The joint embedding property in normal open induction

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Abstract

The models of normal open induction are those discretely ordered rings, integrally closed in their fraction field whose nonnegative part satisfy Peano's induction axioms for open formulas in the language of ordered semirings.

It is known that neither open induction nor the usually studied stronger fragments of arithmetic (where induction for quantified formulas is allowed), have the joint embedding property.

We prove that normal models of open induction have the joint embedding property.

1 Introduction

Models of the theory **Open Induction** (OI for short) are those discretely ordered rings associated to the fragment of Peano Arithmetic based on the induction scheme restricted to quantifier-free formulas (see definition below).

The theory **Normal Open Induction** (NOI for short) is the extension of OI in which we require its models to be normal domains, that is, integrally closed in their fraction field.

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Both theories OI and NOI have the following algebraic characterization. This makes them very different from stronger fragments of arithmetic, and also makes the theory of real closed fields relevant to the study of both OI and NOI.

Theorem 1.1 (Shepherdson) *Let M be a (normal) discretely ordered ring and $RC(M)$ the real closure of its fraction field. Then, M is a model of (normal) open induction if and only if for all $r \in RC(M)$ there is an $a \in M$ such that $|r - a| < 1$.*

Shepherdson used this criterion to show that OI does not prove the normality axiom. He constructed a model of OI in which the equation $X^2 = 2Y^2$ has a nontrivial solution. The model he constructed is decidable (see [11]), this makes OI again different to stronger fragments of Peano Arithmetic as IE_1 (bounded existential induction) which do not have decidable nonstandard models (see [13]).

The models of the theory NOI avoid pathologies such as having $\sqrt{2}$ in their fraction field, indeed $\mathbf{Q}$ is algebraically closed in them (see [3]). Hence requiring normality we get models of OI whose arithmetic gets closer to that of $\mathbf{Z}$ (see [9] for a general introduction to these fragments of Peano Arithmetic).

V. d. Dries showed in [3] that Wilkie's method (see [12]) to construct a model of OI starting from a discretely ordered $\mathbf{Z}$-ring (see definition below) extends to the normal case.

Theorem 1.2 (Wilkie) *Every (normal) discretely ordered $\mathbf{Z}$-ring can be extended to a model of (normal) open induction.*

Our aim is to show that the theory NOI is unique amongst the commonly studied fragments of Peano Arithmetic in the following sense.

Definition. A theory T is said to have the **joint embedding property** (JEP for short) if for every two models M_1 and M_2 of T there exists a model M of T and embeddings

$$M_i \hookrightarrow M \quad (i = 1, 2).$$

Wilkie while studying which Diophantine equations are consistent with OI, proved that there are systems of such equations, each one consistent with this theory, which are mutually inconsistent with it (see [8]). Therefore the theory OI does not have the joint embedding property.

On the other hand in [8] it is proved that any fragment of Peano Arithmetic extending IE_1^- (bounded existential parameter-free induction) fails to have JEP. The theory IE_1^- extends the fragment of Peano Arithmetic canonically associated to NOI (see [9]). Note that fragments of Peano Arithmetic are usually defined in the language of ordered semirings and the theories OI and NOI are defined in the language of ordered rings (see below).

We shall prove the following remarkable fact.

Theorem 1.3 *The theory normal open induction has the joint embedding property.*

Notation and conventions

We shall denote the sets of natural, integer, rational and real numbers by $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{Q}$ and $\mathbf{R}$ respectively. The set of p-adic integers will be denoted by $\mathbf{Z}_p$.

All rings are supposed to be conmutative and with a unit 1. For a set S, S^* denotes the set of nonzero elements of S. For a domain M, $F(M)$ denotes its fraction field and $U(M)$ the set of units of M. For an ordered domain M, $RC(M)$ denotes the real closure of its fraction field in some fixed big real closed field containing M.

A bold face letter such as $\mathbf{x}$ (except those used for the above sets) denotes an n-tuple $(x_1, \ldots, x_n)$ where n should be clear from the context or irrelevant. Also, in this case, if M is any set $\mathbf{x} \in M$ denotes $(x_1, \ldots, x_n) \in M^n$.

2 Preliminaries

Let $\mathcal{L}$ denote the first-order language of ordered rings based on the symbols $0, 1, +, -, \cdot, <$. We shall consider the following $\mathcal{L}$-theories.

DOR: the theory of Discretely Ordered rings. Its models are those ordered rings (hence characteristic zero domains) M which satisfy that for all $a \in M \quad \neg(0 < a < 1)$, that is, they have a copy of $\mathbf{Z}$ as a convex subring.

ZR: the theory of discretely ordered Z-rings. M is a model of ZR if and only if M is a model of DOR and for every $n \in \mathbf{N}$ with $n > 0$ there exists a ring isomorphism from M/nM onto $\mathbf{Z}/n\mathbf{Z}$.

OI: Open Induction. M is a model of OI if and only if M is a model of DOR and for every quantifier-free $\mathcal{L}$-formula $\theta(\mathbf{x}, y)$

$$M \models \forall \mathbf{x} \left((\theta(\mathbf{x}, 0) \wedge \forall y \geq 0 (\theta(\mathbf{x}, y) \rightarrow \theta(\mathbf{x}, y+1)) \rightarrow \forall y \geq 0 \theta(\mathbf{x}, y) \right).$$

N: the axiom of Normality. A domain M is a model of N (or normal) if and only if M is integrally closed in its fraction field. That is, M is normal if for every $n \in \mathbf{N}^*$

$$M \models \forall \mathbf{z} x y (x, y \neq 0 \wedge x^n + z_1 x^{n-1} y + \cdots + z_n y^n = 0 \rightarrow \exists w (x = wy)).$$

Note that we indeed do not need an ordered structure to define normality.

