Vector-valued Hausdorff–Young inequality and applications

J. Garcia-Cuerva, K. S. Kazarian, V. I. Kolyada, and J. L. Torrea

Dedicated to Professor P. L. Ul'yanov on his seventieth birthday

Contents

§1. Introduction

The questions concerning trigonometric series expansions of functions, which go back to the studies of d'Alembert, Euler, and Bernoulli, have been the most important ones in the theory of Fourier series to date. In this field, there is a class of problems that can be stated briefly as follows: estimate the Fourier coefficients of a function via the function itself, and conversely, estimate a function via its Fourier coefficients. Numerous mathematicians have contributed to the solution of these problems. For the space L^2, the main results are Parseval's equality and the Riesz–Fischer theorem. These results are partly extended to spaces L^p for $p \neq 2$ by the Hausdorff–Young and Hardy–Littlewood–Paley theorems. In the case of the L^1-metric, we cannot say anything interesting about the coefficients (except for the fact that they tend to zero). However, for analytic functions from spaces H^1, stronger assertions expressed in the form of the Hardy and Paley inequalities are valid. Furthermore, there are a number of theorems on the behaviour of the Fourier

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coefficients under additional smoothness conditions imposed on a function. The fundamental result in this field is the Bernstein theorem on the absolute convergence of the Fourier series of functions of the class $\operatorname{Lip}\alpha$ ($\alpha > 1/2$).

In the last decades similar questions for classes of functions with values in Banach spaces have been attracting increasing attention. The analysis of vector-valued functions is a comparatively new trend; it dates back to the thirties and has its origins in the papers by Bochner [11], Boas and Bochner [10], and Paley and Zygmund [61]. Vector-valued analysis is related to the classical problems of the theory of functions and often makes it possible to combine different and seemingly unrelated results of the theory on the basis of a common approach. This possibility is associated with the fact that many fundamental operators of real analysis can be interpreted as convolution operators with vector-valued kernels. We find it useful to cite some examples of this kind (although, at first sight, they are far from the main subject of this survey).

On the real line we consider the maximum Hardy–Littlewood operator

$$Mf(x) = \sup_I \frac{1}{|I|} \int_I |f(t)|\,dt, \tag{1.1}$$

where the upper bound is taken over all intervals $I \subset \mathbb{R}$ that contain the point x. It is well known ([74], Ch. 1) that for $1 < p < \infty$ this operator is bounded in $L^p(\mathbb{R})$.

We arrange all positive rational numbers in a sequence $\{\rho_n\}$ and put $I_n = (-\rho_n, \rho_n)$ and $\varphi_n = \dfrac{1}{|I_n|}\chi_{I_n}$ (where χ_E is the characteristic function of a set E). Then for $f \geqslant 0$ we have

$$Mf(x) \leqslant 2 \sup_n \frac{1}{|I_n|} \int_{I_n} f(t)\,dt = 2 \sup_n \int_{\mathbb{R}} \varphi_n(x-t) f(t)\,dt. \tag{1.2}$$

Let $\varphi = \{\varphi_n\}$. For each $x \in \mathbb{R}$ we consider the linear mapping $K(x)$ of the space $X = \mathbb{R}$ into the space $Y = l^\infty$ such that $K(x)\xi = \xi \cdot \varphi(x)$ ($\xi \in \mathbb{R}$). Let

$$Tf(x) = \left\{ \int_{\mathbb{R}} \varphi_n(x-t) f(t)\,dt \right\} = \int_{\mathbb{R}} \varphi(x-t) f(t)\,dt \equiv \int_{\mathbb{R}} K(x-t) f(t)\,dt \tag{1.3}$$

(the last two integrals are interpreted in the sense of Bochner (see § 2)). The operator T takes each function f to a sequence of functions. Here, by virtue of (1.2), we have

$$Mf(x) \leqslant 2\|Tf(x)\|_{l^\infty}$$

for $f \geqslant 0$. Thus, the question about the boundedness in L^p ($1 < p < \infty$) of the semiadditive Hardy–Littlewood operator (1.1) is reduced to the same question for the linear (convolution) operator (1.3) acting from $L^p(\mathbb{R})$ into the space $L^p_Y(\mathbb{R})$ ($Y = l^\infty$).

Convolution operators are the main subject of the theory of Calderón and Zygmund. For the scalar case, the theory was developed in the fifties [17], [18]. The fundamental idea was to derive the boundedness in L^p ($1 < p < \infty$) of the operator

$$Tf(x) = \lim_{\varepsilon \to +0} \int_{|x-t| \geqslant \varepsilon} K(x-t) f(t)\,dt \tag{1.4}$$

from the assumption of its boundedness in L^2 and some smoothness conditions imposed on the kernel K. Later it was discovered that boundedness in L^2 can be replaced by boundedness in L^r for some $r \in [1, \infty]$.

These ideas were extended to the vector-valued case by Benedek, Calderón, and Panzone [6]. Namely, they constructed a similar theory for operators of the form (1.4), where f is a function on $\mathbb{R}^n$ with values in a Banach space X and the kernel K is a function with values in the space $L(X, Y)$ of all linear bounded operators from X into Y (here Y is a Banach space).

Returning to the operator (1.3), we note that its boundedness from $L^\infty(\mathbb{R})$ into $L^\infty_Y(\mathbb{R})$ ($Y = l^\infty$) is obvious. To apply the theory of Calderón and Zygmund, it remains to choose a majorant of K satisfying the corresponding smoothness conditions[1].

As another example, we cite theorems of Littlewood–Paley type ([74], Ch. 4), associated with quite difficult results of the theory of functions. Just as in the case of the maximum operator, it turns out that estimates for the Littlewood–Paley quadratic function in L^p $(1 < p < \infty)$ can be derived from the boundedness in $L^p_{l^2}$ of a special vector-valued operator with a Calderón–Zygmund kernel (see [74], [41], [78]).

Of course, the role of vector-valued analysis in the theory of functions is not exhausted by the cited examples. It is also clear that further investigations in this field are an important and interesting problem and may lead to new unexpected results.

Let us return to the subject of the present paper. Our starting point is the classical Hausdorff–Young theorem, which we now consider in more detail.

Let $f(x) \in L[0, 1]$ be a complex-valued 1-periodic function with Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x}.$$

If $f \in L^2[0, 1]$, then

$$\int_0^1 |f(x)|^2 \, dx = \sum_{n=-\infty}^{+\infty} |c_n|^2 \tag{1.5}$$

(Parseval's equality). On the other hand, if a sequence $\{c_n\}_{n=-\infty}^{+\infty}$ of complex numbers satisfies $\sum |c_n|^2 < \infty$, then there is a function $f(x) \in L^2[0, 1]$ such that the c_n are its Fourier coefficients and (1.5) is valid (the Riesz–Fischer theorem).

These theorem can be partly extended to the spaces L^p for $p \neq 2$. For $1 \leqslant p \leqslant \infty$, we put $p' = p/(p - 1)$.

Theorem 1.1. *Let* $1 \leqslant p \leqslant 2$. *Then the following assertions hold:*

(1) *if* $f \in L^p[0, 1]$ *and*

$$c_n = \int_0^1 f(x) e^{-2\pi i n x} \, dx \quad (n \in \mathbb{Z}), \tag{1.6}$$

[1] For details see the paper [71].

then

$$\left(\sum_{n=-\infty}^{+\infty} |c_n|^{p'}\right)^{1/p'} \leqslant \left(\int_0^1 |f(x)|^p \, dx\right)^{1/p}; \tag{1.7}$$

(2) *if* $\{c_n\}_{n=-\infty}^{+\infty}$ *is a sequence of complex numbers such that*

$$\sum_{n=-\infty}^{+\infty} |c_n|^p < \infty,$$

then there is a function $f \in L^{p'}[0,1]$ *such that* (1.6) *holds and*

$$\left(\int_0^1 |f(x)|^{p'} \, dx\right)^{1/p'} \leqslant \left(\sum_{n=-\infty}^{+\infty} |c_n|^p\right)^{1/p}. \tag{1.8}$$

Let us point out that this theorem does not hold for $p > 2$ (see [82], Ch. 12).

Theorem 1.1 was proved by Young in 1912–13 for the case in which p' is an even positive number. The starting point of Young's proof was a convolution inequality obtained by him. In 1923 Hausdorff extended Young's results to all values $p \in [1,2]$. In the same year F. Riesz proved that an analogue of Theorem 1.1 holds for any uniformly bounded orthonormal system on $[a,b]$,

$$|\varphi_n(x)| \leqslant M \quad (x \in [a,b],\ n \in \mathbb{N}).$$

Theorem 1.2. *Let* $1 \leqslant p \leqslant 2$. *Then the following assertions hold:*

(1) *if* $f \in L^p[a,b]$, *then*

$$\|\{c_k\}\|_{l^{p'}} \leqslant M^{2/p-1} \|f\|_{L^p}, \tag{1.9}$$

where the c_k *are the Fourier coefficients of* f;

(2) *if* $\{c_k\} \in l^p$, *then there is a function* $f \in L^{p'}[a,b]$ *such that the* c_k *are its Fourier coefficients and*

$$\|f\|_{L^{p'}} \leqslant M^{2/p-1} \|\{c_k\}\|_{l^p}. \tag{1.10}$$

In 1926 M. Riesz proposed another proof of this theorem as one of the first applications of his convexity theorem.

An analogue of Theorem 1.1 for the Fourier transform was proved by Titchmarsh [76], [77].

The Fourier transform of a function $f \in L^1(\mathbb{R}^n)$ is defined by the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} \, dx \quad (\xi \in \mathbb{R}^n). \tag{1.11}$$

According to the Plancherel theorem, if $f \in L^2(\mathbb{R}^n)$, then the sequence

$$F_N(\xi) = \int_{|x| \leqslant N} f(x) e^{-2\pi i x \cdot \xi} \, dx \tag{1.12}$$

converges in $L^2(\mathbb{R}^n)$. The limit is denoted by $\widehat{f}(\xi)$ and is called the Fourier transform of f. Furthermore, $\|f\|_2 = \|\widehat{f}\|_2$.

Theorem 1.3. *Let $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant 2$. Then the sequence* (1.12) *converges in the norm of $L^{p'}(\mathbb{R}^n)$, and the limit function satisfies the inequality*

$$\|\widehat{f}\|_{p'} \leqslant \|f\|_p. \tag{1.13}$$

The function $\widehat{f}$ defined in Theorem 1.3 is called the Fourier transform of the function $f \in L^p$, $1 \leqslant p \leqslant 2$.

