

O-minimal fundamental group, homology and manifolds

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Abstract

We compute the definable fundamental group of a definable set in an o-minimal expansion of a field. This is achieved by proving the relevant case of the o-minimal van Kampen theorem. We apply this result to show that if the geometrical realization of a simplicial complex over an o-minimal expansion of a field is a definable manifold of dimension $\neq 4$, then its geometrical realization over the reals is a topological manifold.

1 Introduction

The study of o-minimal structures has two main directions. One is to find new classes of o-minimal structures, as in the fundamental work of Wilkie in [19]. The other one is the study of properties of sets, groups and fields definable in such structures. This paper is concerned with this second aspect: we investigate the properties of definable sets, over an o-minimal expansion of a real closed field, which are invariant under definable homeomorphisms. This line of research can be seen as a continuation and an extension to the o-minimal context of the work of Delfs and Knebusch on semialgebraic homology and homotopy [6, 7]. Along these lines an o-minimal homology theory was developed by Woerheide in [20]. Our first result is an o-minimal version of the van Kampen theorem (Theorem 2.2) which allows us to compute the

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o-minimal fundamental group of a definable set. We show in particular that this group is finitely generated and it admits a combinatorial definition in terms of an abstract simplicial complex as in the classical case (Corollary 2.10). We apply our results to the study of definable manifolds showing that in dimension different from 4 there are no “exotic” definable manifolds which do not arise from classical topological manifolds. The study of (abstract) definable manifolds started with the work of Pillay in [15] where he proved that every group definable in an o-minimal structure admits an abstract definable manifold structure.

Further motivations for the computation of the o-minimal fundamental group can be found in [13] where Peterzil and Starchenko introduce, for their specific purposes, a notion of o-minimal winding number of a plane definable curve, but with *ad hoc* techniques which do not seem to generalize. In our paper [2] an o-minimal Brouwer degree was defined using intersection theory, but only in the differentiable case. The work of Edmundo [10] shows that the computation of the o-minimal fundamental group can find applications to the study of definable groups. Such applications are also discussed in [3].

We assume some familiarity with the basic notions of o-minimality (see van den Dries [8]), category theory and homotopy theory (see Brown [5]). Let \mathbf{M} be an o-minimal expansion of an ordered field (necessarily real closed), that we fix for the rest of the paper. We equip \mathbf{M} with the interval topology and \mathbf{M}^n ($n > 1$), with the product topology. Unless otherwise stated “definable” will mean definable in \mathbf{M} .

Let $\overline{\mathbf{Q}}$ be the algebraic closure of \mathbf{Q} in the real field \mathbf{R} . Then $\overline{\mathbf{Q}}$ can be isomorphically embedded in \mathbf{M} (as a field), so we can assume that $\overline{\mathbf{Q}}$ is contained in \mathbf{M} . Given a semialgebraic set $X \subset \overline{\mathbf{Q}}^n$ we can interpret its defining formula over \mathbf{R} and \mathbf{M} respectively, obtaining the inclusions:

$$X^{\mathbf{R}} \longleftarrow X \longrightarrow X^{\mathbf{M}}$$

We say that the semialgebraic sets $X^{\mathbf{M}}$ and $X^{\mathbf{R}}$ are obtained from X by **extension of the base field** (but keep in mind that \mathbf{M} in general has more structure than just the ordered field structure).

Note that by the triangulation theorem for definable sets, any definable set is definably homeomorphic to a semialgebraic set of the form $X^{\mathbf{M}}$, with X a semialgebraic (actually, piecewise linear) set over $\overline{\mathbf{Q}}$. The idea throughout

this paper is to transfer information from $X^{\mathbf{M}}$ to $X^{\mathbf{R}}$ and viceversa. If \mathbf{M} carries just the field structure we are in the semialgebraic setting and many of the transfer results of this paper were already known in that case. However, these results were proved in most cases by model completeness, a consequence of the Tarski-Seidenberg theorem, which is not available, in general, in an o-minimal expansion of an ordered field. The results of this paper imply that the homology and the fundamental groups of a definable set are invariant under expansions and extensions of o-minimal structures containing a field.

In Section 2 we consider the definable fundamental group (see Definition 2.1) of a definable set and prove the following.

Theorem 1.1 *Let $X \subset \overline{\mathbf{Q}}^n$ be a semialgebraic set over $\overline{\mathbf{Q}}$ and let $x_0 \in \overline{\mathbf{Q}}^n \cap X$. There is a natural isomorphism*

$$\pi^{def}(X^{\mathbf{M}}, x_0) \simeq \pi(X^{\mathbf{R}}, x_0),$$

where $\pi(X^{\mathbf{R}}, x_0)$ is the fundamental group of $X^{\mathbf{R}}$ at x_0 and $\pi^{def}(X^{\mathbf{M}}, x_0)$ is the definable fundamental group of $X^{\mathbf{M}}$ at x_0 .

This theorem allows us to compute the definable fundamental group of an arbitrary definable set (see Corollary 2.10).

