A recursive nonstandard model of normal open induction

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Abstract

Models of normal open induction are those normal discretely ordered rings whose nonnegative part satisfy Peano's axioms for open formulas in the language of ordered semirings. (Where normal means integrally closed in its fraction field.)

In 1964 Shepherdson gave a recursive nonstandard model of open induction. His model is not normal and does not have any infinite prime elements.

In this paper we present a recursive nonstandard model of normal open induction with an unbounded set of infinite prime elements.

1 Introduction

Let L be the language of ordered rings based on the symbols $+, -, \cdot, 0, 1, \leq$. We write $\mathbf{N}^*$ for $\mathbf{N} \setminus \{0\}$. We consider the following sets of axioms in L.

DOR: discretely ordered rings

  ($i.e.$, axioms for ordered rings and $\forall x \neg(0 < x < 1)$).

ZR: discretely ordered $\mathbf{Z}$-rings

  ($i.e.$, DOR and for every $n \in \mathbf{N}^*$ $\forall x \exists q, r (x = qn + r \wedge 0 \leq r < n)$.)

OI: open induction

  ($i.e.$, DOR and for every open L-formula $\psi(x, y)$

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$\forall \vec{x}(\psi(\vec{x},0) \wedge \forall y \geq 0(\psi(\vec{x},y) \rightarrow \psi(\vec{x},y+1)) \rightarrow \forall y \geq 0\psi(\vec{x},y))$.

Besides these three sets of axioms we also consider their *normal* (or integrally closed) counterparts $NDOR$, NZR and NOI respectively, that is, we add to each one of them the following axioms.

N: Normality

($i.e.$ for every $n \in \mathbf{N}^*$, $\forall x, y, z_1, \ldots, z_{n-1}$
$(y \neq 0 \wedge x^n + z_1 x^{n-1} y + \ldots + z_{n-1} x y^{n-1} + y^n = 0 \rightarrow \exists z(yz = x)))$.

The non-negative part of a model of OI can be naturally identified with a model of the fragment $IOpen$ of Peano Arithmetic (PA). It is easy to see that the fragment IE_1 of PA (induction for bounded existential formulas, see [Wm]) is stronger than $IOpen$ + *normality*, where here normality means the axioms N adjusted to the language of semirings.

For M an ordered domain $RC(M)$ will denote the real closure of the fraction field of M. Given two ordered domains $R_1 \subset R_2$ we say that R_1 is an *integral part* of R_2 if R_1 is a discretely ordered ring and for every $\alpha \in R_2$ there is $a \in R_1$ with $a \leq \alpha \leq a+1$.

Models of OI and NOI have been studied by several authors, see [Sh], [W], [D1], [D2], [M-M] [Bg], [Ot1], [Ot2] and [Mo-R].

The following two theorems are the starting point to construct (normal) models of OI.

Theorem 1.1 (Shepherdson [Sh]) *Let M be an ordered domain. M is a model of OI if and only if M is an integral part of $RC(M)$.*

Theorem 1.2 (Wilkie [W]) *Every model of ZR can be extended to a model of OI.*

Note that OI is stronger than ZR. Both theorems extend (in the obvious way) to the normal case. For Shepherdson's theorem this is clear. See [D2], for Wilkie's theorem in the normal case.

Shepherdson (using Theorem 1.1) gave a recursive nonstandard model S of OI, namely

$$S = \left\{ a_n x^{n/q} + \ldots + a_1 x^{1/q} + a_0 \mid n \in \mathbf{N}, q \in \mathbf{N}^*, a_i \in RC(\mathbf{Q}), a_0 \in \mathbf{Z} \right\}$$

with the order determined by making $x > n$ for every $n \in \mathbf{N}$.

The real closure of $\mathbf{Q}(x)$ with $x > n$ for every $n \in \mathbf{N}$ is the field of those Puisseux series in descending powers of x which are real algebraic over $\mathbf{Q}(x)$. Clearly S is an integral part of $RC(\mathbf{Q}(x))$. This is by far not the only recursive model of OI. For any recursive real closed field $K \subseteq \mathbf{R}$ we can similarly construct a recursive model S_K of OI replacing $RC(\mathbf{Q})$ by K in the definition of S.

Shepherdson exhibited his model to show that OI does not prove the normality axioms. It is clear that none of the above S_K's is normal either. In particular $\sqrt{2}$ is rational in Shepherdson's model. On the other hand we know that fragments of arithmetic from IE_1 to PA (are normal and) do not have recursive models (see Tennembaum [T] for PA, McAloon [Mc] for $I\Delta_0$ and Wilmers [Wm] for IE_1). A natural question remains open.

Does NOI have recursive nonstandard models?

In Theorem 4.4 we give a positive answer to this question. Using Wilkie's method we construct a normal integral part M of $R = RC(K(x))$ (with the above notation), for a suitable recursive real closed field $K \subset \mathbf{R}$. Hence this R will have two recursive nonelementary equivalent integral parts S_K and M. Unlike Shepherdson's model M will not be truncation closed, i.e. $a_{l+1}x^{(l+1)/q}+\ldots+a_1x^{1/q}+a_0 \in M$ does not imply $a_lx^{l/q}+\ldots+a_1x^{1/q}+a_0 \in M$.

In another line of research Macintyre and Marker study in [M-M] behaviour of prime elements in models of NOI. They construct (nonrecursive) models of NOI in which the set of infinite prime elements has any given order-type. They pose the following question.

Is there a recursive model of open induction with infinite prime elements?

Neither Shepherdson model S nor any S_K has infinite prime elements.

One might think that the existence of infinite primes in a nonstandard model of open induction M could be used to code a pair of recursively inseparable sets and then prove M is not recursive as in the case of nonstandard models of IE_1. However we shall show it is not the case, giving in turn a positive answer to the above question in the following strong sense.

Theorem 1.3 *There is a recursive nonstandard model of normal open induction with an unbounded set of prime elements.*

This theorem is a corollary of theorems 4.2, 4.3 and 4.4 below.

2 Basic facts

Definition 2.1 Let K be a field and let $K(\!(x^{1/\mathbf{N}})\!)$ denote the field of Puiseux series in descending powers of x with coefficients in K. Namely,

$K(\!(x^{1/\mathbf{N}})\!) = \Big\{\sum_{k\leq m} a_k x^{k/q} : m \in \mathbf{Z},\ q \in \mathbf{N}^*\ a_k \in K\Big\}$.