Let us compare inequalities (1.7) and (1.13). If $f(x) = Ae^{2\pi imx}$ ($m \in \mathbb{Z}$), then (1.7) becomes an equality (Hardy and Littlewood showed that the equality in (1.7) holds only for functions of the above-mentioned type). At the same time, it turns out that for $1 < p < 2$ inequality (1.13) can be improved. Namely, for $1 < p < 2$ one has

$$\|\widehat{f}\|_{p'} \leqslant A_p^n \|f\|_p, \qquad A_p = \bigl(p^{1/p}(p')^{-1/p'}\bigr)^{1/2}. \tag{1.14}$$

This inequality was proved by Babenko [3] for the case in which p' is an even positive number ($n = 1$) and by Beckner [5] for the general case.

Let us return to Theorem 1.1. In the theorem it is supposed that $1 \leqslant p \leqslant 2$. In particular, the theorem states that any sequence from l^p is the sequence of Fourier coefficients of a function from $L^{p'}$. For given $p \in [1, 2]$, the set of all such functions is a subset of $L^{p'}$. Moreover, the closer p is to 2, the larger is the part of the space occupied by the subset. For $p = 2$, this subset coincides with L^2. The situation is different for $p > 2$. It turns out that in this case not every sequence of the class l^p is a sequence of Fourier coefficients. Namely, in the thirties Paley and Zygmund proved that if $\sum_{n=-\infty}^{+\infty} |a_n|^2 = \infty$, then for some arrangement of signs the series

$$\sum_{n=-\infty}^{+\infty} \pm a_n e^{2\pi inx}$$

is not the Fourier series of an integrable function. This assertion is a special case of the more general Paley–Zygmund theorem related to the so-called random trigonometric series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) r_n(t)$$

(here the r_n are the Rademacher functions). The results of Paley and Zygmund stimulated research on random function series $\sum f_n r_n(t)$ (the f_n are functions from some space) and the more general series $\sum x_n r_n(t)$, where $\{x_n\}$ is a sequence of vectors of an arbitrary Banach space X (see [43]).

In the late 60s, interest appeared in trigonometric series with vector-valued coefficients and vector-valued Fourier transforms. Strictly speaking, these subjects were touched on by Bochner [11] as early as the 30s (in particular, Bochner noted that the Plancherel theorem need not hold for Fourier transforms of vector-valued functions).

In 1967, Peetre [62] singled out Banach spaces X such that for some $p \in (1, 2]$ and for any function $f \colon \mathbb{R} \to X$ one has

$$\biggl(\int_{\mathbb{R}} \|\widehat{f}(\xi)\|_X^{p'}\, d\xi\biggr)^{1/p'} \leqslant A_p \biggl(\int_{\mathbb{R}} \|f(t)\|_X^{p}\, dt\biggr)^{1/p}, \qquad p' = \frac{p}{p-1}\,. \tag{1.15}$$

In this case, one says that X has the Fourier type p. In the paper [64], this concept was used in studying interpolation spaces.

If X is a Hilbert space, then X obviously has the Fourier type 2 (more precisely, the Plancherel equality $\|\widehat{f}\|_{L^2_X} = \|f\|_{L^2_X}$ is true). In 1972 Kwapién [50] proved that any space with the Fourier type 2 is isomorphic to a Hilbert space. He derived this result from a similar assertion for spaces whose Rademacher type and cotype are simultaneously equal to 2.

The definitions of Rademacher type and cotype for Banach spaces were introduced in 1972 by Hoffman-Jorgensen [38] (in connection with probability problems on the unconditional convergence of series in Banach spaces) and by Maurey (in papers on the theory of factorization of linear operators). These concepts were immediately applied in numerous papers devoted to the geometry of Banach spaces, probability, and operator theory (see [39], [51], [68]). Considerably fewer papers dealt with the study of the Fourier type of Banach spaces and the corresponding problems of the theory of functions. Essential progress in this field is due to Bourgain [12], [13], who proved the validity of the Hausdorff–Young inequality with respect to the trigonometric system and the Walsh system for Banach spaces with a non-trivial Rademacher type. Weaker inequalities of the same type were obtained by Bourgain [12] for the systems of characters of arbitrary Abelian groups.

The publication of the present survey is due to the fact that although so far a number of general results have already been obtained in the field, these results are scattered in the literature, which does not permit one to see the overall picture at once. At the same time, investigations of vector-valued harmonic analysis and related problems involving inequalities of Hausdorff–Young type have been considerably extended recently. Though it is too early to claim that a unified theory has been constructed, it is necessary to "frame" what has already been done, highlight the most important results and unsolved problems and, on this basis, establish a common approach to further investigations. These are the main purposes of our work. Our approach consists in studying the fundamental concepts of type and cotype of Banach spaces for general uniformly bounded orthonormal systems rather than for the Fourier transform on locally compact Abelian groups. First of all, this enables us to single out general results and principles not associated with specific features of a given system. On the other hand, the study of the whole class of orthonormal systems leads to interesting and non-trivial problems related both to the theory of orthogonal series and the geometry of Banach spaces and harmonic analysis. We hope that this approach will also provide new relations between these theories.

As was mentioned above, a great number of studies are devoted to the concepts of Rademacher type and cotype. From these extensive studies we highlight only the results that are closely related to problems of the theory of functions.

We have tried to make the presentation consistent, systematic, and simple. That is why we sketch the proofs of the basic results. Moreover, for a number of cases, we give new proofs. Some results are apparently presented for the first time.

Finally, we pose unsolved problems—both well-known and those which arose in writing this paper.

§2. General definitions and auxiliary results

Let R be a non-empty set and Λ a σ-algebra of subsets of R. Suppose that a σ-finite measure μ is defined on Λ. We say that (R,Λ,μ) (or, briefly, (R,μ)) is a measure space (see [40], p. 15).

An atom is a μ-measurable set $A\subset R$ with $\mu(A)>0$ such that it does not contain measurable subsets with measure different from zero and $\mu(A)$.

Lemma 2.1. *Let (R,μ) be a measure space. Suppose that a μ-measurable set $E\subset R$ with $\mu(E)>0$ does not contain atoms. Then for any $0<t<\mu(E)$ there is a μ-measurable set $Q\subset E$ with $\mu(Q)=t$.*

The proof can be found, for example, in [81], p. 372.

A real- or complex-valued function f defined on R is said to be μ-measurable if for any open subset G of the real line $\mathbb{R}$ (or the complex plane $\mathbb{C}$) the inverse image $f^{-1}(G)$ is μ-measurable.

We denote by χ_E the characteristic function of a μ-measurable set $E\subset R$.

The distribution function of a μ-measurable function f is defined by the formula

$$\lambda_f(y)\equiv\mu\bigl(\{x\in R:|f(x)|>y\}\bigr)\quad (y\geqslant 0).$$

We denote by $S_0(R,\mu)$ the class of all μ-measurable functions f on R such that they are finite μ-almost everywhere on R and satisfy the condition $\lambda_f(y)<+\infty$ for any $y>0$.

The non-increasing rearrangement of a function $f\in S_0(R,\mu)$ is the function f^* non-increasing on $(0,+\infty)$ and equimeasurable with $|f|$, that is, such that for any $y\geqslant 0$

$$\bigl|\{t>0:f^*(t)>y\}\bigr|=\mu\bigl(\{x\in R:|f(x)|>y\}\bigr).$$

The rearrangement f^* can be defined by the formula (see [7], Ch. 2)

$$f^*(t)=\inf\{y>0:\lambda_f(y)\leqslant t\},\qquad 0<t<\infty. \tag{2.1}$$

We note the following properties of rearrangements (see [49], [7]):

(1) if $0<t<t+s<\infty$, then

$$(f+g)^*(t+s)\leqslant f^*(t)+f^*(s) \tag{2.2}$$

and

$$(fg)^*(t+s)\leqslant f^*(t)f^*(s); \tag{2.3}$$

(2) if $0<p<\infty$, then

$$\int_R|f|^p\,d\mu=\int_0^\infty[f^*(s)]^p\,ds; \tag{2.4}$$

(3) if E is a measurable subset of R with $\mu(E)=t$, then

$$\int_E|f|\,d\mu\leqslant\int_0^t f^*(s)\,ds. \tag{2.5}$$

Let p and r be positive numbers. A function $f \in S_0(R,\mu)$ belongs to a Lorentz space $L^{p,r} \equiv L^{p,r}(R,\mu)$ if

$$\|f\|_{p,r} \equiv \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^r \frac{dt}{t}\right)^{1/r} < \infty.$$

We have $L^{p,p} = L^p, 0 < p < \infty$ (see (2.4)). Next, if $0 < p < \infty$ and $0 < r < q < \infty$, then $L^{p,r} \subset L^{p,q}$; moreover,

$$\|f\|_{p,q} \leqslant c\,\|f\|_{p,r} \tag{2.6}$$

(see [7], p. 217).

We consider the discrete case in which $R = \mathbb{N}$ and $\mu(n) = 1$ for any $n \in \mathbb{N}$. In this case, S_0 is the set of all sequences $\{a_k\}$ of numbers such that $\lim_{k\to\infty} a_k = 0$. A non-increasing rearrangement of a sequence $\{a_k\}$ is denoted by $\{a_k^*\}$. If a sequence $\{|a_k|\}$ has infinitely many elements different from zero, then the rearrangement $\{a_k^*\}$ is obtained by arranging all such elements in non-increasing order. The Lorentz sequence space is denoted by $l^{p,r}$. If $a = \{a_k\}$ and $a_k \to 0$, then

$$\|a\|_{p,r} = \left(\sum_{k=1}^\infty k^{r/p-1}(a_k^*)^r\right)^{1/r}.$$

In what follows we use the following Hardy inequalities.

Lemma 2.2. *Let f be a measurable non-negative function on $(0,\infty)$ and $\alpha > 0$. Then*

(1) *if $1 \leqslant p < \infty$, then the following assertions hold:*

$$\int_0^\infty x^{-\alpha-1}\left(\int_0^x f(t)\,dt\right)^p dx \leqslant C_{p,\alpha} \int_0^\infty x^{-\alpha-1}(xf(x))^p\,dx, \tag{2.7}$$

$$\int_0^\infty x^{\alpha-1}\left(\int_x^\infty f(t)\,dt\right)^p dx \leqslant C_{p,\alpha} \int_0^\infty x^{\alpha-1}(xf(x))^p\,dx; \tag{2.8}$$

(2) *if the function f is monotone decreasing on $(0,\infty)$, then inequalities* (2.7) *and* (2.8) *are valid for any $p > 0$.*

The proof of assertion (1) can be found in [35], Russian p. 296. To prove inequality (2.7) for $0 < p < 1$ for a decreasing function f, we note that in this case the integral on the right-hand side of (2.7) can converge only for $\alpha < p$ (provided that f is not identically zero). Moreover, one can assume that f is bounded and compactly supported. We put $F(x) = \displaystyle\int_0^x f(t)\,dt$. If we integrate by parts and take into account the fact that f is decreasing, then we obtain

$$\begin{aligned}\int_0^\infty x^{-\alpha-1}(F(x))^p\,dx &= \frac{1}{\alpha}\int_0^\infty x^{-\alpha} f(x)\,(F(x))^{p-1}\,dx \\ &\leqslant \frac{1}{\alpha}\int_0^\infty x^{-\alpha+p-1}(f(x))^p\,dx.\end{aligned}$$

We can prove (2.8) for $0 < p < 1$ in a similar manner (assuming that f is decreasing).

