Transfer methods for o–minimal topology

Alessandro Berarducci   Margarita Otero*

Abstract

Let $\mathbf{M}$ be an o-minimal expansion of an ordered field. Let φ be a formula in the language of ordered domains. In this note we establish some topological properties which are transferred from $\varphi^{\mathbf{M}}$ to $\varphi^{\mathbb{R}}$ and vice versa. Then, we apply these transfer results to give a new proof of a result of M. Edmundo – based on the work of A. Strzebonski – showing the existence of torsion points in any definably compact group defined in an o-minimal expansion of an ordered field.

1 Introduction

It is well known that many results of real semialgebraic geometry generalize to any real closed field. A theoretical explanation of this phenomenon is given by the Tarski–Seidenberg transfer principle. In o–minimal geometry, based on an o–minimal expansion $\mathbf{M}$ of an ordered field, we do not have an analogue of the Tarski-Seidenberg theorem, and even in semialgebraic geometry sometimes the concepts involved do not allow to apply this transfer principle. However many results still generalize thanks essentially to the cell decomposition and the triangulation theorems. Our aim in this note is to illustrate the "transfer approach" in o–minimal geometry: once few basic transfer principles are proved, many other results admit an easy transfer from the corresponding result over the reals.

Throughout this paper we fix an o–minimal expansion $\mathbf{M}$ of an ordered field. We equip $\mathbf{M}$ with the interval topology and $\mathbf{M}^n\,(n > 1)$, with the product topology. The word "definable" means definable in $\mathbf{M}$. As usual in homology theory, we consider pairs of topological spaces (X, X_0) with $X_0 \subset X$, and continuos maps $f\colon (X, X_0) \to (Y, Y_0)$ between pairs, namely continuous maps $f\colon X \to Y$ with $f(X_0) \subset Y_0$. We write X instead of $(X, \emptyset)$.

*Partially supported by DGES PB980756/C02/01.

2 Transfer

Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ in the real field $\mathbb{R}$. Then $\overline{\mathbb{Q}}$ can be isomorphically embedded in $\mathbf{M}$ (as a field), so we can assume that $\overline{\mathbb{Q}}$ is contained in $\mathbf{M}$. Given a semialgebraic set $X \subset \overline{\mathbb{Q}}^n$ we can interpret its defining formula over $\mathbb{R}$ and $\mathbf{M}$ respectively, obtaining the inclusions:

$$X^{\mathbb{R}} \longleftarrow X \longrightarrow X^{\mathbf{M}}$$

By the triangulation theorem for definable sets, any closed and bounded definable set $Y \subset \mathbf{M}^n$ is definably homeomorphic to the geometrical realization $|K|^{\mathbf{M}} \subset \mathbf{M}^n$ of a finite simplicial complex K with vertices over $\overline{\mathbb{Q}}$, so in particular Y is definably homeomorphic to a semialgebraic set. More generally any definable set is definably homeomorphic to a semialgebraic set. In the sequel "K" will always denote a finite simplicial complex, and we shall abbreviate $|K|^{\overline{\mathbb{Q}}}$ by $|K|$.

Let X, X_0 be semialgebraic sets with $X_0 \subset X$. The definable (singular) homology groups $H_i^{def}(X^{\mathbf{M}})$, $H_i^{def}(X^{\mathbf{M}}, X_0^{\mathbf{M}})$ and the definable fundamental group $\boldsymbol{\pi}^{def}(X^{\mathbf{M}}, x_0)$ of $X^{\mathbf{M}}$ at $x_0 \in X$ are defined in the obvious way: one replaces in the classical definition continuous maps with definable continuous maps.

The following transfer results hold:

TR1. Connectedness: $X^{\mathbf{M}}$ is definably connected iff $X^{\mathbb{R}}$ is connected.

TR2. Compactness: $X^{\mathbf{M}}$ is definably compact iff $X^{\mathbb{R}}$ is compact.

TR3. Homology: $H_*^{def}(X^{\mathbf{M}}) \cong H_*(X^{\mathbb{R}})$ naturally.

TR4. Fundamental group: $\boldsymbol{\pi}^{def}(X^{\mathbf{M}}, x_0) \cong \boldsymbol{\pi}(X^{\mathbb{R}}, x_0)$ naturally.

TR5. Manifold: Assume $X^{\mathbb{R}}$ is compact and $\dim X^{\mathbb{R}} \neq 4$.
If $X^{\mathbf{M}}$ is a definable manifold, then $X^{\mathbb{R}}$ is a (topological) manifold.