If K a real (algebraically) closed field then $K(\!(x^{1/\mathbf{N}})\!)$ is a real (resp. algebraically) closed field (see [Wk]).

Let $L(x)$ be an ordered field with $x > a$ for all $a \in L$, then the real closure of $L(x)$ is the subfield of $RC(L)(\!(x^{1/\mathbf{N}})\!)$ of those series which are real algebraic over $L(X)$.

We shall use the following notation for subrings of $K(\!(x^{1/\mathbf{N}})\!)$.

$K\Big[x^{\mathbf{Q}}\Big] = \Big\{a_lx^{l/q} + \cdots + a_1x^{1/q} + a_0 : a_0, \cdots, a_l \in K\Big\}$

$K\left[x^{\mathbf{Q}}\right]^{\bullet} = \left\{a_l x^{l/q} + \cdots + a_1 x^{1/q} : q \neq 0 \text{ and } a_1, \cdots, a_l \in K\right\}$
$K\left[x^{\mathbf{Q}}\right]^{\bullet} + \mathbf{Z} = \left\{a + m : a \in K\left[x^{\mathbf{Q}}\right]^{\bullet}, m \in \mathbf{Z}\right\}$.

For instance, Shepherdson's model is $S = K\left[x^{\mathbf{Q}}\right]^{\bullet} + \mathbf{Z}$.

If L is an ordered field $AC(L)$ will denote $RC(L)[\sqrt{-1}]$. If L is an ordered domain $\mathbf{Q}L$ will denote the $\mathbf{Q}$-algebra generated by L.

Definition 2.2 Let R be an ordered field. Let $R_1 \subset R$ and $a \in R$.

We say that a is *infinitesimal (finite, infinite)* if $|a| < 1/n$ for all $n \in \mathbf{N}^*$ ($|a| < n$ for some $n \in \mathbf{N}$, $|a| > n$ for all $n \in \mathbf{N}$, respectively).

We say that a is *infinitely close to (at finite distance from, at infinite distance from)* R_1 if there is $b \in R_1$ such that $|a-b|$ is infinitesimal (there is $b \in R_1$ such that $|a-b|$ is finite, for all $b \in R_1$ $|a-b|$ is infinite, respectively).

We extend above concepts to $AC(R_1)$ via the norm map to $RC(R_1)$

Lemma 2.3 *Let $K \subset \mathbf{R}$ be a real closed field and $L = RC(K(x))$ with $x > n$ for all $n \in \mathbf{N}$. Let M be a (normal) domain included in L. If M is an integral part of $K\left[x^{\mathbf{Q}}\right]$, then M is a (normal) model of OI.*

Proof. By Shepherdson's theorem □

Lemma 2.4 *Let F be an ordered field. Let $L \subset F, L$ a model of ZR, and $\beta \in F$. The following are equivalent:*

1. *β is at infinite distance from L;*

2. *β is at infinite distance from $\mathbf{Q}L$.*

Proof. Let $a \in L$, $k \in \mathbf{Z}^*$ and suppose $|\beta - \frac{a}{k}|$ finite. Since $L \models ZR$, for these a and k there are $r, q \in L$ such that $a = qk + r$ and $0 \leq r < k$. Then $|\beta - q|$ is finite. □

Lemma 2.5 *Let F be an ordered field contained in $\mathbf{R}$. Let L be a discretely ordered ring. Then $F \cap \mathbf{Q}L = \mathbf{Q}$.*

Proof. For the nontrivial inclusion, let $\frac{a}{k} \in F$ with $a \in L$ and $k \in \mathbf{N}$, $k \neq 0$. Since $F \subset \mathbf{R}$ we have $0 \leq |\frac{a}{k}| < m$ for some $m \in \mathbf{N}$, then $0 \leq |a| < mk$. Hence $a \in \mathbf{Z}$ by the discreteness of the order in L. Thus $\frac{a}{k} \in \mathbf{Q}$. □

Next lemma is due to Wilkie (see [W]). He expressed it in slightly different form.

Lemma 2.6 *Let F be an ordered field. Let $L \subset F$ be such that L is a model of ZR. Let $\beta \in F$. If β is not infinitesimally close to $AC(L)$ and β is at infinite distance from L, then $L[\beta]$ is a discretely ordered ring extending L.*

Proof. It suffices to show that if $f(y) \in L[y]$ is a non-constant polynomial, then $f(\beta)$ is infinite. For a contradiction suppose that $f(\beta)$ is finite. Write $f(y) = a(y-\theta_1)\dots(y-\theta_n)$ where $a \in L$ and $\theta_i \in AC(L)$. Then $f(\beta) = a(\beta-\theta_1)\dots(\beta-\theta_n)$. If a product of non-infinitesimal elements is finite, then all the factors must be finite. So β must be at finite distance from all the roots θ_i and a must also be finite, so $a \in \mathbf{Z}$. Then $\frac{1}{a}f(\beta)$ must be also infinite, $\frac{1}{n}\sum_{i=1}^n \theta_i \in \mathbf{Q}L$ and $|\beta - \frac{1}{n}\sum_{i=1}^n \theta_i| \leq \frac{1}{n}\sum_{i=1}^n |\beta-\theta_i|$ is finite. Since $L \models ZR$, by Lemma 2.4 we have β is at finite distance from L, contrary to the assumption. □

Let $\hat{\mathbf{Z}}$ denote the product of the rings $\mathbf{Z}_p$ of p-adic integers.

Lemma 2.7 *Given a ring homomorphism $\varphi: L \to \hat{\mathbf{Z}}$ (L a domain), there is a unique $\mathbf{Z}$-ring L_φ with $L \subset L_\varphi \subset \mathbf{Q}L$ such that φ extends to $\varphi': L_\varphi \to \hat{\mathbf{Z}}$. Moreover $L_\varphi = \{\frac{a}{k} \mid a \in L, k \in \mathbf{Z}^*, k|\varphi(a)\}$.*

Proof. The existence is clear. To prove the uniqueness let us suppose $\varphi: L \to \hat{Z}$ extends to $\varphi_i: L_i \to \hat{\mathbf{Z}}$ with L_i a $\mathbf{Z}$-ring, $L \subset L_i \subset \mathbf{Q}L$ $(i = 1, 2)$. It suffices to show $L_1 \subset L_2$. Let $\frac{a}{k} \in L_1$ $(a \in L, k \in \mathbf{Z}^*)$. Let $c = \varphi_1\left(\frac{a}{k}\right)$. Then $ck = \varphi_1(a) = \varphi_2(a)$. In L_2 let $r, q \in L_2$ be such that $a = qk + r$ with $0 \leq r < k$. Thus $\varphi_2(a) = \varphi_2(q)k + r = ck$, and therefore $r = 0$. We conclude that $q = \frac{a}{k} \in L_2$. □