Lemma 2.3. *Let $\{a_n\}$ be a non-negative sequence and $\alpha > 0$. Then the following assertions hold:*

(1) *if $1 \leqslant p < \infty$, then*

$$\sum_{n=1}^{\infty} n^{-\alpha-1}\Bigl(\sum_{k=1}^{n} a_k\Bigr)^p \leqslant C_{p,\alpha} \sum_{n=1}^{\infty} n^{-\alpha-1}(na_n)^p, \tag{2.9}$$

$$\sum_{n=1}^{\infty} n^{\alpha-1}\Bigl(\sum_{k=n}^{\infty} a_k\Bigr)^p \leqslant C_{p,\alpha} \sum_{n=1}^{\infty} n^{\alpha-1}(na_n)^p; \tag{2.10}$$

(2) *if the sequence $\{a_n\}$ is monotone decreasing, then inequalities* (2.9) *and* (2.10) *hold for any $p > 0$.*

This lemma can readily be derived from the previous one (or can be proved in a similar way).

Let us give some definitions and results related to vector-valued functions.

Let A be a Banach space and (R, μ) a measure space. We consider functions $f\colon R \to A$. A function of the form

$$f = \sum_{j=1}^{N} a_j \chi_{E_j}, \tag{2.11}$$

where $a_j \in A$ and the $E_j \subset R$ are μ-measurable disjoint sets with $\mu(E_j) < \infty$, is called a *μ-simple function*. A function $f\colon R \to A$ is said to be strongly μ-measurable if there is a sequence $\{f_n\}$ of μ-simple functions such that

$$\lim_{n\to\infty} \|f(x) - f_n(x)\| = 0 \text{ for } \mu\text{-almost all } x \in R. \tag{2.12}$$

It is clear that if f is strongly μ-measurable, then the real function $x \mapsto \|f(x)\|$ is μ-measurable on R.

The Bochner integral of the μ-simple function (2.11) is defined by the formula

$$\int_R f(x)\, d\mu(x) = \sum_{j=1}^{N} a_j \mu(E_j).$$

A strongly μ-measurable function $f\colon R \to A$ is said to be μ-integrable in the sense of Bochner on R if there is a sequence $\{f_n\}$ of μ-simple functions satisfying (2.12) such that

$$\lim_{n\to\infty} \int_R \|f(x) - f_n(x)\|\, d\mu(x) = 0.$$

In this case, the sequence $\{\int_R f_n\, d\mu\}$ of integrals converges in the norm of A. The limit of the sequence is called the *Bochner integral* of f over R and is denoted by $\displaystyle\int_R f(x)\, d\mu(x)$ (see [40], Ch. 5).

By the Bochner theorem [11] (see also [40], p. 132), a strongly μ-measurable function f is μ-integrable in the sense of Bochner if and only if the norm $\|f(x)\|$ is μ-integrable.

We denote by $L^p_A \equiv L^p_A(R,\mu)$ $(1 \leqslant p \leqslant \infty)$ the space of all strongly μ-measurable functions $f\colon R \to A$ with finite norm

$$\|f\|_{L^p_A} = \left(\int_R \|f(x)\|^p \, d\mu(x)\right)^{1/p} \quad (1 \leqslant p < \infty),$$
$$\|f\|_{L^\infty_A} = \sup_{x\in R} \operatorname{vrai} \|f(x)\| \quad (p = \infty).$$

In the discrete case $R = \mathbb{N}$, the corresponding space is denoted by l^p_A. Thus, l^p_A is the space of all sequences $\{a_n\}$ $(a_n \in A)$ with finite norm

$$\|\{a_n\}\|_{l^p_A} = \left(\sum_{n=1}^{\infty} \|a_n\|^p\right)^{1/p} \quad (1 \leqslant p < \infty),$$
$$\|\{a_n\}\|_{l^p_A} = \sup_{n\in\mathbb{N}} \|a_n\| \quad (p = \infty).$$

The following lemma can be proved in the standard way (see [22], p. 125).

Lemma 2.4. *Let A be a Banach space, and let $1 \leqslant p < \infty$. Then the set of all μ-simple functions is everywhere dense in $L^p_A(R,\mu)$.*

If A is a Banach space, then we denote by A^* the dual space of all continuous linear functionals on A. If the space A is complex, then multiplication by complex numbers in A^* is defined by the equality

$$(\alpha g)(a) = \overline{\alpha} g(a) \quad (g \in A^*, \ a \in A) \tag{2.13}$$

(see [44], pp. 75, 181, 199). Instead of $g(a)$, where $g \in A^*$ and $a \in A$, we also write $\langle a, g\rangle$.

Proposition 2.5. *Let A be a Banach space, and let $1 < p < \infty$. Then for any function $f \in L^p_A$ we have*

$$\|f\|_{L^p_A} = \sup\left\{\left|\int_R \langle f(x), g(x)\rangle \, d\mu\right| : \|g\|_{L^{p'}_{A^*}} = 1\right\}, \tag{2.14}$$

where the upper bound is taken over all functions $g\colon R \to A^*$ such that $\|g\|_{L^{p'}_{A^*}} = 1$.

This proposition is proved in [23], p. 97. It means that the space L^p_A is isometrically embedded in $(L^{p'}_{A^*})^*$. Note that, generally speaking, these spaces do not coincide (see [23]).

Similarly, a dual assertion holds, by which $L^{p'}_{A^*}$ is isometrically embedded in $(L^p_A)^*$.

Proposition 2.6. *Let A be a Banach space, and let $1 < p < \infty$. Then for any function $g \in L^{p'}_{A^*}$ one has*

$$\|g\|_{L^{p'}_{A^*}} = \sup\left\{\left|\int_R \langle f(x), g(x)\rangle \, d\mu\right| : \|f\|_{L^p_A} = 1\right\}. \tag{2.15}$$

We also consider Lorentz spaces of vector-valued functions. Let $f\colon R \to A$ be a strongly μ-measurable function. The non-increasing rearrangement of the real function $\|f(x)\|$ $(x \in R)$ is denoted by $f^*(t)$ $(0 < t < +\infty)$. If $0 < p, r < \infty$, then we denote by $L_A^{p,r} \equiv L_A^{p,r}(R,\mu)$ the space of all strongly μ-measurable functions $f\colon R \to A$ such that

$$\|f\|_{L_A^{p,r}} \equiv \left(\int_0^\infty [t^{1/p} f^*(t)]^r \frac{dt}{t}\right)^{1/r} < \infty. \tag{2.16}$$

In the discrete case $R = \mathbb{N}$, we consider sequences $\{a_k\}$ $(a_k \in A)$ such that $\lim_{k\to\infty} a_k = 0$. The non-increasing rearrangement of a sequence $\{\|a_k\|\}$ is denoted by $\{a_k^*\}$. The Lorentz space is denoted by $l_A^{p,r}$; this is the space of all sequences $\{a_k\}$ $(a_k \in A)$ such that

$$\|\{a_k\}\|_{l_A^{p,r}} \equiv \left(\sum_{k=1}^\infty k^{r/p-1} (a_k^*)^r\right)^{1/r} < \infty. \tag{2.17}$$

In conclusion we turn our attention to the concept of a vector-valued analytic function.

Let A be a Banach space and D a domain in the complex plane $\mathbb{C}$. A function $f\colon D \to A$ is said to be analytic at a point $z_0 \in D$ if in A there is a limit

$$\lim_{z\to z_0, z\in D} \frac{f(z) - f(z_0)}{z - z_0}\,.$$

If a function f is analytic at each point $z \in D$, then it is called analytic in the domain D. Starting from this definition, one can develop a theory of vector-valued analytic functions almost identical with the familiar one.

In particular, the following analogue of the Hadamard theorem on three lines is true ([22], p. 538). We denote by Ω the strip $\{z = x + iy : 0 < x < 1\}$ in the complex plane and by $\overline{\Omega}$ its closure.

Theorem 2.7. *Let A be a Banach space and $f\colon \overline{\Omega} \to A$ a function bounded and continuous in $\overline{\Omega}$ and analytic in Ω. Then the function*

$$M_\theta = \sup_{-\infty<y<+\infty} |f(\theta + iy)|$$

satisfies the inequality

$$M_\theta \leqslant M_0^{1-\theta} M_1^\theta \quad (0 \leqslant \theta \leqslant 1).$$

§3. Interpolation spaces

In this section we briefly describe the interpolation theory methods that play an important role in the questions we consider.

3.1. The complex interpolation method. The basis of the method is the classical Riesz–Thorin theorem (see [82], Ch. 12).

Let R and S be spaces with measures μ and ν, respectively. Suppose that T is a linear operator whose domain is the space of all μ-simple complex-valued functions on R and whose values are ν-measurable complex-valued functions on S. Let $1 \leqslant p, q \leqslant \infty$. If there is a constant M such that

$$\|Tf\|_{L^q(S,\nu)} \leqslant M\|f\|_{L^p(R,\mu)} \tag{3.1}$$

for all μ-simple functions f on R, then T is said to have the strong type (p, q). The least constant M in (3.1) is called the strong (p, q)-norm (or merely the norm) of the operator T.

Theorem 3.1. *Suppose that* $1 \leqslant p_j, q_j \leqslant \infty$ $(j = 0, 1)$, $0 < \theta < 1$, *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}. \tag{3.2}$$

Let T be a linear operator of the strong types (p_0, q_0) and (p_1, q_1) with the corresponding strong norms M_0 and M_1. Then T has the strong type (p, q), and its strong (p, q)-norm M_θ satisfies the inequality

$$M_\theta \leqslant M_0^{1-\theta} M_1^\theta. \tag{3.3}$$

For the real case this theorem was proved in 1926 by M. Riesz (in this case it is necessary to assume that $p_j \leqslant q_j$, otherwise, an additional factor 2 appears on the right-hand side of (3.3)). In 1938, Thorin extended Theorem 3.1 to complex spaces L^p. The starting point of Thorin's proof is the replacement of the function f by a family of functions depending on the parameter z,

$$F(z) = |f|^{p/p(z)} e^{i \arg f} \quad \big(F(\theta) = f\big), \tag{3.4}$$

where $F(z)$ is a complex-valued function on R for each fixed z, the complex parameter z varies in the strip $0 \leqslant \operatorname{Re} z \leqslant 1$, and $p(z)$ is defined by the first equation in (3.2) with θ replaced by z. The theorem on three lines also plays a very important role in the proof.