In Section 3 we consider the singular definable homology theory for definable sets developed by Woerheide in [20] and as an application of his results we observe (Proposition 3.2) that

$$H_*^{def}(X^{\mathbf{M}}) \simeq H_*(X^{\mathbf{R}}).$$

Finally, in Section 4 (Theorem 4.3) we apply the above theorems together with a result of D. Galewski and R. Stern in [11], to show that if a semialgebraic set $X^{\mathbf{M}}$ is a definable manifold (see Definition 4.1) of dimension $\neq 4$, then $X^{\mathbf{R}}$ is a topological manifold. If the local charts of $X^{\mathbf{M}}$ are semialgebraic, then the result holds also in dimension 4 ([6, Remark 5.4]). Note also that if a semialgebraic subset of \mathbf{M}^n is a definable differentiable manifold, then it is a semialgebraic manifold (one can use appropriate coordinate projections as local charts as in Bochnak, Coste and Roy [4, Corollary 9.3.9], see also van den Dries and Miller [9]). We finish this introduction with some questions:

Question 1. Suppose $X \subset \overline{\mathbf{Q}}^n$ is a semialgebraic set such that $X^{\mathbf{M}}$ is a definable manifold. Is $X^{\mathbf{M}}$ a semialgebraic manifold?

A positive answer to this question would imply (see Remark 4.13) that Theorem 4.3 holds also in dimension 4.

Question 2. Suppose $X, Y \subseteq \overline{\mathbf{Q}}^n$ are semialgebraic sets such that $X^{\mathbf{M}}$ and $Y^{\mathbf{M}}$ are definably homeomorphic definable manifolds. Are $X^{\mathbf{M}}$ and $Y^{\mathbf{M}}$ semialgebraically homeomorphic? Are $X^{\mathbf{R}}$ and $Y^{\mathbf{R}}$ homeomorphic?

This question can be seen as an o-minimal Hauptvermutung. We do not know the answer even when Y is the unit sphere.

Note that an o-minimal Hauptvermutung (when the base field is that of the reals) has been proved by Shiota in [17].

2 The o-minimal fundamental group

For the proof of Theorem 1.1 it is convenient to work with the fundamental groupoid (see e.g. [5]) rather than the fundamental group.

Definition 2.1 (The definable fundamental groupoid) Let X be a definable subset of \mathbf{M}^n . Let $\mathbf{P}^{def} X$ be the set of all the definable continuous maps $a: [0, r] \rightarrow X$ with $r \in \mathbf{M}, r \geq 0$. We call such a a **definable path** in X of length r . The composition of two definable paths a, b of length r and s respectively, is defined in the obvious way and it is a definable path $a + b$ of length $r + s$. Since we allow paths of different lengths $+$ is associative (but in general not commutative). Also, the constant paths are clearly definable. Hence we can consider the category $\mathbf{P}^{def} X$ in which the objects are the points of X and the morphisms are the definable paths.

Two definable paths a and b of the same length r are **definably homotopic relative end points** if there is a definable continuous function $F: [0, r] \times [0, q] \rightarrow X$ for some $q \in \mathbf{M}, q \geq 0$, such that $F(\cdot, 0) = a, F(\cdot, q) = b$, and the functions $F(0, \cdot)$ and $F(r, \cdot)$ are constant (so a and b have the same end points). Two definable paths a, b are **equivalent**, $a \sim b$, if there exist constant definable paths $r, s \in \mathbf{P}^{def} X$ such that $s + a$ is definably homotopic relative end points to $r + b$. It can be verified that equivalent paths of the same length are definably homotopic relative end points.

Let $\pi^{def} X = \mathbf{P}^{def} X / \sim$ be the category whose objects are the points of X and whose morphisms are the equivalent classes of paths in X . Every morphism of $\pi^{def} X$ is an isomorphism, so the category $\pi^{def} X$ is (by definition) a groupoid. We call $\pi^{def} X$ the **definable fundamental groupoid** of X .

Given $A \subset X$ let $\mathbf{P}^{def}XA$ be the subset of $\mathbf{P}^{def}X$ consisting of all paths with endpoints in A . Let $\pi^{def}XA = \mathbf{P}^{def}XA / \sim$ be full subcategory of $\pi^{def}X$ whose objects are the elements of A . We say that A is **definably representative** in X if A meets every definably connected component of X . This is equivalent to say that A meets every definable path connected component of X . If $v \in X$, the **definable fundamental group** $\pi^{def}(X, v)$ is defined as the group $\pi^{def}X\{v\}$ (we can identify a group with a groupoid with a single object).

Let $X \subset \mathbf{M}^n, Y \subset \mathbf{M}^m$ be definable sets. A definable map of pairs $f: (X, A) \rightarrow (Y, B)$ is a definable map $f: X \rightarrow Y$ mapping the subset $A \subset X$ to $B \subset Y$. It can be easily verified that π^{def} is a functor from the category of pairs and definable continuous functions of pairs, to the category of groupoids.

Theorem 2.2 *(The o-minimal van Kampen theorem for inclusion of closed sets). Let $X \subset \mathbf{M}^n$ be a definable set and let X_0, X_1, X_2 be closed definable subsets of X with $X = X_1 \cup X_2$ and $X_0 = X_1 \cap X_2$. Let $A \subset X_0$ be definably representative in X_0, X_1, X_2 . The π^{def} functor applied to the inclusion maps induces a pushout of groupoids:*

$$\begin{array}{ccc}
 \pi^{def}X_0A & \xrightarrow{i_1} & \pi^{def}X_1A \\
 \downarrow i_2 & & \downarrow u_1 \\
 \pi^{def}X_2A & \xrightarrow{u_2} & \pi^{def}XA
 \end{array}$$

To prove the theorem we need some definitions and lemmas.