We shall make use of the following results:

Lemma 2.8 *If $L \models NDOR$ and $\varphi: L \to \hat{\mathbf{Z}}$, then $L_\varphi \models NZR$.*

Proof. See [D2]. □

Lemma 2.9 *If L is normal and y is transcendental over L, then $L[y]$ is normal.*

Proof. See [Bk]. □

Lemma 2.10 *If $\varphi: L \to \hat{\mathbf{Z}}$ is a ring homomorphism (L a domain), $p \in L$ is prime, and $\varphi(p)$ is a unit of $\hat{\mathbf{Z}}$, then p is prime in L_φ.*

Proof. See [M-M]. □

3 The base field

An ordered ring $R = (R, +, \cdot, <)$ is *recursive* if there is an algorithm which, on input $x, y \in R$, computes $x + y$, $x \cdot y$ and decides whether $x \geq y$ or not.

For this to make sense we assume that we have a fixed *representation* of the elements of R as finite strings of symbols, and that there is an algorithm to test whether a finite string of symbols is the representation of some element of R.

In this section we are going to define a recursive real closed field $K \subset \mathbf{R}$ of infinite transcendence degree over $\mathbf{Q}$. We need:

Definition 3.1

1. A real number $\gamma \in \mathbf{R}$ is *recursive* if there is an algorithm which, on input $n \in \mathbf{N}$, computes the first n digits of the decimal expansion of γ.
2. A sequence of real numbers $\{\pi_i \mid i \in \mathbf{N}^*\}$ is a *recursive family of recursive real numbers* if there is an algorithm which on input $i \in \mathbf{N}^*$ and $n \in \mathbf{N}$ computes the first n digits of the decimal expansion of π_i.

The method of Liouville to prove the existence of a transcendental real number actually shows the existence of a transcendental recursive real number. Similarly one can prove that there exists a recursive family $\{\pi_i \mid i \in \mathbf{N}^*\}$ of algebraically independent real numbers.

We fix such a sequence $\{\pi_i \mid i \in \mathbf{N}^*\}$ for the rest of this paper and we define L as the ordered ring $\mathbf{Q}(\{\pi_i \mid i \in \mathbf{N}^*\})$ with the order induced by that of $\mathbf{R}$. We represent a non-zero element $a \in L$ by a pair of relatively prime polynomials over $\mathbf{Q}$. Under this representation we have:

Lemma 3.2 *L is a recursive ordered field.*

Proof. The only difficulty is to show that the order is recursive. It suffices to find an algorithm which given a nonzero polynomial $f(y_1, \ldots, y_n) \in \mathbf{Q}[y_1, \ldots, y_n]$ decides whether $f(\pi_1, \ldots, \pi_n) > 0$. Consider the theory T axiomatized by the axioms for real closed fields, the assertions that $f(\pi_1, \ldots, \pi_n) \neq 0$ for f nonzero, and the various assertions $q < \pi_i < r$, for $q, r \in \mathbf{Q}$, which hold in $\mathbf{R}$. It follows that T is a complete recursive theory in the language of ordered rings with a constant symbol for each π_i. Hence T is decidable and yields the desired algorithm. □

Now define K as the real closure of L in $\mathbf{R}$. We represent a non-zero element a of K by a pair $(g(y), s)$ consisting of the minimal polynomial $g(y) \in L[y]$ of a and a natural number s indicating the relative order of a among the real roots of $g(y)$.

Lemma 3.3 *K is a recursive ordered field.*

Proof. It is a a classic fact in recursive algebra that a recursive ordered field has a recursive real closure. $\square$

4 The construction of the model

Let K be the recursive field defined in the previous section. Let $K_n = RC(\mathbf{Q}(\pi_1, \ldots, \pi_n))$ thus $K = \bigcup_n K_n = RC(\mathbf{Q}(\{\pi_i\})$. We are going to construct a domain $M \subset K[x^{\mathbf{Q}}]$ which is an integral part of $K[x^{\mathbf{Q}}]$. The model M will be generated by elements of the form $\beta_i = p_i(x) + \pi_i$ with $p_i(x) \in K_{i-1}[x^{\mathbf{Q}}]^\bullet$. For such β's we have the following lemma:

Lemma 4.1 *Let $\beta_i = p_i(x) + \pi_i$ with $p_i(x) \in K_{i-1}[x^{\mathbf{Q}}]^\bullet$ $(i = 1, \ldots, n)$. Then $\beta_1, \ldots, \beta_n$ are algebrically independent over $\mathbf{Q}$.*

Proof. Since $p_i(x) \in AC(\mathbf{Q}(x, \pi_1, \ldots, \pi_{i-1}))$ and $\pi_i \notin AC(\mathbf{Q}(x, \pi_1, \ldots, \pi_{i-1}))$, we have $\beta_i \in AC(\mathbf{Q}(x, \pi_1, \ldots, \pi_i))$ and $\beta_i \notin AC(\mathbf{Q}(x, \pi_1, \ldots, \pi_{i-1}))$ $(i = 1, \ldots, n)$. Therefore $\beta_1, \ldots, \beta_n$ are algebrically independent over $\mathbf{Q}$. $\square$

We define $M = \bigcup_n M_n$ as follows. First we fix a recursive enumeration of $K[x^{\mathbf{Q}}]^\bullet$.

<u>Stage 0</u>. $M_0 = \mathbf{Z}$.

<u>Stage $n > 0$</u>. We have the following data:

1. $\beta_i = p_i(x) + \pi_i$ where $p_i(x) \in K_{i-1}[x^{\mathbf{Q}}]$ $(i = 1, \ldots, n)$.
2. A chain of models of NZR, namely, $M_0 \subset M_1 \subset \ldots \subset M_n$ where $M_i = \left\{ \frac{1}{k} f(\beta_1, \ldots, \beta_n) \in \mathbf{Q}[\beta_1, \ldots, \beta_n] \mid k \text{divides} f(1, 1, \ldots, 1) \right\}$.