TR6. Orientability: Assume $X^{\mathbb{R}}$ is a compact manifold.
If $X^{\mathbf{M}}$ is a definably orientable definable manifold, then $X^{\mathbb{R}}$ is an orientable manifold.

Points **TR1 (connectedness)** and **TR2 (compactness)** are well known (see *e.g.* [7]). See also Section 4 for other aspects of **TR2**. We recall that a definable set Y in $\mathbf{M}^n$ (or more generally in an abstract definable manifold) is **definably compact** if for every definable curve $\gamma\colon (0,\varepsilon) \to Y$, $\lim_{t\to\varepsilon}\gamma(t)$ exists and belongs to Y (see [14]).

Point **TR3 (homology)** is proved in [2, §3] (we get the corresponding isomorphism for relative homology). The proof is based on the fact that the definable homology functor satisfies the definable version of the Eilenberg–Steenrod axioms. This last result was proved by A. Woerheide in [20]. When $X = |K|$, the two groups in **TR3** are naturally isomorphic to the simplicial homology group $H_i(K)$. The naturality condition means the following: Consider the following commutative diagram where $X \subset \overline{\mathbb{Q}}^n$, $Y \subset \overline{\mathbb{Q}}^m$ and $f\colon X \to Y$ are semialgebraic.

$$\begin{tikzcd}
X^{\mathbb{R}} \arrow[d, "f^{\mathbb{R}}"'] & X \arrow[l] \arrow[r] \arrow[d, "f"'] & X^{\mathbf{M}} \arrow[d, "f^{\mathbf{M}}"'] \\
Y^{\mathbb{R}} & Y \arrow[l] \arrow[r] & Y^{\mathbf{M}}.
\end{tikzcd}$$

Then the following diagram conmutes for each $i \in \mathbb{Z}$,

$$\begin{tikzcd}
H_i(X^{\mathbb{R}}) \arrow[d, "f_*^{\mathbb{R}}"'] & H_i^{sa}(X) \arrow[l] \arrow[r] \arrow[d, "f_*"'] & H_i^{def}(X^{\mathbf{M}}) \arrow[d, "f_*^{\mathbf{M}}"'] \\
H_i(Y^{\mathbb{R}}) & H_i^{sa}(Y) \arrow[l] \arrow[r] & H_i^{def}(Y^{\mathbf{M}}),
\end{tikzcd}$$

where the vertical arrows are the induced homomorphisms in homology.

Point **TR4 (fundamental group)** is proved in [2, §2]. When $X = |K|$ and x_0 is a vertex of K, both groups are naturally isomorphic to the combinatorial fundamental group $\boldsymbol{\pi}(K, x_0)$. As before, naturality means the isomorphism conmutes with the homomorphism induced by semialgebraic maps.

Point **TR5 (manifold)** is proved in [2, §4] using **TR3, TR4** and a combinatorial characterization of manifolds of dimension $\neq 4$ given by D. Galewski and R. Stern in [9]. We recall that $Y \subset \mathbf{M}^n$ is a (embedded) definable manifold of dimension m, if Y is a union of open sets of Y, which are definably homeomorphic to open subsets of $\mathbf{M}^m$. The topology on Y is induced by the ambient space $\mathbf{M}^n$ and it can be proved that Y has a finite atlas. There is also a notion of abstract definable manifold, in which the topology is given by a finite atlas with definable transition maps (see [15, 13]). The proviso "dim $\neq 4$" is quite annoying, but it will be harmless in our applications. A possible attempt to eliminate the proviso could be based on the o–minimal Hauptvermutung of M. Shiota [16].