Definition 2.3 Let \mathcal{D} be a cell decomposition of \mathbf{M}^2 (in the o-minimal sense) compatible with the rectangle $R = [0, r] \times [0, l]$. Let $\gamma, \gamma' \in \mathbf{P}^{def}R$ be definable paths with the same end points. Let C be a cell of \mathcal{D} . We say that γ and γ' are **C -contiguous** if we can write $\gamma = \alpha + \rho + \beta$ and $\gamma' = \alpha + \rho' + \beta$ where ρ and ρ' are paths in the closure \overline{C} of C . We say that γ, γ' are **\mathcal{D} -contiguous**, if they are C -contiguous for some cell C of \mathcal{D} . Finally we define the relation $\gamma \stackrel{\mathcal{D}}{\sim} \gamma'$ of **\mathcal{D} -equivalence** between paths of $\mathbf{P}^{def}R$ to be the transitive closure of the relation of \mathcal{D} -contiguity.

Lemma 2.4 Let $R = [0, r] \times [0, l]$ and let $s^R, n^R, w^R, e^R \in \mathbf{P}^{def} R$ be the following definable paths:

- $s^R, n^R: [0, r] \rightarrow R$ are defined by $s^R(t) = (t, 0), n^R(t) = (t, l)$;
- $w^R, e^R: [0, l] \rightarrow R$ are defined by $w^R(t) = (0, t), e^R(t) = (r, t)$.

Given any cell decomposition \mathcal{D} of \mathbf{M}^2 compatible with R , the paths $s^R + e^R$ and $w^R + n^R$ are \mathcal{D} -equivalent.

Proof. Fix \mathcal{D} . The proof is by induction on the number k of 1-cells of $[0, r]$ which are images of cells of \mathcal{D} under the projection $p_1: \mathbf{M}^2 \rightarrow \mathbf{M}, (x, y) \mapsto x$. In Figure 1 we have $k = 3$.

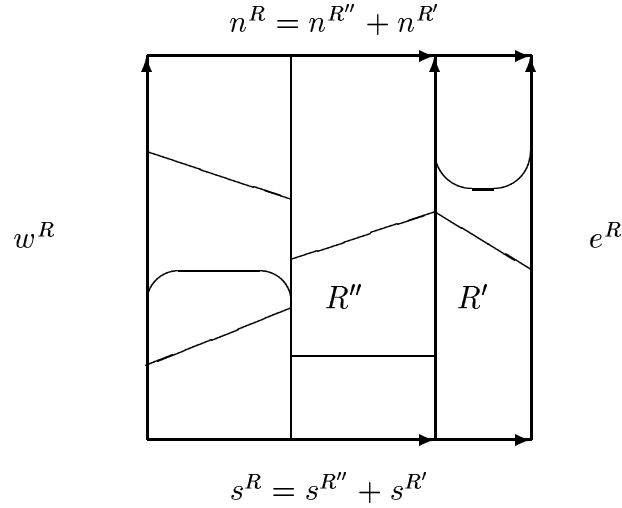


Figure 1. A cell decomposition of the rectangle $R = R'' + R'$

Case $k = 1$. Let C_1, \dots, C_m be the 2-cells of \mathcal{D} contained in R , with C_{j+1} above C_j . Then there is clearly a sequence of definable paths $\gamma_1, \dots, \gamma_{m+1} \in \mathbf{P}^{def} R$ such that γ_j and γ_{j+1} are C_j -contiguous ($1 \leq j \leq m$) and $s^R + e^R = \gamma_1$ and $w^R + n^R = \gamma_{m+1}$.

Case $k > 1$. Let $[r', r]$ be the closure of the rightmost 1-cell of the induced decomposition of $[0, r]$. Since \mathcal{D} is compatible with $R' = [r', r] \times [0, l]$ we can apply case 1 to obtain $s^{R'} + e^{R'} \stackrel{\mathcal{D}}{\sim} w^{R'} + n^{R'}$. Since \mathcal{D} is also compatible with $R'' = [0, r'] \times [0, l]$ we apply induction to obtain $s^{R''} + e^{R''} \stackrel{\mathcal{D}}{\sim} w^{R''} + n^{R''}$. Since $\stackrel{\mathcal{D}}{\sim}$ is a congruence under sums of paths, we get:

$$\begin{aligned}
s^R + e^R &= s^{R''} + s^{R'} + e^{R'} \\
&\stackrel{\mathcal{D}}{\sim} s^{R''} + w^{R'} + n^{R'} \\
&= s^{R''} + e^{R''} + n^{R'} \\
&\stackrel{\mathcal{D}}{\sim} w^{R''} + n^{R''} + n^{R'} \\
&= w^R + n^R
\end{aligned}$$

□

Lemma 2.5 *Let C be a cell of a cell decomposition \mathcal{D} of M^n . Any two paths $\gamma_1, \gamma_2: [0, r] \rightarrow \overline{C}$ with the same end points are definably homotopic relative end points. So in particular \overline{C} is definably simply connected.*

Proof. By induction on n .

If $n = 1$, then \overline{C} is convex, so we can write the definable homotopy $F(t, t') = t\gamma_1(t') + (1 - t)\gamma_2(t')$.

Let $n > 1$. We consider the projection $p_1: M^n \rightarrow M^{n-1}$ on the first $n - 1$ coordinates and $p_2: M^n \rightarrow M$ on the last coordinate. We apply induction to the induced cell decomposition $p_1\mathcal{D}$ of M^{n-1} to conclude that $p_i \circ \gamma_1$ and $p_i \circ \gamma_2$ are definably homotopic relative end points ($i = 1, 2$). Hence $\gamma_1 = (p_1 \circ \gamma_1, p_2 \circ \gamma_1) \sim (p_1 \circ \gamma_2, p_2 \circ \gamma_1) \sim (p_1 \circ \gamma_2, p_2 \circ \gamma_2) = \gamma_2$. □

Proof of Theorem 2.2. Case $A = X$. Consider the following commutative diagram of groupoids.