These ideas served as the starting point of the complex interpolation method, which was independently introduced in works by Calderón, Lions, and Krein (see [79], Ch. 1).

Let A_0 and A_1 be complex Banach spaces linearly and continuously embedded in a Hausdorff topological vector space $\mathcal{A}$. We call two such spaces an *interpolation pair* $\{A_0, A_1\}$. The sum $A_0 + A_1$ is defined as the set of all elements $a \in \mathcal{A}$ representable in the form $a = a_0 + a_1$, where $a_0 \in A_0, a_1 \in A_1$. For each $a \in A_0 + A_1$ we put

$$\|a\|_{A_0+A_1} = \inf\big(\|a_0\|_{A_0} + \|a_1\|_{A_1}\big),$$

where the lower bound is taken over all representations $a = a_0 + a_1$ $(a_0 \in A_0$, $a_1 \in A_1)$. Furthermore, for any $a \in A_0 \cap A_1$, by definition,

$$\|a\|_{A_0 \cap A_1} = \max\big(\|a\|_{A_0}, \|a\|_{A_1}\big).$$

Then $A_0 \cap A_1$ and $A_0 + A_1$ are Banach spaces (see [8], § 2.3).

The following basic definition was introduced by Calderón [16].

Let $\{A_0, A_1\}$ be an interpolation pair. We denote by $\mathcal{F} \equiv \mathcal{F}(A_0, A_1)$ the space of all functions F that are defined in the strip $0 \leqslant \operatorname{Re} z \leqslant 1$ of the complex plane, assume values in $A_0 + A_1$ and have the following properties:

(i) F is bounded and continuous with respect to the norm of $A_0 + A_1$ in the strip $0 \leqslant \operatorname{Re} z \leqslant 1$;

(ii) F is $(A_0 + A_1)$-analytic in the open strip $0 < \operatorname{Re} z < 1$;

(iii) the function $y \mapsto F(j + iy)$ $(y \in \mathbb{R},\ j = 0, 1)$ assumes values in A_j, is A_j-continuous on $\mathbb{R}$, and tends to zero as $|y| \to \infty$.

It is obvious that $\mathcal{F}$ is a vector space. We define a norm on $\mathcal{F}$ by putting

$$\|F\|_{\mathcal{F}} = \max\Bigl(\sup_{y\in\mathbb{R}} \|F(iy)\|_{A_0},\ \sup_{y\in\mathbb{R}} \|F(1+iy)\|_{A_1}\Bigr).$$

The space $\mathcal{F}$ is complete with respect to this norm ([8], Ch. 4).

Definition 3.2. Let $0 < \theta < 1$. We say that an element $a \in A_0 + A_1$ belongs to a space $[A_0, A_1]_\theta \equiv A_\theta$ if there is a function $F \in \mathcal{F}$ such that

$$a = F(\theta).$$

The norm in the space $[A_0, A_1]_\theta$ is defined by the formula

$$\|a\|_{[A_0,A_1]_\theta} = \inf\bigl\{\|F\|_{\mathcal{F}} : F(\theta) = a,\ F \in \mathcal{F}\bigr\}.$$

The space A_θ is complete ([8], Ch. 4).

Lemma 3.3. *Suppose that a function $F \in \mathcal{F}$ has the additional property*

$$F(iy) \in L^{r_0}_{A_0}, \quad F(1+iy) \in L^{r_1}_{A_1} \quad (1 \leqslant r_0, r_1 < \infty).$$

Let $a = F(\theta)$. Then

$$\|a\|_{A_\theta} \leqslant C_\theta\bigl(\|F(iy)\|_{L^{r_0}_{A_0}} + \|F(1+iy)\|_{L^{r_1}_{A_1}}\bigr), \tag{3.5}$$

where C_θ is independent of F.

The lemma is proved in [16].

The most important role is played by the following interpolation property (see [16], [8], 4.1.2).

Theorem 3.4. *Let $\{A_0, A_1\}$ and $\{B_0, B_1\}$ be interpolation pairs of Banach spaces, and let $T\colon A_0 + A_1 \to B_0 + B_1$ be a linear operator. Suppose that the restriction of T to A_i is a bounded operator from A_i to B_i with norm M_i $(i = 0, 1)$. Let $0 < \theta < 1$. Then T is a bounded operator from A_θ to B_θ with norm M_θ satisfying the inequality*

$$M_\theta \leqslant M_0^{1-\theta} M_1^{\theta}.$$

We cite another result of Calderón on the interpolation of spaces $L^p_A(R, \mu)$, where (R, μ) is a given measure space (see [79], 1.18.4, [8], 5.1).

Theorem 3.5. *Suppose that* $\{A_0, A_1\}$ *is an interpolation pair of Banach spaces,* $1 \leqslant p_0, p_1 < \infty$, $0 < \theta < 1$, *and* $1/p = (1-\theta)/p_0 + \theta/p_1$. *Then*

$$\left[L^{p_0}_{A_0}, L^{p_1}_{A_1}\right]_\theta = L^p_{[A_0,A_1]_\theta}, \tag{3.6}$$

and the norms in these spaces are equal. Equality (3.6) *remains valid for* $1 \leqslant p_0 < p_1 = \infty$ *if* $L^\infty_{A_1}$ *is replaced by the closure of the set of simple functions with respect to the norm of the space.*

3.2. The real interpolation method. The idea of the method goes back to the Marcinkiewicz interpolation theorem.

As above, let (R,μ) and (S,ν) be measure spaces. Suppose that T is a mapping of the space $L^p(R,\mu)$ $(1 \leqslant p < \infty)$ into the set of all ν-measurable functions on S (we can consider either real or complex-valued functions). Let $1 \leqslant q < \infty$. We say that T has a weak type (p,q) if there is a constant $C > 0$ such that

$$\left(Tf\right)^*(t) \leqslant C\|f\|_{L^p(R,\mu)} t^{-1/q} \quad (t > 0) \tag{3.7}$$

for any function $f \in L^p(R,\mu)$. We further assume that T possesses the subadditivity property

$$|T(f+g)(x)| \leqslant |Tf(x)| + |Tg(x)|.$$

Theorem 3.6. *Let* $1 < r < \infty$. *Suppose that* T *is a subadditive mapping from* $L^1(\mathbb{R}^n) + L^r(\mathbb{R}^n)$ *into the space of measurable functions on* $\mathbb{R}^n$. *If* T *has the weak type* $(1,1)$ *and a weak type* (r,r), *then for any* $1 < p < r$ *the mapping* T *has the strong type* (p,p),

$$\|Tf\|_p \leqslant C_p \|f\|_p, \qquad f \in L^p(\mathbb{R}^n). \tag{3.8}$$

The theorem remains valid for $r = \infty$ *if we require that* T *act boundedly in* $L^\infty(\mathbb{R}^n)$, $\|Tf\|_\infty \leqslant C\|f\|_\infty$.

The Marcinkiewicz theorem was stated by him without proof in 1939; it was proved by Zygmund in 1956 for the first time (see [8], Ch. 1). The basic idea of the proof consists in the following: to estimate the distribution function $\lambda_{Tf}(y)$ for any $y > 0$, one splits the function f into two summands $f = g + h$ so that $g \in L^1$ and $h \in L^r$; here the splitting is made by "cutting" $|f|$ at the level of y. That idea of "variable splitting" has turned out to be very fruitful and has found numerous applications in analysis. In particular, it has led to the concept of the real interpolation method for Banach spaces. There are various modifications of the method. They involve different ways of splitting the elements but lead to the same interpolation space defined by two indices p and θ, where $1 \leqslant p \leqslant \infty$ and $0 < \theta < 1$. The K-method introduced by Peetre (see [79], 1.3) is the most widespread.

Let $\{A_0, A_1\}$ be an interpolation pair. The *Peetre K-functional* is defined by the formula $(t > 0)$

$$K(t,a;A_0,A_1) = \inf_{a=a_0+a_1} \left\{\|a_0\|_{A_0} + t\|a_1\|_{A_1}\right\}, \qquad a \in A_0 + A_1,$$

where the lower bound is taken over all possible representations $a = a_0 + a_1$ with $a_0 \in A_0$, $a_1 \in A_1$. One often writes $K(t,a)$ instead of $K(t,a;A_0,A_1)$.

Definition 3.7. Let $0 < \theta < 1$ and $1 \leqslant p \leqslant \infty$. The space $(A_0, A_1)_{\theta,p}$ consists of all elements $a \in A_0 + A_1$ such that

$$\|a\|_{\theta,p} = \left\{ \int_0^\infty \left(t^{-\theta} K(t,a)\right)^p \frac{dt}{t} \right\}^{1/p} < \infty \quad (p < \infty), \tag{3.9}$$

$$\|a\|_{\theta,\infty} = \sup_{t>0} t^{-\theta} K(t,a) < \infty \quad (p = \infty). \tag{3.10}$$

The L-method (also introduced by Peetre) is based on a similar idea. The method is equivalent to the K-method but is more flexible in many cases. Without going into details, we cite the equivalence theorem reflecting the essence of the L-method (see [79], 1.4).

Theorem 3.8. *Let* $1 \leqslant r_0, r_1 < \infty$ *and*

$$L(t,a) = \inf_{\substack{a=a_0+a_1 \\ a_j \in A_j}} \left(\|a_0\|_{A_0}^{r_0} + t\|a_1\|_{A_1}^{r_1}\right) \quad (t > 0,\ a \in A_0 + A_1).$$

Further, let $0 < \eta < 1$, $r = (1-\eta)r_0 + \eta r_1$, $\theta = \eta r_1 / r$, *and* $1 \leqslant p \leqslant \infty$.

Then there is a positive constant K *such that for any element* $a \in A_0 + A_1$ *one has*

$$\frac{1}{K}\|a\|_{\theta,p} \leqslant \left(\int_0^\infty \left(t^{-\eta} L(t,a)\right)^{p/r} \frac{dt}{t} \right)^{1/p} \leqslant K\|a\|_{\theta,p}$$

if $1 \leqslant p < \infty$ *and*

$$\frac{1}{K}\|a\|_{\theta,\infty} \leqslant \sup_{t>0} \left(t^{-\eta} L(t,a)\right)^{1/r} \leqslant K\|a\|_{\theta,\infty}.$$

Peetre also developed a J-*method* based on an integral representation of elements. For $t > 0$ and $a \in A_0 \cap A_1$, we put

$$J(t,a) \equiv J(t,a;A_0,A_1) = \max\left(\|a\|_{A_0}, t\|a\|_{A_1}\right).$$

The following equivalence theorem holds ([79], 1.6).

