Phi_m > 0\}$, one has $\Phi_1 > 0$, $\frac{\partial\Phi_m}{\partial\nu} \le 0$ and $\Phi_m = 0$. This forces (using $\mu_m > \mu_1$)

$$
\int_{\{\Phi_m>0\}} \frac{\Phi_m\Phi_1}{r^2}\, dxd\ell = 0,
$$

which in turn implies $\Phi_m \le 0$. Similarly $\Phi_m \ge 0$ and, therefore, $\Phi_m \equiv 0$ for $m \ge N+1$. This completes the proof of the Proposition 7.2. $\qquad\square$

7.3. **Injectivity of** $\mathbb{L}_1$ **on** $\mathcal{C}_{\mu,\nu_1}^{2\gamma+\alpha}$. In the following, we set $N = n-k$ and consider more general equation (5.5). Set

$$
\mathbb{L}_1 = (-\Delta_{\mathbb{R}^n})^\gamma - pA_{N,p,\gamma}u_1^{p-1} \text{ in } \mathbb{R}^n \setminus \mathbb{R}^k.
$$

Proposition 7.3. *Choose the weights* μ, ν_1 *as in Proposition 7.2. The only solution* $\phi \in \mathcal{C}_{\mu,\nu_1}^{2\gamma+\alpha}(\mathbb{R}^n\setminus\mathbb{R}^k)$ *of the linearized equation* $\mathbb{L}_1\phi = 0$ *is the trivial solution* $\phi \equiv 0$.

Proof. The idea is to use the results from Section 3.3 to reduce $\mathbb{L}_1$ to the simpler $\mathcal{L}_1$, taking into account that u_1 only depends on the variable r but not on y. In the notation of Proposition 3.4, define $w = r^{-\frac{N-2\gamma}{2}}\phi$, and w_m its m-th projection over spherical harmonics. Set $\hat{w}_m(\lambda,\omega)$, $\lambda \in \mathbb{R}$, $\omega \in \mathbb{S}^k$ to denote its Fourier-Helgason transform. By observing that the full symbol (3.25), for each fixed ω, coincides with the symbol (3.15), we have reduced our problem to that of Proposition 7.2. This completes the proof. $\qquad\square$

7.4. ***A priori* estimates.** Now we go back to the linearized operator L_ε from (7.2) for the point singularity case $\mathbb{R}^N \setminus \{q_1, \ldots, q_K\}$, or (7.1) for the general $\mathbb{R}^n \setminus \Sigma$, and consider the equation

$$
(7.18) \qquad\qquad L_\varepsilon\phi = h.
$$

For simplicity, we use the following notation for the weighted norms

$$
(7.19) \qquad\qquad \|\phi\|_* = \|\phi\|_{\mathcal{C}_{\mu,\nu}^{2\gamma+\alpha}}, \quad \|h\|_{**} = \|h\|_{\mathcal{C}_{\mu-2\gamma,\nu-2\gamma}^{0,\alpha}}.
$$

Moreover, for this subsection, we assume that μ, ν satisfy

$$\mathrm{Re}(\gamma_0^+) < \mu \le 0, \quad -(n - 2\gamma) < \nu.$$

For this choice of weights we have the following *a priori* estimate:

Lemma 7.4. *Given h with $\|h\|_{**} < \infty$, suppose that ϕ be a solution of (7.18), then there exists a constant C independent of ε such that*

$$\|\phi\|_* \le C\|h\|_{**}.$$

Proof. We will argue by contradiction. Assume that there exists $\varepsilon_j \to 0$, and a sequence of solutions $\{\phi_j\}$ to $L_{\varepsilon_j}\phi_j = h_j$ such that

$$\|\phi_j\|_* = 1, \quad \text{and} \quad \|h_j\|_{**} \to 0 \text{ as } j \to \infty.$$

In the following we will drop the index j without confusion.

We first consider the point singularity case $\mathbb{R}^N \setminus \Sigma$ for $\Sigma = \{q_1, \ldots, q_K\}$. By Green's representation formula one has

$$\phi(x) = \int_{\mathbb{R}^N} G(x, \tilde{x})[h + pA_{N,p,\gamma}\bar{u}_\varepsilon^{p-1}\phi]\, d\tilde{x} =: I_1 + I_2,$$

where G is the Green's function for the fractional Laplacian $(-\Delta_{\mathbb{R}^N})^\gamma$ given by $G(x, \tilde{x}) = C_{N,\gamma}|x - \tilde{x}|^{-(N-2\gamma)}$ for some normalization constant $C_{N,\gamma}$. In the first step, let $x \in \mathbb{R}^N \setminus \bigcup_i B_\sigma(q_i)$. In this case

$$
\begin{aligned}
I_1 &\lesssim \int_{\mathbb{R}^N} G(x, \tilde{x})h(\tilde{x})\, d\tilde{x} \\
&= \int_{\{\mathrm{dist}(\tilde{x},\Sigma)<\frac{\sigma}{2}\}} \cdots + \int_{\{\frac{\sigma}{2}<\mathrm{dist}(\tilde{x},\Sigma)<\frac{|x|}{2}\}} \cdots + \int_{\{\frac{|x|}{2}<\mathrm{dist}(\tilde{x},\Sigma)<2|x|\}} \cdots + \int_{\{\mathrm{dist}(\tilde{x},\Sigma)>2|x|\}} \cdots \\
&\le C\|h\|_{**}(|x|^{-(N-2\gamma)} + |x|^\nu) \\
&\le C\|h\|_{**}|x|^\nu,
\end{aligned}
$$

because of our restriction of ν. Moreover,

$$
\begin{aligned}
I_2 &= \int G(x, \tilde{x})p\bar{u}_\varepsilon(\tilde{x})^{p-1}\phi(\tilde{x})\, d\tilde{x} \\
&= \int_{\{\mathrm{dist}(\tilde{x},\Sigma)<\varepsilon\}} \cdots + \int_{\{\frac{\sigma}{2}>\mathrm{dist}(\tilde{x},\Sigma)>\varepsilon\}} \cdots + \int_{\{\mathrm{dist}(\tilde{x},\Sigma)>\frac{\sigma}{2}\}} \cdots =: I_{21} + I_{22} + I_{23}.
\end{aligned}
$$

Calculate

$$
\begin{aligned}
I_{21} &\le \int_{\{\mathrm{dist}(\tilde{x},\Sigma)<\varepsilon\}} |x - \tilde{x}|^{-(N-2\gamma)}\varrho(\tilde{x})^{-2\gamma}\phi \le \|\phi\|_* \int_{\{\mathrm{dist}(\tilde{x},\Sigma)<\varepsilon\}} |x - \tilde{x}|^{-(N-2\gamma)}\varrho(\tilde{x})^{\mu-2\gamma}\, d\tilde{x}, \\
&\le C\varepsilon^{N+\mu-2\gamma}\|\phi\|_*\varrho(x)^{-(N-2\gamma)},
\end{aligned}
$$

where ϱ is the weight function defined in Section 5.1, and

$$
\begin{aligned}
I_{22} &= \int_{\{\mathrm{dist}(\tilde{x},\Sigma)>\frac{\sigma}{2}\}} |x - \tilde{x}|^{-(N-2\gamma)}\varepsilon^{N(p-1)-2p\gamma}\varrho(\tilde{x})^{-(N-2\gamma)(p-1)}\phi\, d\tilde{x} \\
&\le \|\phi\|_* \int_{\{R>\mathrm{dist}(\tilde{x},\Sigma)>\frac{\sigma}{2}\}} |x - \tilde{x}|^{-(N-2\gamma)}\varepsilon^{N(p-1)-2p\gamma}\varrho(\tilde{x})^{\mu-(N-2\gamma)(p-1)}\, d\tilde{x} \\
&\quad + \|\phi\|_* \int_{\{\mathrm{dist}(\tilde{x},\Sigma)>R\}} |x - \tilde{x}|^{-(N-2\gamma)}\varepsilon^{N(p-1)-2p\gamma}\varrho(\tilde{x})^{\nu-(N-2\gamma)(p-1)} \Big]\, d\tilde{x} \\
&\lesssim \varepsilon^{N(p-1)-2p\gamma}\varrho(x)^{-(N-2\gamma)}\|\phi\|_*.
\end{aligned}
$$

Next for I_{23},

$$I_{23} \lesssim \int_{\{\varepsilon < \mathrm{dist}(\tilde{x}, \Sigma) < \frac{\sigma}{2}\}} |x - \tilde{x}|^{-(N-2\gamma)} \varepsilon^{N(p-1)-2p\gamma} \varrho(\tilde{x})^{-(N-2\gamma)(p-1)} \phi \, d\tilde{x}$$

$$\leq \varepsilon^{N(p-1)-2p\gamma} \varrho(x)^{-(N-2\gamma)} \|\phi\|_* \int_{\{\varepsilon < |\tilde{x}| < \frac{\sigma}{2}\}} |\tilde{x}|^{\mu-(N-2\gamma)(p-1)} \, d\tilde{x}$$

$$\lesssim (\varepsilon^{N(p-1)-2p\gamma} + \varepsilon^{\mu-2\gamma+N}) \|\phi\|_* \varrho(x)^{-(N-2\gamma)}.$$

Combining the above estimates, one has

$$I_2 \leq C(\varepsilon^{N(p-1)-2p\gamma} + \varepsilon^{\mu-2\gamma+N}) \varrho(x)^{-(N-2\gamma)} \|\phi\|_*$$

$$\lesssim (\varepsilon^{N(p-1)-2p\gamma} + \varepsilon^{\mu-2\gamma+N}) \varrho(x)^{\nu} \|\phi\|_*,$$

and thus

$$\sup_{\{\mathrm{dist}(x,\Sigma)>\sigma\}} \{\varrho(x)^{-\nu} |\phi|\} \leq C(\|h\|_{**} + o(1)\|\phi\|_*),$$

which implies, because our initial assumptions on ϕ, that there exists q_i such that

