On divisibility in definable groups

Margarita Otero*
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain
margarita.otero@uam.es

December 10, 2008

Abstract

Let $\mathcal{M}$ be an o–minimal expansion of a real closed field. It is known that a definably connected abelian group is divisible. In this note, we show that a definably compact definably connected group is divisible.

Keywords: o-minimality, definable group, divisibility.
Mathematics Subject Classification 2000: 03C64

Let $\mathcal{M}$ be an o–minimal expansion of a real closed field. A group is said to be definable if both the set and the graph of the group operation are definable in $\mathcal{M}$. By results of Pillay in [10], we know that a definable group can be equipped with a definable manifold topology making the group a topological group. Since topological groups are regular spaces, we can suppose that the manifold topology is induced by that of the ambient space (see theorem 10.1.8 in [6]). In that setting, a definably compact group is a closed and bounded definable group. A definable group is definably connected provided it has no definable subgroups of finite index. A definably connected group which is abelian is also divisible, this is due to Strebonski Theorem on the finiteness of the torsion subgroups (see, *e.g.*, the proof of Theorem 2.1 in [9]).

In this note we observe that, with the available literature in both definable groups and topological groups, the following can be proved.

*Partially supported by GEOR MTM2005-02568 and Grupos UCM 910444

Theorem 1 *Let G be a definably compact definably connected definable group. Then, G is divisible.*

In proving divisibility of the groups we are concern with, the continuous definable maps $p_k\colon G \to G\colon a \mapsto a^k$ for $k > 0$ will play an important role (as in both the Abelian definable case and the classical topological one).

Firstly, we consider the o–minimal cohomology with coefficients in $\mathbb{Q}$, as it is defined in section 3 of [9]. Recall that if X is a definable set then $H^*(X;\mathbb{Q})$ is a finitely dimensional $\mathbb{Q}$–vector space such that $H^m(X;\mathbb{Q}) = 0$ for $m > dim\, X$, and moreover, $H^0(X;\mathbb{Q}) \cong \mathbb{Q}$ provided X is also definably connected. For an element $x \in H^m(X;\mathbb{Q})$, we say x has *degree* m and write $deg\, x = m$. In [9], it is obseved that $H^*(X;\mathbb{Q}) \cong Hom_{\mathbb{Z}}(H_*(X),\mathbb{Q})$, where $H_*(X)$ is the o–minimal homology with coefficients in $\mathbb{Z}$. It will be more convenient for our purposes to take o–minimal homology with coefficients in $\mathbb{Q}$. In this case, we also get $H^*(X;\mathbb{Q}) \cong Hom_{\mathbb{Q}}(H_*(X;\mathbb{Q}),\mathbb{Q})$. For, $H^*(X;\mathbb{Q}) \cong H_*(X) \otimes \mathbb{Q}$ and $Hom_{\mathbb{Q}}(H_*(X) \otimes \mathbb{Q},\mathbb{Q}) \cong Hom_{\mathbb{Z}}(H_*(X),\mathbb{Q})$.

Notice that $H^*(X;\mathbb{Q})$ is a $\mathbb{Q}$-algebra with product defined as follows: let $d\colon X \to X \times X\colon x \mapsto (x,x)$ be the diagonal map, identifying $H^*(X \times X;\mathbb{Q})$ with $H^*(X;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$, via the o–minimal Künneth formula for cohomology, and let $x \cdot y := d^*(x \otimes y)$ (see [9] for details).

Moreover, we have the following result.

Lemma 2 *Let G be a nontrivial definably connected definably compact definable group. Then, there is a unique integer $r > 0$ and elements $y_1,\ldots,y_r$ in the o–minimal $\mathbb{Q}$–cohomology algebra of G such that*

(i) the $deg\, y_i$ is odd (i=1,...,r), and

(ii) $H^*(G;\mathbb{Q})$ is freely generated, as a $\mathbb{Q}$–vector space, by $1 (\in H^0(G;\mathbb{Q}))$ and the monomials $y_{i(1)} \cdot \ldots \cdot y_{i(l)}$ with $1 \le i(1) < \cdots < i(l) \le r$.

Proof. By Corollary 3.6 in [9], we know there is a unique $r \ge 0$ and $y_1,\ldots,y_r$ satisfying the requirements. Now, since G is definably connected, by Theorem 5.2 in [3], the top o–minimal homology group $H_n(G)$ is nontrivial, where $n = dim\, G > 0$. Therefore, $r > 0$. $\square$

We write $len\, x = l$ if x is a monomial of length l, *i.e.*, $x = y_{i(1)} \cdot \ldots \cdot y_{i(l)}$ with $1 \le i(1) < \cdots < i(l) \le r$ (with the notation of the above lemma). In the sequel, we are going to consider the maps p_k, $k > 0$ mentioned above. The computations in [4], for such maps, apply to our o-minimal context and we get the following.

Lemma 3 *Let G be a definably connected definable group. For each $k > 0$, consider the definable continuous map $p_k: G \to G: a \to a^k$, for each $a \in G$. Then, the map $p_k^*: H^*(G;\mathbb{Q}) \to H^*(G;\mathbb{Q})$ sends each monomial x to $k^{len\, x}x$.*

Proof. See Lemma 5.2 in [9]. □

Let X be a definable set of dimension n and let $f: X \to X$ be a continuous definable map. The *Lefschetz number* of f is defined as follows.

$$L(f) = \sum_{m=0}^{n} (-1)^m trace(f_*: H_m(X;\mathbb{Q}) \to H_m(X;\mathbb{Q})).$$

(See [5] for the semialgebraic case, and compare with the definition of the Lefschetz number as an intersection number in the o–minimal differentiable case given in [2].)