$$\begin{array}{ccc}
\pi^{def}X_0 & \xrightarrow{i_1} & \pi^{def}X_1 \\
i_2 \downarrow & & \downarrow u_1 \\
\pi^{def}X_2 & \xrightarrow{u_2} & \pi^{def}X \\
& \searrow v_2 & \swarrow v_1 \\
& & G
\end{array}$$

We must prove the existence of a unique morphism of groupoids $v: \pi^{def}X \rightarrow G$ such that $vu_i = v_i$ for $i = 1, 2$. We shall define $v([a]) = w(a)$ for a suitable $w: \mathbf{P}^{def}X \rightarrow G$. If $a \in \mathbf{P}^{def}X_i \subset \mathbf{P}^{def}X$ ($i = 1, 2$), we are forced to define

$w(a) = v_i([a])$. Let now $a: [0, r] \rightarrow X$ be a definable path in X . By the cell decomposition theorem there is a cell decomposition of $[0, r]$ compatible with the closed subsets $a^{-1}(X_1), a^{-1}(X_2)$ of $[0, r]$. Let $p_0 < p_1 < \dots < p_s$ be the 0-cells of the cell decomposition and let $a_i: [p_{i-1}, p_i] \rightarrow X$ be the restriction of a to $[p_{i-1}, p_i]$ ($0 < i \leq s$). This yields a subdivision $a = a_1 + \dots + a_s$ of a such that the image of each a_i is contained in one of X_1, X_2 (where we are identifying a_i with a reparametrization of a_i from $[0, p_i - p_{i-1}]$ to X). We define $w(a) = w(a_1) + \dots + w(a_s)$ where $w(a_i)$ is defined as above. Using the commutativity of the diagram and the fact that any two subdivisions have a common refinement it is easy to see that w is a well defined morphism.

We now prove that $w(a)$ depends only on the equivalence class of a . (By definition of w this is clear if a is a definable path in X_1 or in X_2 .) Recalling the definition of equivalence of paths, it suffices to prove that $w(a)$ depends only on the definable homotopy class of a relative end points. So fix a definable homotopy relative end points $F: [0, r] \times [0, l] \rightarrow X$ between $a = F(\cdot, 0)$ and the path $b = F(\cdot, l)$. By the cell decomposition theorem there is a cell decomposition \mathcal{D} of \mathbf{M}^2 compatible with $R = [0, r] \times [0, l]$ and the closed subsets $F^{-1}(X_1) \subset R$ and $F^{-1}(X_2) \subset R$. It follows that for each cell C of \mathcal{D} contained in R , $F(\overline{C})$ is included in X_1 or in X_2 . By Lemma 2.4 with its notation, $s^R + e^R \stackrel{\mathcal{D}}{\sim} w^R + n^R$.

Claim 2.6 *If $\gamma_1, \gamma_2 \in \mathbf{P}^{def} R$ are \mathcal{D} -contiguous, then $w(F \circ \gamma_1) = w(F \circ \gamma_2)$.*

Granted the claim, we have $w(F \circ (s^R + e^R)) = w(F \circ (w^R + n^R))$. Since F is a definable homotopy relative end points between a and b , we have $F \circ s^R = a$, $F \circ n^R = b$, and moreover $F \circ e^R$ and $F \circ w^R$ are constant paths. Since w sends constant paths to identities of the groupoid G , we have:

$$\begin{aligned} w(a) &= w(F \circ s^R) \\ &= w(F \circ s^R + F \circ e^R) \\ &= w(F \circ w^R + F \circ n^R) \\ &= w(F \circ n^R) \\ &= w(b) \end{aligned}$$

It remains to prove the claim. Without loss of generality we can assume that γ_1 and γ_2 are C -contiguous for some 2-cell C of \mathcal{D} . So we can write $\gamma_i = \alpha + \rho_i + \beta$ with $\rho_i \in \mathbf{P}^{def} \overline{C}$, $i = 1, 2$. By the choice of \mathcal{D} , $F(\overline{C})$ is contained in X_1 or in X_2 , X_1 say. Therefore both $F \circ \rho_1$ and $F \circ \rho_2$ are paths in X_1 . By Lemma 2.5, ρ_1 and ρ_2 are equivalent paths, hence so are $F \circ \rho_1$

and $F \circ \rho_2$. Thus $w(F \circ \rho_1) = v_1([F \circ \rho_1]) = v_1([F \circ \rho_2]) = w(F \circ \rho_2)$. By additivity of w , $w(F \circ \gamma_1) = w(F \circ \gamma_2)$. This ends the proof for the case $A = X$, noting that the unicity of v is clear from the way w is defined.

The general case follows easily from the case $A = X$ as in [5, 6.7.2, 6.5.13]. The fact that A is definably representative allows one to define a retraction $r: \pi^{def}X \rightarrow \pi^{def}XA$ which induces consistent retractions $r_i: \pi^{def}X_i \rightarrow \pi^{def}X_iA$ ($i = 0, 1, 2$) To define r choose for each $x \in X$ a definable path $\theta(x)$ from a point $r(x)$ of A to x ($x \mapsto r(x)$ will be the functor r on objects) with the proviso that if $x \in X_i$ ($i = 0, 1, 2$) we choose the path in X_i and if $x \in A$ we choose a constant path. Given a definable path a between two points x, y of X let $[a]$ be its equivalence class and define $r([a]) = [\theta(x)] + [a] - [\theta(y)]$. Now the theorem follows from the fact that pushouts are preserved under retractions (see [5, 6.6.7]). \square

Definition 2.7 Let $X \subset \overline{\mathbf{Q}}^n$ be a semialgebraic set and let $A \subset X$. The functor π^{sa} is defined as π^{def} but taking as definable sets the semialgebraic sets. We have natural homomorphisms of groupoids

$$\pi X^{\mathbf{R}}A \longleftarrow \pi^{sa}XA \longrightarrow \pi^{def}X^{\mathbf{M}}A$$

obtained by sending the equivalence class of a semialgebraic path over $\overline{\mathbf{Q}}$ into the equivalence classes of the corresponding paths over \mathbf{R} and \mathbf{M} respectively defined by the same formula. We say they are **homomorphisms obtained by extension of the base field**.