Theorem 3.9. *Let* $1 \leqslant p < \infty$ *and* $0 < \theta < 1$. *For* $a \in (A_0, A_1)_{\theta,p}$, *it is necessary and sufficient that there is a function* $t \to u(t)$ $(t > 0)$ *strongly measurable in* $A_0 + A_1$ *with values in* $A_0 \cap A_1$ *such that*

$$a = \int_0^\infty u(t) \frac{dt}{t} \quad \textit{(convergence in } A_0 + A_1\textit{)} \tag{3.11}$$

and

$$J_{\theta,p}(u) \equiv \left(\int_0^\infty \left(t^{-\theta} J(t, u(t))\right)^p \frac{dt}{t} \right)^{1/p} < \infty. \tag{3.12}$$

Moreover, there is a constant $K > 0$ *such that for any element* $a \in (A_0, A_1)_{\theta,p}$

$$\frac{1}{K}\|a\|_{(A_0,A_1)_{\theta,p}} \leqslant \inf_u J_{\theta,p}(u) \leqslant K\|a\|_{(A_0,A_1)_{\theta,p}}, \tag{3.13}$$

where the lower bound is taken over all possible functions u satisfying the conditions (3.11) *and* (3.12).

The theorem also holds for $p = \infty$ and $0 \leqslant \theta \leqslant 1$ if we put $J_{\theta,\infty}(u) = \sup_{t>0} t^{-\theta} J(t, u(t))$.

The method of averages is also based on an integral representation. Historically, this method (introduced by Lions and Peetre [53]) was the first real interpolation method.

For a Banach space A and $1 \leqslant p < \infty$, we denote by L_A^{*p} the space of all A-valued strongly measurable functions $t \to u(t)$ $(t > 0)$ such that

$$\|u\|_{L_A^{*p}} = \left(\int_0^\infty \|u(t)\|_A^p \frac{dt}{t}\right)^{1/p} < \infty \quad (p < \infty),$$
$$\|u\|_{L_A^{*\infty}} = \sup_{t>0} \|u(t)\|_A < \infty \quad (p = \infty).$$

The essence of the method of averages can be expressed by the following theorem ([79], Ch. 1).

Theorem 3.10. *Let $1 \leqslant p_0, p_1 \leqslant \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$. For $a \in (A_0, A_1)_{\theta,p}$, it is necessary and sufficient that there is a function $t \to u(t)$ $(t > 0)$ strongly measurable in $A_0 + A_1$ with values in $A_0 \cap A_1$ such that*

$$a = \int_0^\infty u(t) \frac{dt}{t} \quad \textit{(convergence in } A_0 + A_1\textit{)} \tag{3.14}$$

and

$$t^{-\theta} u(t) \in L_{A_0}^{*p_0}, \qquad t^{1-\theta} u(t) \in L_{A_1}^{*p_1}. \tag{3.15}$$

Moreover, there is a constant $K > 0$ such that for any element $a \in (A_0, A_1)_{\theta,p}$ one has

$$\frac{1}{K} \|a\|_{(A_0,A_1)_{\theta,p}} \leqslant \inf_u \max\Big\{\|t^{-\theta} u(t)\|_{L_{A_0}^{*p_0}}, \|t^{1-\theta} u(t)\|_{L_{A_1}^{*p_1}}\Big\} \leqslant K \|a\|_{(A_0,A_1)_{\theta,p}},$$

where the lower bound is taken over all functions u satisfying conditions (3.14) *and* (3.15).

If $p_0 = p_1$, then Theorem 3.10 coincides with Theorem 3.9. The point is that we obtain equivalent norms regardless of the choice of p_0 and p_1; it is this fact that makes the mean method more flexible.

At this point, we complete our brief survey of basic real interpolation methods. The fundamental property of the spaces $(A_0, A_1)_{\theta,p}$ is the interpolation property expressed in the following theorem ([79], Ch. 1, [7], p. 301).

Theorem 3.11. *Let $\{A_0, A_1\}$ and $\{B_0, B_1\}$ be interpolation pairs of Banach spaces, and let $T\colon A_0 + A_1 \to B_0 + B_1$ be a linear operator. Suppose that the restriction of T to A_i is a bounded operator from A_i to B_i with a norm M_i $(i = 0, 1)$. Let $0 < \theta < 1$, $1 \leqslant p < \infty$ or $0 \leqslant \theta \leqslant 1$, $p = \infty$. Then T is a bounded operator from $(A_0, A_1)_{\theta,p}$ to $(B_0, B_1)_{\theta,p}$ with a norm $M_{\theta,p}$ satisfying the inequality*

$$M_{\theta,p} \leqslant M_0^{1-\theta} M_1^\theta.$$

The following theorem describes the interpolation of Lorentz spaces (see [8], Ch. 5, [79], 1.18.6). The basic measure space (R, μ) is supposed to be fixed.

Theorem 3.12. *Let* $1 \leqslant p_i < \infty$, $1 \leqslant r_j, r \leqslant \infty$ $(i = 0,1)$, $0 < \theta < 1$, *and* $1/p = (1-\theta)/p_0 + \theta/p_1$. *Next, let* A *be a Banach space. Suppose that* $p_0 \neq p_1$ *(or* $p_0 = p_1$, $1/r = (1-\theta)/r_0 + \theta/r_1$*). Then*

$$\bigl(L_A^{p_0,r_0}, L_A^{p_1,r_1}\bigr)_{\theta,r} = L_A^{p,r}, \tag{3.16}$$

and the norms are equivalent.

Corollary 3.13. *Let* $1 \leqslant p_0, p_1 < \infty$, $p_0 \neq p_1$, $1 \leqslant r \leqslant \infty$, $0 < \theta < 1$, *and* $1/p = (1-\theta)/p_0 + \theta/p_1$. *Then*

$$\bigl(L_A^{p_0}, L_A^{p_1}\bigr)_{\theta,r} = L_A^{p,r}. \tag{3.17}$$

In particular,

$$\bigl(L_A^{p_0}, L_A^{p_1}\bigr)_{\theta,p} = L_A^{p}. \tag{3.18}$$

Remark. Equation (3.17) is also valid for $1 \leqslant p_0 < p_1 = \infty$ if $L_{A_1}^{\infty}$ is replaced by the closure of the set of simple functions with respect to the norm of the space.

Let us note that, in contrast with Theorem 3.5, the space A is supposed to be fixed. This restriction can only be removed in (3.18). Namely, Lions and Peetre [53] (see also [79], 1.18.4) obtained the following result.

Theorem 3.14. *Let* $1 \leqslant p_0, p_1 < \infty$, $0 < \theta < 1$, *and*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

Further, let $\{A_0, A_1\}$ *be an interpolation pair. Then*

$$\bigl(L_{A_0}^{p_0}, L_{A_1}^{p_1}\bigr)_{\theta,p} = L_{(A_0,A_1)_{\theta,p}}^{p}. \tag{3.19}$$

As Cwikel showed [21], generally speaking, there is no similar description of the spaces $(L_{A_0}^{p_0}, L_{A_1}^{p_1})_{\theta,r}$ for $r \neq p$.

3.3. Comparison of real and complex interpolation methods. We consider spaces L^p of scalar functions. By virtue of (3.6) and (3.19), for $1 \leqslant p_0, p_1 < \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0 + \theta/p_1$, we have

$$\bigl(L^{p_0}, L^{p_1}\bigr)_{\theta,p} = \bigl[L^{p_0}, L^{p_1}\bigr]_{\theta} = L^p.$$

In this example, the spaces obtained by means of real and complex interpolation coincide. However, in the general case these methods result in quite different scales of interpolation spaces. Namely, there are interpolation pairs $\{A_0, A_1\}$ such that $[A_0, A_1]_\theta$ and $(A_0, A_1)_{\eta,p}$ are different spaces for any $0 < \theta, \eta < 1$ and $1 \leqslant p \leqslant \infty$ (for example, this is the case for the interpolation of Sobolev spaces [79], 1.9.3).

Let X and Y be Banach spaces; then $X \hookrightarrow Y$ implies that $X \subset Y$ and $\|x\|_Y \leqslant C\|x\|_X$ for any element $x \in X$. If for given $t > 0$ we introduce the new norm $\|x\|_{tX} = t\|x\|_X$ $(x \in X)$ in a Banach space X, then the space thus obtained will be denoted by tX.

The following embeddings hold (Lions and Peetre [53]; [8], Ch. 4).

Theorem 3.15. *Let $\{A_0, A_1\}$ be an interpolation pair, and let $0 < \theta < 1$. Then*

$$(A_0, A_1)_{\theta,1} \hookrightarrow [A_0, A_1]_\theta \hookrightarrow (A_0, A_1)_{\theta,\infty}. \tag{3.20}$$

Proof. 1) Let $a \in (A_0, A_1)_{\theta,1}$. By virtue of Theorem 3.9, a admits the representation

$$a = \int_0^\infty u(t)\frac{dt}{t}$$

with a function u satisfying the corresponding conditions. We consider the Mellin transform of the function $t^{-\theta}u(t)$,

$$F(z) = \int_0^\infty t^{z-\theta}u(t)\frac{dt}{t}. \tag{3.21}$$

We have $a = F(\theta)$. Obviously, $F \in \mathfrak{F}$ and $\|F\|_{\mathfrak{F}} \leqslant C\|a\|_{(A_0,A_1)_{\theta,1}}$. This proves the first embedding in (3.20).

2) Let $a \in [A_0, A_1]_\theta$. Then a admits the representation $a = F(\theta)$, $F \in \mathfrak{F}$. By virtue of the theorem on three lines (see § 2), we have

$$\begin{aligned} K(t,a) &= K(t, F(\theta)) = \|F(\theta)\|_{A_0+tA_1} \\ &\leqslant \sup_y \|F(iy)\|_{A_0}^{1-\theta}\, t^\theta \sup_y \|F(1+iy)\|_{A_1}^\theta \leqslant t^\theta \|F\|_{\mathfrak{F}}. \end{aligned}$$

Therefore, $a \in (A_0, A_1)_{\theta,\infty}$ (see (3.10)) and $\|a\|_{(A_0,A_1)_{(\theta,\infty)}} \leqslant \|F\|_{\mathfrak{F}}$. This implies the second embedding in (3.20).