Point **TR6 (Orientability)** is discussed in Section 5. There are different concepts of orientability in the classical literature. Suppose that $|K|^{\mathbb{R}}$ is a connected manifold of dimension m. We say that $|K|^{\mathbb{R}}$ is orientable via simplices if it is possible to orient all its m dimensional simplices so that their sum is a cycle in $|K|^{\mathbb{R}}$ (with coefficients in $\mathbb{Z}$). We say $|K|^{\mathbb{R}}$ is orientable via local orientations if there is a map s which assigns to each point $p \in |K|^{\mathbb{R}}$ a generator $s(p)$ of the local homology group $H_m(|K|^{\mathbb{R}}, |K|^{\mathbb{R}} - p)$ and it is locally constant (see [12, Appendix A]). In both cases one can conclude that $H_m(|K|^{\mathbb{R}}) \cong \mathbb{Z}$. In the case of orientability via simplices one uses the isomorphism between simplicial and singular homology and in the other case one covers $|K|^{\mathbb{R}}$ by a suitable finite collection of *sufficiently small* compact sets (see [12]). In fact, both definitions are equivalent to the conclusion $H_m(|K|^{\mathbb{R}}) \cong \mathbb{Z}$. In the definable context if we took $H_m^{def}(|K|^{\mathbf{M}}) \cong \mathbb{Z}$ to define the concept of definable orientability (as Delfs and Knebusch do in the semialgebraic case, see [6]), then we would obtain the transfer property (**TR6**) directly from **TR3**. However with this definition it would not be easy to prove that a definably compact definable group is definably orientable. We take instead the local orientations approach (see Section 5) which is more natural for the applications to definable groups. In this case the proof that a definable group is definably orientable is easy (see Corollary 3.4), but the transfer property becomes more difficult: it depends on a proof, in the o–minimal setting, of the existence of a fundamental class induced by a given orientation (Theorem 5.2). M. Edmundo has worked on this problem in the unpublished manuscript [8].

3 Applications

Let us consider the following open problem:

Question 3.1 Let K be a finite simplicial complex of Euler characteristic different from zero. Let $f\colon |K|^{\mathbf{M}} \to |K|^{\mathbf{M}}$ be a definable (in $\mathbf{M}$) continuous map definably homotopic to the identity. Then f has a fixed point.

This is a definable version of Lefschetz fix point theorem for polyhedra. The classical proof uses repeated barycentric subdivisions (see [4]) and can not be adapted to the o–minimal context.

When $\mathbf{M}$ carries only the field structure, we are in the semialgebraic setting and a positive answer follows by transfer from the corresponding true result over the reals. A more general semialgebraic Hopf fixed point theorem has been proved by G. Brumfield in [5].

Another case in which our question admits a positive answer is when $|K|^{\mathbf{M}}$ is an orientable definable manifold. This is proved by M. Edmundo in [8]. Here we give an alternative proof of this fact (Theorem 3.3) using the transfer method. We cannot directly apply transfer because the definable function f need not be semialgebraic. However we will be able to reach the desired conclusion after transfering to $\mathbf{M}$ the following classical result:

Fact 3.2 *Let $X = |K|$ and let Δ be the diagonal in $X \times X$. Suppose $X^{\mathbb{R}}$ is a compact orientable manifold of dimension m. If the map*

$$(id, id)_*^{\mathbb{R}} \colon H_m(X^{\mathbb{R}}) \to H_m(X^{\mathbb{R}} \times X^{\mathbb{R}}, X^{\mathbb{R}} \times X^{\mathbb{R}} - \Delta^{\mathbb{R}})$$

is the zero homorphism. Then, the Euler characteristic of $X^{\mathbb{R}}$ is zero.

(The result follows from Poincaré duality theorem and the Thom isomorphism theorem, see *e.g.* chapters 6 and 7 of [19].)

In the sequel, to apply the results of A. Strzebonski, we shall need the equality between the classical Euler characteristic $\chi(|K|^{\mathbb{R}})$ and the o–minimal Euler characteristic $E(|K|^{\mathbb{R}})$. The o–minimal Euler characteristic of a definable set X is defined as the number of even dimensional open-simplexes minus the number of odd dimensional ones, with respect to a definable triangulation (see [7]). There are significant differences between these notions, for instance $\chi((0,1)) = \chi([0,1]) = E([0,1]) \neq E((0,1))$ (the classical notion is invariant under homotopy equivalence, the o–minimal one under definable bijections). However for sets of the form $|K|^{\mathbb{R}}$ the two notions coincide, because they are both equal to the number of even dimensional (closed) simplexes minus the number of odd dimensional ones.

Theorem 3.3 *Let K be a finite simplicial complex of Euler characteristic different from zero and let $X = |K|$. Suppose that $X^{\mathbf{M}} \subset \mathbf{M}^n$ is a definably oriented definable manifold. Let $f \colon X^{\mathbf{M}} \to X^{\mathbf{M}}$ be a definable continuous map definably homotopic to the identity. Then f has a fixed point.*

Proof. We can assume that $m = \dim X^{\mathbf{M}} \neq 4$, because we can reduce to this case by replacing $X^{\mathbf{M}}$ by $X^{\mathbf{M}} \times X^{\mathbf{M}}$ and f by $f \times f$ (we thank S. Starchenko for this remark).