For the above three $\mathcal{L}$-theories we have their normal counterparts. They are related as follows.

$$\text{NOI} \models \text{NZR} \models \text{NDOR}.$$

Using Shepherdson's criterion is easy to prove that (Normal) Open Induction gives the existence of Euclidean division, so in particular we can divide by standard integers. This makes every model of (N)OI a model of (N)ZR.

Because of theorem 1.2 above and the known characterization of substructures of **Z**-rings we have the following.

Corollary 2.1 (Wilkie) *Let M be a (normal) discretely ordered ring. Suppose that for each prime p there is a ring homomorphism $\varphi_p : M \rightarrow \mathbf{Z}_p$. Then M can be extended to a model of (normal) open induction.*

In what follows, if we have M satisfying the assumptions of the corollary then M_φ will denote the following domain.

$$M_\varphi = \left\{ \frac{a}{n} : a \in M,\ n \in \mathbf{N}^* \text{ and } n | \varphi_p(a) \text{ for each } p \right\}$$

which is a model of (N)ZR

3 Algebraic background

First we recall some basic properties of normal domains (see [1]). Let M be a domain. M^{alg} denotes the algebraic closure of $F(M)$ (inside some big fixed algebraic closed field). M^{int} denote the elements of M^{alg} which are integral over M, that is, they are roots of some monic polynomial with coefficients in M.

Lemma 3.1 *Let M be a domain.*

1. If M is normal and x is transcendental over $F(M)$, then $M[x]$ is also normal.

*2. If M is normal and S is a multiplicative subset of M (*i.e. $0 \notin S$ *and S is closed under multiplication), then* **the localization of M at S**

$$S^{-1}M = \left\{ \tfrac{a}{s} \,:\, a \in M, s \in S \right\}$$

is also normal. In particular, if $\wp$ is a prime ideal of M, then

$$M_\wp = S^{-1}M \text{ with } S = M - \wp$$

is also normal.

3. Let K be an algebraic extension of $F(M)$ and $r \in K$. If $r \in M^{int}$ then the coefficients of the minimal polynomial over $F(M)$ belong to M^{int}.

The proof of theorem 1.3 is based on the fact we have embeddings of ordered fields in fields of formal power series. Here we recall the algebraic notions and properties we shall use later on (see also [7], and [2]).

Definition. Let F be a field. A **valuation ring** V of F is a subring of F such that for all $x \in F$ either $x \in V$ or $x^{-1} \in V$.

Lemma 3.2 *Let F be a field and V a valuation ring of F. Then V is a normal local ring.*

Proof. To see that V is normal take $x \in F$ satisfying

$$X^n + r_1X^{n-1} + \cdots + r_n = 0 \quad \text{with} \quad r_i \in V \ \ (1 \le i \le n).$$

Then $x \in V$ for, suppose is not, by definition of V we have $x^{-1} \in V$, but then $x = -(r_1 + r_2x^{-1} + \cdots + r_n(x^{-1})^{n-1})$, hence $x \in V$, a contradiction.

For the second assertion we must prove V has a unique maximal ideal. Being a domain it suffices to prove the set of nonunits is an ideal (then all ideals will be contained in it). Let $A = V - U(V)$. Firstly, let $a \in A$ and $x \in V$, if $ax \in U(V)$ then $a(ax)^{-1}x = 1$, a contradiction, hence $ax \in A$ for all $x \in V$. Now let $a, b \in A$ then $a + b = a(1 + a^{-1}b) = b(1 + ab^{-1})$ is in A because either $a^{-1}b$ or ab^{-1} is in V.

We write m_v for $V - U(V)$ and k_v for its residue field.

Definition. Let F be a field. A **valuation** v of F is a map

$$v : F^* \longrightarrow \Gamma$$

where Γ is an ordered group (called **the value group** of v), satisfying the following:

i) $v(xy) = v(x) + v(y)$ for all $x, y \in F^*$

ii) $v(x + y) \geq min(v(x), v(y))$ for all $x, y \in F^*$ such that $x + y \neq 0$.

We extend v to F by $v(0) = \infty$ and order $\Gamma \cup \{\infty\}$ making $\infty > \gamma$ for all $\gamma \in \Gamma$, the $+$ in Γ also extends in the obvious way. Note also that if $v(x) \neq v(y)$ then $v(x + y) = min(v(x), v(y))$.

Given a valuation ring V of F, the canonical map $v : F^* \longrightarrow \Gamma_V$, where $\Gamma_V = F^* / U(V)$ is ordered by

$$v(a) \geq v(b) \Leftrightarrow ab^{-1} \in V,$$

is clearly a valuation of F.

Conversely, given $v : F^* \longrightarrow \Gamma$, then $V = \{a \in F : v(a) \geq 0\}$ is a valuation ring of F. And the valuation obtained (as above) from V, v' say, is equivalent to v. This means that there exists an isomorphism λ of the ordered valued group $\Gamma_{v'}$ onto Γ such that $v = \lambda v'$.

Next we recall some relations between orders and valuations of a field.

Definitions.

1. **A valued field** is a pair (F, v) where F is a field and v is a valuation on F. Associated to it we have its valuation ring V, its residue field k_v and its value group Γ_v

Example. The fraction field of $\mathbf{Z}_p$ with the usual p-adic valuation is a valued field, $\mathbf{Z}_p$ is its valuation ring, $\mathbf{F}_p$ its residue field, and $\mathbf{Z}$ its value group. This in particular implies $\mathbf{Z}_p$ is normal and has a unique maximal ideal $p\mathbf{Z}_p$, the set of nonunits.
However, for the valued fields we shall work with, the characteristic of both the base and the residue field is zero.

2. Let $(M, <)$ be an ordered domain, a subring A of M is said to be **convex** (for this order) if and only if for all $x, y \in M$ if $0 \leq x \leq y$ and $y \in A$ then $x \in A$.
As we have said $\mathbf{Z}$ is a convex subring of any discretely ordered ring.