Let us note another embedding, which follows readily from Definition 3.7:

$$(A_0, A_1)_{\theta,p} \hookrightarrow (A_0, A_1)_{\theta,q} \quad \text{if} \quad 1 \leqslant p < q \leqslant \infty. \tag{3.22}$$

§4. Interpolation and the Fourier type of Banach spaces

Theorem 3.15 raises the following question: under what additional conditions on the spaces A_0, A_1 can this theorem be strengthened so that the embedding

$$(A_0, A_1)_{\theta,p} \hookrightarrow [A_0, A_1]_\theta \hookrightarrow (A_0, A_1)_{\theta,q} \quad (1 < p \leqslant q < \infty)$$

will hold? This question was considered by Peetre [62] and led to the concept of the Fourier type of a Banach space. The original idea is contained in the proof of Theorem 3.15: the change of variable $t = e^{-2\pi s}$ ($s \in \mathbb{R}$) in (3.21) for $x = 0$ and $x = 1$ yields the Fourier transforms of the functions

$$f_0(s) = e^{2\pi s\theta}u(e^{-2\pi s}) \quad \text{and} \quad f_1(s) = e^{2\pi s(\theta-1)}u(e^{-2\pi s}).$$

Thus it is clear that the properties of the Fourier transform viewed as an operator defined on the corresponding space of vector-valued functions play an important role in the problem under consideration.

Let A be a complex Banach space. We consider functions $f\colon \mathbb{R} \to A$. The Fourier transform of a function $f \in L^1_A$ is defined by the formula

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x\xi}\,dx, \qquad \xi \in \mathbb{R} \tag{4.1}$$

(where the integral is interpreted in the sense of Bochner).

The following definition, based on the Hausdorff–Young inequality, was introduced by Peetre [62].

Definition 4.1. Let $1 \leqslant p \leqslant 2$. We say that a space A has the Fourier type p if there is a constant K such that for any function $f \in L^p_A$ with compact support one has

$$\|\widehat{f}\|_{L^{p'}_A} \leqslant K\|f\|_{L^p_A}. \tag{4.2}$$

This definition is equivalent to the fact that the Fourier transform defined originally on $L^1_A \cap L^p_A$ by formula (4.1) can be extended to a bounded linear operator from L^p_A to $L^{p'}_A$.

It is clear that any Banach space has the type 1.

The following Peetre theorem [62] answers the question posed at the beginning of this section.

Theorem 4.2. *Let $\{A_0, A_1\}$ be an interpolation pair; suppose that A_j has a Fourier type p_j $(1 < p_j \leqslant 2,\ j = 0,1)$. Next, let $0 \leqslant \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. Then*

$$(A_0, A_1)_{\theta,p} \hookrightarrow [A_0, A_1]_\theta \hookrightarrow (A_0, A_1)_{\theta,p'}. \tag{4.3}$$

Proof. 1) Let $a \in (A_0, A_1)_{\theta,p}$. We use the method of averages. Suppose that the function $t \mapsto u(t) \in A_0 \cap A_1$ is such that

$$a = \int_0^\infty u(t)\,\frac{dt}{t} \quad \text{(convergence in } A_0 + A_1) \tag{4.4}$$

and

$$\|u\|^* \equiv \|t^{-\theta}u\|_{L^{*p_0}_{A_0}} + \|t^{1-\theta}u\|_{L^{*p_1}_{A_1}} < \infty. \tag{4.5}$$

We fix an interval $I \subset (0,\infty)$ and put

$$v = u\,\chi_I, \qquad b = \int_0^\infty v(t)\frac{dt}{t}\,.$$

Further, let

$$G(z) = \int_0^\infty t^{z-\theta}v(t)\frac{dt}{t}\,, \qquad z = x + iy \in \mathbb{C} \tag{4.6}$$

(the Mellin transform). Then $G \in \mathcal{F}(A_0, A_1)$ (G is analytic in $\mathbb{C}$) and $G(\theta) = b$. Let $\varphi_j(t) = t^{j-\theta}v(t)$ $(j = 0,1)$. If in (4.6) we change the variables $t = e^{-2\pi s}$ $(s \in \mathbb{R})$ for $x = 0$ and $x = 1$, then we obtain

$$G(iy) = 2\pi\widehat{\psi}_0(y), \qquad G(1+iy) = 2\pi\widehat{\psi}_1(y),$$

where $\psi_j(s) = \varphi_j(e^{-2\pi s})$ $(j = 0,1)$. By virtue of (4.5), $\psi_j \in L^{p_j}_{A_j}$; here ψ_j has a compact support $(j = 0,1)$. Since A_j has the Fourier type p_j, we have

$$\|G(j+iy)\|_{L^{p'_j}_{A_j}} \leqslant C\|\psi_j\|_{L^{p_j}_{A_j}} = (2\pi)^{-1/p_j}C\|\varphi_j\|_{L^{*p_j}_{A_j}} \qquad (j = 0,1).$$

Hence, by virtue of the inequality (3.5),

$$\|b\|_{[A_0,A_1]_\theta} \leqslant C\|v\|^*, \tag{4.7}$$

where the constant C is independent of the choice of the function u and the interval I. We now put $I_k = [1/k, k]$, $v_k = u\chi_{I_k}$, and $b_k = \int_0^\infty v_k(t)\frac{dt}{t}$. By virtue of (4.4), $b_k \to a$ as $k \to \infty$ in $A_0 + A_1$. Moreover, it follows from (4.7) that $\{b_k\}$ is a Cauchy sequence in $[A_0, A_1]_\theta$ and $\|b_k\|_{[A_0,A_1]_\theta} \leqslant C\|u\|^*$. Therefore, since $[A_0, A_1]_\theta$ is complete, we have $\|a\|_{[A_0,A_1]_\theta} \leqslant C\|u\|^*$. This proves the left embedding in (4.3).

2) Let $a \in [A_0, A_1]_\theta$. We take an arbitrary representation $a = F(\theta)$, where $F \in \mathfrak{F}$. Let $\widetilde{F}(z) = F(z)\Big(\dfrac{\theta+1}{z+1}\Big)^2$, so that $\widetilde{F}(\theta) = a$. Then

$$\widetilde{F}(iy) \in L^{p_0}_{A_0}, \qquad \widetilde{F}(1+iy) \in L^{p_1}_{A_1}. \tag{4.8}$$

We put

$$u(t) = \frac{t^\theta}{2\pi i}\int_{x-i\infty}^{x+i\infty} t^{-z}\widetilde{F}(z)\,dz \quad (t>0,\ 0 \leqslant x \leqslant 1) \tag{4.9}$$

(the inverse Mellin transform). By putting $x = 0$ and $x = 1$, we obtain

$$t^{j-\theta}u(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} t^{-iy}\widetilde{F}(j+iy)\,dy \quad (j = 0, 1).$$

Since A_j has the Fourier type p_j, it follows by virtue of (4.8) that

$$\|t^{j-\theta}u\|_{L^{*p'_j}_{A_j}} \leqslant C\|\widetilde{F}(j+iy)\|_{L^{p_j}_{A_j}} \leqslant c' \sup_y \|F(j+iy)\|_{A_j} \tag{4.10}$$

for $j = 0, 1$. Then, putting $x = \theta$ in (4.9), we have

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{-iy}\widetilde{F}(\theta+iy)\,dy,$$

so that $u(e^{2\pi s}) = (2\pi)^{-1}\widehat{\Phi}(s)$, where $\Phi(y) = \widetilde{F}(\theta+iy)$.

This implies that

$$a = \Phi(0) = \int_{\mathbb{R}} \widehat{\Phi}(s)\,ds = \int_0^\infty u(t)\frac{dt}{t}\,. \tag{4.11}$$

Taking into account (4.10), (4.11) and Theorem 3.10, we obtain the right embedding in (4.3). The proof of the theorem is complete.

Since any Banach space has the type 1, we see that Theorem 3.15 can be viewed as the limiting case of Theorem 4.2.

Remark 4.3. Under the assumptions of Theorem 4.2, neither of the embeddings (4.3) can be strengthened by substituting $r > p$ for p on the left-hand side or $q < p'$ for p' on the right-hand side. This follows easily from Theorems 3.5 and 3.12 and the following proposition.

Proposition 4.4. *Let (Ω, ν) be a measure space, and let $1 < p < \infty$. Then the space $A = L^p(\Omega, \nu)$ has the Fourier type* $\min(p, p')$.

Proof. Let $1 < p \leqslant 2$. Suppose that $f\colon \mathbb{R} \to A$ is a simple function. We consider the Fourier transform

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\, dx.$$

For each given $\xi \in \mathbb{R}$, $\widehat{f}(\xi)$ is a function defined on the space Ω. Applying the generalized Minkowski inequality, we have

$$\begin{aligned}\biggl(\int_{\mathbb{R}} \|\widehat{f}(\xi)\|_A^{p'}\, d\xi\biggr)^{p/p'} &= \biggl(\int_{\mathbb{R}} \biggl(\int_{\Omega} |\widehat{f}(\xi)(y)|^p\, d\nu(y)\biggr)^{p'/p} d\xi\biggr)^{p/p'} \\ &\leqslant \int_{\Omega} \biggl(\int_{\mathbb{R}} |\widehat{f}(\xi)(y)|^{p'}\biggr)^{p/p'} d\nu(y).\end{aligned}$$

By virtue of the Hausdorff–Young inequality,

$$\biggl(\int_{\mathbb{R}} \|\widehat{f}(\xi)(y)\|^{p'}\, d\xi\biggr)^{1/p'} \leqslant \biggl(\int_{\mathbb{R}} |f(x)(y)|^p\, dx\biggr)^{1/p}$$

for each $y \in \Omega$. Therefore, by the Fubini theorem,

$$\biggl(\int_{\mathbb{R}} \|\widehat{f}(\xi)\|_A^{p'}\, d\xi\biggr)^{p/p'} \leqslant \int_{\Omega} d\nu(y) \int_{\mathbb{R}} |f(x)(y)|^p\, dx = \int_{\mathbb{R}} \|f(x)\|_A^p\, dx.$$

Thus, L^p has the Fourier type p for $1 < p \leqslant 2$. The case $p > 2$ can be considered similarly.

Below we shall give a more sophisticated treatment of results related to the concept of the Fourier type. Here we briefly consider further applications of the concept in the theory of interpolation and the theory of function spaces.

Peetre showed in his paper [62] that by using Theorem 4.2 one can obtain some well-known embedding theorems for the Sobolev–Liouville and Besov spaces. We now give the definition of these spaces (see [74], Ch. 5).