Suppose f has no fixed points. Then we have the following commutative diagram of maps of pairs:

$$\begin{tikzcd}
X^{\mathbf{M}} \arrow[rr, "(f,id)"] \arrow[dr] & & (X^{\mathbf{M}}\times X^{\mathbf{M}}, X^{\mathbf{M}}\times X^{\mathbf{M}} - \Delta^{\mathbf{M}}) \\
& (X^{\mathbf{M}}\times X^{\mathbf{M}} - \Delta^{\mathbf{M}},\ X^{\mathbf{M}}\times X^{\mathbf{M}} - \Delta^{\mathbf{M}}), \arrow[ur] &
\end{tikzcd}$$

where $\Delta \subset X \times X$ is the diagonal.

Applying the homology functor to the diagram we obtain:

$$\begin{tikzcd}
H_m^{def}(X^{\mathbf{M}}) \arrow[rr, "(f,id)_*^{\mathbf{M}}"] \arrow[dr] & & H_m^{def}(X^{\mathbf{M}}\times X^{\mathbf{M}}, X^{\mathbf{M}}\times X^{\mathbf{M}} - \Delta^{\mathbf{M}}) \\
& 0. \arrow[ur] &
\end{tikzcd}$$

So $(f,id)_*^{\mathbf{M}}$ is the zero map. Since f is definably homotopic to the identity, $(f,id)_*^{\mathbf{M}} = (id,id)_*^{\mathbf{M}}$. Therefore,

$$(id,id)_*^{\mathbf{M}}\colon H_m^{def}(X^{\mathbf{M}}) \longrightarrow H_m^{def}(X^{\mathbf{M}}\times X^{\mathbf{M}}, X^{\mathbf{M}}\times X^{\mathbf{M}} - \Delta^{\mathbf{M}})$$

is the zero homomorphism.

We can transfer this to the reals by **TR3** and we obtain that

$$(id,id)_*^{\mathbb{R}}\colon H_m(X^{\mathbb{R}}) \longrightarrow H_m(X^{\mathbb{R}} \times X^{\mathbb{R}}, X^{\mathbb{R}} \times X^{\mathbb{R}} - \Delta^{\mathbb{R}})$$

is the zero homomorphism.

Since $X^{\mathbf{M}}$ is a definably compact definably orientable definable manifold of dimension different from 4, by **TR5** and **TR6**, $X^{\mathbb{R}}$ is a compact orientable manifold. Therefore, by Fact 3.2 the Euler characteristic of $X^{\mathbb{R}}$ (hence of $X^{\mathbf{M}}$) is zero, contradicting our assumptions and thus finishing the proof. $\Box$

We now give an application to definable groups. Let $G \subset \mathbf{M}^n$. If G is a definable group, then by a results of A. Pillay in [15] we know that G admits a group topology which makes it into an abstract definable manifold (this will not be in general the topology induced by the ambient space $\mathbf{M}^n$). We say that a definable group G is definably compact if it is so in the abstract manifold topology. Using these facts, together with results of A. Strzebonski in [17], the fixed point theorem (Theorem 3.3) has the following corollary concerning the existence of torsion elements in definable groups, as observed in [1].

Corollary 3.4 *Let $G \subset \mathbf{M}^n$ be a definably compact definable group. Then G has torsion.*

Proof. By [17] a definable group G has an element of order p for each prime p dividing its Euler characteristic. On the other hand we can assume without loss of generality that our definably compact group G is a definable submanifold of $\mathbf{M}^n$ (see [1]). Also, by the triangulation theorem we can assume that $G = |K|^{\mathbf{M}}$ for a finite simplicial complex K. To prove the existence of torsion we can clearly assume that G is infinite. By [17] it will suffice to show that the Euler characteristic of K is zero. Suppose it is not. Now the definably compact definable manifold $|K|^{\mathbf{M}}$ is definably orientable (see definition on Section 5) since, as in the classical case, one can choose a local orientation at a point of a given connected component and extend it to the whole component by left group multiplication. By Theorem 3.3 any definable map $f\colon |K|^{\mathbf{M}} \to |K|^{\mathbf{M}}$ definably homotopic to the identity must have a fixed point. But this is absurd because left multiplication by an element in the definably connected component of the neutral element (which must be non trivial) is definably homotopic the the identity. $\square$

The question of the existence of torsion elements was raised by Y. Peterzil and C. Steinhorn in [14] and it has been positively answered by M. Edmundo in [8] using different methods.