   (Where $f(\beta_1, \ldots, \beta_n)$ ranges over $\mathbf{Z}[\beta_1, \ldots, \beta_n]$.)

<u>Stage $n + 1$</u>. We consider the first element in the enumeration of $K[x^{\mathbf{Q}}]^\bullet$, $p(x)$ say, which is in $K_n[x^{\mathbf{Q}}]$ and it is at infinite distance from M_n. Then we define $p_{n+1}(x) = p(x)$ and $\beta_{n+1} = p_{n+1}(x) + \pi_{n+1}$.

<u>Claim</u> $M_{n+1} = \left\{ \frac{1}{k} f(\beta_1, \ldots, \beta_{n+1}) \in \mathbf{Q}[\beta_1, \ldots, \beta_{n+1}] \mid k | f(1, 1, \ldots, 1) \right\}$ is a model of NZR extending M_n.

Granted the existence of such a $p(x)$ and the claim, we let $M = \bigcup_n M_n$. Since NZR is a $\forall\exists$-theory, $M \models NZR$.

The existence of such a $p(x)$ is clear since $M_n \subset L[x^{\mathbf{Q}}]$ for some finite algebraic extension L of $K_{n-1}[\pi_n]$. Therefore it suffices to get $p(x) \in K_n[x^{\mathbf{Q}}] \setminus L[x^{\mathbf{Q}}]$.

<u>Proof of the claim</u>. We first prove that $M_n[\beta_{n+1}] \models DOR$. Since $M_n \models ZR$ and the distance from β_{n+1} to M_n is infinite (because the $p_{n+1}(x)$ is

at an infinite distance from, M_n), by 2.6 it suffices to prove that β_{n+1} is not infinitely close to $AC(M_n)$. For this note that if $\alpha \in AC(M_n)$, $\alpha = \Sigma_{i<k} c_i x^{i/q}$ ($c_i \in AC(K_n)$) say, and $|\alpha - \beta_{n+1}| < \frac{1}{m}$ for each $m \in \mathbf{N}$, then we must have $c_0 - \pi_{n+1} = 0$. This is impossible since π_{n+1} is transcendental over K_n. Thus $M_n[\beta_{n+1}] \models DOR$.

Now $M_n \models NZR$, so in particular M_n is normal. Since β_{n+1} is transcendental over the quotient field of M_n, $M_n[\beta_{n+1}]$ is also normal by Lemma 2.9. Let now $\varphi\colon M_n[\beta_{n+1}] \to \hat{\mathbf{Z}}$ be determined by $\varphi(\beta_i) = 1$ for $i = 1, \ldots, n+1$. Since $M_n[\beta_{n+1}] \models NDOR$, we can apply Lemma 2.8 and get $M_n[\beta_{n+1}]_\varphi \models NZR$. By Lemma 2.7 $M_{n+1} = M_n[\beta_{n+1}]_\varphi$. Therefore M_{n+1} is a normal $\mathbf{Z}$-ring extending M_n.

Theorem 4.2 *$M \models NOI$.*

Proof. By construction $M \models NDOR$ and $M \subset RC(K(x))$ is an integral part of $K[x^{\mathbf{Q}}]$. Hence by Lemma 2.3, $M \models NOI$. □

Theorem 4.3 *The set of prime elements of M is unbounded.*

Proof. Since β_n is prime in $M_{n-1}[\beta_n]$, by Lemma 2.10 β_n is prime in $M_{n-1}[\beta_n]_\varphi = M_n$. Suppose β_n is prime in M_k ($k \geq n$). Then β_n is prime in $M_k[\beta_{k+1}]$ (since β_{k+1} is transcendental over the quotient field of M_k), hence in $M_{k+1} = M_k[\beta_{k+1}]_\varphi$ (by Lemma 2.7). So β_n is prime in all the M_k's with $k > n$, thus in M.

Since $K\left[x^{\mathbf{Q}}\right]$ is an unbounded subset of $K(\!(x^{1/\mathbf{N}})\!)$ clearly $\{\beta_k : k \in \mathbf{N}^*\}$ is un unbounded subset of M. □

It remains to prove the following.

Theorem 4.4 *The model M is recursive.*

The rest of the paper will be devoted to prove this theorem. The main problem lies in deciding wether a given $p(x) \in K_n\left[x^{\mathbf{Q}}\right]$ is at finite distance from (the already defined) M_n. By Lemma 2.4 this is equivalent to decide if such a $p(x)$ is at finite distance from $\mathbf{Q}[\beta_1, \cdots, \beta_n]$ and this essentially reduces to

> find a recursive bound δ on the total $(\beta_1, \cdots, \beta_n)-$degree of the f's in $\mathbf{Q}[\beta_1, \cdots, \beta_n]$ which might be at finite distance from the given $p(x) \in K_n\left[x^{\mathbf{Q}}\right]$.

In the next section we define valuations on a fixed algebraic closure of K which will allow us to bound the complexity of $\beta_1, \cdots, \beta_n$ and $p(x)$. Then in section 6 we define a function δ of Ackermann growth rate, and bound the degree of f in terms of the complexity of $\beta_1, \cdots, \beta_n$ via δ.

5 Derivations and valuations

Let L and K be as in section 3. Let $F = K(\sqrt{-1})$. Then F is an algebraic closure of L. Let $F^{(i)}$ be the algebraic closure of $\mathbf{Q}(\{\pi_k \mid k \in \mathbf{N}^* \setminus \{i\}\})$ inside F.

For each $i \in \mathbf{N}^*$ we define a derivation $\frac{\partial}{\partial \pi_i}: F \to F$ by $\frac{\partial \pi_j}{\partial \pi_i} = \delta_{ij}$ (this is enough to determine a derivation, see [Bk]).

Next we prove the existence of valuations on F with value group $(\mathbf{Q}, +)$ (with its natural order), which are well behaved with respect to the above derivations. It will be more convenient to work with $\mathbf{Q} \cup \{-\infty\}$ ordered making $-\infty < a$ for every $a \in \mathbf{Q}$ and the following rules for addition: $a + (-\infty) = (-\infty) + a = -\infty$ for all $a \in \mathbf{Q} \cup \{-\infty\}$.