Let $1 \leqslant p \leqslant \infty$ and $r \in \mathbb{N}$. A function $f \in L^p(\mathbb{R}^n)$ belongs to the Sobolev space $W_p^r(\mathbb{R}^n)$ if f has all generalized derivatives $D^\alpha f$ of order $|\alpha| \leqslant r$ belonging to $L^p(\mathbb{R}^n)$. The norm in W_p^r is defined by the formula

$$\|f\|_{W_p^r} = \sum_{\alpha \leqslant r} \|D^\alpha f\|_p.$$

If $1 \leqslant p \leqslant \infty$ and $\alpha > 0$, then the Sobolev–Liouville space $\mathcal{L}_p^\alpha(\mathbb{R}^n)$ is defined as the set of all functions f representable in the form of the convolution $f = G_\alpha * g$, where $g \in L^p(\mathbb{R}^n)$ and G_α is the Bessel kernel of order α. Here

$$\|f\|_{\mathcal{L}_p^\alpha} = \|g\|_p.$$

It is known ([74], Ch. 5) that

$$W_p^r(\mathbb{R}^n) = \mathcal{L}_p^r(\mathbb{R}^n) \quad (1 < p < \infty,\ r \in \mathbb{N}).$$

Let $\alpha > 0$, $1 \leqslant p \leqslant \infty$, and $1 \leqslant q < \infty$.

The Besov space $B_{p,q}^\alpha(\mathbb{R}^n)$ consists of all functions $f \in L^p(\mathbb{R}^n)$ such that the norm

$$\|f\|_{B_{p,q}^\alpha} \equiv \|f\|_p + \left(\int_0^\infty [t^{-\alpha}\omega_k(f;t)_p]^q \frac{dt}{t}\right)^{1/q}$$

is finite, where $k > \alpha$ is an integer and

$$\omega_k(f;t)_p = \sup_{|h|\leqslant t} \left\| \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} f(x+jh) \right\|_p$$

is a modulus of continuity of order k. It is easy to see that

$$B_{p,q}^\alpha \subset B_{p,s}^\alpha \quad \text{with} \quad q < s. \tag{4.12}$$

It is known that

$$(L^p, W_p^r)_{\theta,q} = B_{p,q}^{\theta r} \quad (1 \leqslant p \leqslant \infty,\ 1 \leqslant q < \infty,\ r \in \mathbb{N},\ 0 < \theta < 1) \tag{4.13}$$

and

$$[L^p, W_p^r]_\theta = \mathcal{L}_p^{\theta r} \quad (1 < p < \infty,\ 0 < \theta < 1,\ r \in \mathbb{N}) \tag{4.14}$$

(see [79], Ch. 2). Next, the space $\mathcal{L}_p^\alpha$ is isometric to the space L^p ([74], Ch. 5); by virtue of Proposition 4.4, it follows that W_p^r $(1 < p \leqslant 2)$ has the Fourier type p. By applying Theorem 4.2, we obtain the embeddings

$$B_{p,p}^\alpha \subset \mathcal{L}_p^\alpha \subset B_{p,p'}^\alpha \quad (1 < p \leqslant 2,\ \alpha > 0). \tag{4.15}$$

In a similar manner, we obtain

$$B_{p,p'}^\alpha \subset \mathcal{L}_p^\alpha \subset B_{p,p}^\alpha \quad (2 \leqslant p < \infty,\ \alpha > 0). \tag{4.16}$$

These results are well known (they go back to the paper [54] by Marcinkiewicz and were completely proved by Besov, Lizorkin, and Taibleson; see [74], Ch. 5). The first embedding in (4.15) cannot be improved. However, the second one can be strengthened. Namely,

$$\mathcal{L}_p^\alpha \subset B_{p,2}^\alpha \quad (1 < p \leqslant 2,\ \alpha > 0). \tag{4.17}$$

Similarly,

$$B_{p,2}^\alpha \subset \mathcal{L}_p^\alpha \quad (2 \leqslant p < \infty,\ \alpha > 0) \tag{4.18}$$

(see [74]).

In connection with the last embeddings, we note another paper by Peetre [63], where he considered the concept of the λ-type of a Banach space (related to the unconditional convergence of series in B-spaces). In particular, in that paper new proofs of the embeddings (4.17) and (4.18) are obtained with the use of interpolation.

Useful applications of Theorem 4.2 can also be found in Milman [58]. One of the results of this paper is the equality

$$(W_1^r, W_p^r)_\theta = W_{p_\theta}^r \quad \left(\frac{1}{p_\theta} = \frac{1-\theta}{1} + \frac{\theta}{p},\ 1 < p < \infty,\ r \in \mathbb{N}\right),$$

including the limiting case of the space W_1^r.

§5. The Rademacher type and cotype

Let us consider the system of Rademacher functions

$$r_n(t) = \operatorname{sign} \sin 2^n \pi t \quad (t \in [0,1],\ n \in \mathbb{N}).$$

The Khinchine inequalities (see [82], Ch. 5) express important properties of the system.

Theorem 5.1. *Let $0 < p < \infty$. There is a constant $A_p > 0$ such that for any polynomial $F(t) = \sum_{k=1}^n a_k r_k(t)$ $(a_k \in \mathbb{C})$ one has*

$$\frac{1}{A_p}\biggl(\sum_{k=1}^n |a_k|^2\biggr)^{1/2} \leqslant \biggl(\int_0^1 |F(t)|^p\,dt\biggr)^{1/p} \leqslant A_p\biggl(\sum_{k=1}^n |a_k|^2\biggr)^{1/2}. \tag{5.1}$$

Corollary 5.2. *Let $1 < p < \infty$. There is a constant $B_p > 0$ such that for any complex-valued function $f \in L^p[0,1]$ and any $n \in \mathbb{N}$ one has*

$$\biggl\|\sum_{k=1}^n c_k(f) r_k\biggr\|_p \leqslant B_p \|f\|_p, \qquad c_k(f) = \int_0^1 f(t) r_k(t)\,dt. \tag{5.2}$$

Indeed, if $S_n = \sum_{k=1}^n c_k(f) r_k$, then, by virtue of (5.1),

$$\begin{aligned}\|S_n\|_p^2 &\leqslant A_p^2 \sum_{k=1}^n |c_k(f)|^2 = A_p^2 \int_0^1 f(t)\overline{S_n(t)}\,dt\\ &\leqslant A_p^2 \|f\|_p \|\overline{S}_n\|_{p'} \leqslant B_p \|f\|_p \|S_n\|_p.\end{aligned}$$

This implies (5.2).

We consider polynomials in the Rademacher system with coefficients from an arbitrary Banach space X. Generally speaking, for such polynomials the Khinchine inequality is not valid. However, the following Kahane theorem holds ([43], Ch. 2).

Theorem 5.3. *Let $0 < p < q < \infty$. There is a constant $A_{p,q} > 0$ such that for any Banach space X and any polynomial $F(t) = \sum_{k=1}^n r_k(t) x_k$ $(x_k \in X)$ one has*

$$\|F\|_{L_X^p} \leqslant \|F\|_{L_X^q} \leqslant A_{p,q} \|F\|_{L_X^p}. \tag{5.3}$$

Thus, all L_X^p-norms $(0 < p < \infty)$ of the polynomial F are equivalent. At the same time, in the case of an arbitrary Banach space X, only the trivial estimates of these norms via the coefficients of the polynomial, namely,

$$\|\{x_k\}\|_{l_X^\infty} \leqslant \biggl\|\sum_{k=1}^n r_k x_k\biggr\|_{L_X^1} \leqslant \|\{x_k\}\|_{l_X^1}, \tag{5.4}$$

are valid.

Non-trivial estimates form the basis of the concepts of the Rademacher type and cotype, introduced by Hoffman-Jorgensen and Maurey in 1972.

Definition 5.4. Let $1 \leqslant p \leqslant 2$. We say that a Banach space X has the type p if there is a constant $C > 0$ such that for any vectors $x_1, \dots, x_n \in X$ one has

$$\Big\| \sum_{k=1}^{n} r_k x_k \Big\|_{L^2_X} \leqslant C \Big(\sum_{k=1}^{n} \|x_k\|^p \Big)^{1/p}. \tag{5.5}$$

Definition 5.5. Let $2 \leqslant q \leqslant \infty$. We say that a Banach space X has the cotype q if there is a constant $C > 0$ such that for any vectors $x_1, \dots, x_n \in X$ one has

$$\|\{x_k\}_{k=1}^{n}\|_{l^q_X} \leqslant C \Big\| \sum_{k=1}^{n} r_k x_k \Big\|_{L^2_X}. \tag{5.6}$$

By Theorem 5.3, the L^2_X-norm in these definitions can be replaced by some L^r_X-norm $(0 < r < \infty)$.

Each Banach space has the type 1 and the cotype ∞ (see (5.3) and (5.4)).

Note that for $p > 2$ for any constant C we can choose $x_1, \dots, x_n \in X$ such that (5.5) does not hold. Indeed, let $x_1 = \dots = x_n \neq 0$. Then, by virtue of (5.5), for any $n \in \mathbb{N}$ we have $n^{1/2} \leqslant C n^{1/p}$; therefore, $p \leqslant 2$.

A similar remark is valid for the condition $q \geqslant 2$ in Definition 5.5.

If X is a Hilbert space, then X has the type 2 and the cotype 2.

Indeed, in this case, since the functions r_k are orthogonal, we have

$$\int_0^1 \Big\| \sum_{k=1}^{n} r_k(t) x_k \Big\|^2 dt = \sum_{k=1}^{n} \|x_k\|^2.$$

This assertion is convertible in a sense. Namely, Kwapién [50] established the following important result.

Theorem 5.6. *Let a Banach space X have the type 2 and the cotype 2. Then X is isomorphic to a Hilbert space.*

By virtue of (5.3), this theorem can be stated as follows.

A Banach space X is isomorphic to a Hilbert space if and only if X satisfies the two-sided Khinchine inequalities.

Kwapién [50] also proved that the space X has the type 2 and the cotype 2 simultaneously if and only if

$$C^{-1} \Big(\sum_{k=-n}^{n} \|x_k\|^2 \Big)^{1/2} \leqslant \Big(\int_0^1 \Big\| \sum_{k=-n}^{n} e^{2\pi i k t} x_k \Big\|^2 dt \Big)^{1/2} \leqslant C \Big(\sum_{k=-n}^{n} \|x_k\|^2 \Big)^{1/2},$$

where the x_k are arbitrary vectors from X and $C > 0$ is a constant. The property, in turn, is equivalent to the assertion that X has the Fourier type 2.

Let us return to the general case. The definitions of type and cotype and the Jensen inequality yield the following result.