Theorem 2.8 Let $X \subset \overline{\mathbf{Q}}^n$ be the geometrical realization of a finite simplicial complex K and let $K^0 \subset X$ be the set of vertices of K . The natural homomorphisms obtained by extension of the base field, are isomorphisms:

$$\pi X^{\mathbf{R}}K^0 \xleftarrow{\cong} \pi^{sa}XK^0 \xrightarrow{\cong} \pi^{def}X^{\mathbf{M}}K^0$$

Proof. By induction on the number of simplexes of K of dimension greater than 0.

Case 1. $X = K^0$. Then any path in X is constant and the desired result follows.

Case 2. $X = K^0 \cup |\Delta|^{\overline{\mathbf{Q}}}$, where Δ is the complex consisting of a simplex (of any dimension greater than zero) and all its faces. The desired result follows from the fact that, over any of the three settings $\overline{\mathbf{Q}}$, \mathbf{M} or \mathbf{R} , any two paths in $|\Delta|$ with the same end points are equivalent (by convexity of $|\Delta|$ is it easy to construct the relevant homotopies).

Case 3. $X = |K|^{\overline{\mathbf{Q}}}$ does not fall under the previous cases. Let L be the subcomplex of K obtained by removing a simplex of maximal dimension (so that its proper faces are in L). Then we can write $X = |L|^{\overline{\mathbf{Q}}} \cup |\Delta|^{\overline{\mathbf{Q}}}$ where Δ is the complex consisting of the chosen simplex of maximal dimension and all its faces. Since K^0 is representative in the geometrical realization of any subcomplex of K , by the classical van Kampen Theorem (see [5, Theorem 8.4.2.]), the inclusion maps induce a pushout of groupoids over \mathbf{R} :

$$\begin{array}{ccc}
 \pi K^0 K^0 & \xrightarrow{i_1} & \pi(K^0 \cup |\Delta|^{\mathbf{R}})K^0 \\
 \downarrow i_2 & & \downarrow u_1 \\
 \pi |L|^{\mathbf{R}} K^0 & \xrightarrow{u_2} & \pi(|L|^{\mathbf{R}} \cup |\Delta|^{\mathbf{R}})K^0
 \end{array}$$

By Theorem 2.2 we also obtain the corresponding pushout of groupoids for both settings $\overline{\mathbf{Q}}$ (with π^{sa}) and \mathbf{M} (with π^{def}).

It follows that the groupoids $\pi(|L|^{\mathbf{R}} \cup |\Delta|^{\mathbf{R}})K^0$, $\pi^{sa}(|L|^{\overline{\mathbf{Q}}} \cup |\Delta|^{\overline{\mathbf{Q}}})K^0$ and $\pi^{def}(|L|^{\mathbf{M}} \cup |\Delta|^{\mathbf{M}})K^0$ are determined by the other three groupoids in the corresponding pushout diagram. Since we can apply induction to the other three groupoids (in each of the diagrams) the desired result follows. \square

We recall that a finite simplicial complex K is called **connected** if its geometrical realization $|K|^{\mathbf{R}}$ is connected. This is equivalent to say that $|K|^{\mathbf{M}}$ is definably connected (assuming, for this to make sense, that the vertices of K have coordinates in $\overline{\mathbf{Q}}$).

Lemma 2.9 *Suppose that K is connected and let x_0 be a zero simplex of K . Let $X = |K|^{\overline{\mathbf{Q}}} \subset \overline{\mathbf{Q}}^n$. Then there is a commuting diagram:*

$$\begin{array}{ccccc}
\pi X^{\mathbf{R}} K^0 & \longleftarrow & \pi^{sa} X K^0 & \longrightarrow & \pi^{def} X^{\mathbf{M}} K^0 \\
\downarrow r^{\mathbf{R}} & & \downarrow r & & \downarrow r^{\mathbf{M}} \\
\pi X^{\mathbf{R}} x_0 & \longleftarrow & \pi^{sa} X x_0 & \longrightarrow & \pi^{def} X^{\mathbf{M}} x_0
\end{array}$$

where the vertical arrows are retractions of groupoids and the horizontal arrows are the natural homomorphisms obtained by extension of the base field.

Proof. We define $r = r^{\overline{\mathbf{Q}}}$ as in last part of the proof of Theorem 2.2, but taking semialgebraic paths instead of definable paths. The other two retracts are obtained by extension of the base field. \square

Proof of Theorem 1.1. The arrows in the top row of Lemma 2.9 are isomorphisms by Theorem 2.8. It then follows that the arrows in the bottom row are also isomorphisms and we obtain:

$$\pi^{def}(X^{\mathbf{M}}, x_0) \simeq \pi(X^{\mathbf{R}}, x_0)$$

This proves Theorem 1.1 in the special case when $X^{\mathbf{M}}$ is the geometrical realization of a connected finite simplicial complex with vertices in $\overline{\mathbf{Q}}$. When $X^{\mathbf{M}}$ is an arbitrary semialgebraic set we reduce to this case as follows. We can obviously assume $X^{\mathbf{M}}$ connected since the (definable) fundamental group at x_0 say, depends only on the (definable) connected component containing x_0 . Then we can assume that $X^{\mathbf{M}}$ is bounded since there is a semialgebraic homeomorphism of \mathbf{M}^n onto a bounded set. Moreover we can assume $X^{\mathbf{M}}$ closed, since by [6, Prop. 2.5] every bounded semialgebraic set admits a semialgebraic deformation retraction onto a closed and bounded semialgebraic set. Finally we conclude by an application of the triangulation theorem. \square

We have the following Corollary to Theorem 1.1.