Proposition 5.1 *For every $i \in \mathbf{N}^*$ there is a map d_i (read "π_i-degree")*

$$d_i: F \to \mathbf{Q} \cup \{-\infty\}$$

such that

1. $d_i(0) = -\infty, d_i(1) = 0, d_i(a+b) \le \max(d_i(a), d_i(b))$ *with equality holding if $d_i(a) \neq d_i(b)$, and $d_i(ab) = d_i(a) + d_i(b)$.*
2. *For every $j \in \mathbf{N}^*$, $d_i(\pi_j) = \delta_{i,j}$.*

Moreover for each $a \in F$,

3. $d_i(\frac{\partial a}{\partial \pi_i}) \le d_i(a) - 1$, *with equality holding if $d_i(a) \neq 0$.*
4. *For all $j \in \mathbf{N}^*$, $d_i(\frac{\partial a}{\partial \pi_j}) \le d_i(a)$.*
5. $d_i(a) = 0$ *for every nonzero a with $\frac{\partial a}{\partial \pi_i} = 0$.*

Remark 5.2 1) says that $-d_i$ is a valuation on F, and 5) implies that d_i restricted to $F^{(i)}$ is identically zero, and if $a \in F^{(i)}[\pi_i]$, $a = \sum_{k=0}^{m} a_k \pi_i^k$ with $a_m \neq 0$ say, then $d_i(a) = m$.

Proof of the Proposition. Let $F^{(i)}(\!(\pi_i^{1/\mathbf{N}})\!)$ denote the field of Puisseux series in descending powers of π_i with coefficients in $F^{(i)}$, *i.e.*

$$F^{(i)}(\!(\pi_i^{1/\mathbf{N}})\!) = \left\{ \sum_{j \le m} a_j \pi_i^{j/q} \mid m \in \mathbf{Z}, q \in \mathbf{N}^*, a_j \in F^{(i)} \right\}.$$

The field $F^{(i)}(\!(\pi_i^{1/\mathbf{N}})\!)$ is an algebraically closed field (see [Wk]) and contains $F^{(i)}[\pi_i]$ in a natural way. Therefore for each $i \in \mathbf{N}^*$ we have embeddings

$$\eta_i: F \to F^{(i)}(\!(\pi_i^{1/\mathbf{N}})\!)$$

such that for every $a \in F^{(i)}[\pi_i], \eta_i(a) = a$.

For each $i \in \mathbf{N}^*$ we fix such an embedding η_i and define d_i as follows.

For each $a \in F^*$, if $\eta_i(a) = \sum_{k \le m} a_k \pi_i^{k/q}$ with $a_m \neq 0$ then $d_i(a) = m/q$, and $d_i(0) = -\infty$.

Now we prove that d_i has the required properties.

1) and 2) are clear. 3) and 4) are also clear granted the following claim. claim. Let $a \in F$ with $\eta_i(a) = \sum_{k \le m} a_k \pi_i^{k/q}$, say. Then:

1. For every $j \neq i$, $\eta_i(\frac{\partial a}{\partial \pi_j}) = \sum_{k \le m} (\frac{\partial a_k}{\partial \pi_j}) \pi_i^{k/q}$.

2. $\eta_i(\frac{\partial a}{\partial \pi_i}) = \sum_{k \le m} \frac{k}{q} a_k \pi_i^{k/q-1}$.

We prove 1). The map $\sum_{k \le m} a_k \pi_i^{k/q} \mapsto \sum_{k \le m} (\frac{\partial a_k}{\partial \pi_j}) \pi_i^{k/q}$ is clearly a derivation on $F^{(i)}((\pi_i^{1/\mathbf{N}}))$. The map $\eta_i(a) \mapsto \eta_i(\frac{\partial a}{\partial \pi_j})$ is a derivation on $\eta_i[F]$ which extends to a derivation on $F^{(i)}((\pi_i^{1/\mathbf{N}}))$. Now these two derivations on $F^{(i)}((\pi_i^{1/\mathbf{N}}))$ are identical on $F^{(i)}[\pi_i]$ and therefore they must also coincide on the algebraic closure of $F^{(i)}[\pi_i]$ in $F^{(i)}((\pi_i^{1/\mathbf{N}}))$, and this is $\eta_i[F]$. This proves 1).

The proof of 2) is similar to the proof of 1) considering the derivation on $F^{(i)}((\pi_i^{1/\mathbf{N}}))$defined by the map $\sum_{k \le m} a_k \pi_i^{k/q} \mapsto \sum_{k \le m} \frac{k}{q} a_k \pi^{k/q-1}$. □

We cannot recursively compute $d_i(a)$ since the definition of $d_i(a)$ depends on a non-canonical choice of the embedding η_i. However for our purposes it will be enough the following.

Proposition 5.3 *Given $j \in \mathbf{N}^*$ and $a \in K$ we can recursively compute an upper bound to $d_j(a)$.*

Proof. We can assume that $a \neq 0$. We recall that $a \in K^*$ is represented by its minimal polynomial $g(y) \in L[y]$ and a natural number s indicating that a is the s-th real root of $g(y)$. We write $g(y) = \frac{1}{c} \sum_{i=0}^{d} c_i y^i$ where c and the c_i's belong to $\mathbf{Q}[\{\pi_i \mid i \in \mathbf{N}^*\}]$, $c \neq 0$ and $c_d \neq 0$. Fix $j \in \mathbf{N}^*$ and let m_i be the degree of c_i as a polynomial in π_j $(i = 1, \dots, d)$. Let $\rho_j(a) = \max \left\{ \left| \frac{m_k - m_j}{k-j} \right| \mid 0 \le j < k \le d \right\}$. We claim that $\rho_j(a)$ is the desired upper bound to $d_j(a)$. Indeed $g(a) = 0$ implies $-\infty = d_j(g(a)) = d_j(1/c) \cdot d_j(\sum_{i=0}^{d} c_i a^i)$, hence $d_j(\sum_{i=0}^{d} c_i a^i) = -\infty$. This entails that for some $s, t \in \{0, \dots, d\}$ with $s \neq t$, we have $d_j(c_s a^s) = d_j(c_t a^t)$, *i.e.* $m_s + s d_j(a) = m_t + t d_j(a)$. But then $d_j(a) = \frac{m_s - m_t}{t-s} \le \rho_j(a)$, as desired. □

Finally we extend π_i-degrees and derivations to $F\left[x^{\mathbf{Q}}\right]^\bullet$ as follows.

Definition 5.4 1. For every $i \in \mathbf{N}^*$ we define $\frac{\partial}{\partial \pi_i} \colon F\left[x^{\mathbf{Q}}\right]^\bullet \to F\left[x^{\mathbf{Q}}\right]^\bullet$ by $\frac{\partial}{\partial \pi_i}\left(\sum_{k=0}^{l} a_k x^{k/q}\right) = \sum_{k=0}^{l} \frac{\partial a_k}{\partial \pi_i} x^{k/q}$.

