An additive measure in o-minimal expansions of fields

Alessandro Berarducci
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2, 56127 Pisa
berardu@dm.unipi.it

Margarita Otero*
Departamento de Matemáticas
Universidad Autónoma de Madrid
Cantoblanco,28049 Madrid
margarita.otero@uam.es

Abstract

Given an o-minimal structure M which expands a field, we define, for each positive integer d, a real valued additive measure on a Boolean algebra of subsets of M^d and we prove that all the definable sets included in the finite part $Fin(M^d)$ of M^d are measurable. When the domain of M is $\mathbb{R}$ we obtain the Lebesgue measure, but restricted to a proper subalgebra of that of the Lebesgue measurable sets (the Jordan measurable sets). Our measure has good logical properties, being invariant under elementary extensions and under expansions of the language. In the final part of the paper we consider the problem of defining an analogue of the Haar measure for definably compact groups.

1 Introduction

Many of the tools of differential geometry are available in any o-minimal structure expanding a field (see for instance [1]). However, a satisfactory theory of measure and integration seems to be lacking in the o-minimal context, especially if we want to take into consideration non-archimedean structures.

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In this paper we propose a notion of measure which, while missing some of the properties of the Lebesgue measure (*e.g.*, σ–additivity), still has reasonable properties (*e.g.*, finite additivity) and makes sense in any o-minimal expansion M of a field. More precisely, fixing M as above, we define, for each positive integer d, a real valued (*not M–valued*) additive measure $\mu = \mu^{(d)}$ on a Boolean algebra $\mathbb{S}^{(\mathrm{d})}(\mathrm{M})$ of subsets of M^d which includes all the sets of the form $A \cap Fin(M)^d$, where A is definable and $Fin(M)$ is the set of all finite elements of M, namely the set of elements which are smaller in absolute value than some positive integer. In analogy with Lebesgue measure, $\mu^{(d)}$ assigns measure 1 to $[0,1]^d$ and is invariant under translations by elements in $Fin(M^d)$. In particular if the domain of M is $\mathbb{R}$ we have $Fin(M) = M$ and we get the Jordan measure (which coincides with the Lebesgue measure but it is defined on a smaller algebra of sets, for instance the rational numbers, as a subset of M, will not be measurable).

Our measure is invariant under elementary extensions in the sense that if $M \prec M'$ and $\phi(\vec{x})$ is a formula with parameters in M, then $\phi^M(\vec{x}) \cap Fin(M^d)$ has the same measure than $\phi^{M'}(\vec{x}) \cap Fin(M'^d)$ (relative to the respective measures of M and M'). Note that this property is incompatible with σ-additivity as can be seen passing to a countable elementary substructure M_0 of M (in M_0 the set $[0,1]^d$ is countable, so in the presence of σ-additivity it would have measure zero). It will be obvious from the definition of $\mu^{(d)}$ that an expansion of the language affects neither the measure nor the class of measurable sets.

This paper is organized as follows. In section 2 we define the measure $\mu^{(d)}$ on a suitable Boolean ring $\mathcal{Q}^{(d)}(M)$ (closed under finite unions and difference, but not under complements) and we prove that for every $N \in \mathbb{N}$, every definable set included in $[-N,N]^d$ is measurable. In section 3 we extend the measure to a Boolean algebra $\mathbb{S}^{(\mathrm{d})}(\mathrm{M})$, and we give a definition of integral. In section 4 we show that if M is $\aleph_1$–saturated, the standard part map st: $Fin(M^d) \to \mathbb{R}^d$ is measure preserving, if we put on the target the Lebesgue measure. In section 5 we consider the open problem of defining a left invariant additive measure on every definably compact group, in analogy with the Haar measure on compact groups. This problem arose in connection with a conjecture of Anand Pillay, aiming at understanding the similarities between definably compact groups and Lie groups.

2 A measure on $\mathbb{Q}$–bounded sets

Let $n, d \in \mathbb{N}$ with $n, d > 0$. Let $B \subset M^d$, we say that B is a $1/n$–**box** of dimension d if B is a set of the form $[q_1, r_1) \times \cdots \times [q_d, r_d)$ where each $[q_i, r_i)$ is a half-open interval with rational end-points of the form $[k/n, k + 1/n)$ for some $k \in \mathbb{Z}$ (depending on i). We say that B is a **box** of dimension d if B is a $1/n$–**box** of dimension d for some $n > 0$. Note that two distinct $1/n$–boxes of the same dimension are necessarily disjoint. A subset P of M^d is $1/n$–**polyrectangle** of dimension d if P is a finite union of $1/n$–boxes of dimension d. We say that P is a **polyrectangle** of dimension d if P is a $1/n$–polyrectangle of dimension d, for some $n > 0$. Any subset of M^d of the form $[q_1, r_1) \times \cdots \times [q_d, r_d)$ with $q_i, r_i \in \mathbb{Q}$ and $q_i < r_i$ for $i = 1, \ldots, d$ is a polyrectangle, in fact it is a $1/n$–polyrectangle for unbounded many n's in $\mathbb{N}$.