4 Covering by proper balls

In this Section we prove some results we shall need for the transfer principle **TR6 (orientation)**. They will also clarify the role of **TR2 (compactness)**. The correspondence between definable compactness and compactness in **TR2** does not ensure that the basic properties of compactness have a natural definable correspondence. For instance it is clearly not the case, if $\mathbf{M}$ is non-archimedean, that any definable open cover of a definably compact set has a finite subcover. We shall however prove that a definably compact definable manifold can be covered by finitely many proper balls (Corollary 4.4).

Let Y be a definable set. A subset B of Y is called **definable proper m-ball** (in Y) if there is a definable homeomorphism taking the closure (in Y) of B onto the standard unit closed ball $D^m \subset \mathbf{M}^m$ and taking the frontier (in Y) of B onto the unit sphere S^{m-1}.

Let K be a finite simplicial complex and $|K|^{\mathbf{M}} \subset \mathbf{M}^m$ its geometrical realization in $\mathbf{M}^m$. An **open-simplex** $\sigma \subset \mathbf{M}^m$ of K is (the geometrical realization of) a simplex of K minus its proper faces. So $|K|^{\mathbf{M}}$ is a disjoint union of open-simplexes. Note that a vertex of K is an open-simplex. Given a vertex x of K, the **star** $St(x, K)$ is the union of all the open-simplexes of K having x as a vertex. The **link** $Lk(x, K)$ is the union of all the simplexes included in the closure of $St(x, K)$ which do not contain x.

Lemma 4.1 *Let $x \in |K|^{\mathbf{M}} \subset \mathbf{M}^m$ be a vertex of the simplicial complex K. If $St(x, K)$ is open in $\mathbf{M}^m$, then it is a proper m-ball (in $\mathbf{M}^m$).*

Proof. It suffices to observe that radial projection from x gives a homeomorphism from a small sphere $S^{m-1} \subset St(x, K)$ onto $Lk(x, K)$, and that $\overline{St(x, K)}$ is the cone over $Lk(x, K)$ with vertex x. $\square$

Lemma 4.2 *A definable bounded open subset A of $\mathbf{M}^m$ is a finite union of definable proper m-balls (in $\mathbf{M}^m$).*

Proof. Let $\psi: \overline{A} \to |K|^{\mathbf{M}} \subset \mathbf{M}^m$ be a triangulation of the definably compact set $\overline{A}$ compatible with the open set $A \subset \mathbf{M}^m$. The set $O = \psi(A)$ is a union of open-simplexes of K and it is open in $|K|^{\mathbf{M}}$. Moreover O, being definably homeomorphic to the open set A, is open in $\mathbf{M}^m$ ("invariance of domain", see [20] or [10]). We must show that O is a finite union of definable proper m-balls (so A will also be a finite union of definable proper m-balls because $\psi_{|A}$ extends to a definable homeomorphism between closed sets). Let $x \in |K|^{\mathbf{M}}$ be a vertex of a subdivision K' of K. If $x \in O$ then $St(x, K')$ is an open subset of O, so it is open in $\mathbf{M}^m$. By Lemma 4.1 $St(x, K')$ is a definable proper m-ball. To finish we prove that, for K' the barycentric subdivision of K, every point of O is contained in one such ball. So let $y \in O$ and let σ be the unique open-simplex of K included in O and containing y. Then y belongs to $St(\hat{\sigma}, K')$ where $\hat{\sigma} \in K'$ is the barycenter of σ. $\square$

Theorem 4.3 *Let $Y \subset \mathbf{M}^n$ be a definable manifold of dimension m and let $N \subset Y$ be a definably compact subset. Then there is a finite family of definable proper m-balls in Y covering N. Moreover, given a finite atlas for Y, we can choose the family of proper m-balls so that the closure (in Y) of each of them is contained in an open set of the atlas.*

Proof. Let $\{U_i \mid i = 1, \ldots, s\}$ be a finite atlas for Y and let $\phi_i: U_i \to \phi_i(U_i)$ be a definable homeomorphism onto a definable open subset of $\mathbf{M}^m$. Since Y has the induced topology from $\mathbf{M}^n$, it is definably normal (any two disjoint definable closed sets have disjoint definable open neighbourhoods). Hence (see for instance [1, Lemma 10.5]) there are open sets V_i covering Y with $\overline{V_i} \subset U_i$ $(i = 1, \ldots, s)$, where $\overline{V_i}$ is the closure of V_i in Y. We have $N = \bigcup_i (N \cap V_i)$. Since $N \cap \overline{V_i}$ is a definably compact subset of U_i, it has an open neighbourhood O_i with $\overline{O_i} \subset U_i$ and $\overline{O_i}$ definably compact. (If not, for every $\epsilon > 0$ there is a point $x(\epsilon)$ outside of U_i and at distance $< \epsilon$ from $N \cap \overline{V_i}$; by o–minimality we can assume that the limit for $\epsilon \to 0$ of $x(\epsilon)$ exists and we easily get a contradiction.) To finish the proof it suffices to show that O_i is a finite union of definable proper m-balls. Since $\overline{O_i}$ is definably compact, so is its image $\phi_i(\overline{O_i})$. It follows that the open set $\phi_i(O_i) \subset \phi_i(\overline{O_i})$ is bounded, hence by Lemma 4.2 it is a finite union of definable proper m-balls. Since $\phi_{i|O_i}$ is a definable homeomorphism between O_i and $\phi_i(O_i)$ which extends to the respective closures, it follows that this homeomorphism and its inverse preserves properness of definable m-balls. So O_i is also a finite union of definable proper m-balls. $\square$