Corollary 2.10 *Let X be a definable subset of \mathbf{M}^n and let $x_0 \in X$. Then there is a finite simplicial complex K , $v_0 \in K^0$ and an isomorphism*

$$\pi^{def}(X, x_0) \simeq \pi(|K|^{\mathbf{R}}, v_0).$$

In particular, $\pi^{def}(X, x_0)$ is a finitely generated and finitely related group which admits a combinatorial definition depending only on K as in the classical case.

Proof. Every definable set is homeomorphic to a semialgebraic set definable over \mathbf{Q} , so that the theorem applies. The last statement follows by the classical result. \square

3 Definable homology

A homology theory for semialgebraic sets has been studied by Delfs and Knebusch (see [6] and [7]). Here we consider a (singular) homology theory developed by Woerheide [20] for definable sets over \mathbf{M} . We give below the relevant definitions, omitting those details which are obvious adaptations of the corresponding classical notions in singular homology.

Definition 3.1 Given a definable set $X \subset \mathbf{M}^n$ we consider the abelian group $S_k^{def}(X)$ freely generated by the singular definable simplexes (=definable continuous maps) $\sigma: \Delta_k \rightarrow X$, where $\Delta_k = \{(t_0, \dots, t_k) \in \mathbf{M}^k \mid \sum_i t_i = 1, t_i \geq 0\}$ is the standard k -dimensional simplex in \mathbf{M} . The boundary operator $\delta: S_{k+1}^{def}(X) \rightarrow S_k^{def}(X)$ is defined as in the classical case making $S_*^{def}(X) = \bigoplus_k S_k^{def}(X)$ into a chain complex. Similarly one defines the definable chain complex of a pair of definable sets: $S_*^{def}(X, Y) = S_*^{def}(X)/S_*^{def}(Y)$. The graded group $H_*^{def}(X) = \bigoplus_k H_k^{def}(X)$ is defined as the homology of the complex $S_*^{def}(X)$. Similarly $H_*^{def}(X, Y)$ is the homology of $S_*^{def}(X, Y)$. A definable continuous map f induces a morphism f_* of homology groups, making H_*^{def} into a functor as in the classical case.

Woerheide proved that the functor H_*^{def} satisfies the Eilenberg-Steenrod axioms for the category of definable sets over \mathbf{M} and therefore, as he observed, we have an isomorphism between simplicial and definable singular homology for definable closed and bounded sets.

If X is a semialgebraic subset of \mathbf{M} , besides the groups $H_i^{def}(X)$ we may also consider the groups $H_i^{sa}(X)$, defined exactly as above but with “singular semialgebraic simplexes” instead of “singular definable simplexes”. Since the semialgebraic subsets of \mathbf{M} can be considered as the definable sets of the real

closed field underlying \mathbf{M} , we obtain again a functor H_*^{sa} which satisfies the Eilenberg-Steenrod axioms, provided we restrict ourselves to the category of semialgebraic sets and maps.

Proposition 3.2 *Let $X \subset \overline{\mathbf{Q}}^n$ be a semialgebraic set. Then the inclusion homomorphisms*

$$S_*(X^{\mathbf{R}}) \longleftarrow S_*^{sa}(X) \longrightarrow S_*^{def}(X^{\mathbf{M}})$$

induce natural isomorphisms in homology:

$$H_*(X^{\mathbf{R}}) \xleftarrow{\cong} H_*^{sa}(X) \xrightarrow{\cong} H_*^{def}(X^{\mathbf{M}}).$$

Proof. Special case: If $X = |K|_{\overline{\mathbf{Q}}}$ is a simplicial complex, then each of the three singular homologies (classical, semialgebraic, definable) is naturally isomorphic to the simplicial homology of K . In the classical setting the proof of this fact can be found in [12, Lemma 34.2]. In the other two settings the proof is analogous using the Eilenberg-Steenrod axioms.

In the general case we proceed as in the proof of Theorem 1.1. We reduce to the special case noting that deformation retracts induce isomorphisms in homology. \square

4 Definable manifolds

Definition 4.1 A definable set $X \subset \mathbf{M}^k$ is a **definable manifold** of dimension n if every point of X has a definable open neighbourhood (with the induced topology of \mathbf{M}^k) definably homeomorphic to an open subset of \mathbf{M}^n .

