

A descending chain condition for groups definable in \mathcal{o} -minimal structures

Alessandro Berarducci
Dipartimento di Matematica
Università di Pisa
56127 Pisa, Italy

Margarita Otero*
Departamento de Matemáticas
Universidad Autónoma de Madrid
28049 Madrid, Spain

Yaa'cov Peterzil†
Department of Mathematics
Haifa University
Haifa, Israel

Anand Pillay‡
Department of Mathematics
University of Illinois
Illinois 61801, USA

March 23, 2007

Abstract

We prove that if G is a group definable in a saturated \mathcal{o} -minimal structure, then G has no infinite descending chain of type-definable subgroups of bounded index. Equivalently, G has a smallest (necessarily normal) type-definable subgroup G^{00} of bounded index and G/G^{00} equipped with the “logic topology” is a compact Lie group. These results give partial answers to some conjectures of the fourth author.

MSC: 03C64, 22E15

Keywords: \mathcal{o} -minimality, compact groups, type-definable subgroups, descending chain conditions

1 Introduction and preliminaries

Classical mathematical objects may arise from purely logical considerations, once the appropriate context is chosen. In this paper the classical objects are compact Lie groups, and the logical context is that of groups definable

*Partially supported by BFM-2002-04797

†Partially supported by funds from the NSF Focused Research Grant DMS-0100979

‡Partially supported by NSF grants DMS-0300639 and the FRG DMS-0100979

in sufficiently saturated \mathcal{o} -minimal structures. (See [5] for the notion of a saturated structure, and see [18] for that of an \mathcal{o} -minimal structure.) There has been a long series of papers highlighting the connections and analogies between groups definable in \mathcal{o} -minimal structures and Lie groups. See for example papers by Pillay, Peterzil, Starchenko, Steinhorn, Edmundo, Razenj, Strzebonski, Berarducci and Otero ([1], [2], [3], [4], [6], [7], [10], [11], [12], [13], [16], [17]). In the case where the underlying ordered structure of the \mathcal{o} -minimal structure M is $(\mathbb{R}, <)$, then already in [13], it was shown that any group definable in M has definably the structure of a Lie group. However, to posit the reals in advance is to beg the question. It is more natural, from the purely model-theoretic viewpoint to take M to be a κ -saturated \mathcal{o} -minimal structure, for some large cardinal κ . In this case we have the general notion of a bounded type-definable equivalence relation E on a definable set X . Bounded means that there are strictly less than κ many equivalence classes, and type-definable means defined by a conjunction of fewer than κ many formulas (with parameters). The quotient set X/E can be equipped with the “logic topology”: the closed sets of X/E are those whose preimage in X is type-definable. Then X/E is a compact topological space. As a special case, consider a definable group G and a type-definable normal subgroup H of G of bounded index. Under the logic topology G/H is a compact topological group. In [14], the fourth author raised some conjectures about the structure of G/H , when M is \mathcal{o} -minimal. The gist of these conjectures is that for any definable group G in M there is a smallest type-definable subgroup G^{00} of G of bounded index, and that the quotient G/G^{00} is a compact Lie group. Moreover if G is “definably compact” then the dimension of G/G^{00} as a Lie group should equal the \mathcal{o} -minimal dimension of G . Intuitively G^{00} should be seen as the “intrinsic” infinitesimal subgroup of G and the projection $G \rightarrow G/G^{00}$ should be seen as the “intrinsic” standard part map. In [14] the conjectures were proved in two extreme cases, when G has \mathcal{o} -minimal dimension 1, and when G is “definably simple”. In this paper we prove the *first part* of the conjectures in full generality:

Theorem 1.1. *Let G be a group definable in a saturated \mathcal{o} -minimal structure M . Then G has the DCC on type-definable subgroups of bounded index, and if G^{00} is the smallest such, then G/G^{00} , equipped with the logic topology, is a compact Lie group. Moreover if G has no proper definable subgroups of finite index, then G/G^{00} is connected.*

We also give some consequences such as:

Corollary 1.2. *Let G be as in the hypotheses of Theorem 1.1. Suppose G is commutative and suppose H is a type-definable subgroup of G of bounded*

index which is torsion-free. Then $H = G^{00}$. In particular if G is a commutative torsion-free definable group then $G = G^{00}$. Moreover any definable torsion-free subgroup of G is contained in G^{00} .