In particular we obtain:

Corollary 4.4 *Let $Y \subset \mathbf{M}^n$ be a definably compact definable manifold of dimension m. Then Y is a finite union of definable proper m-balls.*

The above corollary may seem an immediate consequence of the triangulation theorem, since a polyhedron admits a finite covering given by the stars of its vertices. However the following remark shows that these stars may not be proper balls, even in the classical case.

Remark 4.5 (see [18]) If the polyhedron $|K|^{\mathbb{R}}$ has the link of each vertex homeomorphic to the sphere S^{m-1}, then $|K|^{\mathbb{R}}$ is a topological manifold. The converse holds in dimension ≤ 3 but it is not true in general. A counterexample is the double suspension $\Sigma\Sigma P$ of Poincaré dodecahedral space P. The space P is obtained by identifying opposite faces of a solid dodecahedron after a twist of $2\pi/10$. The double suspension $\Sigma\Sigma P$ is a manifold (homeomorphic to S^5) but the link of its two suspension points is ΣP, which is not a sphere. In fact ΣP is not even a manifold because to be such the links of the vertices should be simply connected and the link of a suspension point is P which is not simply connected. Therefore we have a manifold $\Sigma\Sigma P$ such that the stars of some vertices (the suspension points) are not homeomorphic to proper balls, as otherwise the links would be homeomorphic to spheres.

5 Orientation

In this section we prove **TR6**.

Let Y be a definable manifold of dimension m. A **definable orientation** on Y is a map s which assigns to each point $p \in Y$ a generator $s(p)$ of the local definable homology group $H_m^{def}(Y, Y-p)$ and it is locally constant in the following sense. For each point there is a definably compact neighbourhood (of the point) N and a class $\zeta_N \in H_m^{def}(Y, Y-N)$ such that for each $p \in N$ the natural homomorphism $H_m^{def}(Y, Y-N) \to H_m^{def}(Y, Y-p)$, induced by the inclusion map $(Y, Y-N) \to (Y, Y-p)$, sends ζ_N into $s(p)$.

Note that a definable orientation is not a definable map. It is defined as in the classical case except that the homology groups are the definable ones and the neighbourhoods are definably compact.

Let $X^{\mathbf{M}}$ be a definably compact definably oriented definable manifold, and assume that $X^{\mathbb{R}}$ is a manifold. To prove that an orientation on $X^{\mathbf{M}}$ induces an orientation on $X^{\mathbb{R}}$ (transfer result **TR6**), the difficulty is that a priori the orientation on $X^{\mathbf{M}}$ could be locally constant only on neighbourhoods which are infinitesimally small. To see that this is not the case we use Theorem 4.3. We need some preliminaries.

Given an oriented definable manifold Y of dimension m, a **definable fundamental class** of Y is an element $\zeta \in H_m^{def}(Y)$ such that for each $p \in Y$ the natural homomorphism $H_m^{def}(Y) \to H_m^{def}(Y, Y-p)$ sends ζ to $s(p)$, where s is the given orientation.

Let S^n be the unit n-dimensional sphere. The following result was proved by A. Woerheide in [20] and used as the key Lemma to prove the o–minimal version of the Jordan-Brouwer separation theorem.

Fact 5.1 *Let N be a definable subset of $(S^n)^{\mathbf{M}}$ definably homeomorphic to a k-dimensional cube I^k, with $k \leq n$. Then $(S^n)^{\mathbf{M}} - N$ has the same homology groups of a point. That is,*

$$H_q^{def}((S^n)^{\mathbf{M}} - N) = \begin{cases} 0 & q \neq 0 \\ \mathbb{Z} & q = 0. \end{cases}$$

Note that Fact 5.1 would follow at once if one could prove that $(S^n)^{\mathbf{M}} - N$ is definably contractible to a point, as it happens when $k = 0$. We leave this as an open question. A positive answer would have no classical analogue due to the possibility of wild embeddings of I^k into S^n (see [11, §4.6]).