For each $d > 0$, the set $\mathcal{PT}^{(d)}(M)$ of polyrectangles of dimension d of M forms a **Boolean ring** (*i. e.,* a non empty class closed under finite unions and differences, but not necessarily under complements). Let $\mu^{(d)} \colon \mathcal{PT}^{(d)} \to \mathbb{R}^{\geq 0}$ denote the additive measure on the Boolean ring of polyrectangles of dimension d which assigns value $1/n^d$ to a $1/n$–box of dimension d.

A subset A of M^d is said to be $\mathbb{Q}$–**bounded** if there is a positive $q \in \mathbb{Q}$ such that $A \subset [-q, q]^d$. We say than a point x of M^d is $\mathbb{Q}$–**bounded** if the set $\{x\}$ is $\mathbb{Q}$–bounded. The set $\mathcal{Q}^{(d)}(M)$ of $\mathbb{Q}$–bounded subsets of M^d forms a Boolean ring containing $\mathcal{PT}^{(d)}(M)$.

2.1. Definition. Given $d > 0$ and $A \in \mathcal{Q}^{(d)}(M)$ we define:
The **outer measure** of A: $(\mu^{(d)})^*(A) = \mathrm{Inf}\{\mu^{(d)}(P) \colon P \supset A,\ P \in \mathcal{PT}^{(d)}(M)\}$.
The **inner measure** of A: $(\mu^{(d)})_*(A) = \mathrm{Sup}\{\mu^{(d)}(P) \colon P \subset A,\ P \in \mathcal{PT}^{(d)}(M)\}$.

Note that $(\mu^{(d)})^*(P) = (\mu^{(d)})_*(P) = \mu^{(d)}(P) \in \mathbb{Q}^{\geq 0}$ for polyrectangles of dimension d and that $(\mu^{(d)})^*(A)$, $(\mu^{(d)})_*(A) \in \mathbb{R}^{\geq 0}$ for each $A \in \mathcal{Q}^{(d)}(M)$.

In what follows we will omit the superscript "(d)" for the measure functions if the dimension is clear from the context.

2.2. Lemma. *Let $d > 0$. Let μ^*, $\mu_* \colon \mathcal{Q}^{(d)}(M) \to \mathbb{R}^{\geq 0}$ be the outer and inner measure functions on $\mathcal{Q}^{(d)}(M)$, respectively. Let $A, D \in \mathcal{Q}^{(d)}(M)$. Then the following holds.*

(i) $\mu_*(A) \leq \mu^*(A)$.

(ii) *Monotonicity of the outer measure: If $A \subset D$ then $\mu^*(A) \leq \mu^*(D)$.*

(iii) *Subadditivity of the outer measure:*
$\mu^*(A \cup D) \leq \mu^*(A) + \mu^*(D)$.

(iv) *If $A \cap D = \emptyset$ then $\mu_*(A) + \mu_*(D) \leq \mu_*(A \cup D)$.*

(v) *If $A \subset P$ with $P \in \mathcal{PT}^{(d)}(M)$ then $\mu_*(A) = \mu^{(d)}(P) - \mu^*(P \setminus A)$.*

(vi) *Let $m > 0$. If $P \in \mathcal{PT}^{(m)}(M)$ then $(\mu^{(d+m)})^*(A \times P) = (\mu^{(d)})^*(A)\mu^{(m)}(P)$. In particular $(\mu^{(d+1)})^*(A \times [0,1)) = (\mu^{(d)})^*(A)$.*

Proof. For (iii) it suffices to prove that for every $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ we have $\mu^*(A \cup D) \leq \mu^*(A) + \mu^*(D) + \varepsilon$. Fix $\varepsilon > 0$ then there are $P, Q \in \mathcal{PT}^{(d)}(M)$ with $P \supset A$ and $Q \supset D$ such that $\mu(P) < \mu^*(A) + \varepsilon/2$ and $\mu(Q) < \mu^*(D) + \varepsilon/2$. Then we conclude by subadditivity of μ. The proof of the rest is similar: for (iv) we use the additivity of μ and for (vi) the fact that $\mu^{(d+m)}(P \times Q) = \mu^{(d)}(P)\mu^{(m)}(Q)$ for polyrectangles P and Q. $\square$