Notice that by [12] if G is commutative and not definably compact, then G contains a torsion-free definable subgroup H such that G/H is definably compact. By the Corollary any such H is contained in G^{00} .

We should mention that the general question of the structure of arbitrary type-definable groups in o -minimal structures has not yet been examined in full generality, and appears to be quite interesting.

In the remainder of this introduction we will give precise definitions, recall the relevant results from [14], and make a few additional observations. In section 2, we discuss definable connectedness and local connectedness, and prove a crucial lemma: in the o -minimal context, any type-definable bounded index subgroup H of a definable, definably connected group G , contains a “canonical” type-definable bounded index normal (in G) subgroup N such that G/N is both connected and locally connected. In section 3, we deduce the main theorem, by reducing to the case of commutative groups and a countable language, as well as invoking the following result of Pontryagin [15]:

Fact 1.3. *Let G be a connected, locally connected, second countable, compact commutative group. Then G is the direct sum (as a topological group) of at most countably many copies of S^1 (the compact Lie group \mathbb{R}/\mathbb{Z}).*

For now let us fix M to be a saturated structure (not necessarily o -minimal) of cardinality κ , where κ is say inaccessible, and strictly greater than the cardinality of the language of M . By a *definable set* in M we mean a subset of some M^n defined by an L -formula with additional parameters from M . By a *type-definable set* in M we mean a subset of M^n which is the intersection of $< \kappa$ -many definable sets. We say that X is (type-)definable over A if X can be defined by a formula (set of formulas) with parameters from A . If X is a (possibly type-)definable set and E is a type-definable equivalence relation on X , then we call E *bounded* if $|X/E| < \kappa$. We should make it clear the role of the inaccessible cardinal κ is just cosmetic and is present only because of the model-theoretic convention of working in a “universal domain”. The real meaning of boundedness of a type-definable equivalence relation E on a type-definable set X , is that the set X/E of equivalence classes does not get bigger when one passes to an elementary extension M' of M .

By a definable group we mean a definable set G equipped with a group operation which is definable. By a type-definable group G we mean a type-definable set equipped with a group operation whose graph is type-definable. This actually implies that there is a definable function whose restriction to $G \times G$ is the given group operation.

Remark 1.4. (i) Let X be a definable set and E a type-definable equivalence relation on X . Then there is a family $(E_i : i \in I)$ ($|I| < \kappa$) of equivalence relations on X , with each E_i being defined by at most a countable set of formulas and with $E = \bigcap_i E_i$.

(ii) Suppose H is a type-definable subgroup of a definable group G . Then there are type-definable subgroups H_i of G for $i \in I$ (with $|I| < \kappa$) such that each H_i is defined by at most a countable set of formulas, and with $H = \bigcap_{i \in I} H_i$.

Proof. (i) Suppose that E is defined by $\{\phi_j(x, y) : j \in J\}$. We may assume that this set of formulas is closed under finite conjunctions and that for each $j \in J$, $\models \forall x \in X(\phi_j(x, x))$ and $\models \forall x, y \in X(\phi_j(x, y) \rightarrow \phi_j(y, x))$.

Now fix some $\phi_{j_0}(x, y)$, with $j_0 \in J$. Then by compactness and our assumptions, we can find $j_1 \in J$ such that $\models \forall x, y, z \in X(\phi_{j_1}(x, y) \wedge \phi_{j_1}(y, z) \rightarrow \phi_{j_0}(x, z))$. Continue to find $j_2, j_3, \dots \in J$, such that $\{\phi_{j_n}(x, y) : n < \omega\}$ defines an equivalence relation. This construction is clearly enough to prove (i).

