The $\mu-$invariant of symmetric Z-homology spheres.

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We prove here in a progressive way that:

The μ invariant of a 3-dimensional Z-homology sphere with a periodic reversing orientation selfhomeomorphism whose period is bigger than two is zero.

Recall that the $\mu-$invariant of a 3-dimensional Z_2-homology sphere M is defined by using a 4-dimensional manifold W^4, whose boundary is M, such that $H_1(W^4, Z_2) = 0$ and the quadratic intersection form in $H_2(W^4, Z_2)/Tor$ is even by means of the signature of the intersection quadratic form in $H_2(W^4, Z_2)/Tor$ as:

$$\mu M = -\frac{\sigma W^4}{16} \quad mod\ 1$$

It is well defined by Rochlin's theorem. [R]

We can establish that the $\mu-$invariant of a 3-dimensional Z_2-homology sphere N with a reversing orientation selfhomeomorphism is zero or 1/2, since in this case, the conected sum of N with N is equal to the connected sum of N with $-N \approx N$, which is the boundary of $(N - B^3) \times I$, a 4-dimensional manifold with $H_1(W^4, Z_2) = 0$ and null quadratic intersection form, so $2\mu N = 0$.

It was proved by Birman[B] and also by Galewski and Stern[G] and by Hsiang and Pao[H], separately, that the $\mu-$invariant of a 3-dimensional Z-homology sphere M with a periodic reversing orientation selhomeomorphism h of period 2 is zero.

We see here that when the period of h is bigger than two, the $\mu-$invariant of a 3-dimensional Z-homology sphere M is zero.

We prove first:

The $\mu-$invariant of a 3-dimensional Z-homology sphere M with a periodic reversing orientation selfhomeomorphism h of period four is zero.

Proof:

The set of fixed points of h is formed by two separate points [D] and the fixed points set of h^2 is a knot K by Smith theory [F], when M is a $Z-$homology sphere. This knot contains both fixed points of h, and the selfhomeomorpism h leaves it invariant, since $h^2(h(x)) = h(h^2(x)) = h(x)$, reversing its orientation. (K is amphicaeiral by h).

Let's call $N = M/h^2$, then N is also a Z-homology sphere, because the projection on the first homology group is an epimorphism and because of Poincare duality. The selfhomeomorphism h projects to a selfhomeomorphism h in N, that will be designed by the same letter because there is no place to confusion.

M is the double cover of N, branched over the knot K.

Let F be a Seifert surface of K.

We construct a bordism B_σ between M and two disjoint copies of N, by considering $N \times I$, and by making the 2-cover B_σ of $N \times I$, branched over $F \times [0, 1/2]$, from two copies of

$$N \times I - bicollar(F \times [0, 1/2]) \approx N \times I - F \times [-1, 1] \times [0, 1/2),$$

by identifying in both copies, the boundaries of

$$bicollar((F \times [0, 1/2)) - F \times (-1, 1) \times \{0\} :$$

If x^1 is the point $x \in F$ in the first copy and x^2 is the point $x \in F$ in the second copy, and being (δ means boundary)

$$\begin{gathered}(\delta(F \times [-1, 1] \times [0, 1/2)) - F \times (-1, 1) \times \{0\} = \\ F \times \{-1, 1\} \times [0, 1/2) \cup (F) \times (-1, 1) \times \{1/2\}\end{gathered}$$

we identify

$$(x^1, -1, t) \in (F \times \{-1\} \times [0, 1/2)) \text{ with } (x^2, 1, t) \in (F \times \{1\} \times [0, 1/2))$$

and

$$(x^1, 1, t) \in (F \times \{1\} \times [0, 1/2)) \text{ with } (x^2, -1, t) \in (F \times \{-1\} \times [0, 1/2))$$

We also identify $(x^1, s, 1/2)$ with $(x^2, -s, 1/2)$ $\forall x \in F \times (-1, 1) \times \{1/2\}$.

The boundary of B_σ is the disjoint union of M and two copies of N.

By a Mayer-Vietoris sequence [D]:

$$\begin{gathered}\to H_2(F) \to H_2(N \times I) \oplus H_2(N \times I) \to H_2(B_\sigma) \to \\ \to H_1(F) \to H_1(N \times I) \oplus H_1(N \times I) \to H_1(B_\sigma) \to \\ \to H_0(F) \to H_0(N \times I) \oplus H_0(N \times I) \to H_0(B_\sigma) \to 0.\end{gathered}$$

we get

$H_1(B_\sigma)Z_2 = 0$ and $H_2(B_\sigma)$ is a free abelian group with 2g generators, where every generator corresponds to an a_i, generator of $H_1(F)$.

If the quadratic form in $H_2(B_\sigma)$ is even, $\mu M = (-\frac{1}{16}\sigma(B_\sigma)+2\mu N)\ mod 1$.