2.3. Definition. Let $d > 0$. Let A be in $\mathcal{Q}^{(d)}(M)$. We say that A is a **measurable** set if $(\mu^{(d)})_*(A) = (\mu^{(d)})^*(A)$.

polyrectangles are measurable so for each $d > 0$, we can extend $\mu^{(d)}$ to measurable sets in $\mathcal{Q}^{(d)}(M)$ by

$$\mu^{(d)}(A) := (\mu^{(d)})_*(A) = (\mu^{(d)})^*(A).$$

Note that $A \in \mathcal{Q}^{(d)}(M)$ is measurable if for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ there are $P, Q \in \mathcal{PT}^{(d)}(M)$ with $Q \subset A \subset P$ such that $\mu(P) - \mu(Q) < \varepsilon$. Using Lemma 2.2 we obtain:

2.4. Proposition. *For each $d > 0$ the class of measurable sets of $\mathcal{Q}^{(d)}(M)$ forms a Boolean ring and μ is an additive measure on this Boolean ring.*

2.5. Theorem. *Let $d > 0$. Every $\mathbb{Q}$–bounded definable subset of M^d is measurable. Moreover if A is a $\mathbb{Q}$–bounded definable set with $Int(A) = \emptyset$, then $\mu^{(d)}(A) = 0$.*

Note that the converse of the second assertion fails: a set of infinitesimal diameter has outer measure zero, and yet it can have non-empty interior. We shall prove the theorem by induction on the dimension of the ambient space d. We need some preliminaries.

2.6. Lemma. *Let $a, b, c, d \in \mathbb{Q}$ with $a < b$ and $c < d$. Let $f\colon (a,b) \to [c,d]$ be a definable function. If $f \in \mathcal{C}^2((a,b))$ with sign(f') and sign(f'') constant, then for every $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$, $|f'| \leq \frac{2(d-c)}{\varepsilon}$ on $[a+\varepsilon, b-\varepsilon]$. In particular, Graph$(f'_{|[a+\varepsilon,b-\varepsilon]})$ is $\mathbb{Q}$–bounded.*

Proof. Assume f' and f'' are nonnegative on (a,b) (the other cases are treated similarly). Fix $\varepsilon > 0$ in $\mathbb{Q}$ and consider $\xi \in [b-\varepsilon, b-\varepsilon/2]$ such that $0 \leq f'(\xi) = \frac{f(b-\varepsilon/2)-f(b-\varepsilon)}{\varepsilon/2} \leq \frac{2(d-c)}{\varepsilon}$. Since f'' is nonnegative on (a,b) we have $0 \leq f'(x) \leq f'(\xi)$, for all $x \leq \xi$. $\square$

2.7. Lemma. *Let $d > 0$. Assume Theorem 2.5 is true for d and let $C \subset M^d$ be a cell. Let $f\colon C \to M$ be a definable function with Graph$(f) \in \mathcal{Q}^{(d+1)}(M)$. Then $(\mu^{(d+1)})^*(Graph(f)) = 0$.*

Proof. We distinguish two cases depending on C being an open cell or not. In the second case C has not interior and the case is handled by induction. We deal with the first case. We first consider a cell decomposition $\mathcal{D}$ of C such that the for each open cell $D \in \mathcal{D}$ (i.e. each cell of maximal dimension) and for every $i = 1, \ldots, d$, $\frac{\partial f}{\partial x_i}_{|D}$ and $\frac{\partial^2 f}{\partial x_i^2}_{|D}$ exist, are continuous, and have constant sign. Since $C = \bigcup_{D \in \mathcal{D}} D$, by subadditivity of the outer measure $\mu^*(Graph(f)) \leq \sum_{D\in\mathcal{D}} \mu^*(Graph(f_{|D}))$. Therefore, without loss of generality we can assume that *sign* $\left(\frac{\partial f}{\partial x_i}\right)$ and *sign* $\left(\frac{\partial^2 f}{\partial x_i^2}\right)$ exist and are constant on C. Let $N_0 \in \mathbb{N}$ be such that $Im\, f \subset [-N_0, N_0]$.

Our next goal is a reduction to the case in which C is replaced by a box. Since $C \in \mathcal{Q}^{(d)}(M)$ and it is definable, by hypothesis C is measurable. Therefore for every positive $\varepsilon \in \mathbb{Q}$, there is a polyrectangle P of dimension d (possibly empty), such that $P \subset C$ and $\mu(C \setminus P) < \varepsilon$. Let m be such that P is a $1/m$-polyrectangle and write $P = \bigcup_{B\subset P} B$, where the B's are $1/m$-boxes (so the union is disjoint). Since $Graph(f_{|C\setminus P}) \subset (C \setminus P) \times [-N_0, N_0]$, by Lemma 2.2 $\mu^*(Graph(f)) \leq \sum_{B\subset P} \mu^*(Graph(f_{|B})) + 2N_0\varepsilon$. So it suffices to prove that the graph of f restricted to a box B has outer measure zero, where we can moreover assume that the first and second partial derivatives of f exist and have constant sign on an open set containing B.