The proof of (ii) is similar. □

Fact 1.5. (See [9] and [14].)

(i) Let X be a definable set and E a bounded type-definable equivalence relation on X . Let $\pi : X \rightarrow X/E$ be the canonical surjection. Then X/E is a compact (Hausdorff) topological space under the “logic topology” in which $Z \subseteq X/E$ is defined to be closed if $\pi^{-1}(Z)$ is type-definable.

(ii) If moreover G is a definable group and H is a normal type-definable subgroup of bounded index then G/H with the logic topology is a compact topological group.

(iii) In the context of (ii) suppose that G has no definable proper subgroup of finite index. Then G/H is connected.

Remark 1.6. Here are two equivalent descriptions of the logic topology on X/E given above:

(i) $Z \subseteq X/E$ is closed iff $Z = \pi(Y)$ for some type-definable subset Y of X .

(ii) Suppose M_0 is some model (elementary substructure of M) such that X is definable over M_0 and E is type-definable over M_0 . Then $Z \subseteq X/E$ is

closed iff $\pi^{-1}(Z)$ is type-definable over M_0 . Hence the compact space X/E has a basis of cardinality at most $|M_0| + |L|$, where L is the language of M .

Proof. (i) Suppose $Y \subseteq X$ is type-definable, then so is $\pi^{-1}(\pi(Y))$.
(ii) If $a, b \in X$ and $tp(a/M_0) = tp(b/M_0)$ then, since E is bounded, $E(a, b)$ (see for example [9] Lemma 1.6). Hence if $Z \subseteq X/E$, $\pi^{-1}(Z)$ is M_0 -invariant, so if it is type-definable it must be type-definable over M_0 . Now suppose $x \in X$ and U is an open subset of X/E containing x/E . Then $\pi^{-1}(U)$ is the union of some small ($< \kappa$) collection of definable sets. By compactness (saturation of M) there is some finite subunion Y such that the E -class of x is contained in Y . So $\pi(Y^c)$ is closed and does not contain x/E . Thus x/E is in the interior of $\pi(Y)$. As Y is an M_0 -definable set, we see that X/E has a basis determined by M_0 -definable sets, so of cardinality at most $|M_0| + |L|$. \square

Lemma 1.7. *Let X, E and $\pi : X \rightarrow X/E$ be as in Fact 1.5. Let $a \in X$ and let aE denote its E -equivalence class as a subset of X . Suppose Y is a definable subset of X such that $aE \subseteq Y$. Then $\pi(a) = a/E$ is in the interior of $\pi(Y)$.*

Proof. By Remark 1.6(i), $Z = \pi(X \setminus Y)$ is a closed set in X/E which does not contain $\pi(a)$. So the complement of Z is an open set in X/E and $Z \cap \pi(Y)$ contains $\pi(a)$. \square

Fact 1.8. *Suppose G is a definable group, and H is a type-definable subgroup of G of bounded index. Then $N = \bigcap_{g \in G} H^g$ is normal in G , of bounded index in G , and type-definable over any set of parameters over which H and G are.*

Proof. G/N acts faithfully on G/H so has bounded cardinality. Thus $N = \bigcap_{g \in I} H^g$ for some subset I of G of cardinality $< \kappa$. As N is fixed by automorphisms which fix G and H , the last part follows. \square

Fact 1.9. (See [14].) *Suppose G is a definable group. Then the following are equivalent:*

- (i) G has the DCC on type-definable subgroups of bounded index, namely there is no infinite descending chain $H_1 > H_2 > \dots$ of type-definable subgroups of G , each of bounded index in G ,
- (ii) G has a smallest (necessarily normal) type-definable subgroup of bounded index, which we call G^{00} , and G/G^{00} , equipped with the logic topology, is a compact Lie group.