We write now how the elements of $H_2(B_\sigma)$ determined by nulhomologous closed curves in F are:

We call [a] the element of $H_2(B_\sigma)$ determined by a, representative closed curve from $H_1(F)$, nulhomologous in N, which bounds a Seifert surface $F_a \subset N$;

Given a closed curve $a \subset F \subset N$, we call

$$a^+ = a \times \{1\} \subset F \times \{1\} \subset \delta(bicollar(F)) \subset N$$

and $F_{a^+} \subset N$ the Seifert surface of a^+

$$a^- = a \times \{-1\} \subset F \times \{-1\} \subset \delta(bicollar(F)) \subset N$$

and $F_{a^-} \subset N$ the Seifert surface of a^-

We denote by $F^1_{a^+} \subset N$, the Seifert surface of a^+ in the first copy of $N \times I$, at any level $\{t\}$ and by $F^2_{a^+} \subset N$, the Seifert surface of a^+ in the second copy, at any level, $(F_{a^+} \subset N \subset N \times I)$.

Then,

$$[a] = F^1_{a^+} \times \{1/2\} \cup a^+ \times [0,1/2) \cup a^- \times [0,1/2) \cup F^2_{a^-} \times \{1/2\}$$

and also,

$$[a] = F^1_{a^-} \times \{3/4\} \cup a^- \times [0,3/4) \cup a^+ \times [0,3/4) \cup F^2_{a^+} \times \{3/4\}.$$

Then, we have for a pair $([a_i],[a_j])$, where a_i, a_j are closed curves in F, generators of $H_1(F)$, nulhomologous in N:

$$\begin{gathered}
[a_i]\cdot[a_j] = \\
(F^1_{a_i^+} \times \{1/2\} \cup a_i^+ \times [0,1/2) \cup a_i^- \times [0,1/2) \cup F^2_{a_i^-} \times \{1/2\}) \cap \\
(F^1_{a_j^-} \times \{3/4\} \cup a_j^- \times [0,3/4) \cup a_j^+ \times [0,3/4) \cup F^2_{a_j^+} \times \{3/4\}) = \\
= F^1_{a_i^+} \times 1/2 \cap a_j^- \times 1/2 + F^1_{a_i^-} \times 1/2 \cap a_j^+ \times 1/2 = \\
\text{(here } lk \text{ means linking number)} \\
= lk(a_i^+, a_j^-) + lk(a_i^-, a_j^+) = lk(a_i^+, a_j) + lk(a_i, a_j^+)
\end{gathered}$$

The intersection quadratic form matrix in $H_2(B_\sigma)/Tor$ is, then, given by a matrix whose entries are:

$$(lk(a_i^+, a_j) + lk(a_j^+, a_i)).$$

so, it is even.

Now we prove that this matrix has signature zero, because the knot K_σ is amphicaeiral:

In fact, as the knot K verifies $h(K) = -K$, the bordism B_σ can be constructed also by doing the double cover of $N \times I$ branched over $h(F) \times [0, 1/2)$. Then, another matrix for the intersection quadratic form Q in B_σ can be calculated from the basis $\{h(a_1), h(a_2), \cdots, h(a_{2g-1}), h(a_{2g})\} \subset h(F)$, (which gives a different basis of $H_2(B_\sigma)$), and, as $(h(a))^+ = h(a^-)$ for every curve in F, because h reverses orientation; we have:

$$lk(h(a_i))^+, h(a_j)) = lk(h(a_i^-), h(a_j)) =$$
$$= -lk(a_i^-, a_j) = -lk(a_i, a_j^+) = -lk(a_j^+, a_i)$$

$$lk(h(a_j))^+, h(a_i)) = lk(h(a_j^-), h(a_i)) =$$
$$= -lk(a_j^-, a_i) = -lk(a_j, a_i^+) = -lk(a_i^+, a_j)$$

By adding, we get as matrices for Q two opposite matrices which should have the same signature, therefore, zero.

Then, $\mu M = 0 + 2\mu N = 0$, (since N is a Z-homology sphere with a reversing orientation selfhomeomorphim, $\mu N = 0 \ or \ 1/2$). Therefore, μM is zero.

We get also that:

The $\mu-$invariant of an 3-dimensional Z-homology sphere M with a periodic reversing orientation selfhomeomorphism h of period 2^r, $r > 1$ is zero.

Proof:

In fact, what we have proved along the previous processus is that the μ-invariant of a Z-homology sphere which is a two branched cover over an amphichaeiral knot in a symmetric Z-homology sphere is zero.

The manifold M in the enounciate is a manifold which is a two branched cover over an amphichaeiral knot $K = \{x | h^{r-1}(x) = x\}$ by Smith theory [F] amphichaeiral for h, because it is kept by h, and contains the fixed points of h, in a symmetric Z-homology sphere: $M/h^{2^{r-1}}$.

We have got, together with the first result from Birman [B], Galewski and Stern [G], Hsiang and Pao [H], that **The $\mu-$invariant of a 3-dimensional Z-homology sphere M with a periodic reversing orientation self-homeomorphism h whose period is any power of 2, is zero.**

Then, we can settle that:

The $\mu-$invariant of a 3-dimensional Z-homology sphere M with a periodic reversing orientation selfhomeomorphism h is zero.

This result follows now from the consideration that any number n bigger than 2 can be written $n = m2^r$ where m is an odd number and $r > 1$. Then h^m is a reversing orientation selfhomeomorphism with period 2^r, $r > 1$, in M.

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