The $\mu-$invariant of symmetric Z_2-homology spheres.

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We prove here in a progressive way that:

The μ invariant of a 3-dimensional Z_2-homology sphere with a periodic reversing orientation selfhomeomorphism whose period is bigger than two is zero.

Recall that the $\mu-$invariant of a 3-dimensional Z_2-homology sphere M is defined by using an acycled 4-dimensional manifold W^4, whose boundary is M, by means of the signature of the intersection quadratic form in $H_2(W^4)$:

$$\mu M = -\frac{\sigma W^4}{16} \quad mod\ 1$$

We can establish that the $\mu-$invariant of a 3-dimensional Z_2-homology sphere with a reversing orientation selfhomeomorphism is zero or 1/2, since in this case, the conected sum of N with N is equal to the connected sum of N with $-N$ which is the boundary of $(N - B^3) \times I$, 4-dimensional manifold with null quadratic intersection form, so $2\mu N = 0$.

It was proved by Birman and also by Galewski and Stern and by Hsiang and Pao, separately, that the $\mu-$invariant of a 3-dimensional Z_2-homology sphere M with a periodic reversing orientation selfhomeomorphism h of period 2 is zero.

We see here that the $\mu-$invariant of M is zero when the period of h is bigger than two.

First we prove:

The $\mu-$invariant of a 3-dimensional Z_2-homology sphere M with a periodic reversing orientation selfhomeomorphism h of period four is zero.

Proof:

The set of fixed points of h is formed by two separate points and the fixed points set of h^2 is a knot, by Smith theory, when M is a Zhomology sphere. This knot contains the two fixed points of h, and the selfhomeomorpism h leaves invariant the knot, reversing its orientation. The same is true when M is a Z_2-homology sphere; we do the proof:

Let's call $N = M/h^2$, then N is also a Z_2-homology sphere. The self-homeomorphism h projects to a selfhomeomorphism h in N, that we will design with the same letter because there is no place to confusion.

M is the double cover of N, branched over the knot K, (amphicaeiral by h).

When M is a Z_2-homology sphere, $N = M/h^2$ is also a Z_2-homology sphere, and $H_1(N) = Z_{2m_1+1} \oplus Z_{2m_2+1} \oplus \cdots \oplus Z_{2m_r+1}$, for some integer numbers: $m_1, m_2, \cdots, m_r$.

Our knot K does not necessarily bounds a bicollared Seifert surface in N, if it is not nulhomologous, but some odd multiple of K: ((2s+1)K) is nulhomologous and making the connected sum $\sum N$ of 2s+1 copies of N with itself, by the fixed points of h, in such a way that the connected sum is compatible with h, we get also the connected sum of K with itself 2s+1 times, which is nulhomologous in $\sum N$ and we do the proof:

We call $\sum N$ the connected sum of 2s+1 copies of N. The manifold $\sum N$ is also a Z_2-homology sphere, with a reversing orientation selfhomeomorphism, so its μ-invariant is zero or 1/2. We call K_σ the knot connected sum of 2s+1 copies of K; (the knot K_σ is amphicaeiral for h).

The connected sum of M with itself 2s+1 times ($\sum M$), is a double cover of $\sum N$ branched over K_σ.

We construct a bordism B_σ between $\sum M$ and two disjoint copies of $\sum N$, by considering $(\sum N) \times I$, and making the 2-cover B_σ of $(\sum N) \times I$, branched over $F \times [0, 1/2)$, from two copies of

$$\sum N \times I - bicollar(F \times [0, 1/2)) \approx \sum N \times I - F \times [-1, 1] \times [0, 1/2),$$

by identifying in the copies, in a crossed way, the boundaries of

$$bicollar((F \times [0, 1/2))less(F - K_\sigma) \times (-1, 1) \times \{0\} :$$

If x^1 is the point $x \in F$ in the first copy and x^2 is the point $x \in F$ in the second copy, (δ meaning boundary) and being

$$(\delta(F \times [-1, 1] \times [0, 1/2)) - (F - K_\sigma) \times (-1, 1) \times \{0\} =$$
$$F \times \{-1, 1\} \times [0, 1/2) \cup (F) \times (-1, 1) \times \{1/2\}$$

we identify

$(x^1, -1, t) \in (F \times \{-1\} \times [0, 1/2))$ with $(x^2, 1, t) \in (F \times \{1\} \times [0, 1/2))$

and

$(x^1, 1, -s) \in (F \times \{1\} \times [0, 1/2))$ with $(x^2, -1, s) \in (F \times \{-1\} \times [0, 1/2))$

We identify also $(x^1, s, 1/2)$ with $(x^2, -s, 1/2) \quad \forall x \in F \times (-1, 1) \times \{1/2\}$.

The boundary of B_σ is the disjoint union of $\sum M$ and two copies of $\sum N$.

By a Mayer-Vietoris sequence, $H_2(B_\sigma)$ is a direct sum of $H_2(N)$ with itself $2(2s+1)$ times plus a free abelian group of 2g generators, where every generator corresponds to a c_i, generator of $H_1(F)$, for which, some $(2n_i + 1)c_i$ is nulhomologous,

We write now how are the elements of $H_2(B_\sigma)$ determined by nulhomologous closed curves contained in F, with Seifert surface in $\sum N$:

We call [a] the element of $H_2(B_\sigma)$ determined by a, representative closed curve from $H_1(F)$, nulhomologous in $\sum N$, which bounds a Seifert surface $F_a \subset \sum N$;
Given a closed curve $a \subset F \subset \sum N$, we call

$a^+ = a \times \{1\} \subset F \times \{1\} \subset bicollar(F) \subset \sum N$ and $F_{a^+} \subset \sum N$ the Seifert surface of a^+
$a^- = a \times \{-1\} \subset F \times \{-1\} \subset bicollar(F) \subset \sum N$, and $F_{a^-} \subset \sum N$ the Seifert surface of a^-