Lemma 1.10. *Suppose G is a definable group, and N a definable normal subgroup of G . Suppose that both N and G/N have the DCC on type-definable subgroups of bounded index. Then so does G .*

Proof. Suppose that $H_1 < H_2$ are type-definable subgroups of G of bounded index in G (and H_1 is a proper subgroup of H_2). Then $H_i \cap N$ and $H_i N/N$ are type-definable subgroups of $N, G/N$ respectively, of bounded index. Moreover, either $H_1 \cap N$ is a proper subgroup of $H_2 \cap N$, or $H_1 N/N$ is a proper subgroup of $H_2 N/N$. This is enough. \square

We now specialize to the case that $M = (M, <, \dots)$ is o -minimal (and still saturated). If G is a definable group in M then by [13], G can be definably equipped with the structure of a “definable M -manifold” in such a manner that multiplication and the inverse map become continuous. In particular G is equipped with the structure of a topological group, although because the underlying order on M is very far from that of \mathbb{R} , the topology is very disconnected. We want to be able to apply the results on groups definable in o -minimal structures to quotient groups G/N . So we make a tacit assumption of “elimination of imaginaries”, which will hold for example if M is an expansion of an ordered group. For simplicity of notation we will assume that any definable group G we consider, is embedded in M^n in such a way that the M -manifold topology on G agrees with the induced topology from the ambient space M^n , so in particular multiplication on G is continuous in the topology induced from M^n . We may call this topology on G the M -topology. (It should be said that this assumption is *not* essential, and all our proofs hold if we work instead with the “definable M -manifold topology” on G .) By [13] any definable subgroup of G is closed, G has the DCC on definable subgroups, and G has no proper definable subgroup of finite index if and only if G is “definably connected”, namely has no proper definable clopen subset.

Fact 1.11. ([14]) *Suppose H is a normal, type-definable subgroup of G of bounded index. Then*

- (i) *H is an open subgroup of G in the M -topology (so G/H equipped with the quotient topology is a discrete group).*
- (ii) *The canonical surjective homomorphism $\pi : G \rightarrow G/H$ is continuous (where G has its M -topology and G/H the logic topology).*

Finally recall the conjectures from [14].

Conjecture (i) If G is a group definable in an o -minimal structure, then G has a smallest type-definable subgroup G^{00} of bounded index in G and the

- quotient G/G^{00} equipped with the logic topology is a compact Lie group.
- (ii) If G is definably compact then the o -minimal dimension of G equals the dimension of G/G^{00} as a Lie group.
 - (iii) If G is commutative then G^{00} is divisible and torsion-free.

We will prove part (i) of the Conjecture in this paper. In [14], the full conjectures were proved in the two extreme cases: $\dim(G) = 1$, and G definably simple (noncommutative). However at the present time we still do not know whether for G an arbitrary definably compact, definably connected, commutative definable group, G^{00} is even a proper subgroup of G .

2 Definable connectedness and local connectedness

For this section and the remainder of the paper we work in a (saturated) o -minimal structure M .

Definition 2.1. *Let $X \subset M^n$ be a type-definable set. We equip X with the induced topology from M^n . By a relatively definable subset of X we mean something of the form $Y \cap X$ where $Y \subseteq M^n$ is definable. We say that X is definably connected if there do not exist disjoint relatively definable open subsets Z_1, Z_2 of X whose union is X .*

Lemma 2.2. *Let $X \subseteq M^n$ be type-definable. Then X is definably connected if and only if $X = \bigcap_{i \in I} X_i$ for some directed family $(X_i : i \in I)$ of definable, definably connected sets, and where $|I| < \kappa$.*

Proof. Suppose the right-hand side holds and suppose for a contradiction that X is not definably connected. So there are definable open subsets Y_1, Y_2 of M^n such that

- (i) $X \cap Y_1 \cap Y_2 = \emptyset$, and
- (ii) $X \cap (Y_1 \cup Y_2)^c = \emptyset$.

By compactness, (i) and (ii) hold with X_i in place of X , for some $i \in I$, contradicting definable connectedness of X_i .