Note that the matrix of $f^*: H^m(X;\mathbb{Q}) \to H^m(X;\mathbb{Q})$ is the transpose of the matrix of $f_*: H_m(X;\mathbb{Q}) \to H_m(X;\mathbb{Q})$ ($f^* = Hom(f_*, \mathbb{Q})$). Hence, we also get $L(f) = \sum_{m=0}^{n} (-1)^m trace(f^*: H^m(X;\mathbb{Q}) \to H^m(X;\mathbb{Q}))$.

The next step is to make use of the a definable version of the Lefschetz fix point theorem, see theorem 1.1 in [8], and proposition 2 in [5] for the semialgebraic case.

Theorem 4 *Let X be a closed and bounded definably connected set. If $f: X \to X$ is a continuous definable map and $L(f) \neq 0$. Then, f has a fix point.*

Corollary 5 *Let G be a definably compact definable group. If $f: G \to G$ is a continuous definable map and $L(f) \neq 0$. Then, there is an element $b \in G$ such that $f(b) = b$.*

Finally, we shall follow Brown in [4] to compute $L(p_k)$ for each $k \geq 2$, and prove Theorem 1.

Lemma 6 *Let G be a definably connected definably compact definable group. Then, for each $k \geq 2$, $L(p_k) = (1-k)^r$, where r is as in Lemma 2.*

Proof. Let $\{1, x_1, \ldots, x_s\}$ be a basis of the $\mathbb{Q}$–vector space $H^*(G;\mathbb{Q})$, where the x_i's are monomials. By Lemma 3, $p_k^*(x_i) = k^{len\, x_i} x_i$, $(i = 1, \ldots, s)$, and $p_k^*(1) = 1$. Then, the matrix of $p_k^*: H^m(G;\mathbb{Q}) \to H^m(G;\mathbb{Q})$ is either

0 (if $H^m(G;\mathbb{Q})=0$) or a diagonal matrix with entry $k^{len\,x_i}$ corresponding to each x_i in $H^m(G;\mathbb{Q})$. Therefore, $L(p_k)=\sum_{i=1}^{s}(-1)^{deg\,x_i}k^{len\,x_i}+1$. On the other hand, the x_i's are monomials (products of the y_j's of Lemma 2) and the y_j's are of odd degree, so that $deg\,x_i\equiv len\,x_i\ (mod\,2)$, and hence, $L(p_k)=\sum_{i=1}^{s}(-1)^{len\,x_i}k^{len\,x_i}+1$. Since there are $\begin{pmatrix} r \\ l \end{pmatrix}$ monomials of length l, we get $L(p_k)=\sum_{l=1}^{r}\begin{pmatrix} r \\ l \end{pmatrix}(-1)^l k^l+1=(1-k)^r$. □

Proof of Theorem 1: Fix both $a\in G$ and $k(\geq 2)$. We shall prove the existence of an element $b\in G$ such that $b^k=a$. Let $f\colon G\to G\colon c\mapsto c^{k+1}a^{-1}$, for all $c\in G$. Since G is definably connected, there is a definable path $\gamma\colon[0,1]\to G$ such that $\gamma(0)=a^{-1}$ and $\gamma(1)=e$, where e is the neutral element of G. Let

$$F\colon[0,1]\times G\to G\colon(t,c)\mapsto F(t,c)\colon=c^{k+1}\gamma(t).$$

Clearly, F is a definable homotopy between the maps $F(0,-)=f$ and $F(1,-)=p_{k+1}$. Hence, the induced cohomology morphisms f^* and $(p_{k+1})^*$ (both from $H^*(G;\mathbb{Q})$ to $H^*(G;\mathbb{Q})$)) coincide. Hence, $L(f)=L(p_{k+1})$, and by Lemma 6, $L(p_{k+1})=(-k)^r(\neq 0)$. By Corollary 5, there is an element b in G such that $b^{k+1}a^{-1}=b$, as required. □

After this note was written, two alternative proofs of theorem 1 have been given in [7] and [1], both papers make reference to a preprint version of this note.

References

[1] A. Berarducci *Zero-groups and maximal tori* in: *Logic Colloquium 2004* LNL 29 (2008) 33-46

[2] A. Berarducci and M. Otero, *Intersection theory for o–minimal manifolds*, Annals of Pure and Applied Logic 107 (2001) 87–119

[3] A. Berarducci and M. Otero, *Transfer methods for o–minimal topology*, J. Symbolic Logic 68 (2003) 785-794

[4] R. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Company (1971)

[5] G. Brumfield, *A Hopf fixed point theorem for semialgebraic maps*, in: *Real Algebraic Geometry*, Proc. Rennes 1991, LNM 1524 (1992)

[6] L. van den Dries, *Tame Topology and o–minimal structures*, LMS 248, Cambridge Univ. Press (1998)

[7] M. Edmundo, *A remark on divisibility of definable groups*, Math. Log Q. 51 (2005) 639-641

[8] M. Edmundo, *A fixed point theorem in o-minimal structures*, Ann. Inst. Fourier (Grenoble) 57 (2007), no. 5, 1441–1450

[9] M. Edmundo and M. Otero, *Definably compact abelian groups*, J. Math. Log. 4 (2004), no. 2, 163–180.

[10] A. Pillay, *On groups and fields definable in o–minimal structures*, J. Pure Appl. Algebra 53 (1988) 239–255