(7.20)
$$\sup_{\{|x-q_i|<\sigma\}} |x - q_i|^{-\mu} |\phi| \geq \frac{1}{2}.$$

In the second step we study the region $|x - q_i| < \sigma$. Without loss of generality, assume $q_i = 0$. Recall that we are writing $\phi = I_1 + I_2$. On the one hand,

$$I_1 = \int_{\mathbb{R}^N} G(x, \tilde{x}) h(\tilde{x}) \, d\tilde{x}$$

$$= \int_{\{|\tilde{x}|>2\sigma\}} \cdots + \int_{\{|\tilde{x}|<\frac{|x|}{2}\}} \cdots + \int_{\{\frac{|x|}{2}<|\tilde{x}|<2|x|\}} \cdots + \int_{\{2|x|<|\tilde{x}|<2\sigma\}} \cdots$$

$$\leq c\|h\|_{**} \Big[\int_{\{|\tilde{x}|>2\sigma\}} |x - \tilde{x}|^{-(N-2\gamma)} |\tilde{x}|^{\nu-2\gamma} \, d\tilde{x} + \int_{\{|\tilde{x}|<\frac{|x|}{2}\}} |x - \tilde{x}|^{-(N-2\gamma)} |\tilde{x}|^{\mu-2\gamma} \, d\tilde{x}$$

$$+ \int_{\{\frac{|x|}{2}<|\tilde{x}|<2|x|\}} |x - \tilde{x}|^{-(N-2\gamma)} |\tilde{x}|^{\mu-2\gamma} \, d\tilde{x} + \int_{\{2|x|<|\tilde{x}|<2\sigma\}} |x - \tilde{x}|^{-(N-2\gamma)} |\tilde{x}|^{\mu-2\gamma} \, d\tilde{x} \Big]$$

$$\leq C\|h\|_{**} |x|^{\mu}.$$

On the other hand, for I_2, recall that ϕ is a solution to

$$(-\Delta_{\mathbb{R}^N})^{\gamma} \phi - p A_{N,p,\gamma} \bar{u}_{\varepsilon}^{p-1} \phi = h.$$

Define $\bar{\phi}(\tilde{x}) = \varepsilon^{-\mu} \phi(\varepsilon \tilde{x})$, then $\bar{\phi}$ satisfies

$$(-\Delta_{\mathbb{R}^N})^{\gamma} \bar{\phi} - p A_{N,p,\gamma} u_1^{p-1} \bar{\phi} = \varepsilon^{2\gamma-\mu} h(\varepsilon \tilde{x}).$$

By the assumption that $\|h\|_{**} \to 0$, one has that the right hand side tends to 0 as $j \to \infty$. Since $|\bar{\phi}(\tilde{x})| \leq C\|\phi\|_* |\tilde{x}|^{\mu}$ locally uniformly, and by regularity theory, $\bar{\phi} \in \mathcal{C}_{loc}^{\eta}(\mathbb{R}^N \setminus \{0\})$ for some $\eta \in (0,1)$, thus passing to a subsequence, $\bar{\phi} \to \phi_{\infty}$ locally uniformly in any compact set, where $\phi_{\infty} \in \mathcal{C}_{\mu,\mu}^{\alpha+2\gamma}(\mathbb{R}^N \setminus \{0\})$ is a solution of

(7.21)
$$(-\Delta_{\mathbb{R}^N})^{\gamma} \phi_{\infty} - p A_{N,p,\gamma} u_1^{p-1} \phi_{\infty} = 0 \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

(to handle the non-locality we may pass to the extension in a standard way). Since $\mu \leq 0$, it satisfies the condition in Proposition 7.2, from which we get that $\phi_{\infty} \equiv 0$, so $\bar{\phi} \to 0$.

Now we go back to the calculation of I_2. Here we use the change of variable $x = \varepsilon x_1$.

$$
\begin{aligned}
I_2 &= \int_{\{|\tilde{x}|<\sigma\}} p|x - \tilde{x}|^{-(N-2\gamma)} \bar{u}_\varepsilon^{p-1} \phi \, d\tilde{x} = \varepsilon^\mu \int_{\{|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} |x_1 - \tilde{x}|^{-(N-2\gamma)} u_1^{p-1}(\tilde{x}) \bar{\phi}(\tilde{x}) \, d\tilde{x} \\
&= \varepsilon^\mu \Big[\int_{\{|\tilde{x}|<\frac{1}{R}\}} \cdots + \int_{\{\frac{1}{R}<|\tilde{x}|<R\}} \cdots + \int_{\{R<|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} \cdots \Big] =: J_1 + J_2 + J_3,
\end{aligned}
$$

for some positive constant R large enough to be determined later. For J_1, fix x, when $\varepsilon \to 0$ one has

$$
\begin{aligned}
J_1 &= \varepsilon^\mu \int_{\{|\tilde{x}|<\frac{1}{R}\}} |x_1 - \tilde{x}|^{-(N-2\gamma)} u_1^{p-1}(\tilde{x}) \bar{\phi}(\tilde{x}) \, d\tilde{x} \le \varepsilon^\mu \|\phi\|_* |x_1|^{-(N-2\gamma)} \int_{\{|\tilde{x}|<\frac{1}{R}\}} |\tilde{x}|^{\mu-2\gamma} \, d\tilde{x} \\
&\le C R^{-(N-2\gamma+\mu)} \|\phi\|_* |x|^\mu.
\end{aligned}
$$

For J_2 we use the fact that in this region $\bar{\phi} \to 0$, so

$$
\begin{aligned}
J_2 &= \varepsilon^\mu \int_{\{\frac{1}{R}<|\tilde{x}|<R\}} |x_1 - \tilde{x}|^{-(N-2\gamma)} u_1^{p-1}(\tilde{x}) \bar{\phi}(\tilde{x}) \, d\tilde{x} \\
&= o(1)\varepsilon^\mu \int_{\{\frac{1}{R}<|\tilde{x}|<R\}} \frac{1}{|x_1 - \tilde{x}|^{N-2\gamma}} \, d\tilde{x} = o(1)\varepsilon^\mu |x_1|^{-(N-2\gamma)} = o(1)|x|^\mu,
\end{aligned}
$$

and finally,

$$
\begin{aligned}
J_3 &= \varepsilon^\mu \int_{\{R<|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} |x_1 - \tilde{x}|^{-(N-2\gamma)} u_1^{p-1}(\tilde{x}) \bar{\phi}(\tilde{x}) \, d\tilde{x} \\
&= \varepsilon^\mu \|\phi\|_* \Big[\int_{\{R<|\tilde{x}|<\frac{|x_1|}{2}\}} |x_1 - \tilde{x}|^{-(N-2\gamma)} |\tilde{x}|^\mu u_1^{p-1} \, d\tilde{x} + \int_{\{\frac{|x_1|}{2}<|\tilde{x}|<2|x_1|\}} |x_1 - \tilde{x}|^{-(N-2\gamma)} u_1^{p-1} |\tilde{x}|^\mu \, d\tilde{x} \\
&\quad + \int_{\{2|x_1|<|\tilde{x}|<\frac{\sigma}{\varepsilon}\}} |x_1 - \tilde{x}|^{-(N-2\gamma)} u_1^{p-1} |\tilde{x}|^\mu \, d\tilde{x} \Big] \\
&\le C\varepsilon^\mu |x_1|^\mu \|\phi\|_* |x_1|^{-\tau} \le o(1)\|\phi\|_* |x|^\mu
\end{aligned}
$$

for some $\tau > 0$.

Combining all the above estimates, one has

$$
\||x|^{-\mu} I_1\| \le o(1)(\|\phi\|_* + 1),
$$

which implies

$$
\|\phi\|_* \le o(1)\|\phi\|_* + o(1) + \|h\|_{**} = o(1).
$$

This is a contradiction with (7.20).

For the more general case $\mathbb{R}^n \setminus \Sigma$ when Σ is a smooth k-dimensional sub-manifold, the argument is similar as above, the only difference is that one arrives to the analogous to (7.21) in the estimate for I_2 near Σ:

$$
(-\Delta_{\mathbb{R}^n})^\gamma \phi_\infty - p u_1^{p-1} \phi_\infty = 0 \quad \text{in } \mathbb{R}^n \setminus \mathbb{R}^k.
$$

After the obvious rescaling by the constant $A_{N,p,\gamma}$, where $N = n - k$, one uses Remark 5.8 and the injectivity result in Proposition 7.3 instead of the one in Proposition 7.2. This completes the proof of Lemma 7.4. $\qquad\square$

8. Fredholm properties - surjectivity

Our analysis here follows closely the one in [70] for the local case. These lecture notes are available online but, unfortunately, yet to be published.

For the rest of the paper, we will take the pair of dual weights $\mu, \tilde{\mu}$ such that $\mu + \tilde{\mu} = -(N - 2\gamma)$ and $\nu + \tilde{\nu} = -(n - 2\gamma)$ satisfying

$$-\frac{2\gamma}{p-1} < \tilde{\mu} < \text{Re}(\gamma_0^-) \leq -\frac{N-2\gamma}{2} \leq \text{Re}(\gamma_0^+) < \mu < 0,$$
(8.1)
$$-(n-2\gamma) < \tilde{\nu} < 0.$$

In order to consider the invertibility of the linear operators (7.1) and (7.2), defined in the spaces

$$L_\varepsilon : \mathcal{C}_{\tilde{\mu},\tilde{\nu}}^{2\gamma+\alpha} \to \mathcal{C}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}^{0,\alpha},$$

it is simpler to consider the conjugate operator

$$\tilde{L}_\varepsilon(w) := f_1^{-1} L_\varepsilon(f_2 w), \quad \tilde{L}_\varepsilon : \mathcal{C}_{\tilde{\mu}+\frac{N-2\gamma}{2},\tilde{\nu}+\frac{n-2\gamma}{2}}^{2\alpha+\gamma} \to \mathcal{C}_{\tilde{\mu}+\frac{N-2\gamma}{2},\tilde{\nu}+\frac{n-2\gamma}{2}}^{2\alpha+\gamma},$$
(8.2)

where f_2 is a weight $\varrho^{-\frac{n-2\gamma}{2}}$ near infinity, and $\varrho^{-\frac{N-2\gamma}{2}}$ near the singular set Σ, while f_1 is $\varrho^{-\frac{n+2\gamma}{2}}$ near infinity, and $\varrho^{-\frac{N+2\gamma}{2}}$ near the singular set. Recall that ϱ is defined in Section 5.1. In addition, in each neighborhood use Fermi coordinates as in formula (8.4) below. This conjugate operator is better behaved in weighted Hilbert spaces and simplifies the notation in the proof of Fredholm properties.