Conversely, suppose X is type-definable and definably connected. We may assume X to be $\bigcap_{i \in I} Y_i$ where the Y_i are definable and the family is directed. Pick a point $x_0 \in X$. Let Z_i be the definably connected component of Y_i which contains x_0 . Then $X \subset Z_i$ as X is definably connected. It follows that $X = \bigcap_{i \in I} Z_i$ and note that the family $(Z_i : i \in I)$ is still directed. \square

Theorem 2.3. *Let X be type-definable. Then X is the disjoint union of $< \kappa$ -many maximal definably connected type-definable subsets of X , which we call the definably connected components of X . If X is type-definable over a set A , then so is each of its definably connected components.*

Proof. Assume X to be type-definable over A . We can write $X = \bigcap_{i \in I} X_i$, where $\{X_i \mid i \in I\}$ is a directed family of definable (over A) sets with $|I| < \kappa$. Let $X_{i,0}, \dots, X_{i,n(i)}$ be the definably connected components of X_i , each of which is also A -definable. For each $x \in X$ let $f_x \in \prod_{i \in I} \{0, \dots, n(i)\}$ be such that $x \in \bigcap_{i \in I} X_{i,f_x(i)}$. Then the family $\{X_{i,f_x(i)} \mid i \in I\}$ is directed. Let $Y_{f_x} = \bigcap_{i \in I} X_{i,f_x(i)}$. By Lemma 2.2, each Y_{f_x} is definably connected. Moreover since each f_x belongs to $\prod_{i \in I} \{0, \dots, n(i)\}$, the family of such functions has cardinality $\leq 2^{|I|} < \kappa$. So $F = \{Y_f \mid \exists x \in X f = f_x\}$ is a partition of X into a bounded number of type-definable definably connected sets. It remains to prove that each $Y_f \in F$ is a maximal definably connected type-definable subset of X . To this aim it suffices to prove that every definably connected type-definable subset C of X is contained in one and only one member of F . This follows from the observation that for each $i \in I$, C must be contained in one and only one definably connected component $X_{i,f(i)}$ of X_i . So, for f defined in this way, $C \subset Y_f$. \square

Remark 2.4. *If X is a type-definable set and $f : X \rightarrow X$ a definable homeomorphism, then f will permute the definably connected components of X .*

We state the next result for an arbitrary type-definable subgroup H of a definable group G , although we will only be applying when H has bounded index in G .

Theorem 2.5. *Suppose that G is a definable group, and H a type-definable subgroup of G (not necessarily of bounded index). Then the definably connected component of H which contains the identity is a normal (type-definable) subgroup of H of bounded index in H , which we call H^0 . If H is normal in G , then H^0 is normal in G too.*

Proof. If $x \in H^0$ then as multiplication by x is a definable homeomorphism of H , and $xH^0 \cap H^0 \neq \emptyset$, by Remark 2.4 we see that $xH^0 = H^0$. Similarly, considering the definable homeomorphism $x \rightarrow x^{-1}$, $H^0 = (H^0)^{-1}$. Conjugation by $h \in H$ is also a definable homeomorphism of H so $H^0 = hH^0h^{-1}$ for any $h \in H$. Thus H^0 is a normal subgroup of H . Similarly one shows that if $H \triangleleft G$, then $H^0 \triangleleft G$. Note that the translates of H^0 in H are precisely

the definably connected components of H , hence by Theorem 2.3, H^0 has bounded index in H . \square

We now work towards proving that if H is a bounded index type-definable definably connected normal subgroup of the definable group G then G/H is “locally connected”. First a lemma relating definable connectedness and connectedness.