We denote by $F^1_{a^+} \subset \sum N$, the Seifert surface of a^+ in the first copy of $N \times I$, at any level $\{t\}$ and by $F^2_{a^+} \subset \sum N$, the Seifert surface of a^+ in the second copy, $(F_{a^+} \subset \sum N \subset \sum N \times I)$.
Then,

$$[a] = F^1_{a^+} \times \{1/2\} \cup a^+ \times [0, 1/2) \cup a^- \times [0, 1/2) \cup F^2_{a^-} \times \{1/2\}$$

and also,

$$[a] = F^1_{a^-} \times \{3/4\} \cup a^- \times [0, 3/4) \cup a^+ \times [0, 3/4) \cup F^2_{a^+} \times \{3/4\}.$$

Then, we have for a pair $([a_i] = [(2n_i + 1)c_i, [a_j] = (2n_j + 1)c_j)$, where a_i, a_j are closed curves in F, generators of $H_1(F)$, nulhomologous in $\sum N$:

$$[a_i] \cap [a_j] =$$
$$(F^1_{a_i^+} \times \{1/2\} \cup a_i^+ \times [0,1/2) \cup a_i^- \times [0,1/2) \cup F^2_{a_i^-} \times \{1/2\}) \cap$$
$$(F^1_{a_j^-} \times \{3/4\} \cup a_j^- \times [0,3/4) \cup a_j^+ \times [0,3/4) \cup F^2_{a_j^+} \times \{3/4\}) =$$
$$(lk \text{ meaning linking number})$$
$$= lk(a_i^+, a_j^-) + lk(a_i^-, a_j^+) = lk(a_i^+, a_j) + lk(a_i, a_j^+)$$

The intersection quadratic form matrix in $H_2(B_\sigma)$ is, then, given by a matrix whose entries are:

$$(lk(a_i^+, a_j) + lk(a_j^+, a_i)) =$$
$$= (lk((2n_i+1)c_i^+, (2n_j+1)c_j + lk((2n_j+1)c_j^+), (2n_i+1)c_i)).$$

Now we prove that this matrix has signature zero, because the knot K_σ is amphicaeiral:

In fact, as the knot K verifies $h(K) = -K$, the bordism B_σ can be constructed also by doing the double cover of $N \times I$ branched over $h(F) \times [0,1/2)$. Then, another matrix for the intersection quadratic form Q in B_σ can be calculated from the basis $\{h(c_1), h(c_2), \cdots, h(c_{2g-1}), h(c_{2g})\} \subset h(F)$, (which gives a different basis of $H_2(B_\sigma)$), and, as $(h(a))^+ = h(a^-)$ for every curve in F, because h reverses orientation, we have:

$$lk(h(a_i))^+, h(a_j)) = -lk(h(a_i^-), h(a_j)) = -lk(a_i^-, a_j) = -lk(a_i, a_j^+) =$$
$$= -lk(a_j^+, a_i)$$

$$lk(h(a_j))^+, h(a_i)) = -lk(h(a_j^-), h(a_i)) = -lk(a_j^-, a_i) = -lk(a_j, a_i^+) =$$
$$= -lk(a_i^+, a_j)$$

By adding the previous terms, we get as matrices for Q two opposite matrices which should have the same signature, therefore, zero.

Then, the $\mu-$invariant of $\sum M$ is equal to the signature of the intersection quadratic form in $H_2(B_\sigma)$ plus $2\mu-$invariant$\sum N = 0$, because $\sum N$

is Z_2-homology sphere with a reversing orientation selfhomeomorphim, ($\mu \sum N = 0$ *or* $1/2$), so

$$0 = \mu \sum M = (2s+1)\mu M \Longrightarrow \mu M = 0$$

because M is Z_2-homology sphere with a reversing orientation selfhomeomorphim, ($\mu M = 0$ *or* $1/2$) therefore, the μ-invariant is defined module 1.

In an analogous way, we can prove that:
The $\mu-$invariant of a 3-dimensional Z_2-homology sphere M with a periodic reversing orientation selfhomeomorphism h of period 2^3 is zero.

Proof:
The set of fixed points of h is formed by two separate points and the fixed points set of h^4 is a knot by Smith theory, when M is a Z_2-homology sphere. This knot contains the two fixed points of h, and the selfhomeomorpism h leaves invariant the knot, reversing its orientation.

Let's call $N = M/h^4$. The selfhomeomorphism h projects to a selfhomeomorphism h in N, that we will design with the same letter because there is no place to confusion.
When M is a Z_2-homology sphere, $N = M/h^4$ also is a Z_2-homology sphere. M is a double cover of N, branched over the knot K.
Repeating the previous procedure for M y N, we get that the μ-invariant of M is zero.

With the same procedure we get that:

The $\mu-$invariant of an 3-dimensional Z_2-homology sphere M with a periodic reversing orientation selfhomeomorphism h of period 2^r, $r > 1$ is zero.
For that, we consider $N = M/h^{2^{r-1}}$ and repeat the previous procedure.

We have got, together with the first result from Birman, Galewski and Stern, Hsiang and Pao, that **The $\mu-$invariant of a 3-dimensional Z-homology sphere M with a periodic reversing orientation selfhomeomorphism h whose period is any power of 2, is zero.**

Then, we can settle that:

The $\mu-$invariant of a 3-dimensional Z_2-homology sphere M with a periodic reversing orientation selfhomeomorphism h is zero.

This result follows now from the consideration that any number n bigger than 2 can be written $n = m2^r$ where m is an odd number and $r > 1$. Then M has h^m, a reversing orientation selfhomeomorphism with period 2^r.

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