8.1. Fredholm properties.
Fredholm properties for extension problems related to this type of operators were considered in [61, 62].

In the notation of Section 5.1, and following the paper [63], we define the weighted Lebesgue space $L_{\delta,\vartheta}^2(\mathbb{R}^n \setminus \Sigma)$. These are L_{loc}^2 functions for which the norm

$$\|\phi\|_{L_{\delta,\vartheta}^2(\mathbb{R}^n \setminus \Sigma)}^2 = \int_{\mathbb{R}^n \setminus B_R} |\phi|^2 \varrho^{-2\gamma-2\vartheta} \, dz + \int_{B_R \setminus \mathcal{T}_\sigma} |\phi|^2 \, dz + \int_{\mathcal{T}_\sigma} |\phi|^2 \varrho^{N-1-2\gamma-2\delta} \, dr dy d\theta$$
(8.3)

is finite. Here $dr dy d\theta$ denotes the corresponding measure in Fermi coordinates $r > 0$, $y \in \Sigma$, $\theta \in \mathbb{S}^{N-1}$. One defines accordingly, for $\gamma > 0$, weighted Sobolev spaces $W_{\delta,\vartheta}^{2\gamma,2}$ with respect to the vector fields from Remark 5.3 (see [61] for the precise definitions).

The seemingly unusual normalization in the integrals in (8.3) is explained by the change of variable $\phi = f_2 w$. Indeed,

$$\|w\|_{L_{\delta,\vartheta}^2}^2 = \int_{-\infty}^{-\log R} \int_{\mathbb{S}^{n-1}} |w|^2 e^{2\vartheta\tilde{t}} \, d\tilde{\theta} d\tilde{t} + \int_{\{\text{dist}(\cdot,\Sigma) > \sigma, |z| < R\}} |w|^2 \, dz$$
(8.4)
$$+ \int_{-\log \sigma}^{+\infty} \int_\Sigma \int_{\mathbb{S}^{N-1}} |w|^2 e^{2\delta t} \, d\theta dy dt.$$

We have

Lemma 8.1. *For the choice of parameters*

$$-\delta < \tilde{\mu} + \frac{N-2\gamma}{2}, \quad -\vartheta > \tilde{\nu} + \frac{n-2\gamma}{2},$$

we have the continuous inclusions

$$\mathcal{C}_{\tilde{\mu},\tilde{\nu}}^{2\gamma+\alpha}(\mathbb{R}^n \setminus \Sigma) \hookrightarrow L_{-\delta,-\vartheta}^2(\mathbb{R}^n \setminus \Sigma).$$

The spaces $L_{\delta,\vartheta}^2$ and $L_{-\delta,-\vartheta}^2$ are dual with respect to the natural pairing

$$\langle \phi_1, \phi_2 \rangle_* = \int_{\mathbb{R}^n} \phi_1 \phi_2,$$

for $\phi_1 \in L^2_{\delta,\vartheta}$, $\phi_2 \in L^2_{-\delta,-\vartheta}$. Now, let $\tilde{L}_\varepsilon$ be the operator defined in (8.2). It is a densely defined, closed graph operator (this is a consequence of elliptic estimates). Then, relative to this pairing, the adjoint of

$$(8.5) \qquad\qquad \tilde{L}_\varepsilon : L^2_{-\delta,-\vartheta} \to L^2_{-\delta,-\vartheta}$$

is precisely

$$(8.6) \qquad\qquad (\tilde{L}_\varepsilon)^* = \tilde{L}_\varepsilon : L^2_{\delta,\vartheta} \to L^2_{\delta,\vartheta}.$$

Now we fix μ, $\tilde{\mu}$, ν, $\tilde{\nu}$ as in (8.1), and choose $-\delta < 0$ slightly smaller than $\tilde{\mu} + \frac{N-2\gamma}{2}$ and $-\vartheta < 0$ just slightly larger than $\tilde{\nu} + \frac{n-2\gamma}{2}$ so that, in particular,

$$(8.7) \qquad \begin{aligned} &-\tfrac{2\gamma}{p-1} + \tfrac{N-2\gamma}{2} < -\delta < \tilde{\mu} + \tfrac{N-2\gamma}{2} < 0 < \mu + \tfrac{N-2\gamma}{2} < \delta < \tfrac{N-2\gamma}{2}, \\ &-\tfrac{n-2\gamma}{2} < \tilde{\nu} + \tfrac{n-2\gamma}{2} < -\vartheta < 0 < \vartheta < \tfrac{n-2\gamma}{2}, \end{aligned}$$

and we have the inclusions from Lemma 8.1. In addition, we can choose δ, ϑ different from the corresponding indicial roots. Higher order regularity is guaranteed by the results in Section 6.2. We will show:

Proposition 8.2. *Let $\delta \in (-\frac{N+2\gamma}{2} - 2\gamma, \frac{N-2\gamma}{2})$ and $\vartheta \in (-\frac{n-2\gamma}{2}, \frac{n-2\gamma}{2})$ be real numbers satisfying* (8.7).

Assume that $w \in L^2_{\delta,\vartheta}$ is a solution to

$$\tilde{L}_\varepsilon w = \tilde{h} \ \text{on} \ \mathbb{R}^n \setminus \Sigma$$

for $\tilde{h} \in L^2_{\delta,\vartheta}$. Then we have the a priori *estimate*

$$(8.8) \qquad\qquad \|w\|_{L^2_{\delta,\vartheta}} \lesssim \|\tilde{h}\|_{L^2_{\delta,\vartheta}} + \|w\|_{L^2(\mathcal{K})},$$

where $\mathcal{K}$ is a compact set in $\mathbb{R}^n \setminus \Sigma$. Translating back to the original operator L_ε, if ϕ is a solution to $L_\varepsilon \phi = h$ in $\mathbb{R} \setminus \Sigma$, then (8.8) *is rewritten as*

$$(8.9) \qquad\qquad \|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}} \lesssim \|h\|_{L^2_{\delta,\vartheta}} + \|\phi\|_{L^2(\mathcal{K})}.$$

As a consequence, L_ε has good Fredholm properties. The same is true for the linear operators from (8.5) *and* (8.6).

Proof. The proof here goes by subtracting suitable parametrices near Σ and near infinity thanks to Theorem 6.1. Then the remainder is a compact operator. For simplicity we set $\varepsilon = 1$.

We first consider the point singularity case, i.e., $k = 0$, $n = N$.

Step 1: (Localization) Let us study how the operator $L_1 : L^2_{\delta+2\gamma,\vartheta+2\gamma} \to L^2_{\delta,\vartheta}$ is affected by localization, so that it is enough to work with functions supported near infinity and near the singular set.

In the first step, assume that the singularity happens only at $r = \infty$ (but not at $r = 0$). We would like to patch a suitable parametrix at $r = \infty$. Let χ be a cut-off function such that $\chi = 1$ in $\mathbb{R}^N \setminus B_R$, $\chi = 0$ in $B_{R/2}$. Let $\mathcal{K} = B_{2R}$, and set

$$h_1 := L_1(\chi \phi) = \chi L_1 \phi + [L_1, \chi]\phi,$$

where $[\cdot, \cdot]$ denotes the commutator operator. Contrary to the local case, the commutator term does not have compact support, but can still give good estimates in weighted Lebesgue spaces by carefully controlling the the tail terms. Let

$$I(x) := [L_1, \chi]\phi(x) = (-\Delta_{\mathbb{R}^N})^\gamma(\chi\phi)(x) - \chi(x)(-\Delta_{\mathbb{R}^N})^\gamma\phi(x) = k_{N,\gamma} \int_{\mathbb{R}^N} \frac{\chi(x) - \chi(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \phi(\tilde{x}) \, d\tilde{x}.$$

Let us bound this integral in $L^2_{\delta,\vartheta}$.

We first consider the case $|x| \ll 1$. Note that

$$I(x) = \int_{B_{2R} \setminus B_{R/2}} \frac{\chi(x) - \chi(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \phi(\tilde{x}) \, d\tilde{x} + \int_{\mathbb{R}^N \setminus B_{2R}} \frac{\phi(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x}$$

$$\leq C R^{\frac{2\vartheta + 2\gamma - N}{2}} \|\phi\|_{L^2_{\delta+2\gamma, \vartheta+2\gamma}} + \|\phi\|_{L^2(\mathcal{K})}.$$

We have that $\|I\|_{L^2_\delta(B_1)}$ can be bounded by $o(1)\|\phi\|_{L^2_{\delta+2\gamma, \vartheta+2\gamma}} + \|\phi\|_{L^2(\mathcal{K})}$ for large R if $\vartheta < \frac{N-2\gamma}{2}$ and $\delta < \frac{N-2\gamma}{2}$.

Next, for $|x| \geq 2R$, we need to add the weight at infinity and calculate $\|I\|_{L^2_\vartheta(\mathbb{R}^N \setminus B_{2R})}$. But

$$I(x) = \int_{B_{R/2}} \frac{\phi(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x} + \int_{B_{2R} \setminus B_{R/2}} \frac{\chi(x) - \chi(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \phi(\tilde{x}) \, d\tilde{x}$$

$$\leq C \|\phi\|_{L^2_{\delta+2\gamma, \vartheta+2\gamma}} R^{\frac{2\vartheta + 6\gamma + N}{2}} |x|^{-(N+2\gamma)} \text{ if } \delta > -\frac{N+2\gamma}{2} - 2\gamma.$$

One has that $\|I\|_{L^2_\vartheta(\mathbb{R}^N \setminus B_R)} = o(1)\|\phi\|_{L^2_{\delta+2\gamma, \vartheta+2\gamma}}$ if $\vartheta > -\frac{N+2\gamma}{2} - 2\gamma$.