Lemma 2.6. *Suppose G is a definable group, and H a type-definable normal subgroup of bounded index in G . Let $\pi : G \rightarrow G/H$ be the canonical surjection. Suppose $X \subseteq G$ is type-definable and definably connected. Then $\pi(X) \subset G/H$ is connected (when G/H is equipped with the logic topology).*

Proof. Note that $\pi(X)$ is closed by Remark 1.6(i). Suppose that $\pi(X)$ is not connected. Then $\pi(X)$ is the disjoint union of two closed sets Z_1 and Z_2 . Let $Y_i = \pi^{-1}(Z_i)$ for $i = 1, 2$. The Y_i are type-definable, and by 1.11 (ii) are closed in G . X is the disjoint union of its closed subsets $Y_1 \cap X$ and $Y_2 \cap X$. By the compactness theorem, both $Y_1 \cap X$ and $Y_2 \cap X$ are relatively definable subsets of X . This gives a contradiction to the definable connectedness of X . \square

Definition 2.7. (*[8], Chapter 3.*)

- (i) A topological space is said to be locally connected at a point x if every open set U containing x contains an open connected neighbourhood of x .
- (ii) A topological space is said to be locally connected if it is locally connected at each point.
- (iii) A topological space is said to be connected *im kleinen* at a point x if for every open neighbourhood U of x there is an open neighbourhood V of x contained in U , such that for every $y \in V$ there is some connected subset of U containing both x and y .

A space may be connected *im kleinem* at x without being locally connected at x . However a space is locally connected at each point (thus locally connected) if and only if it is connected *im kleinem* at each point. The latter is the definition of local connectedness given by Pontryagin ([15], Chapter V, 36, A). The proof that the two definitions are equivalent can be found in Chapter 3 of [8]. We will use the following consequence:

Proposition 2.8. *A topological space is locally connected if and only if for every point x and open set U containing x , there is a connected (not necessarily open) neighbourhood V of x , with $V \subseteq U$.*

Proof. Left implies right is clear. On the other hand, the right hand side clearly implies that the space is connected *im kleinem* at every point: Given an open set U containing x , let V be as in the right-hand side. Let V° be the interior of V . Then V° is an open neighbourhood of x contained in U and V witnesses that for every $y \in V^\circ$ there is a connected set containing x and y . \square

Theorem 2.9. *Let G be a definable group, and H a normal, type-definable, definably connected subgroup of bounded index. Then G/H with the logic topology is locally connected.*

Proof. As G/H is a topological group, by Proposition 2.8 it is enough to prove that every open neighbourhood U of the identity $1/H$ in G/H contains a connected neighbourhood of $1/H$. Let $\pi : G \rightarrow G/H$ be the canonical surjection. Then $\pi^{-1}(U)$ is a union of (boundedly many) definable subsets of G . As $\pi^{-1}(U)$ contains H , by compactness (or saturation), there is a definable set $Y \subset \pi^{-1}(U)$ such that $H \subseteq Y$. Let Y_1 be the definably connected component of Y which contains 1 . As H is definably connected $H \subset Y_1$. By 1.7, $1/H$ is contained in the interior of $\pi(Y_1)$. Clearly $\pi(Y_1) \subseteq U$. Moreover by Lemma 2.6, $\pi(Y_1)$ is connected. Hence $\pi(Y_1)$ is the required set. \square

So if G^{00} exists (namely if there exists a smallest type-definable subgroup of G of bounded index), then G^{00} is definably connected and G/G^{00} is locally connected. Moreover if G is definably connected, then G/G^{00} is connected.

3 Proofs of main results.

Again M is a saturated \mathcal{o} -minimal structure.

We will first give

PROOF OF THEOREM 1.1.

We fix G a group definable in the saturated \mathcal{o} -minimal structure M and we will prove that G has no infinite descending chain of type-definable subgroups of bounded index. By Fact 1.9, this will yield the full statement of Theorem 1.1. The proof goes through a few steps and reductions. Clearly we may assume from the beginning that G is definably connected (equivalently has no definable proper subgroup of finite index).

STEP 1. We may assume that G is commutative.