3. Let $(F, <)$ be an ordered field and A a subring of F. **The convex hull of** A in F is

$$CH(A, F) = \{r \in F \,:\, \exists a \in A(|r| < a)\}.$$

Clearly $CH(A, F)$ is a valuation ring of F and its maximal ideal is

$$m_v = \{r \in F \,:\, \forall a \in A \; a > 0(|r| < 1/a)\}.$$

4. Given an ordered field $(F, <)$, a valuation v of F is said to be **compatible** with $<$ if and only if

$$\forall x, y \in F(0 \leq x < y \rightarrow v(x) \geq v(y)).$$

Clearly v is compatible with $<$ if and only if V is convex for $<$.

5. A valuation ring V of a field F is said to be **real** if k_v is real, *i.e.*, orderable.

Now we are ready to state the relations between orders in a valued field and its residue field. See [2] (page 219) for details.

Proposition 3.1 *(i) Given $F \models OF$ and a convex valuation ring V of F, there is a unique order on k_v satisfying*

$$\forall x \in U(V) \ (x + m_v > 0 \text{ if and only if } x > 0 \text{ in } F).$$

(ii) Given a real valuation ring V of a field F, for every order $<$ on k_v there is at least one order on F compatible with v such that the unique order on k_v defined as in (i) coincides with $<$.

Remarks.
1. It is the convexity of V that makes the order on k_v (in *(i)*) well defined. And *(i)* implies that every convex valuation ring is real.
2. If V in *(i)* is CH($\mathbf{Q}$,F) then this unique order in k_v is **Archimedean** *i.e.*,

$$\forall x, y \in k_v^* \ \exists n \in \mathbf{N} \quad \text{such that} \quad n|x| > |y|.$$

In this case the value group Γ_v is called **the group of the Archimedean classes** of F.

We shall make special use of this value group in the proof of theorem 1.3.
3. The above relations give us the following natural characterization of discrete ordered extensions:
Let $D \models DOR$ and M a domain extending D. Then there is a discrete order on M extending that of D if and only if there is a real valuation ring V of $F(D)$ such that

$$V \supset CH(\mathbf{Q}, F(D)) \text{ and } V \cap M = \mathbf{Z}$$

For, suppose first $D \subset M \models$DOR, then $V = CH(\mathbf{Q}, F(M))$ satisfies the above conditions because $\mathbf{Z}$ is convex in M. Conversely, let $<_k$ be an order on k_v, get $<$ on $F(D)$ as in *(ii)* of the Proposition, call $<_D$ the order on D, note that $<$ extends $<_D$ for $V \supset CH(\mathbf{Q}, F(D))$ so if $a, b \in D$ with $0 <_D a <_D b$ then $0 <_D a/b <_D 1$, hence $a/b \in V$, so $v(a) \geq v(b)$ and v is compatible with $<$, therefore $a < b$ in M. Also $0 < a < 1$ for some $a \in M$ implies $a \in V \cap M$ (by compatibility again), but then $a \in \mathbf{Z}$, a contradiction.

Now we move our attention to fields of formal power series.
Definition. Let k be a field and $(\Gamma, +)$ an ordered group. **A formal power series on** Γ **over** k is a map $f : \Gamma \to k \quad f(\gamma) = a_\gamma$, such that the support of $f \quad$ (*Supp* $f = \{\gamma : a_\gamma \neq 0\}$) is well ordered.
We denote f by

$$f(t) = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma.$$

We define $+$ and $\cdot$ for power series $f(t) = \sum_{\gamma \in \Gamma} a_\gamma t^\gamma$ and $g(t) = \sum_{\gamma \in \Gamma} b_\gamma t^\gamma$ on Γ over k, in the obvious manner by

$$f(t) + g(t) = \sum_{\gamma \in \Gamma} (a_\gamma + b_\gamma) t^\gamma \quad \text{and} \quad f(t)g(t) = \sum_{\gamma \in \Gamma} \left(\sum_{\eta + \delta = \gamma} a_\eta b_\delta \right) t^\gamma$$

In [5] it is proved that the set of formal powers series on Γ over k together with these two operations is a field (note that the well ordered support is essential to make $\cdot$ well defined). We denote it by $k((t^\Gamma))$ and identify k with kt^0.

In $k((t^\Gamma))$ there is a natural valuation

$$v(f) = min\, Supp\, f \quad (min\, \emptyset = \infty).$$

We write $cv(f) = a_\eta$ if $\eta = v(f)$. Any valued field with value group Γ and residue field k of characteristic zero can be embedded (as valued field) in $k((t^\Gamma))$, moreover $k((t^\Gamma))$ is a maximal valued field, *i.e.*, it has no proper valued extensions with the same residue field and the same value group. The following well known proposition will be essential for the proof of Theorem 1.3. See [10] and [5] for details.

Proposition 3.2 1. *Let k be a* RCF *and Γ a divisible group then*

$$k((t^\Gamma)) \models RCF.$$

2. *For any ordered field F, there are (noncanonical) order-embeddings*

$$F \hookrightarrow \mathbf{R}((t^\Gamma))$$

where Γ is the divisible hull of the group of Archimedean classes of F.

Proof.
1. Let K denote $k((t^\Gamma))$. First note that K is a real field. Suppose $\sum_{i=1}^n f_i^2 = 0$. Let $\eta = min\{v(f_i) : i = 1, \dots, n\}$. And let $I \subset \{1, \dots, n\}$ be such that $\forall j \in I \quad v(f_j) = \eta$. Then $\sum_{i \in I}(cv(f_i))^2 = 0$. Since k is real this implies $cv(f_i) = 0$ for all $i \in I$. Hence $f_i = 0$ for all i $(1 \le i \le n)$ with $v(f_i)$ minimal. This implies $f_i = 0$ $(i \le i \le n)$.