2. For every $i \in \mathbf{N}^*$ we define $d_i'\colon F\left[x^{\mathbf{Q}}\right] \to \mathbf{Q} \cup \{-\infty\}$ by:
   $d_i'\left(\sum_{k=0}^{l} a_k x^{k/q}\right) = \max\{d_i(a_k) \mid 0 \le k \le l\}$.

The two propositions of this section remains true, *mutatis mutandis*, for these $\frac{\partial'}{\partial \pi_i}$ and d_i'. We verify that the π_i-degree of the product is the sum of the π_i-degrees. Let $\alpha = d_i'(f(x))$ and $\beta = d_i'(g(x))$. Write $f(x) = \Sigma_{i=0}^{m} a_i x^{i/q}$ and $g(x) = \Sigma_{i=0}^{m} b_i x^{i/q}$. Let l and k be maximal such that a_l and b_k have d_i equal to α and β respectively. Then for every other pair of indexes u, v with $u+v = l+k$, $d_i(a_u \cdot b_v) < d_i(a_l \cdot b_k) = \alpha+\beta$, and therefore the coefficient of $x^{(l+k)/q}$ in the product has π_i-degree $\alpha+\beta$ and we are done. The verification of the other properties is easy.

We shall continue writing $\frac{\partial}{\partial \pi_i}$ and d_i instead of $\frac{\partial'}{\partial \pi_i}$ and d_i' respectively.

6 The Ackermann bound

Let $K \subset \mathbf{R}$ be as defined in section 3. In this section we consider the following problem.

Let $k \in \mathbf{N}^*$, $\pi_1, \ldots, \pi_k \in K$ (algebrically independent over $\mathbf{Q}$), $\gamma_i = g_i(x) + \pi_i$ where $g_i(x) \in K_{i-1}\left[x^{\mathbf{Q}}\right]^\bullet$ (hence $\gamma_1, \ldots, \gamma_k$ are algebrically independent over $\mathbf{Q}$) and $\mathbf{Q}[\gamma_1, \ldots, \gamma_k] \cap K = \mathbf{Q}$.

Given $p(x) \in K\left[x^{\mathbf{Q}}\right]^\bullet$, determine algorithmically a bound $\delta \in \mathbf{N}$ such that for every $f(\gamma_1, \ldots, \gamma_k) \in \mathbf{Q}[\gamma_1, \ldots, \gamma_k]$, if $\exists r \in K$ such that $f(\vec{\gamma}) = p(x) + r$, then the total degree of f as a polynomial in $\gamma_1, \ldots, \gamma_k$ ($= deg(f)$) is $\le \delta$.

Remark 6.1 1. If $f_1(\vec{\gamma}), f_2(\vec{\gamma}) \in \mathbf{Q}[\gamma_1, \ldots, \gamma_k]$ and $\exists r_1, r_2 \in K$ such that $f_i(\vec{\gamma}) = p(x) + r_i$ $(i = 1, 2)$, then $deg(f_1) = deg(f_2)$ (by the assumption $\mathbf{Q}[\vec{\gamma}] \cap K = \mathbf{Q}$ and the fact that $\gamma_1, \ldots, \gamma_k$ are algebrically independent over $\mathbf{Q}$). So a bound on $deg(f)$ exists. The problem is to find such a δ algorithmically from $\gamma_1, \ldots, \gamma_k$ and $p(x)$.

2. If $f(\gamma_1, \ldots, \gamma_k) \in \mathbf{Q}[\gamma_1, \ldots, \gamma_k]$, then $f(\vec{\gamma})$ is a polynomial in π_k with coefficients in $K_{k-1}\left[x^{\mathbf{Q}}\right]$. Hence for $p(x) \in K\left[x^{\mathbf{Q}}\right]^\bullet$ and $r \in K$ to satisfy $f(\vec{\gamma}) = p(x) + r$, we must have $r = f(\pi_1, \ldots, \pi_k)$ (setting $x = 0$) and therefore $p(x) = f(\vec{\gamma}) - r$ must be a polynomial in π_k with coefficients in $K_{k-1}\left[x^{\mathbf{Q}}\right]$. Moreover the degree of p in π_k must be equal to $d_i(p)$ (see remark in the previous section).

In principle the bound we want to get depends on the degrees of $p(x)$, $g_1(x)$, …,$g_k(x)$ as fractional polynomials in x, and on the algebraic relations among their coefficients. We take care of these relations via the π_i-degrees introduced in the previous section. We shall actually get a bound δ depending only on the π_i-degrees of $p(x), g_1(x), \ldots, g_k(x)$ and not on their degrees in x.

Theorem 6.2 *There is a recursive function* $\delta\colon \mathbf{N}^* \times \mathbf{N} \times \mathbf{N}^* \to \mathbf{N}$ *such that for all* $k, l, m \in \mathbf{N}$, *with* $k, m > 0$, *for all* $\gamma_i = g_i(x) + \pi_i$ *with* $g_i(x) \in K_{i-1}\left[x^{\mathbf{Q}}\right]^{\bullet}$ $(1 \le i \le k)$, *and for all* $p(x) \in K\left[x^{\mathbf{Q}}\right]^{\bullet}$ *we have:*
if

1) $\mathbf{Q}[\gamma_1, \ldots, \gamma_k] \cap K = \mathbf{Q}$,
2) $d_j(g_i(x)) \le m$ *and* $d_j(p(x)) \le m$ $(1 \le j < i \le k)$,

and

3) $d_j(p(x)) \le l$,

then for all $r \in K$ *and* $f(\vec{\gamma}) \in \mathbf{Q}[\gamma_1, \ldots, \gamma_k]$,

$$f(\gamma_1, \ldots, \gamma_k) = p(x) + r \textit{ implies } deg(f) \le \delta(k, l, m).$$

Note that if $i \le j$, then $d_j(p_i(x)) = 0$. A priori the π_i-ranks can assume positive or negative rational values, so it might seem strange that we can work with a function δ which only assumes non-negative integer values. However it turns out that in the relevant cases $p(x)$ will be a polynomial in π_k and therefore its π_k-degree will be a non-negative integer.