Now let $1 \leq |x| < 2R$, and calculate $\|I\|_{L^2(B_{2R} \setminus B_1)}$. Again, we split

$$I(x) = \int_{B_{R/2}} \frac{\chi(x)\phi(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x} + \int_{B_{2R} \setminus B_{R/2}} \frac{\chi(x) - \chi(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \phi(\tilde{x}) \, d\tilde{x} + \int_{\mathbb{R}^N \setminus B_{2R}} \frac{(\chi(x) - 1)\phi(\tilde{x})}{|x - \tilde{x}|^{N+2\gamma}} \, d\tilde{x}$$

$$=: I_{21} + I_{22} + I_{23}.$$

Similar to the above estimates, we can get that the L^2 norm can be bounded by

$$\|\phi\|_{L^2(\mathcal{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma, \vartheta+2\gamma}}$$

if $\vartheta < \frac{N-2\gamma}{2}$. Thus we have shown that

$$\|h_1\|_{L^2_{\delta,\vartheta}} \lesssim \|h\|_{L^2_{\delta,\vartheta}} + \|\phi\|_{L^2(\mathcal{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma, \vartheta+2\gamma}},$$

so localization does not worsen estimate (8.9).

In addition, the localization at $r = 0$ is similar. One just needs to interchange the role of r and $1/r$, and ϑ by δ. Similar to the estimates in Step 5 below, one can get that the error caused by the localization can be bounded by $\|\phi\|_{L^2(\mathcal{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma, \vartheta+2\gamma}}$ if $-\frac{N+2\gamma}{2} - 2\gamma < \vartheta < \frac{N-2\gamma}{2}$, $-\frac{N+2\gamma}{2} - 2\gamma < \delta < \frac{N-2\gamma}{2}$.

Step 2: (The model operator) After localization around one of the singular points, say $q_1 = 0$, the operator L_1 can be approximated by the model operator $\mathcal{L}_1$ from (7.3), or by its conjugate given in (7.5). Moreover, recalling the notation (7.6) for the potential term and its asymptotics (7.7), it is enough to show that

$$(8.10) \qquad \qquad \|w\|_{L^2_\delta} \lesssim \|\tilde{h}\|_{L^2_\delta},$$

if $w = w(t, \theta)$ is a solution of

$$(8.11) \qquad \qquad P^{go}_\gamma w - \kappa w = \tilde{h}, \quad t \in \mathbb{R}, \theta \in \mathbb{S}^{N-1},$$

that has compact support in $t \in (0, \infty)$. Here we have denoted $\kappa = pA_{N,p,\gamma}$.

Now project over spherical harmonics, so that $w = \sum_m w_m(t)E_m(\theta)$, and w_m satisfies

$$P^m_\gamma w_m - \kappa w_m = h_m, \quad t \in \mathbb{R}.$$

Our choice of weights (8.7) implies that there are no additional solutions to the homogeneous problem and that we can simply write our solution as (6.3), in Fourier variables. Then

(8.12)
$$\int_{\mathbb{R}} e^{2\delta t} |w_m(t)|^2\, dt = \int_{\mathbb{R}} |w_m(\xi + \delta i)|^2\, d\xi = \int_{\mathbb{R}} \frac{1}{|\Theta_\gamma^m(\xi + \delta i) - \kappa|^2} |\hat{h}_m(\xi + \delta i)|^2\, d\xi$$
$$\leq C \int_{\mathbb{R}} |\hat{h}_m(\xi + \delta i)|^2\, d\xi = \int_{\mathbb{R}} e^{2\delta t} |h_m(t)|^2\, dt,$$

where we have used (6.14). (note that there are no poles on the $\mathbb{R} + \delta i$ line). Estimate (8.10) follows after taking sum in m and the fact that $\{E_m\}$ is an orthonormal basis.

For the estimate near infinity, we proceed in a similar manner, just approximating the potential by $\tau = 0$.

Step 3: (Compactness) Let $\{w_j\}$ be a sequence of solutions to $\tilde{L}_1 w_j = \tilde{h}_j$ with $\tilde{h}_j \in L^2_{\delta,\vartheta}$. Assume that we have the uniform bound $\|w_j\|_{L^2_{\delta,\vartheta}} \leq C$. Then there exists a subsequence, still denoted by $\{w_j\}$, that is convergent in $L^2_{\delta,\vartheta}$ norm. Indeed, by the regularity properties of the equation, $\|w_j\|_{W^{2\gamma,2}_{\delta,\vartheta}} \leq C$, which in particular, implies a uniform $W^{2\gamma,2}$ in every compact set $\mathcal{K}$. But this is enough to conclude that $\{w_j\}$ has a convergent subsequence in $W^{2\gamma,2}(\mathcal{K})$. Finally, estimate (8.8) implies that this convergence is also true in $L^2_{\delta,\vartheta}$, as we claimed.

Step 4: (Fredholm properties for $\tilde{L}_1$) This is a rather standard argument. First, assume that the kernel is infinite dimensional, and take an orthonormal basis $\{w_j\}$ for this kernel. Then, by the claim in Step 3, we can find a Cauchy subsequence. But, for this,

$$\|w_j - w_{j'}\|^2 = \|w_j\|^2 + \|w_{j'}\|^2 = 2,$$

a contradiction.

Second, we show that the operator has closed range. Let $\{w_j\}$, $\{\tilde{h}_j\}$ be two sequences such that

(8.13)
$$\tilde{L}_1 w_j = \tilde{h}_j \quad \text{and} \quad \tilde{h}_j \to h \text{ in } L^2_{\delta,\vartheta}.$$

Since $\mathrm{Ker}\,\tilde{L}_1$ is closed, we can use the projection theorem to write $w_j = w_j^0 + w_j^1$ for $w_j^0 \in \mathrm{Ker}\,\tilde{L}_1$ and $w_j^1 \in (\mathrm{Ker}\,\tilde{L}_1)^\perp$. We have that $\tilde{L}_1 w_j^1 = \tilde{h}_j$.

Now we claim that this sequence is uniformly bounded, i.e., $\|w_j^1\|_{L^2_{\delta,\vartheta}} \leq C$ for every j. By contradiction, assume that $\|w_j^1\|_{L^2_{\delta,\vartheta}} \to \infty$ as $j \to \infty$, and rescale

$$\tilde{w}_j = \frac{w_j^1}{\|w_j^1\|_{L^2_{\delta,\vartheta}}}$$

so that the new sequence has norm one in $L^2_{\delta,\vartheta}$. From the previous remark, there is a convergent subsequence, still denoted by $\{\tilde{w}_j\}$, i.e., $\tilde{w}_j \to \tilde{w}$ in $L^2_{\delta,\vartheta}$. Moreover, we know that $\tilde{L}_1 \tilde{w} = 0$. However, by assumption we have that $w_j^1 \in (\mathrm{Ker}\,\tilde{L}_1)^\perp$, therefore so does $\tilde{w}$. We conclude that $\tilde{w}$ must vanish identically, which is a contradiction with the fact that $\|w_j^1\|_{L^2_{\delta,\vartheta}} = 1$. The claim is proved.

Now, using the remark in Step 3 again, we know that there exists a convergent subsequence $w_j^1 \to w^1$ in $L^2_{\delta,\vartheta}$. This w^1 must be regular, so we can pass to the limit in (8.13) to conclude that $\tilde{L}_1(w^1) = h$, as desired.

Step 5. Now we consider the case that Σ is a sub-manifold of dimension k, and study the localization near a point in $z_0 \in \Sigma$. In Fermi coordinates $z = (x, y)$, this is a similar estimate to that of (5.8).

First let χ be a cut-off function such that $\chi(r) = 1$ for $r \leq d$ and $\chi(r) = 0$ for $r \geq 2d$. Define $\tilde{\chi}(z) = \chi(\mathrm{dist}(z, \Sigma))$, and consider $\tilde{\phi} = \tilde{\chi}\phi$. Using Fermi coordinates near Σ and around a point

$z_0 \in \Sigma$, that can be taken to be $z_0 = (0,0)$ without loss of generality, then, for $z = (x,y)$ satisfying $|x| \ll 1, |y| \ll 1$, by checking the estimates in the proof of Lemma 5.7, one can get that

$$(8.14) \qquad (-\Delta_z)^\gamma \tilde{\phi}(z) = (1 + |x|^{\frac{1}{2}})(-\Delta_{\mathbb{R}^n \setminus \mathbb{R}^k})^\gamma \tilde{\phi}(x,y) + |x|^{-\gamma}\tilde{\phi} + \mathcal{R}_2$$

where

$$\mathcal{R}_2 = \int_\Sigma \int_{\{|x|^\beta < |\tilde{x}| < 2d\}} \frac{\tilde{\phi}}{|\tilde{x}|^{N+2\gamma}} \, d\tilde{x}dy + |x|^{-\beta(N+2\gamma)} \int_\Sigma \int_{\{|\tilde{x}| < |x|^\beta\}} \tilde{\phi} \, d\tilde{x}dy$$

$$+ |x|^{\beta k} \int_\Sigma \int_{\{|x|^\beta < |\tilde{x}| < 2d\}} \frac{\tilde{\phi}}{|\tilde{x}|^{n+2\gamma}} \, d\tilde{x}dy$$

$$\leq \|\tilde{\phi}\|_{L^2_{\delta+2\gamma}} |x|^{-\frac{1}{4}(N-2\gamma-2\delta)},$$

where we have used Hölder inequality and that $\beta = \frac{1}{2}$. Here $\|\tilde{\phi}\|_{L^2_{\delta+2\gamma}}$ is the weighted norm near Σ. One can easily check that the L^2_δ norm of $\mathcal{R}_2$ and $|x|^{-\gamma}\tilde{\phi}$ are bounded by $o(1)\|\phi\|_{L_{\delta+2\gamma,\vartheta+2\gamma}}$ for small d if $\delta < \frac{N-2\gamma}{2}$.