Let now L/K be an algebraic extension. The natural valuation v on K extends to a valuation v' on L with value group Γ' and residue field k' such that the quotient Γ'/Γ a torsion group and the field extension k'/k algebraic. Therefore $\Gamma = \Gamma'$ for, if $a \in \Gamma'$ then there is $n \in \mathbf{N}^*$ such that $na \in \Gamma$, and Γ is divisible so $a \in \Gamma$. Also since $k \models$RCF and k'/k is algebraic either $k' = k(i)$ or $k' = k$. The maximality of K implies L algebraically closed in the first case, and $L = K$ in the second. This proves $K \models$RCF.

2. Let $V = CH(\mathbf{Q}, F)$ the convex hull of $\mathbf{Q}$ in F and Γ_v the group of Archimedean classes. Let $<$ be the unique order on k_v such that

$$\forall x \in U(V) \quad x + m_v > 0 \quad \text{if and only if} \quad x > 0 \text{ in } F$$

(see Proposition 3.1).
For any embedding of F into $k_v((t^{\Gamma_v}))$, as valued fields, there is a unique order in $k_v((t^{\Gamma_v}))$ induced by the above order on k_v, namely

$$f > 0 \quad \text{if and only if} \quad cv(f) > 0 \text{ in } k_v.$$

Now, the compatibility of v with respect to $<$ on F makes this embedding an order-embedding. On the other hand, k_v is Archimedean hence a subfield of the reals, and Γ_v (being torsion-free) is embeddable in its divisible hull.

Remarks.
1. With the notation of 2 (in the above proof), note that if $F = F(M)$ for some DOR, M then

$$v(f) < 0 \text{ for all } f \in M - \mathbf{Z}$$

2. The only case where the embedding in 2 of the Proposition is canonical is when F is Archimedean, and then $V = F$, and $\Gamma_v = \{0\}$, so F is a subfield of $\mathbf{R}$.

4 Some facts about normal discretely ordered rings

The domains we usually consider have 1 and -1 as their only units. To prove that a domain is normal is usually easier if it contains a field. The next lemma shows that for substructures of models of OI suffices to prove normality for the $\mathbf{Q}$-algebra generated by the relevant domain.

Lemma 4.1 *Let M be a discretely ordered ring and $\varphi_p : M \to \mathbf{Z}_p$ a ring homorphism for each p. Suppose that the localization of M at $\mathbf{Z}$, $\mathbf{Z}^{-1}M$, is normal, then M_φ is also normal.*

Proof. Let $x \in F(M_\varphi) = F(\mathbf{Z}^{-1}M) = F(M)$ and x integral over M_φ. And let

$$f(X) = X^n + a_1 X^{n-1} + \cdots + a_n \in M_\varphi[X]$$

with $f(x) = 0$. Now, each $\varphi_p : M \to \mathbf{Z}_p$ extends uniquely to M_φ by

$$\varphi_p\left(\frac{a}{n}\right) = \frac{\varphi_p(a)}{n} \in \mathbf{Z}_p.$$

Since $\mathbf{Z}^{-1}M \supset M_\varphi$ and the first domain is normal by hypothesis, x being integral over M_φ, must belong to $\mathbf{Z}^{-1}M$, $x = a/m$ say, with $a \in M$ and

$m \in \mathbf{N}$. Then $\frac{\varphi_p(a)}{m}$ is integral over $\mathbf{Z}_p$ and belongs to its fraction field. Now, each $\mathbf{Z}_p$ is normal,hence $\frac{\varphi_p(a)}{m}$ is in $\mathbf{Z}_p$ for each p. Therefore, $a/m \in M_\varphi$.

Lemma 4.2 *Let $M \models NOI$ and $M' \models NDOR$ extending M. Then $F(M)$ is algebraically closed in $F(M')$.*

Proof. Let $F = F(M)$, $F' = F(M')$, and $F^{alg} = RC(M)[i]$ the algebraic closure of F. We have to prove that $F^{alg} \cap F' \subset F$.

Let $a, b \in M'$ be such that $a/b \in F^{alg}$. Then there is $c \in M$ such that

$$\tfrac{ca}{b} \in M^{int} \, (\subset (M')^{int}).$$

Since $\frac{ca}{b} \in F'$ by normality $\frac{ca}{b} \in M'$.

On the other hand $\frac{ca}{b} \in F^{alg} \cap F' \subset RC(M)$ and $M \models$OI, so there is a d in M with $\left|\frac{ca}{b} - d\right| < 1$. Now, $\frac{ca}{b}, d \in M'$ which is a DOR. Hence $\frac{ca}{b} = d \, (\in M)$, therefore $\frac{a}{b} \in F$.

Note that we do not need M to be normal, but M being a model of OI and having $M' \models$NDOR extending it, it is forced to be normal.

Lemma 4.3 *Let $M \models NOI$. Let $M' \models NDOR$ extending M and let*

$$f_1, \ldots, f_s \in F(M').$$

If $f_1, \ldots, f_s \in F(M')$ are linearly independent over $F(M)$ then they are also linearly independent over $F(M)^{alg}$.

Proof. Suppose $\sum_{i=1}^{s} f_i \alpha_i = 0$ for some $\alpha_1, \ldots, \alpha_s \in F(M)^{alg}$ not all zero.
Take $\alpha \in F(M)^{alg}$ such that

$$F(M)(\alpha) = F(M)(\alpha_1, \ldots \alpha_s)$$

and let $h(X) \in F(M)[X]$ be the minimum polynomial of α over $F(M)$. Then

$$[F(M')(\alpha) : F(M')] = [F(M)(\alpha) : F(M)].$$

Because otherwise $h(X)$ splits in $F(M')[X]$. But then the coefficients of its factors would be in $F(M')$ and algebraic over $F(M)$, hence by the previous lemma, in $F(M)$, a contradiction.