We are going to define $\delta(k, l, m)$ inductively on k and then on l. For the induction process it will be relevant to bound $d_j(f)$ in terms of $deg(f)$. This is clear since $d_j(f) \le m \cdot deg(f)$ (with the above notation).

Definition 6.3 Let $\delta\colon \mathbf{N}^* \times \mathbf{N} \times \mathbf{N}^* \to \mathbf{N}$ be recursively defined as follows:

1. $\delta(1, l, m) = l + 1$;
2. $\delta(k + 1, 0, m) = \delta(k, m, m)$;
3. $\delta(k + 1, l + 1, m) = \delta(k, m(\delta(k + 1, l, m) + 1), m(\delta(k + 1, l, m) + 1))$.

Note that $\delta(x, y, z) \ge \max\{y, 1\}$ for all $x, y, z \in \mathbf{N}$.

Proof of theorem. By induction on k and secondary induction on l.

$\underline{k = 1}$. Let $f(\gamma_1) = p(x) + r$ with $d_j(p(x)) \le l$. By Remark 6.1 $p(x)$ is a polynomial in π_1 of degree $\le l$ with coefficients in $K_0\left[x^{\mathbf{Q}}\right]$. Therefore $\frac{\partial^{l+1} p}{\partial \pi_1^{l+1}} = 0$. On the other hand $\frac{\partial \gamma_1}{\partial \pi_1} = 1$ and $\frac{\partial a}{\partial \pi_1} = 0$ for all $a \in K_0\left[x^{\mathbf{Q}}\right]$, so $\frac{\partial^{l+1} f}{\partial \gamma_1^{l+1}} = \frac{\partial^{l+1} f}{\partial \pi_1^{l+1}} = q$ where $q = \frac{\partial^{l+1} r}{\partial r^{l+1}} \in K$. Since $\mathbf{Q}[\gamma_1] \cap K = \mathbf{Q}$, q must be in $\mathbf{Q}$. Hence $deg(f) \le l + 1 = \delta(1, l, m)$.

$\underline{k > 1,\ l = 0}$. Let $f(\gamma_1, \ldots, \gamma_k) = p(x) + r$ with $d_j(p(x)) \le 0$, $d_j(g_i) \le m, d_j(p) \le m$ for $1 \le j < i \le k$. By Remark 6.1, $p(x)$ is a polynomial in π_k of degree ≤ 0 with coefficients in $K_{k-1}\left[x^{\mathbf{Q}}\right]$, so $p(x) \in K_{k-1}\left[x^{\mathbf{Q}}\right]$. Therefore

$\frac{\partial p}{\partial \pi_k} = 0$. Since $\frac{\partial \gamma_k}{\partial \pi_k} = 1$, $\frac{\partial \gamma_i}{\partial \pi_k} = 0$ for $i < k$ and $\frac{\partial a}{\partial \pi_k} = 0$ for all $a \in K_{k-1}\left[x^{\mathbf{Q}}\right]$, we have $\frac{\partial f}{\partial \gamma_k} = \frac{\partial f}{\partial \pi_k} = q$ where $q = \frac{\partial r}{\partial \pi_k} \in \mathbf{Q}$ (as $\mathbf{Q}[\gamma_1, \ldots, \gamma_k] \cap K = \mathbf{Q}$). Hence $f = h_0(\gamma_1, \ldots, \gamma_{k-1}) + q\gamma_k = p(x) + r$. Thus $h_0(\gamma_1, \ldots, \gamma_{k-1}) = (p(x) - qg_k) + r'$ with $r' \in K$. To apply induction on k we need to bound the π_j-degrees of $p(x) - qg_k$. For $j \leq k-1$ we have $d_j(p(x) - qg_k) \leq \max\{d_j(p(x)), d_j(g_k)\} \leq m$. By induction on k, $deg(h_0) \leq \delta(k-1, m, m)$. Hence $deg(f) \leq \max\{deg(h_0), 1\} \leq \delta(k-1, m, m) = \delta(k, 0, m)$.

$\underline{k > 1, l > 0}$. Let $f(\gamma_1, \ldots, \gamma_k) = p(x) + r$ with $d_j(p(x)) \leq l$, $d_j(g_i) \leq m$, $d_j(p(x)) \leq m$ for $1 \leq j < i \leq k$. By Remark 6.1, p is a polynomial in π_k of degree $\leq l$. Hence similarly to above we have $\frac{\partial f}{\partial \gamma_k} = \frac{\partial p}{\partial \pi_k} + r'$ with $r' \in K$. By the proposition of section 5 $d_k(\frac{\partial p}{\partial \pi_k}) \leq l - 1$ and $d_j(\frac{\partial p}{\partial \pi_k}) \leq d_j(p) \leq m$ for $j < k$. By induction on l, $deg(\frac{\partial f}{\partial \gamma_k}) \leq \delta(k, l-1, m)$. Hence $f = h_0(\gamma_1, \ldots, \gamma_{k-1}) + g(\gamma_1, \ldots, \gamma_k) = p(x) + r$ with $deg(g) = def(\frac{\partial f}{\partial \gamma_k}) + 1 \leq \delta(k, l-1, m) + 1$. Thus $h_0 = (p(x) - g(\gamma_1, \ldots, \gamma_k) + g(\pi_1, \ldots, \pi_k)) + r'$ with $r' \in K$ and $p' = (p(x) - g(\vec{\gamma}) + g(\vec{\pi})) \in K_{k-1}\left[x^{\mathbf{Q}}\right]^{\bullet}$. As above to apply induction we need to bound the π_j-degrees of p'. For $j \leq k-1$ we have $d_j(p') \leq \max\{d_j(p(x)), d_j(g(\vec{\gamma})), d_j(g(\vec{\pi}))\} \leq \max\{m, m \cdot deg(g), deg(g)\} = m \cdot deg(g) \leq m \cdot [\delta(k, l-1, m) + 1]$. Finally by induction on k, $deg(h_0) \leq \delta(k-1, m \cdot [\delta(k, l-1, m) + 1], m \cdot [\delta(k, l-1, m) + 1]) = \delta(k, l, m)$. Hence $deg(g) \leq \delta(k, l, m)$. □

7 The recursivity of the model

We are now ready to prove Theorem 4.4, namely the model M defined in section 4 is recursive. We follow the notation of section 4.