Note that by definable manifold we mean definable submanifold of \mathbf{M}^k rather than abstract definable manifolds as in [15], [8] and [14]. This is not a great loss of generality since any Hausdorff abstract definable manifold can be embedded in \mathbf{M}^k for some k (for regular semialgebraic manifolds see Robson [16], for the o-minimal case see [8], and see [2], [10] for a proof that Hausdorff implies regular in this setting). Note also that there is not need to require

that a definable manifold has finitely many charts (as it is instead required for abstract definable manifolds), as the following proposition shows.¹

Proposition 4.2 *Let $X \subset \mathbf{M}^k$ be a definable manifold of dimension n . Then X has a finite atlas.*

Proof. Let (φ, K) be a definable triangulation of \mathbf{M}^k compatible with X . Let K' be the barycentric subdivision of K . Then (ψ, K') , where $\psi = \varphi: \mathbf{M}^k \rightarrow |K'| = |K|$, is a definable triangulation of \mathbf{M}^k compatible with X . Moreover if K_1 is the subcomplex of K' such that $|K_1| = \psi(X)$, then $\psi(X)$ is covered by the stars in K_1 of the vertices of K_1 (see Definition 4.6, below). Let $S = |St(v)|$ be such a star. Then $\psi^{-1}(S)$ is open in X . To finish the proof it suffices to show that $\psi^{-1}(S)$ is definably homeomorphic to an open subset of \mathbf{M}^n . Let then $x = \psi^{-1}(v)$ and (U, ϕ) be a chart of X with $x \in U$. Since S is star-convex we can shrink S around v to get S' definably homeomorphic to S and with $\psi^{-1}(S') \subset U$. Now, $\psi^{-1}(S')$ is open in X and since it sits in a single chart it is definably homeomorphic to an open set of \mathbf{M}^n . \square

As an application of the results of the previous sections we prove:

Theorem 4.3 *Let K be a finite simplicial complex with vertices in $\overline{\mathbf{Q}}^k$ and let $|K|^{\mathbf{M}} \subset \mathbf{M}^k$ and $|K|^{\mathbf{R}} \subset \mathbf{R}^k$ be geometrical realizations of K over \mathbf{M} and \mathbf{R} respectively. If $|K|^{\mathbf{M}}$ is a definable manifold of dimension $n \neq 4$, then $|K|^{\mathbf{R}}$ is a topological manifold of dimension n .*

The strategy to prove Theorem 4.3 is the following: from the fact that $|K|^{\mathbf{M}}$ is a definable manifold we derive as much homology and homotopy information as we can, working in the definable category. Then we transfer this information to $|K|^{\mathbf{R}}$ using the results of the previous sections. In dimension ≤ 3 the homology information will suffice because of the following classical result (see Aleksandrov [1, §1, Ch. XIII, Vol.3]):

Fact 4.4 *If $|K|^{\mathbf{R}}$ is a homology manifold of dimension $n \leq 3$, then $|K|^{\mathbf{R}}$ is a topological manifold of dimension n .*

In dimension ≥ 5 we need the following result of D. E. Galewski and R. J. Stern in [11, Theorem 1.5] (see also [18, page 121]):

¹We thank an anonymous referee to point out this fact.

Fact 4.5 *If $|K|^{\mathbf{R}}$ is a homology manifold of dimension $n \geq 5$ such that the link of each point of $|K|^{\mathbf{R}}$ is connected and simply connected, then $|K|^{\mathbf{R}}$ is a topological manifold of dimension n .*

We recall the definition of the notion of link and homology manifold with the obvious adaptations to the definable setting.

Definition 4.6 Let K be a finite simplicial complex. For each point $x \in |K|^{\mathbf{M}}$ the **closed star** $\overline{St}(x)^{\mathbf{M}}$ of x (with respect to K), is the union of all the closed simplexes of K which contain x . The **open star** $St(x)^{\mathbf{M}}$ is the union of all the open simplexes of K whose closure contains x (an open simplex is obtained from a closed one by removing its proper faces). The **link** $Lk(x)^{\mathbf{M}}$ is the set $\overline{St}(x)^{\mathbf{M}} - St(x)^{\mathbf{M}}$.

Several definitions of homology manifold can be found in the literature. According to Munkres [12, §63] a topological space X is a **homology manifold** of dimension n if all the local homology groups $H_i(X, X - x)$, for $x \in X$ and $i \in \mathbf{Z}$, are isomorphic to $H_i(\mathbf{R}^n, \mathbf{R}^n - 0)$ (which is infinite cyclic for $i = n$, and zero otherwise). In Galewski and Stern [11] a polyhedron X is called a homology manifold if there exists a triangulation $|K|^{\mathbf{R}}$ of X such that for every subdivision K' of K and every vertex x of K' , the link $Lk(x)^{\mathbf{R}}$ (with respect to K') has the same homology groups of the sphere S^{n-1} . If X is a polyhedron this definition of homology manifold is equivalent to the one in Munkres. In order to apply the results in [11] we need this equivalence both in the classical and in the definable category. We shall give a proof in the latter case. To this aim we define:

Definition 4.7 A definable set $X \subset \mathbf{M}^k$ is a **homology definable manifold** of dimension n if all the local homology groups $H_i^{def}(X, X - x)$, $x \in X$, are isomorphic to $H_i^{def}(\mathbf{M}^n, \mathbf{M}^n - 0)$ ($i \in \mathbf{Z}$).

Note that by Proposition 3.2 for every i ,

$$H_i^{def}(\mathbf{M}^n, \mathbf{M}^n - 0) \simeq H_i(\mathbf{R}^n, \mathbf{R}^n - 0) \simeq \tilde{H}_{i-1}(S^{n-1}) \simeq \tilde{H}_{i-1}^{def}(S^{n-1}(\mathbf{M}))$$

where \tilde{H}_* denotes reduced homology and $S^{n-1}(\mathbf{M}) \subset \mathbf{M}^n$ is the unit sphere of dimension $n - 1$ over \mathbf{M} .