Assuming the domain of f is a box B, let $B = [a_1, b_1) \times \cdots \times [a_d, b_d)$ with $a_i < b_i$ in $\mathbb{Q}$. Fix $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$ and let $B_\varepsilon = [a_1+\varepsilon, b_1-\varepsilon) \times \cdots \times [a_d+\varepsilon, b_d-\varepsilon)$. By Lemma 2.6, $\left|\frac{\partial f}{\partial x_i}\right| \leq \frac{2N_0}{\varepsilon}$ on B_ε. Let $N \in \mathbb{N}$ with $N > \frac{2N_0}{\varepsilon}$. Then for all

$x, y \in B_\varepsilon$ we have $|fx - fy| < Nd\|x - y\|$, where $\|z\| = max\{|z_i| \colon 1 \leq i \leq d\}$ for $z = (z_1, \ldots, z_d) \in M^d$. Therefore $Graph(f_{|B_\varepsilon})$ cannot intersect more than Nd $1/n$–boxes of dimension $d+1$ with the same base (a $1/n$–box of dimension d in B_ε). On the other hand, for unbounded many n's in $\mathbb{N}$, B_ε is a $1/n$–polyrectangle which is the disjoint union of $\mu(B_\varepsilon)n^d$ $1/n$–boxes of dimension d; for such n's, the set $Graph(f_{|B_\varepsilon})$ is contained in the disjoint union of $\mu(B_\varepsilon)n^d Nd$ $1/n$–boxes of dimension $d+1$ and hence $\mu^* Graph(f_{|B_\varepsilon}) \leq \frac{1}{n^{d+1}} Ndn^d \mu(B_\varepsilon)$. Since for a fixed ε we have $\lim\limits_{n \to \infty} \frac{Nd}{n}\mu(B_\varepsilon) = 0$, we get $\mu^*(Graph(f_{|B_\varepsilon})) = 0$. We conclude that $\mu^*(Graph(f_{|B})) = 0$ by first observing that $\mu^*(Graph(f_{|B \setminus B_\varepsilon})) \leq \mu(B \setminus B_\varepsilon)2N_0$ and $\lim\limits_{\varepsilon \to 0} \mu(B \setminus B_\varepsilon)2N_0 = 0$, and then applying subadditivity of the outer measure. $\square$

2.8. Corollary. *Let $d > 0$. Assume Theorem 2.5 is true for d and let $C \subset [-N_0, N_0]^{d+1}$ with $N_0 \in \mathbb{N}$, be a cell. Then $(\mu^{(d+1)})^*(bd(C)) = 0$, where $bd(C)(= cl(C) \setminus Int(C))$ denotes the boundary of C.*

Proof. We consider the two cases: $C = Graph(f)$ and $C = (f, g)_D (= \{(x, z) \in D \times M \colon f(x) < z < g(x)\})$, where $f, g \colon D \to M$ are definable continuous functions and $D \subset [-N_0, N_0]^d$ is a cell. By induction hypothesis $\mu(bd(D)) = 0$. In the first case $bd(C) \subset Graph(f) \cup (bd(D) \times [-N_0, N_0])$, hence the result follows by Lemma 2.7 and Lemma 2.2. In the second case, note that $bd(C) \subset Graph(f) \cup Graph(g) \cup (bd(D) \times [-N_0, N_0])$, and we conclude by the first case, Lemma 2.7 and Lemma 2.2. $\square$

Proof of Theorem 2.5. By Proposition 2.4 it suffices to prove the theorem for cells.

Suppose $d = 1$ and let C be a cell contained in $[N_0, N_0]$ with $N_0 \in \mathbb{N}$. Both cases C a point and C an open interval are treated similarly. We consider the second one: $C = (a, b)$ with $a, b \in M$ and $a < b$. Fix $\varepsilon \in \mathbb{Q}$ with $\varepsilon > 0$. Let $r = Sup\{q \in \mathbb{Q} \colon q \leq a\} \in \mathbb{R}$ and $s = \mathrm{Inf}\{q \in \mathbb{Q} \colon q \geq b\} \in \mathbb{R}$. Let $q_1, q_2 \in \mathbb{Q}$ with $q_1 < r$, $q_2 > s$, $r - q_1 < \varepsilon/3$ and $q_2 - s < \varepsilon/3$. Then $r \leq s$ and $P = [q_1 + \varepsilon/3, q_2 - \varepsilon/3) \subset (a, b) \subset [q_1, q_2) = Q$. In both cases $P = \emptyset$ and $P \neq \emptyset$, we get $\mu(P) - \mu(Q) < \varepsilon$, so (a, b) is measurable.