Next we consider the effect of the localization. Let

$$I_1(z) = [L_1, \tilde{\chi}]\phi = k_{n,\gamma} \int_{\mathbb{R}^n} \frac{\tilde{\chi}(z) - \tilde{\chi}(\tilde{z})}{|z - \tilde{z}|^{n+2\gamma}} \phi(\tilde{z}) \, d\tilde{z}.$$

For $|z| \gg 1$, one has

$$I_1(z) \lesssim |z|^{-(n+2\gamma)} \int_{\mathcal{T}_{2d}} \phi(\tilde{z}) \, d\tilde{z} \lesssim d^{\delta + \frac{N}{2} + 3\gamma} \|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}} |z|^{-(n+2\gamma)}.$$

Adding the weight at infinity we get that $\|I_1\|_{L^2_\vartheta(\mathbb{R}^n \setminus B_R)}$ can be bounded by $o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}}$ for d small and R large whenever $\delta > -\frac{N+2\gamma}{2} - 2\gamma$, $\vartheta > -\frac{n+2\gamma}{2} - 2\gamma$.

For $|x| \ll 1$, one has

$$I_1(z) \lesssim \int_{\mathcal{T}_{2d} \setminus \mathcal{T}_d} \frac{\phi}{|z - \tilde{z}|^{n+2\gamma}} \, d\tilde{z} + \int_{\mathcal{T}_{2d}^c} \frac{\phi}{|z - \tilde{z}|^{n+2\gamma}} \, d\tilde{z}.$$

One can check that the L^2_δ term can be bounded by

$$\|I_1\|_{L^2_\delta} \leq C[\|\phi\|_{L^2(\mathcal{K})} + R^{\frac{-n+2\gamma+2\vartheta}{2}} \|\phi\|_{L^2_{\vartheta+2\gamma}(B^c_R)}]$$

$$\leq C[\|\phi\|_{L^2(\mathcal{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}}]$$

for d small and R large if $\delta < \frac{N-2\gamma}{2}$, $\vartheta < \frac{n-2\gamma}{2}$.

For $z \in \mathcal{K}$, the estimate goes similarly for $\delta > -\frac{N+2\gamma}{2} - 2\gamma$. In conclusion, we have

$$(8.15) \qquad \|I_1\|_{L^2_{\delta,\vartheta}} \leq C \left[\|\phi\|_{L^2(\mathcal{K})} + o(1)\|\phi\|_{L^2_{\delta+2\gamma,\vartheta+2\gamma}} \right].$$

Note that this estimate only uses the values of the function ϕ when $|y| \ll 1$. Indeed, by checking the arguments in Lemma 5.7, the main term of the expansion for the fractional Laplacian in (8.14) comes from I_{11}, i.e. for $|y| \ll 1$. The contribution when $|y| > |x|^\beta$ is included in the remainder term $|x|^\gamma \tilde{\phi} + \mathcal{R}_2$. The localization around the point $z_0 = (0,0)$ is now complete.

Step 6. Next, after localization, we can replace (8.11) by

$$P^{g_k}_\gamma w - \tau w = \tilde{h}, \quad \text{in } \mathbb{S}^{N-1} \times \mathbb{H}^{k+1},$$

and w is supported only near a point $z_0 \in \partial \mathbb{H}^{k+1}$, that can be taken arbitrarily. We first consider the spherical harmonic decomposition for $\mathbb{S}^{N-1}$ and recall the symbol for each projection from Theorem 3.5.

The L^2_δ estimate follows similarly as in the case of points, but one uses the Fourier-Helgason transform on hyperbolic space instead of the usual Fourier transform as in Theorem 3.5. Note, however, that the hyperbolic metric in (3.22) is written in half-space coordinates as $\frac{dr^2+|dy|^2}{r^2} = dt^2 + e^{2t}|dy|^2$, so in order to account for a weight of the form r^δ one would need to use this transform written in rectangular coordinates. This is well known and comes Kontorovich-Lebedev formulas ([84]). Nevertheless, for our purposes it is more suitable to use this transform in geodesic polar coordinates as it is described in Section 10.2. To account for the weight, we just recall the following relation between two different models for hyperbolic space $\mathbb{H}^{k+1}$, the half space model with metrics $\frac{dr^2+|dy|^2}{r^2}$ and the hyperboloid model with metric $ds^2 + \sinh s\, g_{\mathbb{S}^k}$ in geodesic polar coordinates:

$$\cosh s = 1 + \frac{|y|^2 + (r-2)^2}{4r}.$$

Since we are working locally near a point $z_0 \in \partial\mathbb{H}^{k+1}$, we can choose $y = 0$ in this relation, which yields that $e^{-\delta t} = r^\delta = 2^\delta e^{-\delta s}$. Thus we can use a weight of the form $e^{-\delta s}$ in replacement for $e^{-\delta t}$.

One could redo the theory of Section 6 using the Fourier-Helgason transform instead. Indeed, after projection over spherical harmonics, and following (10.14), we can write for $\zeta \in \mathbb{H}^{k+1}$,

$$w_m(\zeta) = \int_{\mathbb{H}^{k+1}} \mathcal{G}(\zeta, \zeta')\bar{h}(\zeta')\, d\zeta',$$

where the Green's function is given by

$$\mathcal{G}(\zeta, \zeta') = \int_{-\infty}^{+\infty} \frac{1}{\Theta_\gamma^m(\lambda) - \tau} k_\lambda(\zeta, \zeta')\, d\lambda.$$

The poles of $\frac{1}{\Theta_\gamma^m(\lambda)-\tau}$ are well characterized; in fact, they coincide with those in the point singularity case.

But instead, we can take one further reduction and consider the projection over spherical harmonics in $\mathbb{S}^k$. That is, in geodesic polar coordinates $\zeta = (s, \varsigma)$, $s > 0$, $\varsigma \in \mathbb{S}^k$, we can write $w_m(s, \varsigma) = \sum_j w_{m,j}(s)E_j^{(k)}(\varsigma)$, where $E_j^{(k)}$ are the eigenfunctions for $-\Delta_{\mathbb{S}^k}$. Moreover, note that the symbol (3.25) is radial, so it commutes with this additional projection.

Now we can redo the estimate (8.12), just by taking into account the following facts: first, one also has a simple Plancherel formula (10.11). Second, for a radially symmetric function, the Fourier-Helgason transform takes the form of a simple spherical transform (10.12). Third, the spherical function Φ_λ satisfies (10.13) and we are taking a weight of the form $e^{\delta s}$. Finally, the expression for the symbol (3.25) is the same as in the point singularity case (3.15).

This yields estimate (8.8) from which Fredholm properties follow immediately. $\square$

Remark 8.3. We do not claim that our restrictions on δ, ϑ in Proposition 8.2 are the sharpest possible (indeed, we chose them in the injectivity region for simplicity), but these are enough for our purposes.

Gathering all restrictions on the weights we obtain:

Corollary 8.4. *The operator in* (8.6) *is injective, both in* $\mathbb{R}^N \setminus \{q_1, \ldots, q_K\}$ *and* $\mathbb{R}^n \setminus \bigcup \Sigma_i$. *As a consequence, its adjoint* $(\tilde{L}_\varepsilon^*)^* = \tilde{L}_\varepsilon$ *given in* (8.5) *is surjective.*

Proof. Lemma 7.4 shows that, after conjugation, $\tilde{L}_\varepsilon$ is injective in $\mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu}+\frac{N-2\gamma}{2}, \tilde{\nu}+\frac{n-2\gamma}{2}}$. By regularity estimates and our choice of δ, ϑ from (8.7), we immediately obtain injectivity for (8.6). Since, thanks to the Fredholm properties,

$$\mathrm{Ker}(\tilde{L}_\varepsilon^*)^\perp = \mathrm{Rg}(\tilde{L}_\varepsilon),$$

the Corollary follows. $\square$

8.2. **Uniform estimates.** Now we return to the operator L_ε defined in (7.2), the adjoint of

$$L_\varepsilon : L^2_{-\delta,-\vartheta} \to L^2_{-\delta-2\gamma,-\vartheta-2\gamma}$$

is just

$$L^*_\varepsilon : L^2_{\delta+2\gamma,\vartheta+2\gamma} \to L^2_{\delta,\vartheta}.$$

From the above results, one knows that L^*_ε is injective and L_ε is surjective.

Fixing the isomorphisms

$$\pi_{2\delta,2\vartheta} : L^2_{-\delta,-\vartheta} \to L^2_{\delta,\vartheta},$$

we may identify the adjoint L^*_ε as

$$L^*_\varepsilon = \pi_{-2\delta,-2\vartheta} \circ L_\varepsilon \circ \pi_{2\delta,2\vartheta} : L^2_{-\delta+2\gamma,-\vartheta+2\gamma} \to L^2_{-\delta,-\vartheta}.$$

Now we have a new operator

$$\mathbf{L}_\varepsilon = L_\varepsilon \circ L^*_\varepsilon : L_\varepsilon \circ \pi_{-2\delta,-2\vartheta} \circ L_\varepsilon \circ \pi_{2\delta,2\vartheta} : L^2_{-\delta+2\gamma,-\vartheta+2\gamma} \to L^2_{-\delta-2\gamma,-\vartheta-2\gamma}.$$

This map is an isomorphism. Hence there exists a bounded two sided inverse

$$\mathbf{G}_\varepsilon : L^2_{-\delta-2\gamma,-\vartheta-2\gamma} \to L^2_{-\delta+2\gamma,-\vartheta+2\gamma}.$$

Moreover, $G_\varepsilon = L^*_\varepsilon \circ \mathbf{G}_\varepsilon$ is right inverse of L_ε which map into the range of L^*_ε. We will fix our inverse to be this one.

From Corollary 8.4

$$\mathbf{G}_\varepsilon : \mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma} \to \mathcal{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}$$

and

$$G_\varepsilon : \mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma} \to \mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}$$

are bounded.

We are now in the position to prove uniform surjectivity. It is a consequence of the following two results:

Lemma 8.5. If $u \in \mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}$ and $v \in \mathcal{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}$ solve equations $L_\varepsilon u = 0$, $u = L^*_\varepsilon v$, then $u \equiv v \equiv 0$.