Proof. By [10] G has a sequence of definable normal definably connected

subgroups $1 = N_0 < N_1 < \dots < N_k$ such that each N_{i+1}/N_i is commutative and such that G/N_k is semisimple, namely has no proper infinite definable normal commutative subgroups. By [10] again, any definable connected semisimple group is, after quotienting by its finite centre, an almost direct product of finitely many definably simple (noncommutative) groups. But by [14], Theorem 1.1 is true for definably simple groups, thus clearly G/N_k has the DCC on type-definable subgroups of bounded index. So by Lemma 1.10, it suffices to prove that each N_{i+1}/N_i has the DCC on type-definable subgroups of bounded index. This completes Step 1.

We now assume for a contradiction that G is definably connected, commutative and has an infinite descending chain $H_1 > H_2 > \dots$ of type-definable subgroups of bounded index.

STEP 2. We may assume that the language of the structure M is countable, and that each H_i is (type)-defined over a countable elementary substructure M_0 of M (over which G is defined too).

Proof. By Remark 1.4 (ii) we may assume that each H_i is defined by at most a countable set of formulas. Let L' be a countable sublanguage of the language L of M such that the ordering is in L' and such that G is L' -definable (with parameters) and that each of the countably many formulas involved in the definitions of the H_i ($i < \omega$) is an L' -formula with parameters from M . So we may assume $M = M|L'$, and clearly there is a countable elementary substructure M_0 such that both G and the H_i 's are (type-) defined over M_0 .

Now let H be the smallest subgroup of G of bounded index which is type-definable over M_0 . (This clearly exists and was called in [14] $G_{M_0}^{00}$.)

CLAIM 3. H is divisible and definably connected.

Proof. By [17] G has only finitely many elements of order n for any n . Thus the kernel of the map $n : G \rightarrow G$ (taking x to nx) is finite and hence this map is surjective. So G is divisible. Now for any n , nH is clearly of bounded index in $nG = G$. Moreover nH is also type-definable over M_0 . Hence $nH = H$ for all n . We have proved that H is divisible. By Theorem 2.3 and Theorem 2.5 the definably connected component of H which contains the identity, is also a type-definable subgroup of G of bounded index, type-defined over M_0 , so equals H . Hence H is definably connected.

CLAIM 4. G/H (equipped with the logic topology) is a compact Lie group.

Proof. By Remark 1.6(ii), the compact group G/H has a countable basis. Moreover, by Theorem 2.9 G/H is locally connected and, by Fact 1.5.(iii),

connected. By Fact 1.3, G/H is a direct sum of (at most countably) copies of the 1-dimensional compact Lie group S^1 . To prove CLAIM 4, it suffices to show that G/H has only finitely many elements of order 2 (for then G/H has to be a direct sum of finitely many copies of S^1 , thus a compact Lie group). But if $a \in G$ and $a + H$ has order 2 in G/H then, as H is divisible (by CLAIM 3), the coset $a + H$ must contain an element of order 2 in G . But again by Strzebonski's result mentioned above, G has only finitely many elements of order 2. So Claim 4 is established.

Now the H_i/H form an infinite descending chain of closed subgroups of G/H , contradicting the fact that G/H is a compact Lie group. This final contradiction completes the proof of Theorem 1.1.

We complete the paper with some corollaries of Theorem 1.1.

Corollary 3.1. *Suppose G is definable in M . Suppose H is a type-definable subgroup of G of bounded index. Then H is definably connected by finite. Namely H^0 has finite index in H .*

Proof. H contains G^{00} . H/G^{00} is a closed subgroup of the compact Lie group G/G^{00} , so is a compact Lie group itself, thus has a connected component $(H/G^{00})^0$ say, of finite index. The preimage of $(H/G^{00})^0$ in G is then a type-definable subgroup K of H of finite index. We claim that K is definably connected. If not then K can be partitioned into two relatively definable open sets Y_1, Y_2 . Since G^{00} is definably connected by Theorem 2.5, each Y_i is a union of translates of G^{00} . But then the $\pi(Y_i)$ disconnect $(H/G^{00})^0$, a contradiction. \square

Question. Suppose H is a type-definable subgroup (not necessarily of bounded index) of a definable group G . Is it the case that H is definably connected by finite?