Lemma 4.8 *Let $X \subset \mathbf{M}^k$ be a closed and bounded definable set. Then X is a homology definable manifold of dimension n if and only if there exists a triangulation $X \simeq |K|^{\mathbf{M}}$ of X such that for every subdivision K' of K and every vertex x of K' , $H_i^{def}(Lk(x)^{\mathbf{M}}) \simeq H_i^{def}(S^{n-1}(\mathbf{M}))$ ($i \in \mathbf{Z}$), where $Lk(x)^{\mathbf{M}}$ is the link with respect to K' .*

Proof. Since X is closed and bounded, X is definably homeomorphic to a set of the form $|K|^{\mathbf{M}}$, so we may suppose $X = |K|^{\mathbf{M}}$. We claim that for each $x \in |K|^{\mathbf{M}}$ we have $H_i^{def}(|K|^{\mathbf{M}}, |K|^{\mathbf{M}} - x) \simeq \tilde{H}_{i-1}^{def}(Lk(x)^{\mathbf{M}})$ for every i . By the excision axiom $H_i^{def}(|K|^{\mathbf{M}}, |K|^{\mathbf{M}} - x) \simeq H_i^{def}(\overline{St}(x)^{\mathbf{M}}, \overline{St}(x)^{\mathbf{M}} - x)$. Now $Lk(x)^{\mathbf{M}}$ is a definable deformation retract of $\overline{St}(x)^{\mathbf{M}} - x$ by radial projection from x . Therefore we have $H_i^{def}(\overline{St}(x)^{\mathbf{M}}, \overline{St}(x)^{\mathbf{M}} - x) \simeq H_i^{def}(\overline{St}(x)^{\mathbf{M}}, Lk(x)^{\mathbf{M}})$. The claim follows by the long exact homology sequence of the pair observing that $\overline{St}(x)^{\mathbf{M}}$ has zero reduced homology (being definably contractible to a point). The end of the proof is easy observing that the claim also shows that the homology of the link is independent of the triangulation. \square

Remark 4.9 Although we shall not need it, we mention the following further property of homology manifold since it is the one referred to in [18, page 121]: in a homology manifold of dimension n the link of each simplex of dimension $i < n$ has the same homology of S^{n-i-1} , while the link of a simplex of dimension n is empty (see [12, Theorem 63.2]).

If $|K|^{\mathbf{R}}$ is a manifold, it is also a homology manifold. One obtains the definable version of this fact with the same proof:

Lemma 4.10 *If K is a finite simplicial complex such that $|K|^{\mathbf{M}}$ is a definable manifold of dimension n , then $|K|^{\mathbf{M}}$ is a homology definable manifold of dimension n .*

Proof. Since every $x \in |K|^{\mathbf{M}}$ has a definable neighbourhood definably homeomorphic to \mathbf{M}^n , by the excision axiom we obtain the isomorphism $H_i^{def}(|K|^{\mathbf{M}}, |K|^{\mathbf{M}} - x) \simeq H_i^{def}(\mathbf{M}^n, \mathbf{M}^n - 0)$. \square

The notion of homology manifold transfers from \mathbf{M} to \mathbf{R} :

Lemma 4.11 *If K is a finite simplicial complex with the coordinates of the vertices in $\overline{\mathbf{Q}}$, then $|K|^{\mathbf{M}}$ is a homology definable manifold of dimension n if and only if $|K|^{\mathbf{R}}$ is a homology manifold of dimension n .*

Proof. Having at disposal Proposition 3.2 the proof is the same as in the semialgebraic case [6, Proposition 5.3]. One needs to observe that the link of x can be defined using as parameters only the vertices of K and not the point x itself. It then follows by Proposition 3.2 that $Lk(x)^{\mathbf{M}}$ and $Lk(x)^{\mathbf{R}}$ have the same homology (using definable homology in the first case). One concludes using the characterization of homology manifold in terms of the links (see Lemma 4.8). \square

Proof of Theorem 4.3. Let $|K|^{\mathbf{M}}$ be a definable manifold of dimension n . Then by the previous lemmas $|K|^{\mathbf{R}}$ is a homology manifold of dimension n . If $n \leq 3$ this suffices to conclude that $|K|^{\mathbf{R}}$ is a topological manifold by Fact 4.4. If $n \geq 5$ by Fact 4.5 we need to prove that for each $x \in |K|^{\mathbf{R}}$ the link $Lk(x)^{\mathbf{R}}$ is connected and simply connected. Reasoning as in the classical case one can show that there is a definable homotopy equivalence between $Lk(x)^{\mathbf{M}}$ and $S^{n-1}(\mathbf{M})$ (see [6, Corollary 5.2]). It follows that $Lk(x)^{\mathbf{M}}$ is definably connected and definably simply connected. Then clearly $Lk(x)^{\mathbf{R}}$ is connected, and by Theorem 1.1 it is also simply connected. \square

By the triangulation theorem we obtain:

Corollary 4.12 *Let $X \subset \overline{\mathbf{Q}}^k$ be a closed and bounded semialgebraic set. If $X^{\mathbf{M}}$ is a definable manifold of dimension $n \neq 4$, then $X^{\mathbf{R}}$ is a topological manifold of dimension n .*

Remark 4.13 For semialgebraic manifolds the situation is much simpler as observed in [6, Remark 5.4]: Let $X \subset \overline{\mathbf{Q}}^k$ be a semialgebraic set. Then $X^{\mathbf{M}}$ is a semialgebraic manifold of dimension n if and only if $X^{\mathbf{R}}$ is a semialgebraic manifold of dimension n . Both directions are analogous. If $X^{\mathbf{M}}$ is a semialgebraic manifold of dimension n then by Proposition 4.2, $X^{\mathbf{M}}$ has a finite atlas of semialgebraic charts. Hence we can transfer to \mathbf{R} the statement “every point in $X^{\mathbf{M}}$ has a neighbourhood homeomorphic to an open subset of \mathbf{M}^n through one the semialgebraic charts”.

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