Suppose now the theorem is true for d and let $C \subset [-N_0, N_0]^{d+1}$ be a cell. To get the equality between the outer and inner measure, it suffices to prove that for every $\varepsilon \in \mathbb{Q}$, with $\varepsilon > 0$, $\mu^*(C) < \mu_*(C) + \varepsilon$. Fix such an ε. By Corollary 2.8 there is a polyrectangle $P \subset [-N_0, N_0]^{d+1}$ such that $bd(C) \subset P$ and $\mu(P) < \varepsilon$. Consider the polyrectangle $[-N_0, N_0]^{d+1} \setminus P$ and let Q be the union of the boxes in $[-N_0, N_0]^{d+1} \setminus P$ contained in C. Then $P \cup Q$ is a polyrectangle containing C, and so $\mu^*(C) < \mu(P \cup Q) < \mu(Q) + \varepsilon < \mu_*(C) + \varepsilon$, and hence C is measurable.

It remains to show that if $Int(C) = \emptyset$ then we get $\mu(C) = 0$. We distinguish two cases: $C = Graph(f)$ and $C = (f, g)_D$ where $f, g \colon D \to M$ are definable continuous function and $D \subset [-N_0, N_0]^d$ is a cell. In the first case we apply Lemma 2.7. In the second case we must have $Int(D) = \emptyset$ (with Int taken in M^d) and since $\mu(C) \leq 2N_0 \mu(D)$ we conclude $\mu(C) = 0$ by applying the induction hypothesis. □

2.9. Proposition. *For each $d > 0$, the measure function $\mu = \mu^{(d)}$ is invariant under $\mathbb{Q}$–bounded translations,* i.e., *if $A \in \mathcal{Q}^{(d)}(M)$ is measurable and $x \in M^d$ is $\mathbb{Q}$–bounded then $x + A$ is measurable and $\mu(x + A) = \mu(A)$.*

It is not difficult to give a direct proof of the Proposition. For a short alternative proof based on the notion of standard part see section 4.

As already remarked in the introduction, $\mu^{(d)}$ is invariant under elementary extensions and expansions of the language.

2.10. Remark. If we let $d > 0$ vary, the class of measurable sets is closed under cartesian products, but *not* under projections. To see this last one take for instance, $A = (\mathbb{Q} \times \mathbb{Q}) \cap \Delta_{[0,1]^2}$, where $\Delta_{[0,1]^2} \in \mathcal{Q}^{(2)}(M)$ is the diagonal on $[0, 1]^2$, then A is measurable and $\mu(A) = 0$ (for each $n > 0$, A can be covered by a polyrectangle of measure $1/n$), but $\pi(A)$ is $\mathbb{Q} \cap [0, 1] \in \mathcal{Q}^{(1)}(M)$, which is not measurable.

3 Extension of the measure to a Boolean algebra

We extend the measure $\mu^{(d)}$ to a Boolean algebra containing the Boolean ring of the $\mathbb{Q}$-bounded measurable sets as follows.

3.1. Definition. Let $d > 0$. Let $Fin(M^d)$ denote the convex hull of $\mathbb{Q}^d$ in M^d. Given $A \subset Fin(M^d)$ we say that A is **measurable** if for all $n \in \mathbb{N}$ the $\mathbb{Q}$–bounded set $A_n := A \cap [-n, n]^d$ is measurable in the sense of Definition 2.3. In this case, we define $\mu^{(d)}(A) := \lim_{n \to \infty} \mu^{(d)}(A_n) \in \mathbb{R} \cup \{\infty\}$.

Clearly both definitions 3.1 and 2.3 coincide for $\mathbb{Q}$–bounded sets. Note that if M is not Archimedean and $A \subset Fin(M^d)$ is definable, then A is $\mathbb{Q}$–bounded. Also that if A is measurable the limit that defines its measure exists in $\mathbb{R} \cup \{\infty\}$.