PROOF OF COROLLARY 1.2.

So G is a definable commutative group, H is a type-definable torsion-free subgroup of bounded index, and we want to prove that $H = G^{00}$. Suppose not. Then H properly contains G^{00} and H/G^{00} is a non-trivial compact Lie group. Thus H/G^{00} has an element $a + G^{00}$ of some finite order $n > 1$. As in the proof of Claim 3 in the proof of Theorem 1.1, G^{00} is divisible. Thus

the coset $a + G^{00}$ contains an element b of H with $nb = 0$, contradicting H being torsion-free.

For the last statement, let H be a definable torsion-free subgroup of G . Then $H \cap G^{00}$ is a torsion-free type-definable subgroup of H of bounded index, so it coincides with H .

Acknowledgements. The first two authors thank Dikran Dikranjan for useful information on compact groups.

References

- [1] A. Berarducci and M. Otero, Intersection theory for \mathcal{o} -minimal manifolds, *Annals of Pure and Applied Logic*, 107 (2001), 87-119.
- [2] A. Berarducci and M. Otero, \mathcal{o} -minimal fundamental group, homology and manifolds, *J. London Math. Soc.* (2), 65 (2002), 257-270.
- [3] A. Berarducci and M. Otero, Transfer methods for \mathcal{o} -minimal topology, *Journal of Symbolic Logic* 68 (2003) 785-794.
- [4] A. Berarducci and M. Otero, An additive measure in \mathcal{o} -minimal expansions of fields, *Quarterly Journal of Mathematics* 55 (2004), 411-419.
- [5] C.C. Chang and H. J. Keisler, *Model Theory*, North-Holland, Amsterdam-London 1973.
- [6] M. Edmundo, Solvable groups definable in \mathcal{o} -minimal structures, *J. Pure and Applied Algebra*, 185 (2003), 103-145.
- [7] M. Edmundo, \mathcal{o} -minimal cohomology and definably compact definable groups, Preprint, 96 pp.
- [8] J. G. Hocking and G. S. Young, *Topology*, Dover Publications Inc. (first published by Addison-Wesley 1961), New York, 1988.
- [9] D. Lascar and A. Pillay, Hyperimaginaries and automorphism groups, *Journal of Symbolic Logic*, 66 (2001), 127-143.
- [10] Y. Peterzil, A. Pillay, and S. Starchenko, Definably simple groups in \mathcal{o} -minimal structures, *Trans. AMS*, 352 (2000), 4397-4419.
- [11] Y. Peterzil, A. Pillay, and S. Starchenko, Simple algebraic and semialgebraic groups over real closed fields, *Trans. AMS*, 352 (2000), 4421-4450.

- [12] Y. Peterzil and C. Steinhorn, Definable compactness and definable subgroups of o -minimal groups, *J. London Math. Soc* 59 (1999), 769-786.
- [13] A. Pillay, On groups and fields definable in o -minimal structures, *J. Pure and Applied Algebra*, 53 (1988), 239-255.
- [14] A. Pillay, Type-definability, compact Lie groups, and o -minimality, *J. Math. Logic*, 4 (2004), 147-162
- [15] L. Pontryagin, *Topological Groups*, Princeton University Press, Princeton 1946.
- [16] V. Razenj, One dimensional groups over an o -minimal structure, *Annals of Pure and Applied Logic*, 53 (1991), 269-277.
- [17] A. Strzebonski, Euler characteristic in semialgebraic and other o -minimal structures, *J. Pure and Applied Algebra*, 86 (1994), 173-201.
- [18] L. van den Dries, *Tame topology and o -minimal structures*, LMS Lecture Notes Series, 248, Cambridge Univ. Press, 1998.

E-mail addresses: berardu@dm.unipi.it (A. Berarducci), margarita.otero@uam.es (M. Otero), kobi@math.haifa.ac.il (Y. Peterzil), pillay@math.uiuc.edu (A. Pillay).