Therefore the powers of α which form a basis of $F(M)(\alpha)$ as a $F(M)$-vector space also form a basis of $F(M')(\alpha)$ as a $F(M')$-vector space. Let m be their dimension and

$$\alpha_i = \sum_{j=0}^{m-1} g_{ij} \alpha^j$$

with $g_{ij} \in F(M)$. Now, $0 = \sum\limits_{i=1}^{s} f_i\alpha_i = \sum\limits_{i=1}^{s} f_i \left(\sum\limits_{j=0}^{m-1} g_{ij}\alpha^j\right) = \sum\limits_{j=0}^{m-1} \left(\sum\limits_{i=1}^{s} g_{ij} f_i\right) \alpha^j$. Therefore for each j $(0 \leq j \leq m-1)$

$$\sum_{i=1}^{s} g_{ij} f_i = 0.$$

Hence $f_1, \ldots, f_s$ are linearly dependent over $F(M)$.

5 The joint embedding property in normal open induction

We begin with the key lemma for the proof of theorem 1.3

Lemma 5.1 *Let M_1 and M_2 be two discretely ordered rings. Suppose M_1 is normal and $M_1 \cap M_2 = \mathbf{Z}$. Then there is a discrete order on $M_1 \otimes_{\mathbf{Z}} M_2$ and embeddings of ordered domains*

$$h_i : M_i \hookrightarrow M_1 \otimes_{\mathbf{Z}} M_2 \ \ (i = 1, 2).$$

Proof. Let $F_1 = F(M_1)$ and $F_2 = F(M_2)$. By Lemma 4.2, applied to $\mathbf{Z}$ and M_1, $\mathbf{Q}$ is algebraically closed in F_1. Then F_1 and F_2 are linearly disjoint over $\mathbf{Q}$, *i.e.*, $F_1 \otimes_{\mathbf{Q}} F_2$ is a domain (see Lemma 3 on page 391 in [7] noting that since we are in characteristic zero, algebraically closed means regular).
On the other hand

$$M_1 \otimes_{\mathbf{Z}} M_2 \subset \mathbf{Z}^{-1}(M_1 \otimes_{\mathbf{Z}} M_2) \simeq \mathbf{Z}^{-1} M_1 \otimes_{\mathbf{Q}} \mathbf{Z}^{-1} M_2 \subset F_1 \otimes_{\mathbf{Q}} F_2,$$

naturally. Hence $M_1 \otimes_{\mathbf{Z}} M_2$ is a domain (see Theorem 1 on page 386 in [7]). Also since $\mathbf{Z}$ is a Dedekind domain we can identify the rings M_1 and M_2 with the subrings $M_1 \otimes_{\mathbf{Z}} 1$ and $1 \otimes_{\mathbf{Z}} M_2$ of $M_1 \otimes_{\mathbf{Z}} M_2$. Therefore we can define:

(a) for every $a \in M_1$ $\quad a \otimes 1 < 0 \quad$ if and only if $\quad a < 0$ in M_1;

(b) for every $b \in M_2$ $\quad 1 \otimes b < 0 \quad$ if and only if $\quad b < 0$ in M_2.

In the rest of this proof we shall make the following identifications:

(1) M_1 with $M_1 \otimes_{\mathbf{Z}} 1$ as ordered domains;

(2) M_2 with $1 \otimes_{\mathbf{Z}} M_2$ as ordered domains;

(3) $M_1 \otimes_{\mathbf{Z}} M_2$ with its image in $F_1 \otimes_{\mathbf{Q}} F_2$ as domains.

Now consider the language $\mathcal{L}' = \mathcal{L} \cup C_1 \cup C_2$, where C_1 and C_2 are two new sets of constants for elements of M_1 and M_2 respectively such that $C_1 \cap C_2 = \mathbf{Z} - \{0, 1\}$ (as usual we are identifying the constants for elements of M_1 and M_2 with the elements themselves). Let $\mathcal{L}'' = \mathcal{L} - \{<\}$.

Consider the following sets of sentences in $\mathcal{L}'$:

- $\Delta(M_i)$: Open diagram of M_i written in $\mathcal{L} \cup C_i \quad (i = 1, 2)$;
- $\Delta(M_1 \otimes_{\mathbf{Z}} M_2) = \left\{ \varphi\left(\sum\limits_i a_{i_1} b_{i_1}, \ldots, \sum\limits_i a_{i_n} b_{i_n}\right) : \varphi \right.$ open formula of $\mathcal{L}''$ such that $\left. M_1 \otimes_{\mathbf{Z}} M_2 \models \varphi\left(\sum\limits_i a_{i_1} \otimes b_{i_1}, \ldots, \sum\limits_i a_{i_n} \otimes b_{i_n}\right)\right\}$;
- OD : the set of $\mathcal{L}'$ open sentences obtained from the axioms of ordered domains by replacing each universal sentence $\forall \mathbf{x} \varphi(\mathbf{x})$ in $\mathcal{L}$ by the set of $\mathcal{L}'$-open formulas
$$\left\{ \varphi\left(\sum_i a_{i_1} b_{i_1}, \ldots, \sum_i a_{i_n} b_{i_n}\right) : \sum a_{i_j} \otimes b_{i_j} \in M_1 \otimes_{\mathbf{Z}} M_2 \right\};$$
- $D = \left\{ \neg\left(0 < \sum\limits_i a_i b_i < 1\right) : \sum\limits_i a_i \otimes b_i \in M_1 \otimes_{\mathbf{Z}} M_2 \right\}$.

Let $T = \Delta(M_1) \bigcup \Delta(M_2) \bigcup \Delta(M_1 \otimes_{\mathbf{Z}} M_2) \bigcup$ OD $\bigcup$ D.