Theorem 5.2 *If $X^{\mathbf{M}}$ is a definably compact oriented definable manifold, then $X^{\mathbf{M}}$ has one and only one fundamental class for the given orientation.*

Proof. (We follow the approach in Lemma A7 and Theorem A8 of [12], so we only highlight the changes we have to make in the o–minimal context.) We prove the stronger result: if N is a definably compact subset of a definable manifold Y of dimension m with an orientation s, then there is one and only one relative homology class $\zeta_N \in H_m^{def}(Y, Y-N)$ such that for each $p \in N$ the inclusion of pairs $(Y, Y-N) \to (Y, Y-p)$ induces an homomorphism in homology sending ζ_N to $s(p) \in H_m^{def}(Y, Y-p)$. The fundamental class is then obtained taking $Y = X^{\mathbf{M}} = N$.

Case 1. $Y = \mathbf{M}^n$ and N is definably homeomorphic to the geometrical realization of a closed simplex. In this case we show that the inclusion of pairs $(Y, Y-N) \to (Y, Y-p)$ induces an isomorphism in homology. First note that up to definable homeomorphisms, we may consider $\mathbf{M}^n$ as the open southern hemisphere of $S = (S^{n+1})^{\mathbf{M}}$. By Fact 5.1, $S-N$ and $S-p$ have both trivial homology. Hence $H_q^{def}(S-N, S-p) = 0$, for all q. Thus $H_q^{def}(\mathbf{M}^n - N, \mathbf{M}^n - p) = 0$, for all q. It then follows that the inclusion of $\mathbf{M}^n - N$ into $\mathbf{M}^n - p$ induces an isomorphism in homology and hence we can conclude that the inclusion of pairs $(\mathbf{M}^n, \mathbf{M}^n - N) \to (\mathbf{M}^n, \mathbf{M}^n - p)$ induces an isomorphism in homology.

Case 2. Y is arbitrary and $N = N_1 \cup N_2$, where the result is known to be true for N_1, N_2 and for $N_1 \cap N_2$. Then one reason as in [12, Lemma A7, Theorem A8] using the relative Mayer-Vietoris sequence (before proving the existence of ζ_N one must prove the unicity).

Case 3. $Y = \mathbf{M}^n$ and N is a definably compact subset of Y. By the triangulation theorem N is the union of a finite family closed under intersections of sets which are definably homeomorphic to geometrical realizations of closed simplexes. The result then follows by induction using Case 1 and Case 2.

Case 4. $N \subset Y$ is contained in a definable proper m-ball of Y. Since a definable proper m-ball is definably homeomorphic to $\mathbf{M}^m$ we reduce to Case 3 by the excision axiom.

Case 5. $N \subset Y$ is arbitrary. The idea is to write N as a suitable finite union $N = \bigcup_{i=1}^k N_i$ of closed sets, in such a way that we can conclude the proof by induction using Case 2 for the induction step and Case 4 for the base case. By Theorem 4.3 there is a finite family $\{U_i \mid i = 1, \ldots, k\}$ of definable proper m-balls in Y covering N. Let f_i be a definable homeomorphism between U_i and the standard open unit ball $B \subset \mathbf{M}^{m+1}$. For $\varepsilon > 0$, let $B(\varepsilon) \subset B$ be the closed ball of radius $1-\varepsilon$, with the same center as B. Let $U_i(\varepsilon) \subset U_i$ be the f_i-preimage of $B(\varepsilon)$. We claim that for ε small enough the closed sets $U_i(\varepsilon)$ $(i = 1, ..., k)$, cover N. To see this, suppose for a contradiction that for each $\varepsilon > 0$ there is a point $x(\varepsilon) \in N$ which is not contained in $\bigcup_i U_i(\varepsilon)$. By o–minimality and definable compactness we can assume that the limit $x = \lim_{\varepsilon \to 0} x(\varepsilon)$ exists in N. But x must belong to some U_i, hence also to some $U_i(\varepsilon)$. This contradiction proves the claim. Now fix ε as given by the claim, and consider a triangulation of Y compatible with N and each $U_i(\varepsilon)$. This induces a partition of Y into open curvilinear simplexes (an open simplex is a simplex minus its proper faces). Since N is closed, it can be written as the union $\bigcup_{i=1}^{s} N_i$ of those *closed* curvilinear simplexes of the triangulation which are contained in N. By construction each N_i is contained in a definable proper m-ball U_i, and the family $\{N_i \mid i = 1, \dots, k\}$ is closed under finite intersections (as the intersection of two simplexes of a triangulation is either empty or a common face of both). We can thus conclude by induction on k using Case 2 and Case 4. $\square$