Since the β_i's are algebraically independent over $\mathbf{Q}$, it suffices to decide the order. To achieve this we have to identify which elements of $K\left[x^{\mathbf{Q}}\right]$ we have chosen to define the β_i's and use the recursiveness of the operations and the order in $K\left[x^{\mathbf{Q}}\right]$ to decide whether a polynomial in the β_i's with coefficients in $\mathbf{Q}$ is > 0 or not.

Therefore it will be enough to prove the following.

Lemma 7.1 *Given M_n we can recursively find β_{n+1}*

Proof. Given M_n and hence $\beta_i = p_i(x) + \pi_i$ $(i = 1, \cdots, n)$. We first get the least element of a fixed recursive enumeration of $K\left[x^{\mathbf{Q}}\right]$ which is in $K_n\left[x^{\mathbf{Q}}\right]$, and then check if this element, $p(x)$ say, is at infinite distance from M_n. If so we take $\beta_{n+1} = p(x) + \pi_{n+1}$. If it is not the case, we consider the next element in $K\left[x^{\mathbf{Q}}\right]$ (which is in $K_n\left[x^{\mathbf{Q}}\right]$).
(As we comment in section 4 this process ends since we always have elements in $K_n\left[x^{\mathbf{Q}}\right]$ which are at infinite distance from M_n.)

Therefore it remains to prove we can recursively check that the distance from $p(x) \in K_n \left[x^{\mathbf{Q}}\right]$ to M_n is finite. We do this as follows. Note first that by Lemma 2.4 this is equivalent to: $p(x)$ is at finite distance from $\mathbf{Q}[\beta_1, \ldots, \beta_n]$. Now we use Proposition 5.3 to compute a common upper bound, m say, of $d_j(p_i(x))$ and $d_j(p(x))$ for $(1 \leq j \leq n)$ and for $(j < i)$.

Then by Theorem 6.2 the fact $p(x)$ is at finite distance from $\mathbf{Q}[\beta_1, \ldots, \beta_n]$ must be witnessed by a polynomial in $\beta_1, \ldots, \beta_n$ of total degree less or equal than $\delta(n, m, m)$. Let $\delta(n, m, m) = s$ and $p(x) = \sum_{j=1}^{m} c_j x^{j/q}$. We have, $p(x)$ is at finite distance from $\mathbf{Q}[\beta_1, \ldots, \beta_n]$ if and only if there are $a_{(i)} \in \mathbf{Q}$ such that

$$\sum_{|(i)| \leq s} a_{(i)} \beta_i^{i_1} \cdots \beta_n^{i_n} = \sum_{j=1}^{m} c_j x^{j/q}$$

where $(i) = (i_1, \ldots, i_n)$ and $|(i)| = i_1 + \ldots + i_n$.

Equating coefficients of $x^{j/q}$ we get that the above is equivalent to the existence of $a_{(i)} \in \mathbf{Q}$ with $|(i)| \leq s$ such that for each j (running in a finite set) $\sum_{|(i)| \leq s} a_{(i)} d_{(i),j} = c_j$, with $c_j = 0$ for $j > m$. The $d_{(i),j}$'s can be computed from the β_i's. Next we compute a primitive element, c say, of $\{d_{(i),j}; c_j | (i), j\}$ over $\mathbf{Q}(\pi_1, \ldots, \pi_n)$, and its degree, k say (such c can be taken as a linear $\mathbf{Q}$-combination of $\{d_{(i),j}; c_j | (i), j\}$). And then compute the coordinates of the elements of $\{d_{(i),j}; c_j | (i), j\}$ in the basis $\{1, c, \ldots, c^{k-1}\}$. Then, getting rid of the denominators and equating coefficients of monomials in $\pi_1, \ldots, \pi_n$ we can check by Cramer's rule if the relevant $\mathbf{Q}$-linear system has a (unique) solution. If it has a solution, $p(x)$ is at finite distance from M_n. If it has not a solution we take this $p(x)$ to define β_{n+1}. $\square$

P.S. After the paper was written we learned M. Moniri has constructed (see [Mn]) recursive models of open induction of prescribed finite trancendence degree > 1 with cofinal twin primes. However his models are *not* normal.

References

[B] Birch, B., Notes on algebraic number theory. Univ.of Oxford (1990)

[Bg] Boughattas, S., L'induction pour les formules ouverts n'est pas finiment axiomatisable, in C. R. Ac. Paris, s.I(1989)297-300

[Bk] Bourbaki, N., **Commutative Algebra** Hermann, Paris(1969)

[D1] v.d.Dries, L., Which curves over **Z** have points with coordinates in a discretely ordered ring?, Trans. Am. Math. Soc. 264(1980)33-56

[D2] v. d. Dries, L., Some model theory and number theory for models of weak systems of arithmetic, in L.Pacholski *et al.*(eds.), **Model**

Theory of Algebra and Arithmetic, Lect. Notes in Math. 834, Springer Ver.(1980)346-362

[E] Edwards, H., **Galois Theory**, Springer Ver.(1984)

[M-M] Macintyre, A. and Marker, D., Primes and their residues rings in models of open induction, Annals of Pure and Applied Logic 43 (1989) 57–77

[Mc] McAloon, K., On the complexity of models of arithmetic, Jour. Symb. Logic 47(1982)403-415

[Mn] Moniri, M., Recursive models of open induction of prescribed finite transcendence degree > 1 with cofinal twin primes. To appear.

[Mo-R] Mourgues, M.H., and Ressayre J.-P., Tout corp reel clos possede une partie entiere, in C. R. Ac. Paris, s.I(1992)93-96

[Ot1] Otero,M.,On diophantine equations solvable in models of open induction, Jour. Symb. Logic 55(1990)779-786

[Ot2] Otero,M., The joint embedding property in normal open induction, Ann. Pure Appl. Logic 60(1993)275-290

[Sh] Shepherdson, J., A nonstandard model for a free variable fragment of number theory, Bull. L'Acad, Pol. Sci. 12(1964)79-86

[T] Tennenbaum, S., Non-Archimedean models for arithmetic, Not. Am. Math. Soc.(1959)270

[Wd] v.D.Waerden,B.L., **Modern Algebra II**, Fred. Ungar. New York(1950)

[Wk] Walker, R., **Algebraic Curves**, Princeton Univ. Press(1950)

[W] Wilkie, A., Some results and problems on weak systems of arithmetic, in A. Macintyre *et al.*(eds.), **Logic Colloquium'77**, North-Holland (1978)285-296

[Wm] Wilmers,G., Bounded existential induction, Jour. Symb. Logic 50(1985)72-90