Assume T is consistent. Let M' be a model of T. Let M be the substructure of M' generated by the constants. Then $M \models \Delta(M_1 \otimes_{\mathbf{Z}} M_2)$. Hence the map
$$\begin{array}{ccc} M & \longrightarrow & M_1 \otimes_{\mathbf{Z}} M_2 \\ \sum\limits_i a_i b_i & \mapsto & \sum\limits_i a_i \otimes b_i \end{array}$$
is a bijection. On the other hand $M \models$ OD $\bigcup$ D. This induces a discrete order in the domain $M_1 \otimes_{\mathbf{Z}} M_2$. Finally $M \models \Delta(M_1) \bigcup \Delta(M_2)$. Therefore there are induced order embeddings
$$M_i \hookrightarrow M_1 \otimes_{\mathbf{Z}} M_2 \qquad (i = 1, 2).$$
Therefore to prove the lemma it suffices to prove the consistency of T. To do this we apply compactness. Let T_0 be a finite subset of T. And let S_0 be
$$\{\sum_i a_i \otimes b_i \in M_1 \otimes_{\mathbf{Z}} M_2 : \text{the term } \sum_i a_i b_i \text{ occurs in some formula of } T_0\} \cup \{0\}$$

Claim. Let Γ_i be the group of Archimedean classes of F_i for $i = 1, 2$. Let Γ be the direct sum of Γ_1 and Γ_2 with lexicographic inverse order. Then there are order embeddings

$$\psi_i : F_i \hookrightarrow \mathbf{R}((t^{\Gamma_i})) \subset \mathbf{R}((t^{\Gamma})) \quad (i = 1, 2)$$

such that if

$$\psi : F_1 \otimes_{\mathbf{Q}} F_2 \to \mathbf{R}((t^{\Gamma}))$$

is the induced homomorphism (*i.e.* $\psi\left(\sum a_i \otimes b_i\right) = \sum \psi_1(a_i)\psi_2(b_i)$), then:

(I) ψ is injective on S_0;

(II) $\neg\left(0 < \sum \psi_1(a_i)\psi_2(b_i) < 1\right)$ for every $\neg\left(0 < \sum a_i b_i < 1\right)$ occurring in $D \cap T_0$.

We first prove that the Claim implies the consistency of T_0, and then we prove the Claim. Let

$$S^i = \{c \in M_i : \text{the constant } c \text{ occurs in some formula of } T_0\} \quad (i = 1, 2).$$

Let (ψ_1, ψ_2) be any pair of embeddings and ψ the induced homomorphism. By the identifications **(1)**, **(2)** and **(3)** above we have that

$$\psi\left(S^i\right) \models \Delta(M_i) \cap T_0 \quad (i = 1, 2)$$

with the order induced by that of $\mathbf{R}((t^{\Gamma}))$. For $\psi\left(S^i\right) = \psi_i\left(S^i\right)$. Hence if in the rest of this proof we want to enlarge S_0 by adding a finite set of $\mathcal{L}'$-sentences, which is already in $\Delta(M_i)$ to $\Delta(M_i) \cap T_0$, we can always do so.

On the other hand the sentences of $\Delta(M_1 \otimes_{\mathbf{Z}} M_2)$ are essentially of one of the following types

$$\sum a_i b_i \neq 0 \quad \text{and} \quad \sum a_i b_i = 0.$$

Without loss of generality we may suppose all of them are of the first type:

(*) $\forall a_i \in M_1, \forall b_i \in M_2 \ \ (1 \leq i \leq l)$

$$\sum_{i=1}^{l} a_i \otimes b_i = 0 \iff \exists \mathbf{c} \in M_1, \ \mathbf{m} \in \mathbf{Z} \text{ and } s \in \mathbf{N} \text{ such that}$$

$$\sum_{i=1}^{l} m_{ij} b_i = 0 \ \ (1 \leq j \leq s) \quad \text{and} \quad a_i = \sum_{j=1}^{s} m_{ij} c_j \ \ (1 \leq i \leq l).$$

This can be proved by flatness of M_1 and M_2 as $\mathbf{Z}$-modules (see [1] page 45).

Therefore we can transfer all $\sum a_i b_i = 0$ occurring in $\Delta(M_1 \otimes_{\mathbf{Z}} M_2) \cap T_0$ to $\Delta(M_i) \cap T_0$, $i = 1, 2$ adding to the latter sentences witnessing the right hand side of the above equivalence.

T_0 is consistent:
Firstly $\psi\,(S_0) \models \Delta(M_1 \otimes_{\mathbf{Z}} M_2)$. For, $0 \in S_0$ and by **(I)** of the Claim ψ is injective. Also by the injectivity of ψ we can define an order in S_0 by

$$\sum a_i \otimes b_i > 0 \ \text{ in } \ S_0 \qquad \text{iff} \qquad \sum \psi_1(a)\psi_2(b) > 0 \ \text{ in } \ \mathbf{R}((t^\Gamma)).$$

Hence, $\psi\,(S_0) \models \mathrm{OD}\ \cap T_0$. Also by **(II)** of the Claim we have

$$\psi\,(S_0) \models \mathrm{D}\ \cap T_0.$$

This implies $\psi\,(S_0) \models T_0$, which proves the consistency of T, and hence the Lemma.

Proof of the Claim:
First note that from (*) above we get:

(i) $\forall m \in \mathbf{Z}$, $\mathbf{a}, c \in M_1$ and $\mathbf{b} \in M_2$,

$$\text{if} \quad m \textstyle\sum a_i \otimes b_i = c \quad \text{then} \quad m | c \quad \text{in} \quad M_1;$$

(ii) for all $\sum_{i \in I} a_i \otimes b_i \in M_1 \otimes_{\mathbf{Z}} M_2$ there is $J \subset I$ and $m \in \mathbf{Z}$ such that

$$m \textstyle\sum_I a_i \otimes b_i = \sum_J a_i \otimes b_i'$$

where the a_i $(i \in J)$ are linearly independent over $\mathbf{Q}$ and the b_i' $(i \in J)$ are $\mathbf{Z}$-linear combinations of the b_i $(i \in I)$.