Theorem 5.3 *Assume $X^{\mathbb{R}}$ is a manifold of dimension m. If $X^{\mathbf{M}}$ is an orientable definable manifold, then $X^{\mathbb{R}}$ is an orientable manifold.*

Proof. Fix an orientation on $X^{\mathbf{M}}$. We can assume $X^{\mathbf{M}}$ is definably connected and $m > 0$. By Theorem 5.2 $X^{\mathbf{M}}$ has a fundamental class ζ, so $H_m^{def}(X^{\mathbf{M}}) \neq 0$. By transfer result **TR3**, $H_m(X^{\mathbb{R}}) \neq 0$. Since, by **TR1**, $X^{\mathbb{R}}$ is connected, this implies that $H_m(X^{\mathbb{R}})$ is infinite cyclic, and a generator of this group will induce an orientation on $X^{\mathbb{R}}$. $\square$

References

[1] A. Berarducci and M. Otero, Intersection theory for o–minimal manifolds, Annals of Pure and Applied Logic 107 (2001) 87–119

[2] A. Berarducci and M. Otero, O–minimal fundamental group, homology and manifolds, J. London Math. Soc. 65 (2002) 257–270

[3] A. Berarducci and M. Otero, Some examples of transfer methods for o-minimal structures, The Proceedings of the meeting "Spring Stage on Logic, Algebra and Geometry", Caserta, Italy, 21-24 March 2000, Preprint

[4] R. Brown, **Elements of Modern Topology**, McGraw-Hill, London 1968

[5] G. W. Brumfield, A Hopf fixed point theorem for semialgebraic maps, in: **Real Algebraic Geometry**, Proc. Rennes 1991, LNM 1524, Springer–Verlag, 1992.

[6] H. Delfs and M. Knebusch, On the homology of algebraic varieties over real closed fields, J. reine u.angew. Math 335 (1982) 122–163

[7] L. van den Dries, **Tame Topology and o–minimal structures**, London Math. Soc. Lect. Notes Series 248, Cambridge Univ. Press 1998

[8] M. J. Edmundo, O–minimal cohomology and definably compact definable groups, Preprint, 2000 (revised version 2001)

[9] D. E. Galewski and R. J. Stern, Classification of simplicial triangulations of topological manifolds, Annals of Mathematics 111 (1980) 1–34

[10] J. Johns, An open mapping for o–minimal structures, Journal of Symbolic Logic 66 (2001) 1817–1820

[11] J. G. Hocking and G. S. Young, **Topology**, New York, Dover Publ. 1988

[12] J. W. Milnor and J. D. Stasheff, **Characteristic Classes**, Annals of Mathematics Studies, Princeton Univ. Press, Princeton 1974

[13] M. Otero, Y. Peterzil and A. Pillay, Groups and rings definable in o–minimal expansions of real closed fields, Bull. of London Math. Soc. 28 (1996) 7–14

[14] Y. Peterzil and C. Steinhorn, Definable compacteness and definable subgroups of o–minimal groups, J. of London Math. Soc.(2) 59 (1999) 769–786

[15] A. Pillay, On groups and fields definable in o–minimal structures, J. Pure Appl. Algebra 53 (1988) 239–255

[16] M. Shiota, **Geometry of subanalytic and semialgebraic sets**, progress in mathematics, Birkhäuser, Boston 1997

[17] A. Strzebonski, Euler characteristic in semialgebraic and other o–minimal structures, J. Pure Appl. Algebra 96 (1994) 173–201

[18] W. P. Thurston, **Three–dimensional Geometry and Topology**, Princeton University Press, 1997.

[19] J. W. Vick, **Homology Theory**, Springer–Verlag, 1994.

[20] A. Woerheide, **O–minimal homology**, PhD Thesis, University of Illinois at Urbana–Champaign, 1996

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI PISA
VIA BUONARROTI 2, 56127 PISA, ITALY
E-mail: berardu@dm.unipi.it

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD AUTÓNOMA DE MADRID
CANTOBLANCO, 28049 MADRID, SPAIN
E-mail: margarita.otero@uam.es