For each $d > 0$, the measurable subsets of $Fin(M^d)$ forms a Boolean algebra $\mathbb{S}^{(\mathrm{d})}(\mathrm{M})$ and the nonnegative extended real valued function μ defined on $\mathbb{S}^{(\mathrm{d})}(\mathrm{M})$ is an additive measure with $\mu([0,1]^d) = 1$ and $\mu(Fin(M^d)) = \infty$. However, $\mathbb{S}^{(\mathrm{d})}(\mathrm{M})$ is not a σ–algebra, *e.g.*, $\mathbb{Q} \notin \mathbb{S}^{(1)}$. Note also that sets of infinitesimal width have measure 0, for example the set $A = Fin(M) \times [0, \rho] \subset Fin(M^2)$, with $\rho < 1/m$ for every $m \in \mathbb{N}^*$, is measurable and $\mu^{(2)}(A) = 0$.

For each $d > 0$, we call the triple $(Fin(M^d), \mathbb{S}^{(\mathrm{d})}(\mathrm{M}), \mu)$ the **measure space** of M.

3.2. Proposition. *For each $d > 0$, the following holds.*

(i) *Definable subsets of $Fin(M^d)$ are measurable. More generally, if $A \subset M^d$ is definable, then $A \cap Fin(M^d)$ is measurable.*

(ii) *The set $\mathbb{S}^{(\mathrm{d})}(\mathrm{M})$ and the function μ on $\mathbb{S}^{(\mathrm{d})}(\mathrm{M})$ are invariant under $\mathbb{Q}$–bounded translations.*

Proof. By Theorem 2.5 and Proposition 2.9. □

3.3. Remark. Once we have a measure we can define an integral as follows: if $A \subset M^d$ and $f \colon M^d \to M$ has non-negative values, we define

$$\int_A f d\mu = \mu^{(d+1)}([0, f)_A),$$

where $[0, f)_A$ is the set $\{(x, y) : x \in A, 0 \leq y < fx\}$. This of course makes sense under the assumption that $[0, f)_A \in \mathbb{S}^{(d+1)}$. Similarly one can define the integral of a function with either non-positive values or with both positive and negative values, in the usual way. The integral defined in this way has many of the usual properties of the Lebesgue integral, and can be used for instance to define the area of a parametrized surface by the usual formulas of the calculus. However, some of the usual properties fail or do not make sense: for instance, we may have a set A with $\mu(A) > 0$ and a function f with $f > 0$ on A, but $\displaystyle\int_A f d\mu = 0$; another case is the formula

$\int_A (q_1 f + q_2 g) d\mu = q_1 \int_A f d\mu + q_2 \int_A g d\mu$, which is true if the coefficients q_1, q_2 are rationals, but it does not make sense for arbitrary coefficients in $Fin(M)$ since the integral is real valued (one can however obtain a correct formula using the standard part of the coefficients, in the sense of Definition 4.1). For similar reasons, an appropriate formulation of Fubini's theorem would require some care.

4 Standard part and Lebesgue measure

We define the notion of **closed box** by replacing –in the definition of box– the semi-closed intervals by closed intervals. Equivalently a closed box is the closure of a box. If $B \subset M^d$ is a (closed) box, we denote by $B^{\mathbb{R}} \subset \mathbb{R}^d$ the (closed) box in $\mathbb{R}^d$ defined by the same rational intervals.

4.1. Definition. We define the **standard part** map $\mathrm{st}\colon Fin(M^d) \to \mathbb{R}^d$ by: $y = \mathrm{st}(x)$ if for every closed box B we have $x \in B$ if and only if $y \in B^{\mathbb{R}}$.

4.2. Proposition. *If M is $\aleph_1$-saturated, the image under $\mathrm{st}\colon Fin(M^d) \to \mathbb{R}^d$ of a closed box $B \subset M^d$ of dimension d, is the closed box $B^{\mathbb{R}} \subset \mathbb{R}^d$.*

Proof. The inclusion $\mathrm{st}(B) \subset B^{\mathbb{R}}$ follows at once from the definition of standard part. To prove the equality let $y \in B^{\mathbb{R}}$. We can write $\{y\} = \bigcap_{i \in \mathbb{N}} B_i^{\mathbb{R}}$ where $\{B_i^{\mathbb{R}} \mid i \in \mathbb{N}\}$ is an enumeration of all the closed boxes of dimension d containing y. The conjunction of all the formulas $x \in B_i$ defines a type in M which, by saturation, is realized by some $x \in \mathcal{Q}^{(d)}(M)$. For this x we have $\mathrm{st}(x) = y$ and $x \in B$. $\square$

Note that the assumption that the boxes are closed is essential: indeed if M is non-archimedean and $a < b$ are rational, the image under the standard map of the semi-closed interval $[a, b)^M$ is the closed interval $[a, b]^{\mathbb{R}}$ (as b is the standard part of any point $x \in [a, b)^M$ infinitesimally close to b).