Proof. Suppose u,v satisfy the given system, then one has $L_\varepsilon L^*_\varepsilon v = 0$. Consider $\tilde{w} = \pi_{2\delta,2\vartheta} v$. Multiply the equation by w; integration by parts in $\mathbb{R}^n$ yields

$$0 = \int w L_\varepsilon \circ \pi_{-2\delta,-2\vartheta} \circ L_\varepsilon w = \int \pi_{-2\delta,-2\vartheta} |L_\varepsilon w|^2.$$

Thus $L_\varepsilon w = 0$. Moreover, since $v \in \mathcal{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}$, one has $w \in \mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu}+2\gamma+2\delta_{\tilde{\mu}},\tilde{\nu}+2\gamma+2\delta_{\tilde{\nu}}} \hookrightarrow \mathcal{C}^{2\gamma+\alpha}_{\mu',\nu'}$ for some $\mu' > \mathrm{Re}(\gamma_0^+), \nu' > -(n-2\gamma)$, thus by the injectivity property, one has $w \equiv 0$. We conclude then that $u \equiv v \equiv 0$. $\qquad\square$

Lemma 8.6. *Let G_ε be the bounded inverse of L_ε introduced above, then for ε small, G_ε is uniformly bounded, i.e. for $h \in \mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}$, if $u \in \mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}, v \in \mathcal{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}$ solve the system $L_\varepsilon u = h$ and $L^*_\varepsilon v = u$, then one has*

$$\|u\|_{\mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}(\mathbb{R}^n \setminus \Sigma)} \le C\|h\|_{\mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}(\mathbb{R}^n \setminus \Sigma)}$$

for some $C > 0$ independent of ε small.

Proof. The proof is similar to the proof of Lemma 7.4. So we just sketch the proof here and point out the differences. It is by contradiction argument. Assume that there exists $\{\varepsilon^{(n)}\} \to 0$ and a sequence of functions $\{h^{(n)}\}$ and solutions $\{u^{(n)}\}$, $\{v^{(n)}\}$ such that

$$\|u\|_{\mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}(\mathbb{R}^n\setminus\Sigma)} = 1, \quad \|h\|_{\mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}(\mathbb{R}^n\setminus\Sigma)} \to 0,$$

and solve the equation

$$L_\varepsilon u = h, \ L_\varepsilon^* v = u.$$

Here note that, for simplicity, we have dropped the superindex (n). Then using the Green's representation formula, following the argument in Proposition 7.4, one can show that

$$\sup_{\{\text{dist}(x,\Sigma)>\sigma\}} \{\varrho(x)^{-\tilde{\nu}}|u|\} \leq C(\|h\|_{\mathcal{C}^{0,\alpha}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}} + o(1)\|u\|_{\mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}}),$$

which implies that there exists q_i such that

(8.16)
$$\sup_{\{|x-q_i|<\sigma\}} |x - q_i|^{-\tilde{\mu}}|u| \geq \frac{1}{2}.$$

In the second step we study the region $\{|x - q_i| < \sigma\}$. Without loss of generality, assume $q_i = 0$. Define the rescaled function as $\bar{u} = \varepsilon^{-\tilde{\mu}}u(\varepsilon x)$ and similarly for $\bar{v}$ and $\bar{h}$. Similarly to the argument in 7.4, $\bar{u}$ will tend to a limit u_∞ that solves

$$(-\Delta)^\gamma u_\infty - pA_{N,p,\gamma}u_1^{p-1}u_\infty = 0 \quad \text{in } \mathbb{R}^N.$$

If we show that this limit vanishes identically, $u_\infty \equiv 0$, then we will reach a contradiction with (8.16).

For this, we wish to show that $\bar{v}$ also tends to a limit. If

(8.17)
$$\|v\|_{\mathcal{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\nu}+2\gamma}} \leq C_0\|u\|_{\mathcal{C}^{2\gamma+\alpha}_{\tilde{\mu},\tilde{\nu}}},$$

then it is true that the limit exists. If not, we can use the same contradiction argument to show that after some scaling, $\bar{v}$ will tend to a limit $v_\infty \in \mathcal{C}^{4\gamma+\alpha}_{\tilde{\mu}+2\gamma,\tilde{\mu}+2\gamma}$ which solves

$$L_1^* v_\infty = 0.$$

This implies that $v \equiv 0$. This will give a contradiction and yield that (8.17) holds for some constant C_0.

By the above analysis we arrive at the limit problem, in which u_∞, v_∞ solve

$$L_1 u_\infty = 0, \ L_1^* v_\infty = u_\infty \quad \text{in } \mathbb{R}^N.$$

Thus $L_1 L_1^* v_\infty = 0$. Multiply the equation by v_∞ and integrate, one has $L_1^* v_\infty = 0$, which implies that $v_\infty \equiv 0$. So also $u_\infty \equiv 0$. Then, following the argument in Lemma 7.4, one can get a contradiction. So the uniform surjectivity holds for all ε small. $\square$

9. Conclusion of the proof

If ϕ is a solution to

$$(-\Delta_{\mathbb{R}^n})^\gamma(\bar{u}_\varepsilon + \phi) = |\bar{u}_\varepsilon + \phi|^p \quad \text{in } \mathbb{R}^n \setminus \Sigma,$$

we first show that $\bar{u}_\varepsilon + \phi$ is positive in $\mathbb{R}^n \setminus \Sigma$.

Indeed, for z near Σ, there exists $R > 0$ such that if $\varrho(z) < R\varepsilon$, then

$$c_1\varrho(z)^{-\frac{2\gamma}{p-1}} < \bar{u}_\varepsilon < c_2\varrho(z)^{-\frac{2\gamma}{p-1}}$$

for some $c_1, c_2 > 0$. Since $\phi \in \mathcal{C}^{2,\alpha}_{\tilde{\mu},\tilde{\nu}}$, we have $|\phi| \leq c\varrho(z)^{\tilde{\mu}}$. But $\tilde{\mu} > -\frac{2\gamma}{p-1}$, so it follows that $\bar{u}_\varepsilon + \phi > 0$ near Σ. Since $\bar{u}_\varepsilon + \phi \to 0$ as $|z| \to \infty$, by the maximum principle (see for example Lemma 4.13 of [15]), we see that $\bar{u}_\varepsilon + \phi > 0$ in $\mathbb{R}^n \setminus \Sigma$, so it is a positive solution and it is singular at all points of Σ.

Next we will prove the existence of such ϕ. For this, we take an additional restriction on $\tilde{\nu}$

$$-(n-2\gamma) < \tilde{\nu} < -\frac{2\gamma}{p-1}.$$

9.1. **Solution with isolated singularities $(\mathbb{R}^N \setminus \{q_1, \ldots, q_K\})$.** We first treat the case where Σ is a finite number of points. Recall that equation

$$(-\Delta_{\mathbb{R}^N})^\gamma (\bar{u}_\varepsilon + \phi) = A_{N,p,\gamma} |\bar{u}_\varepsilon + \phi|^p \text{ in } \mathbb{R}^N \setminus \{q_1, \ldots, q_K\}$$

is equivalent to the following:

$$(9.1) \qquad\qquad L_\varepsilon(\phi) + Q_\varepsilon(\phi) + f_\varepsilon = 0,$$

where f_ε is defined in (5.2), L_ε is the linearized operator from (7.1) and Q_ε contains the remaining higher order terms. Because of Lemma 8.6, it is possible to construct a right inverse for L_ε with norm bounded independently of ε. Define

$$(9.2) \qquad\qquad F(\phi) := G_\varepsilon[-Q_\varepsilon(\phi) - f_\varepsilon],$$

then equation (9.1) is reduced to

$$\phi = F(\phi).$$

Our objective is to show that $F(\phi)$ is a contraction mapping from $\mathcal{B}$ to $\mathcal{B}$, where

$$\mathcal{B} = \{\phi \in \mathcal{C}_{\tilde{\mu},\tilde{\nu}}^{2\gamma+\alpha}(\mathbb{R}^n \setminus \Sigma) : \|\phi\|_* \leq \beta \varepsilon^{N-\frac{2p\gamma}{p-1}}\}$$

for some large positive β.

In this section, $\|\cdot\|_*$ is the $\mathcal{C}_{\tilde{\mu},\tilde{\nu}}^{2\gamma+\alpha}$ norm, and $\|\cdot\|_{**}$ is the $\mathcal{C}_{\tilde{\mu}-2\gamma,\tilde{\nu}-2\gamma}^{0,\alpha}$ norm where $\tilde{\mu}, \tilde{\nu}$ are taken as in the surjectivity section.

First we have the following lemma:

Lemma 9.1. *We have that, independently of ε small,*

$$\|Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)\|_{**} \leq \frac{1}{2l_0} \|\phi_1 - \phi_2\|_*$$

for all $\phi_1, \phi_2 \in \mathcal{B}$, where $l_0 = \sup \|G_\varepsilon\|$.

Proof. The estimates here are similar to Lemma 9 in [64]. For completeness, we give here the proof.

With some abuse of notation, in the following paragraphs the notation $\|\cdot\|_*$ and $\|\cdot\|_{**}$ will denote the weighted $\mathcal{C}^0$ norms and not the weighted $\mathcal{C}^\alpha$ norms (for the same weights) that was defined in (7.19).