To see **(i)** note that $\sum_{i=1}^{l} a_i \otimes mb_i - c = 0$ implies $\sum_{i=1}^{l} m_{ij} m b_i - m_{l+1,j} = 0$ for all $(1 \le j \le s)$ and $c = \sum_{j=1}^{s} m_{l+1,j} c_j$, for some $c_j \in M_1$ and $m_{ij} \in \mathbf{Z}$. The first equality implies $m | m_{l+1,j}$ $(1 \le j \le s)$, and then the second $m|c$. Also **(ii)** is clear.

Now we obtain from S_0 and T_0, two finite sets S_1 and T_1 such that:

$$T_1 = A \cup B, \quad A \subset \Delta(M_1) \cup \Delta(M_2), \quad \text{and} \quad S_1 \supset S_0$$

just using the following construction:

- For each pair of elements in S_0, $\sum a_i \otimes b_i \neq \sum c_i \otimes d_i$ put

  $$\textstyle\sum a_i \otimes b_i - \sum c_i \otimes d_i$$

  in S_1;

- For each $\sum a_i b_i \neq 0$ ($\neg(0 < \sum a_i b_i < 1)$ respectively) occurring in T_0 and each nonzero $\sum a_i \otimes b_i$ of S_1, first get c_i linearly independent over $\mathbf{Q}$ such that $\sum c_i \otimes d_i = m \sum a_i \otimes b_i$ and put this in A using **(*)** above. Then,
  - if all $d_i \in \mathbf{Z}$ then $\sum c_i d_i = c$ is in $\Delta(M_1)$ for some $c \in M_1$, put this and $c = m \sum a_i \otimes b_i$ in A, also put $c \neq 0$ (respectively $ma = c$ and $\neg(0 < a < 1)$, where a is obtained using (i)) in A;
  - if $d_i \in M_2 - \mathbf{Z}$, for some i, then put
$$\sum c_i d_i \neq 0 \text{ (respectively } \neg(0 < \sum c_i d_i < m)) \text{ in } B;$$
- Put all new elements of $M_1 \otimes_{\mathbf{Z}} M_2$ which have been used above, in S_1.

So it is clear that any ψ satisfying the following:

(I') if $\sum a_i b_i \neq 0$ is in B then $\psi\left(\sum a_i b_i\right) \neq 0$
and
(II') if $\neg(0 < \sum a_i b_i < m)$ is in B then $\neg(0 < \psi\left(\sum a_i b_i\right) < m)$

will also satisfy **(I)** and **(II)**, above.

Existence of ψ: Fix any ψ_2 and suppose there is no ψ_1 such that the induced ψ satisfies **(I')** and **(II')**. Let S_2 be the subset of S_1 formed by those elements which occur in B.
$$S_2 = \left\{ \sum_{i=1}^{l_j} a_{ij} \otimes b_{ij} : \ (1 \leq j \leq s) \right\}.$$
By construction, for each j, we have:
– $\{a_{1j}, \dots, a_{l_j j}\}$ linearly independent over $\mathbf{Q}$;
and
– there is $i \in \{1, \dots, l_j\}$ with $|b_{ij}| > n$ for each $n \in \mathbf{N}$.
Let $k_{ij} = cv(\psi_2(b_{ij}))$ $(1 \leq i \leq l_j)$ $(1 \leq j \leq s)$, $\eta_{ij} = v_2(\psi_2(b_{ij})) \in \Gamma_2$ and $\eta_j = min\{\eta_{ij} : 1 \leq i \leq l_j\}$. Without loss of generality we may suppose $1, \dots, s_j$ $(\leq l_j)$ are the indexes with $\eta_{ij} = \eta_j$ $(1 \leq i \leq s_j)$.

For ψ_1 any embedding $F_1 \hookrightarrow \mathbf{R}((t^{\Gamma_1}))$ and each $f_j = \sum_{i=1}^{l_j} \psi_1(a_{ij})\psi_2(b_{ij})$, if
$$f_j = 0 \quad \text{or} \quad 0 < f_j < m \quad \text{for some } m \in \mathbf{N}$$
then
$$\sum_{i=1}^{s_j} k_{ij}\psi_1(a_{ij}) = 0.$$

For $a_{ij} \in M_1$ implies $v_1(\psi_1(a_{ij})) \leq 0$, $b_{ij} \in M_2$ and for each j at least one $b_{ij} \notin \mathbf{Z}$, so
$$v_2(\psi_2(b_{ij})) \leq 0 \ \ (1 \leq ij \leq s_j) \text{ and } \eta_j < 0.$$
Now, $|f_j| \leq m$ implies $v(f_j) \geq 0$, hence all terms in f_j with negative exponent must cancel, the order in Γ is lexicographic inverse, so
$$\delta + \eta_j < 0 \quad \text{for all} \quad \delta \in \Gamma_1,$$
hence $\sum_{i=1}^{s_j} k_{ij} k_{\delta_{ij}} = 0$ for each coefficient $k_{\delta_{ij}}$ of $\psi_1(a_{ij})$.

Therefore we have for each embedding
$$\psi_1 : R_1 \hookrightarrow \mathbf{R}((t^{\Gamma_1})),$$
where R_1 is the subset of M_1 occurring in S_1, there is a j with $(1 \leq j \leq s)$, such that $\sum_{i=1}^{s_j} k_{ij} \psi_1(a_{ij}) = 0$.