We denote by $\lambda^{(d)}$ **the Lebesgue measure** on $\mathbb{R}^d$ and we drop the "d" when the dimension is clear from the context. The next theorem shows that the standard part map is measure preserving.

4.3. Theorem. *Let M be $\aleph_1$-saturated. Let $A \in \mathcal{Q}^{(d)}(M)$ be measurable. Then $\mathrm{st}(A)$ is a Lebesgue measurable subset of $\mathbb{R}^d$ whose Lebesgue measure $\lambda(\mathrm{st}(A))$ is equal to $\mu(A)$.*

Proof. If A is either a box or a closed box of dimension d the result to be proved follows easily from Proposition 4.2 together with the observation that the measure of a box is equal to the measure of its closure (with respect to either λ or μ).

If A is a polyrectangle of dimension d we write it as a disjoint union of boxes and use the additivity of the measure. We have to be careful since if B, C are disjoint boxes their standard parts need not be disjoint. This however is not a problem since in any case the intersection $\mathrm{st}(B) \cap \mathrm{st}(C)$ has Lebesgue measure zero (being the intersection of the closures of two disjoint boxes).

In the general case we reduce to the case of polyrectangles using the fact that a set $A \in \mathcal{Q}^{(d)}(M)$ is measurable if for every positive $\varepsilon \in \mathbb{R}$ there are $P, Q \in \mathcal{PT}^{(d)}(M)$ with $Q \subset A \subset P$ such that $\mu(P) - \mu(Q) < \varepsilon$. $\square$

In the light of the above result one may wonder whether one can take the equality $\mu(A) = \lambda(\mathrm{st}(A))$ as a definition of μ (assuming that M is $\aleph_1$-saturated). However, the problem with this definition is that the additivity of the measure would be difficult to prove (since the standard parts of two disjoint sets are not necessarily disjoint). Indeed, the additivity amounts to the following:

4.4. Corollary. *Let $A, B \in \mathcal{Q}^{(d)}(M)$ be disjoint measurable sets. Then $\mathrm{st}(A) \cap \mathrm{st}(B)$ is a subset of $\mathbb{R}^d$ of Lebesgue measure zero.*

Proof. Using the additivity of μ and λ we obtain:

$$\begin{aligned}
\mu(A) + \mu(B) &= \mu(A \cup B) \\
&= \lambda(\mathrm{st}(A \cup B)) \\
&= \lambda(\mathrm{st}(A) \cup \mathrm{st}(B)) \\
&= \lambda(\mathrm{st}(A)) + \lambda(\mathrm{st}(B)) - \lambda(\mathrm{st}(A) \cap \mathrm{st}(B)) \\
&= \mu(A) + \mu(B) - \lambda(\mathrm{st}(A) \cap \mathrm{st}(B))
\end{aligned}$$

$\square$

Note that the corollary can be used to prove non-definability results: for instance if A is the relative complement of B in $[0,1]^d$ and $\mathrm{st}(A) \cap \mathrm{st}(B)$ has positive Lebesgue measure, then A is not definable (as it is not μ-measurable). It is not difficult to find examples of such sets (one can easily find examples with $\mathrm{st}(A) \cap \mathrm{st}(B) = [0,1]^d$).

We finish by giving a proof of translation invariance of μ, using the translation invariance of the Lebesgue measure λ.

Proof of Proposition 2.9. Passing to an elementary extension we can assume that M is $\aleph_1$-saturated. We have:

$$\begin{aligned}\mu(x+A) &= \lambda(\mathrm{st}(x+A) \\ &= \lambda(\mathrm{st}(x)+\mathrm{st}(A)) \\ &= \lambda(\mathrm{st}(A)) \\ &= \mu(A)\end{aligned}$$

$\square$

5 On a conjecture of Pillay

Any compact topological group has a Haar measure, namely a left-invariant additive measure on a σ-algebra containing the Borel sets (see for instance [2]). This may lead to:

5.1. Conjecture. *It is possible to define a left-invariant additive measure on the Boolean algebra of all definable subsets of a definably compact group.*

(See [3] for the concept of definably compact.)

The conjecture can be verified for various specific groups. For example we have:

5.2. Theorem. *Conjecture 5.1 is true for the d-dimensional torus $\mathbb{T}^d$, defined as the set $[0,1)^d \subset M^d$ with the group operation $*$ defined as follows: $x * y = z$ if there is an $n \in \mathbb{Z}^d$ with $x + y = z + n$.*

Proof. It suffices to define the measure as the restriction of μ to the definable subsets of $[0,1)^d$. Left-invariance is obtained by Proposition 2.9. $\square$

Conjecture 5.1 is related to the following conjecture of A. Pillay [4], which was the original motivation for our investigation.