First we show that there exists $\tau > 0$ such that for $\phi \in \mathcal{B}$, we have

$$|\phi(x)| \leq \frac{1}{4} \bar{u}_\varepsilon(x) \quad \text{for all} \quad x \in \bigcup_{i=1}^k B(q_i, \tau).$$

Indeed, from the asymptotic behaviour of u_1 in Proposition 2.1 we know that

$$c_1 |x|^{-\frac{2\gamma}{p-1}} < u_{\varepsilon_i}(x) < c_2 |x|^{-\frac{2\gamma}{p-1}} \text{ if } |x| < R\varepsilon_i,$$

$$c_1 \varepsilon_i^{N-\frac{2p\gamma}{p-1}} |x|^{-(N-2\gamma)} < u_{\varepsilon_i}(x) < c_2 \varepsilon_i^{N-\frac{2p\gamma}{p-1}} |x|^{-(N-2\gamma)} \text{ if } R\varepsilon_i \leq |x| < \tau.$$

The claim follows because $\phi \in \mathcal{B}$ implies that

$$|\phi(x)| < c\beta \varepsilon^{N-\frac{2p\gamma}{p-1}} \varrho(x)^{\tilde{\mu}}.$$

Next, since $|\frac{\phi}{\bar{u}_\varepsilon}| \leq \frac{1}{4}$ in $B(q_i, \tau)$, by Taylor' expansion,

$$|Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)| \leq c|\bar{u}_\varepsilon|^{p-2}(|\phi_1| + |\phi_2|)|\phi_1 - \phi_2|.$$

Thus for $x \in B(q_i, \tau)$, we have

$$\varrho(x)^{2\gamma - \tilde{\mu}} |Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)| \leq c\varrho(x)^{\tilde{\mu} + \frac{2\gamma}{p-1}} (\|\phi_1\|_* + \|\phi_2\|_*) \|\phi_1 - \phi_2\|_*$$

$$\leq c\tau^{\tilde{\mu} + \frac{2\gamma}{p-1}} \beta \varepsilon^{N - \frac{2\gamma}{p-1}} \|\phi_1 - \phi_2\|_*.$$

The coefficient in front can be taken as small as desired by choosing ε small. Outside the union of the balls $B(q_i, \tau)$ we use the estimates

$$\bar{u}_\varepsilon(x) \leq c\varepsilon^{N - \frac{2p\gamma}{p-1}} |x|^{-(N - 2\gamma)} \quad \text{and} \quad |\phi| \leq c\varepsilon^{N - \frac{2p\gamma}{p-1}} |x|^{\tilde{\nu}},$$

where c depends on τ but not on ε nor ϕ.

For $\varrho \geq \tau$ and $|x| \leq R$, we can neglect all factors involving $\varrho(x)$, so

$$|Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)| \leq c(|\bar{u}_\varepsilon|^{p-1} + |\phi|^{p-1})|\phi_1 - \phi_2| \leq c\varepsilon^{(p-1)(N - \frac{2p\gamma}{p-1})} |\phi_1 - \phi_2|$$

$$\leq c\varepsilon^{p(N - 2\gamma) - N} \|\phi_1 - \phi_2\|_*,$$

for which the coefficient can be as small as desired since $p > \frac{N}{N - 2\gamma}$.

Lastly, for $|x| \geq R$, in this region $\bar{u}_\varepsilon = 0$, so

$$\varrho(x)^{2\gamma - \tilde{\nu}} |Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)| \leq c\varrho(x)^{2\gamma - \tilde{\nu}} (\phi_1^{p-1} + \phi_2^{p-1}) |\phi_1 - \phi_2|$$

$$\leq c\varrho(x)^{2\gamma - \tilde{\nu} + p\tilde{\nu}} \varepsilon^{N(p-1) - 2p\gamma} \|\phi_1 - \phi_2\|_*,$$

and here the coefficient can be also chosen as small as we wish because $\tilde{\nu} < -\frac{2\gamma}{p-1}$ implies that $2\gamma - \tilde{\nu} + p\tilde{\nu} < 0$.

Combining all the above estimates, one has

$$\|Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)\|_{**} \leq \frac{1}{2l_0} \|\phi_1 - \phi_2\|_*$$

as desired.

Now we go back to the original definition of the norms $\| \cdot \|_*$, $\| \cdot \|_{**}$ from (7.19). For this, we need to estimate the Hölder norm of $Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)$. First in each $B(q_i, \tau)$,

$$\nabla Q_\varepsilon(\phi) = p\Big((\bar{u}_\varepsilon + \phi)^{p-1} - \bar{u}_\varepsilon^{p-1} - (p-1)\bar{u}_\varepsilon^{p-1}\phi\Big)\nabla\bar{u}_\varepsilon + p((\bar{u}_\varepsilon + \phi)^{p-1} - \bar{u}_\varepsilon^{p-1})\nabla\phi,$$

and similarly as before, we can get that

$$\varrho(x)^{2\gamma + 1 - \tilde{\mu}} |\nabla(Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2))| \leq c\varepsilon^{N - \frac{2p\gamma}{p-1}} \|\phi_1 - \phi_2\|_*.$$

Moreover, for $\tau < \varrho(x) < R$,

$$|\nabla(Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2))| \leq c\varepsilon^{p(N - 2\gamma) - N} \|\phi_1 - \phi_2\|_*.$$

Lastly, for $\varrho(x) > R$,

$$\nabla Q_\varepsilon(\phi_1 - \phi_2) = p\phi_1^{p-1} \nabla(\phi_1 - \phi_2) + p\nabla\phi_2(\phi_1^{p-1} - \phi_2^{p-1}),$$

which yields

$$\varrho^{-\tilde{\nu} + 2\gamma + 1} |\nabla Q_\varepsilon(\phi_1 - \phi_2)|$$

$$\leq \varrho^{2\gamma + 1 - \tilde{\nu}} \Big[(\|\phi_1\|_* + \|\phi_2\|_*)^{p-1} \varrho^{(p-1)(\tilde{\nu} - 2\gamma)} \|\phi_1 - \phi_2\|_* \varrho^{\tilde{\nu} - 2\gamma - 1} + \|\phi_2\|_* \varrho^{\tilde{\nu} - 2\gamma - 1} \|\phi_1^{p-1} - \phi_2^{p-1}\|_* \Big]$$

$$\leq c\varepsilon^{N - \frac{2p\gamma}{p-1}} \|\phi_1 - \phi_2\|_*.$$

This completes the desired estimate for $Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)$ and concludes the proof of the lemma. $\square$

Recall that $\|f_\varepsilon\|_{**} \leq C_0 \varepsilon^{N - \frac{2p\gamma}{p-1}}$ for some $C_0 > 0$ from (5.4). Then the lemma above gives an estimate for the map (9.2). Indeed,

$$\|F(\phi)\|_* \leq l_0[\|Q_\varepsilon(\phi)\|_{**} + \|f_\varepsilon\|_{**}] \leq l_0\|Q_\varepsilon(\phi)\|_{**} + l_0 C_0 \varepsilon^{N - \frac{2p\gamma}{p-1}}$$

$$\leq \frac{1}{2}\|\phi\|_* + l_0 C_0 \varepsilon^{N - \frac{2p\gamma}{p-1}} \leq \beta \varepsilon^{N - \frac{2p\gamma}{p-1}},$$

and

$$\|F(\phi_1) - F(\phi_2)\|_* \leq l_0\|Q_\varepsilon(\phi_1) - Q_\varepsilon(\phi_2)\|_{**} \leq \frac{1}{2}\|\phi_1 - \phi_2\|_*$$

if we choose $\beta > 2l_0 C_0$. So $F(\phi)$ is a contraction mapping from $\mathcal{B}$ to $\mathcal{B}$. This implies the existence of a solution ϕ to (9.1).

9.2. The general case $\mathbb{R}^n \setminus \Sigma$, Σ a sub-manifold of dimension k.

For the more general case, only minor changes need to be made in the above argument. The most important one comes from Lemma 5.7 and it says that the weight parameter μ must now lie in the smaller interval:

$$(9.3) \qquad -\frac{2\gamma}{p-1} < \tilde{\mu} < \min\left\{\gamma - \tfrac{2\gamma}{p-1}, \tfrac{1}{2} - \tfrac{2\gamma}{p-1}, \operatorname{Re}(\gamma_0^-)\right\}.$$

In this case, we only need to replace the exponent $N - \frac{2p\gamma}{p-1}$ in the above argument by $q = \min\{\frac{(p-3)\gamma}{p-1} - \tilde{\mu}, \frac{1}{2} - \gamma + \frac{(p-3)\gamma}{p-1} - \tilde{\mu}\}$, then $q > 0$ if $\tilde{\mu}$ is chosen to satisfy (9.3). We get a solution to (9.1), and this concludes the proof of Theorem 1.1.

10. Appendix

10.1. Some known results on special functions.

Lemma 10.1. [3, 82] *Let $z \in \mathbb{C}$. The hypergeometric function is defined for $|z| < 1$ by the power series*

$$_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}.$$

It is undefined (or infinite) if c equals a non-positive integer. Some properties are

    i. *The hypergeometric function evaluated at $z = 0$ satisfies*

$$(10.1) \qquad _2F_1(a+j, b-j; c; 0) = 1; \; j = \pm 1, \pm 2, \ldots$$

    ii. *If $|arg(1 - z)| < \pi$, then*

$$(10.2) \qquad \begin{aligned} _2F_1(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \, _2F_1(a,b;a+b-c+1;1-z) \\ &+ (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \, _2F_1(c-a,c-b;c-a-b+1;1-z). \end{aligned}$$

    iii. *The hypergeometric function is symmetric with respect to first and second arguments, i.e*

$$(10.3) \qquad _2F_1(a,b;c;z) = \, _2F_1(b,a;c;z).$$

    iv. *Let $m \in \mathbb{N}$. The m-derivative of the hypergeometric function is given by*

$$(10.4) \qquad \frac{d^m}{dz^m}\left[(1-z)^{a+m-1} \, _2F_1(a,b;c;z)\right] = \frac{(-1)^m (a)_m (c-b)_m}{(c)_m}(1-z)^{a-1} \, _2F_1(a+m,b;c+m;z).$$

Lemma 10.2. [3, 82] *Let $z \in \mathbb{C}$. Some well known properties of the Gamma function $\Gamma(z)$ are*

$$(10.5) \qquad\qquad \Gamma(\bar{z}) = \overline{\Gamma(z)},$$

$$(10.6) \qquad\qquad \Gamma(z+1) = z\Gamma(z),$$

$$(10.7) \qquad\qquad \Gamma(z)\Gamma\left(z + \tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z).$$

It is a meromorphic function in $z \in \mathbb{C}$ and its residue at each poles is given by

$$(10.8) \qquad\qquad \mathrm{Res}(\Gamma(z), -j) = \frac{(-1)^n}{j!}, \quad j = 0, 1, \ldots.$$

Let $\psi(z)$ denote the Digamma function defined by

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz} = \frac{\Gamma'(z)}{\Gamma(z)}.$$

This function has the expansion

$$(10.9) \qquad\qquad \psi(z) = \psi(1) + \sum_{l=0}^{\infty}\left(\frac{1}{l+1} - \frac{1}{l+z}\right).$$

Let $B(z_1, z_2)$ denote the Beta function defined by

$$B(z_1, z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1 + z_2)}.$$

If z_2 is a fixed number and $z_1 > 0$ is big enough, then this function behaves

$$B(z_1, z_2) \sim \Gamma(z_2)(z_1)^{-z_2}.$$

10.2. **A review of the Fourier-Helgason transform on Hyperbolic space.** Consider hyperbolic space $\mathbb{H}^{k+1}$, parameterized with coordinates ζ. It can be written as a symmetric space of rank one as the quotient $\mathbb{H}^{k+1} \approx \frac{SO(1,k+1)}{SO(k+1)}$. Fourier transform on hyperbolic space is a particular case of the Helgason-Fourier transform on symmetric spaces. Some standard references are [9, 56, 83]; we mostly follow the exposition of Chapter 8 in [45].