Let $\theta(\mathbf{x})$ be the open formula of $\mathcal{L}$ such that $\theta(\mathbf{a})$ is the conjunction of all the open formulas of $\Delta(M_1)$ which occur in T_1.

Then, in $\mathbf{R}((t^{\Gamma_1}))$, we have that for each $\mathbf{a} \in M_1$ with $M_1 \models \theta(\mathbf{a})$:

$$\bigvee_{j=1}^{s} \sum_{i=1}^{s_j} k_{ij} \psi_1(a_{ij}) = 0$$

and $k_{ij} \neq 0$ for all $(1 \leq i \leq s_j)$ and $1 \leq j \leq s$.
Since $M_1 \models \theta(\mathbf{a})$ if and only if $\mathbf{R}((t^{\Gamma_1})) \models \theta(\psi(\mathbf{a}))$, we have

$$\mathbf{R}((t^{\Gamma_1})) \models \forall \mathbf{x} \left[\theta(\mathbf{x}) \to \bigvee_{j=1}^{s} \left(\sum_{i=1}^{s_j} k_{ij} x_{ij} = 0 \right) \right]$$

with $k_{ij} \in \mathbf{R}^*$ $(1 \leq i \leq s_j)$ and $(1 \leq j \leq s)$.
Let $\varphi(\mathbf{x}, \mathbf{y})$ be $\theta(\mathbf{x}) \to \bigvee_{j=1}^{s} \left(\sum_{i=1}^{s_j} y_{ij} x_{ij} = 0 \right)$ then
$$\mathbf{R}((t^{\Gamma_1})) \models \exists \mathbf{y} \left[\bigwedge_{i,j} y_{ij} \neq 0 \wedge \forall \mathbf{x} \varphi(\mathbf{x}, \mathbf{y}) \right].$$
By completeness of RCF this last sentence is also true in $RC(\mathbf{Z})$.
Get $k_{ij} \in RC(\mathbf{Z})^*$ witnessing this fact. So by model-completeness of RCF
$$RC(M_1) \models \forall \mathbf{x} \varphi(\mathbf{x}, \mathbf{k}).$$
Get $\mathbf{a} \in M_1$ (as above) with $M_1 \models \varphi(\mathbf{a})$ then there is a j with $(1 \leq j \leq s)$ such that
$$\sum_{i=1}^{s_j} k_{ij} a_{ij} = 0$$

with $k_{ij} \in RC(\mathbf{Z})^*$. But $a_{1j}, \ldots, a_{s_j j}$ are linearly independent over $\mathbf{Q}$, hence by Lemma 4.3 (applied to the extension $\mathbf{Z} \subset M_1$) they are also linearly independent over $\mathbf{Q}^{alg} \supset RC(\mathbf{Z})$. This is a contradiction.

Proof of Theorem 1.3:
It suffices to prove it for NZR. Let $M_1, M_2 \models$NZR. Without loss of generality we may suppose $M_1 \cap M_2 = \mathbf{Z}$, and then use the Lemma to get a discrete order on $M_1 \otimes_{\mathbf{Z}} M_2$ extending those of M_1 and M_2.

Now, each M_i $(i = 1, 2)$ is normal hence $\mathbf{Z}^{-1}M_i$ is also normal. We have already seen $\mathbf{Z}^{-1}M_1 \otimes_{\mathbf{Q}} \mathbf{Z}^{-1}M_2$ is a domain. Hence it is also normal (see Lemma 1 on page 400 of [7]). So $\mathbf{Z}^{-1}(M_1 \otimes_{\mathbf{Z}} M_2)$ is normal. On the other hand both M_1 and M_2 are $\mathbf{Z}$-rings, hence for each p we have ${\varphi_p}^{(i)} : M_i \rightarrow \mathbf{Z}_p$.

Let

$$\varphi_p : M_1 \otimes_{\mathbf{Z}} M_2 \rightarrow \mathbf{Z}_p$$

be canonically induced by ${\varphi_p}^{(1)}$ and ${\varphi_p}^{(2)}$, for each p. Then by Lemma 4.1 we have

$$(M_1 \otimes_{\mathbf{Z}} M_2)_{\varphi} \models \text{NZR}.$$

Therefore, by Theorem 1.2, $(M_1 \otimes_{\mathbf{Z}} M_2)_{\varphi}$ can be embedded in a model of NOI.

Open question

Related to the result above there is a natural question.

Does NOI *have the amalgamation property?*

In the proof of the JEP we have made essential use of the fact that each model of normal open induction is an end extension of $\mathbf{Z}$ and a flat $\mathbf{Z}$-module, this is false in general for extentions of models of NOI.

On the other hand Wilkie's counterexample for JEP in OI uses the fact $\mathbf{Q}$ is not algebraically closed in the fraction field of his models. Lemma 3.2 tells us we cannot adapt Wilkie's example to get the failure of amalgamation for NOI.

Let us also note that we have amalgamation for fraction fields of models of NOI.

Let F, F_1 and F_2 be fraction fields of three models of NOI with F_1 and F_2 extending F. Firstly, by lemma 4.2, F is algebraically closed in F_1 and F_2, since we are in characteristic zero this means both extensions are regular.

Therefore we can take copies of F_1 and F_2 linearly disjoint over F. Hence, by lemma 2.5 in [4], there is an order in F_1F_2 extending the orders of F_1 and F_2. On the other hand the fields extension F_1F_2/F is regular (see page 58 in [6]). Since $F/\mathbf{Q}$ is also regular, we have $F_1F_2/\mathbf{Q}$ regular, that is, $\mathbf{Q}$ is algebraically closed in F_1F_2. This implies that F_1F_2 can be embedded in the fraction field of a model of NOI (see [3]).

P. S.: We shall give a negative answer to the above question in a forthcoming paper.

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