5.3. Conjecture. *(Pillay) Let G be a definably compact group over a saturated o-minimal structure M of sufficiently large cardinality κ (say κ inaccessible). Then G has a subgroup G^{00} with the following properties:*

1. *G^{00} is the smallest type–definable subgroup of G of bounded index (i.e. of index $< \kappa$).*

2. G/G^{00} *is a compact Lie group provided we put on it the following* logic topology*: a subset* A *of* G/G^{00} *is closed iff* $\pi^{-1}(A)$ *is a type–definable subset of* G *(i.e. it is the intersection of* $< \kappa$ *definable subsets), where* $\pi\colon G \to G/G^{00}$ *is the quotient map.*

3. The o-minimal dimension of G *is equal to the dimension of* G/G^{00} *as a Lie group.*

As we learned from Anand Pillay, the logic topology is defined in such a way that the quotient map $G \to G/G^{00}$ behaves similarly to the standard part map $\mathrm{st}\colon Fin(M^d) \to \mathbb{R}^d$. More precisely one can observe:

5.4. Proposition. *Assume that* M *is* κ*-saturated, with* κ *large enough so that the standard part map* $\mathrm{st}\colon \mathcal{Q}^{(d)}(M) \to \mathbb{R}^d$ *is surjective. Then a bounded set* $A \subseteq \mathbb{R}^d$ *is closed if and only if* $\mathrm{st}^{-1}(A)$ *is type–definable.*

Proof. Let M^{00} be the kernel of $\mathrm{st}\colon Fin(M) \to \mathbb{R}$, and observe that M^{00} is type–definable. Now let $A \subset \mathbb{R}^d$ be a bounded set.

If $A \subset \mathbb{R}^d$ is closed, then A is an intersection of a countable family $\{\overline{P_i}^{\mathbb{R}} \mid i \in \mathbb{N}\}$, where each P_i is a polyrectangle. It follows that $\mathrm{st}^{-1}(A) = \{x \mid \bigwedge_{i\in\mathbb{N}}(x \in P_i + M^{00})\}$, so $\mathrm{st}^{-1}(A)$ is type–definable.

Conversely if $\mathrm{st}^{-1}(A)$ is type–definable, then $\mathrm{st}^{-1}(A)$ is the intersection of a collection of sets X_i $(i \in I)$ defined over a common set of $< \kappa$ parameters. We claim that A is closed. To this aim suppose $y \in \overline{A}$. We must show that $y \in A = \mathrm{st}(\mathrm{st}^{-1}(A))$. So we must find an $x \in \bigcap_i X_i$ such that $\mathrm{st}(x) = y$. To this aim it suffices to show that the type $\bigwedge_i x \in X_i \wedge \bigwedge_j x \in B_j$ is consistent, where $\{B_j \mid j \in \mathbb{N}\}$ is the collection of all the closed boxes B with $y \in Int(B^{\mathbb{R}})$. By compactness it suffices to prove the consistency of the type $p(x) = \bigwedge_i x \in X_i \wedge x \in B$ where B is a closed box with $y \in Int(B^{\mathbb{R}})$. Now since y was assumed to be in the closure of A, certainly $B^{\mathbb{R}}$ meets A. So by surjectivity of the standard map there is x' such that $\mathrm{st}(x') \in B^{\mathbb{R}} \cap A$. This x' realizes the type p. □

The relation between the two conjectures is given by the fact that, granted Conjecture 5.3, one could try to define the measure of a definable subset of G as the Haar measure of its image in G/G^{00}.

In connection with these problems it may be useful to investigate the following notion, as was suggested to us by Pillay and Hrushowski on occasion of the model theory meeting in honor of Daniel Lascar (Paris, June 6-7, 2003).

5.5. Definition. A definable subset A of a definably compact group G is **big** if and only if finitely many translates of A cover G.

It would be reasonable to expect that if the union of two definable sets is big, then one of the two is big. With the help of Theorem 5.2 we can prove that this is true in the case of the torus $\mathbb{T}^d$:

5.6. Proposition. *A definable subset A of the torus $\mathbb{T}^d$ is big if and only if it has positive measure. So if the union of two definable subsets is big, then one of the two is big.*

Proof. If A is big, then $\mathbb{T}$ is covered by finitely many translates of A, so by (sub)additivity of the measure one of them has positive measure. But then by left-invariance each translate of A has positive measure, and so does A itself.
Conversely if A has positive measure, then it contains a non–empty polyrectangle. But polyrectangles are obviously big in $\mathbb{T}$, and therefore so does A. $\square$

Clearly the validity of Conjecture 5.1 would imply a generalization of the proposition to any definably compact group.

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