Hyperbolic space $\mathbb{H}^{k+1}$ may be defined as the upper branch of a hyperboloid in $\mathbb{R}^{k+2}$ with the metric induced by the Lorentzian metric in $\mathbb{R}^{k+2}$ given by $-d\zeta_0^2 + d\zeta_1^2 + \ldots + d\zeta_{k+1}^2$, i.e., $\mathbb{H}^{k+1} = \{(\zeta_0, \ldots, \zeta_{k+1}) \in \mathbb{R}^{k+2} : \zeta_0^2 - \zeta_1^2 - \ldots - \zeta_{k+1}^2 = 1, \ \zeta_0 > 0\}$, which in polar coordinates may be parameterized as

$$\mathbb{H}^{k+1} = \{\zeta \in \mathbb{R}^{k+2} : \zeta = (\cosh s, \varsigma \sinh s), \ s \geq 0, \ \varsigma \in \mathbb{S}^k\},$$

with the metric $g_{\mathbb{H}^{k+1}} = ds^2 + \sinh^2 s\, g_{\mathbb{S}^k}$. Under these definitions the Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}^{k+1}} = \partial_{ss} + k\frac{\cosh s}{\sinh s}\partial_s + \frac{1}{\sinh^2 s}\Delta_{\mathbb{S}^k},$$

and the volume element is

$$d\mu_\zeta = \sinh^k s\, ds\, d\varsigma.$$

We denote by $[\cdot, \cdot]$ the internal product induced by the Lorentzian metric, i.e.,

$$[\zeta, \zeta'] = \zeta_0\zeta_0' - \zeta_1\zeta_1' - \ldots - \zeta_{k+1}\zeta_{k+1}'.$$

The hyperbolic distance between two arbitrary points is given by $\mathrm{dist}(\zeta, \zeta') = \cosh^{-1}([\zeta, \zeta'])$, and in the particular case that $\zeta = (\cosh s, \varsigma \sinh s)$, $\zeta' = O$,

$$\mathrm{dist}(\zeta, O) = s.$$

The unit sphere $\mathbb{S}^{N-1}$ is identified with the subset $\{\zeta \in \mathbb{R}^{k+2} : [\zeta, \zeta] = 0, \zeta_0 = 1\}$ via the map $b(\varsigma) = (1, \varsigma)$ for $\varsigma \in \mathbb{S}^k$.

Given $\lambda \in \mathbb{R}$ and $\omega \in \mathbb{S}^k$, let $h_{\lambda,\omega}(\zeta)$ be the generalized eigenfunctions of the Laplace-Beltrami operator. This is,

$$\Delta_{\mathbb{H}^{k+1}} h_{\lambda,\omega} = -\left(\lambda^2 + \tfrac{k^2}{4}\right) h_{\lambda,\omega}.$$

These may be explicitly written as

$$h_{\lambda,\omega}(\zeta) = [\zeta, b(\omega)]^{i\lambda - \frac{k}{2}} = (\cosh s - \sinh s \langle \varsigma, \omega \rangle)^{i\lambda - \frac{k}{2}}, \quad \zeta \in \mathbb{H}^{k+1}.$$

In analogy to the Euclidean space, the Fourier transform on $\mathbb{H}^{k+1}$ is defined by

$$\hat{u}(\lambda, \omega) = \int_{\mathbb{H}^{k+1}} u(\zeta)\, h_{\lambda,\omega}(\zeta)\, d\mu_\zeta.$$

Moreover, the following inversion formula holds:

$$(10.10) \qquad u(\zeta) = \int_{-\infty}^{\infty} \int_{\mathbb{S}^k} \bar{h}_{\lambda,\omega}(\zeta) \hat{u}(\lambda, \omega) \frac{d\omega\, d\lambda}{|c(\lambda)|^2},$$

where $c(\lambda)$ is the Harish-Chandra coefficient:

$$\frac{1}{|c(\lambda)|^2} = \frac{1}{2(2\pi)^{k+1}} \frac{|\Gamma(i\lambda + (\frac{k}{2}))|^2}{|\Gamma(i\lambda)|^2}.$$

There is also a Plancherel formula:

$$(10.11) \qquad \int_{\mathbb{H}^{k+1}} |u(\zeta)|^2\, d\mu_\zeta = \int_{\mathbb{R} \times \mathbb{S}^{N-1}} |\hat{u}(\lambda, \omega)|^2 \frac{d\omega\, d\lambda}{|c(\lambda)|^2},$$

which implies that the Fourier transform extends to an isometry between the Hilbert spaces $L^2(\mathbb{H}^{k+1})$ and $L^2(\mathbb{R}_+ \times \mathbb{S}^k, |c(\lambda)|^{-2} d\lambda\, d\omega)$.

If u is a radial function, then $\hat{u}$ is also radial, and the above formulas simplify. In this setting, it is customary to normalize the measure of $\mathbb{S}^k$ to one in order not to account for multiplicative constants. Thus one defines the spherical Fourier transform as

$$(10.12) \qquad \hat{u}(\lambda) = \int_{\mathbb{H}^{k+1}} u(\zeta) \Phi_{-\lambda}(\zeta)\, d\mu_\zeta,$$

where

$$\Phi_\lambda(\zeta) = \int_{\mathbb{S}^k} h_{-\lambda,\omega}(\zeta)\, d\omega$$

is known as the elementary spherical function. In addition, (10.10) reduces to

$$u(\zeta) = \int_{-\infty}^{\infty} \hat{u}(\lambda) \Phi_\lambda(\zeta) \frac{d\lambda}{|c(\lambda)|^2}.$$

There are many explicit formulas for $\Phi_\lambda(\zeta)$ (we also write $\Phi_\lambda(s)$, since it is a radial function). In particular, $\Phi_{-\lambda}(s) = \Phi_\lambda(s) = \Phi_\lambda(-s)$, which yields regularity at the origin $s = 0$. Here we are interested in its asymptotic behavior. Indeed,

$$(10.13) \qquad \Phi_\lambda(s) \sim e^{(i\lambda - \frac{k}{2})s} \quad \text{as} \quad s \to +\infty.$$

It is also interesting to observe that

$$\widehat{\Delta_{\mathbb{H}^{k+1}} u} = -\left(\lambda^2 + \tfrac{k^2}{4}\right) \hat{u}.$$

We define the convolution operator as

$$u * v(\zeta) = \int_{\mathbb{H}^{k+1}} u(\zeta') v(\tau_\zeta^{-1} \zeta')\, d\mu_{\zeta'},$$

where $\tau_\zeta : \mathbb{H}^{k+1} \to \mathbb{H}^{k+1}$ is an isometry that takes ζ into O. If v is a radial function, then the convolution may be written as

$$u * v(\zeta) = \int_{\mathbb{H}^{k+1}} u(\zeta')v(\text{dist}(\zeta, \zeta')) \, d\mu_{\zeta'},$$

and we have the property

$$\widehat{u * v} = \hat{u}\,\hat{v},$$

in analogy to the usual Fourier transform.

On hyperbolic space there is a well developed theory of Fourier multipliers. In L^2 spaces everything may be written out explicitly. For instance, let $m(\lambda)$ be a multiplier in Fourier variables. A function $\hat{u}(\lambda, \omega) = \hat{m}(\lambda)u_0(\lambda, \omega)$, by the inversion formula for the Fourier transform (10.10) and expression (10.2), may be written as

(10.14)
$$
\begin{aligned}
u(x) &= \int_{-\infty}^{\infty} \int_{\mathbb{S}^k} m(\lambda)\hat{u}_0(\lambda, \omega)\bar{h}_{\lambda,\omega}(\zeta) \frac{d\omega \, d\lambda}{|c(\lambda)|^2} \\
&= \int_{-\infty}^{\infty} \int_{\mathbb{H}^{k+1}} m(\lambda)u_0(\zeta')k_\lambda(\zeta, \zeta') \, d\mu_{\zeta'} \, d\lambda,
\end{aligned}
$$

where we have denoted

$$k_\lambda(\zeta, \zeta') = \frac{1}{|c(\lambda)|^2} \int_{\mathbb{S}^k} \bar{h}_{\lambda,\omega}(\zeta) \, h_{\lambda,\omega}(\zeta') \, d\omega.$$

It is known that k_λ is invariant under isometries, i.e.,

$$k_\lambda(\zeta, \zeta') = k_\lambda(\tau\zeta, \tau\zeta'),$$

for all $\tau \in SO(1, k+1)$, and in particular,

$$k_\lambda(\zeta, \zeta') = k_\lambda(\text{dist}(\zeta, \zeta')),$$

so many times we will simply write $k_\lambda(\rho)$ for $\rho = \text{dist}(\zeta, \zeta')$. We recall the following formulas for k_λ:

Lemma 10.3. [45, Lemmas 8.5.2 y 8.5.3] *For $k + 1 \geq 3$ odd,*

$$k_\lambda(\rho) = c_k \left(\frac{\partial_\rho}{\sinh \rho} \right)^{\frac{k}{2}} (\cos \lambda\rho),$$

and for $k + 1 \geq 2$ even,

$$k_\lambda(\rho) = c_k \int_\rho^\infty \frac{\sinh \tilde{\rho}}{\sqrt{\cosh \tilde{\rho} - \cosh \rho}} \left(\frac{\partial_{\tilde{\rho}}}{\sinh \tilde{\rho}} \right)^{\frac{k+1}{2}} (\cos \lambda\tilde{\rho}) \, d\tilde{\rho}.$$

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