Proposition 5.7. *Let X be a Banach space. Then the following assertions hold:*

(i) *if X has a type p_0, $1 < p_0 \leqslant 2$, then X has the type p for any $1 \leqslant p \leqslant p_0$;*

(ii) *if X has a cotype q_0, $2 \leqslant q_0 < \infty$, then X has the cotype q for any $q_0 \leqslant q \leqslant \infty$.*

We introduce the following definition.

Definition 5.8. For any Banach space X, we put

$$\begin{aligned} p_X &= \sup\{p \leqslant 2 : X \text{ has the type } p\}, \\ q_X &= \inf\{q \geqslant 2 : X \text{ has the cotype } q\}. \end{aligned}$$

If $p_X > 1$ $(q_X < \infty)$, then X is said to have a non-trivial type (respectively, a non-trivial cotype).

As an example, we consider the spaces L^p (see [51], p. 70).

Proposition 5.9. *Let $X = L^p(\Omega, \mu)$, $1 \leqslant p < \infty$. Then X has the type* $\min(p, 2)$ *and the cotype* $\max(p, 2)$.

Proof. Let $1 \leqslant p \leqslant 2$ and $x_1, \dots, x_n \in X$. By applying the Fubini theorem and the Khinchine inequalities, we obtain

$$\Big\|\sum_{k=1}^{n} r_k x_k\Big\|_{L^p_X} \asymp \bigg(\int_\Omega \Big(\sum_{k=1}^{n} |x_k(\xi)|^2\Big)^{p/2} d\mu(\xi)\bigg)^{1/p}, \tag{5.7}$$

whence

$$\Big\|\sum_{k=1}^{n} r_k x_k\Big\|_{L^p_X} \leqslant C\Big(\sum_{k=1}^{n} \|x_k\|^p_{L^p}\Big)^{1/p},$$

so that X has the type p. On the other hand, by virtue of (5.7) and the Minkowski inequality,

$$\begin{aligned} \Big\|\sum_{k=1}^{n} r_k x_k\Big\|^p_{L^p_X} &\geqslant C\bigg(\sum_{k=1}^{n}\Big(\int_\Omega |x_k(\xi)|^p \, d\mu(\xi)\Big)^{2/p}\bigg)^{p/2} \\ &= C\Big(\sum_{k=1}^{n} \|x_k\|^2_{L^p}\Big)^{p/2} \qquad (C > 0); \end{aligned}$$

therefore, X has the cotype 2. The case $2 < p < \infty$ can be considered in a similar way.

If a measure μ is such that Ω consists of finitely many μ-atoms, then the space $L^p(\Omega, \mu)$ $(1 \leqslant p \leqslant \infty)$ is finite-dimensional and has the type 2 and the cotype 2. In the other cases, however, the type and cotype values obtained in Proposition 5.9 are sharp.

Proposition 5.10. *Let $1 \leqslant p < \infty$ and $X = L^p(\Omega, \mu)$, where the space Ω is not a union of finitely many μ-atoms. Then*

$$p_X = \min(p, 2), \qquad q_X = \max(p, 2).$$

Proof. It follows from our condition and Lemma 2.1 that for any $n \in \mathbb{N}$ there are pairwise disjoint μ-measurable subsets $E_1, \dots, E_n \subset \Omega$ with $\mu(E_k) \equiv \mu_k > 0$, $k = 1, \dots, n$. We put $x_k(\xi) = \chi_{E_k}(\xi)\mu_k^{-1/p}$, $\xi \in \Omega$. Then $\|x_k\|_{L^p} = 1$ and $\| \sum_{k=1}^n r_k(t)x_k\|_{L^p} = n^{1/p}$ for any binary irrational $t \in [0, 1]$. Let $1 \leqslant p \leqslant 2$. If

$$\left\| \sum_{k=1}^n r_k x_k \right\|_{L^2_X} \leqslant C \left(\sum_{k=1}^n \|x_k\|^s_{L^p} \right)^{1/s}$$

for some $s \geqslant 1$ (where the constant C is independent of n), then $n^{1/p} \leqslant Cn^{1/s}$ and, since n is arbitrary, we have $s \leqslant p$. Taking into account Proposition 5.9, we obtain $p_X = p$ and $q_X = 2$. Similarly, for $p > 2$ we have $q_X = p$ and $p_X = 2$.

Remark 5.11. If $X = L^\infty[0, 1]$ or $X = c_0$, then $p_X = 1$ and $q_X = \infty$.

Indeed, in the space $X = L^\infty[0, 1]$ we choose the vectors $x_k = r_k$ $(k = 1, \dots, n)$. For any binary irrational point $t \in [0, 1]$, we have

$$\left\| \sum_{k=1}^n r_k(t) x_k \right\|_X = n. \tag{5.8}$$

If $p > 1$, then (5.5) does not hold for sufficiently large n. Thus, $p_X = 1$. Considering $x_k(\xi) = \chi_{[(k-1)/n, k/n)}(\xi)$ $(k = 1, \dots, n)$, we also make sure that $q_X = \infty$.

In the case $X = c_0$, for any fixed $n \in \mathbb{N}$ we put

$$x_k = \sum_{j=1}^{2^n} \varepsilon_k^{(j)} e_j \quad (k = 1, \dots, n),$$

where $\varepsilon_k^{(j)}$ is a value of the function r_k on the interval $(\frac{j-1}{2^n}, \frac{j}{2^n})$. We again obtain equality (5.8), which implies that $p_X = 1$. We obtain $q_X = \infty$ by taking $x_k = e_k$, where e_k is the sequence whose elements are all zero except for the kth, which is equal to 1.

There is a certain symmetry (duality) between the concepts of type and cotype. Indeed, using Proposition 2.6 and the orthogonality of the Rademacher functions, it is easy to prove the following assertion.

Proposition 5.12. *Let a Banach space X have a type p, $1 < p \leqslant 2$. Then the dual space X^* has the cotype p'.*

The converse is not true: the space c_0 has only the type 1 and the dual space l^1 has the cotype 2.

From the fact that X has a cotype q, $2 \leqslant q < \infty$, it does not follow that X^* has the type q' (as an example, we can take $X = l^1$).

Spaces with a non-trivial type play an important role in many problems of geometry of Banach spaces and Fourier vector-valued analysis. It was established in [56], [67] that the non-triviality of type implies that the space possesses certain geometric and analytic convexity properties.

Definition 5.13. Let $1 \leqslant p \leqslant \infty$. We say that a Banach space X uniformly contains l_n^p if for any $\lambda > 1$ and any $n \in \mathbb{N}$ there are elements $x_1, \dots, x_n \in X$ such that for any numbers $\alpha_1, \dots, \alpha_n$ one has

$$\Big(\sum_{k=1}^n |\alpha_k|^p\Big)^{1/p} \leqslant \Big\|\sum_{k=1}^n \alpha_k x_k\Big\| \leqslant \lambda\Big(\sum_{k=1}^n |\alpha_k|^p\Big)^{1/p} \quad (1 \leqslant p < \infty),$$

$$\max_{1\leqslant k\leqslant n} |\alpha_k| \leqslant \Big\|\sum_{k=1}^n \alpha_k x_k\Big\| \leqslant \lambda \max_{1\leqslant k\leqslant n} |\alpha_k| \quad (p = \infty).$$

It is clear that the elements x_k satisfy the inequalities $1 \leqslant \|x_k\| \leqslant \lambda$.

Maurey and Pisier [56] proved the following theorem.

Theorem 5.14. *If X is a Banach space, then*

(i) *$p(X) > 1$ if and only if X does not uniformly contain l_n^1;*
(ii) *$q(X) < \infty$ if and only if X does not uniformly contain l_n^∞.*

It is not hard to show that X uniformly contains l_n^1 if and only if X^* uniformly contains l_n^1 (see [30], [31]). We restrict our consideration to the real case. Let X uniformly contain l_n^1. Then for any $n \in \mathbb{N}$ and any $\lambda > 1$ there are vectors $x_1, \dots, x_{2^n}$ such that $\|x_i\| \leqslant \lambda$ and

$$\Big\|\sum_{i=1}^{2^n} \alpha_i x_i\Big\| \geqslant \sum_{i=1}^{2^n} |\alpha_i|$$

for any numbers $\alpha_1, \dots, \alpha_{2^n}$. Let $\delta^{(i)} = \{\delta_1^{(i)}, \dots, \delta_n^{(i)}\}$ $(i = 1, \dots, 2^n)$ be all possible systems of n numbers equal to ± 1. On the subspace Y spanned by the linearly independent vectors $x_1, \dots, x_{2^n}$, we define linear functionals y_k^* by putting

$$\langle x, y_k^*\rangle = \lambda \sum_{i=1}^{2^n} \delta_k^{(i)} \alpha_i, \qquad x = \sum_{i=1}^{2^n} \alpha_i x_i \quad (k = 1, \dots, n).$$

It is clear that $1 \leqslant \|y_k^*\| \leqslant \lambda$. We extend y_k^* to the whole space X preserving the norm. Since

$$\Big\langle x_i, \sum_{k=1}^n \beta_k y_k^* \Big\rangle = \lambda \sum_{k=1}^n \delta_k^{(i)} \beta_k,$$

we have

$$\Big\|\sum_{k=1}^n \beta_k y_k^*\Big\| \geqslant \sum_{k=1}^n |\beta_k| \quad (\beta_k \in \mathbb{R}).$$

Therefore, X^* uniformly contains l_n^1. The converse can be proved in a similar way.

Thus, from Theorem 5.14 we derive the following assertion.

Proposition 5.15. *If X is a Banach space, then $p_X > 1$ if and only if $p_{X^*} > 1$.*

The following proposition is an easy consequence of Theorem 5.14.

Proposition 5.16. *If $p_X > 1$, then $q_X < \infty$.*

The converse is not true (if $X = l^1$, then $p_X = 1$ and $q_X = 2$). Moreover, even if we suppose that $q_X < \infty$ and $q_{X^*} < \infty$, we cannot claim that $p_X > 1$ (see [67]).

Returning to the duality between the concepts of type and cotype, we demonstrate that Proposition 5.12 can be strengthened.

Proposition 5.17. *Let a Banach space X have a type p, $1 < p \leqslant 2$. Then for any r, $1 < r < \infty$, there is a constant $C > 0$ such that for each function $g \in L^r_{X^*}[0,1]$ the sequence $\{c_k(g)\}$ of its Fourier coefficients with respect to the Rademacher system satisfies the inequality*

$$\left(\sum_{k=1}^{\infty} \|c_k(g)\|_{X^*}^{p'}\right)^{1/p'} \leqslant C\|g\|_{L^r_{X^*}}. \tag{5.9}$$

Proof. By virtue of Proposition 2.6, for any $\varepsilon > 0$ and any $n \in \mathbb{N}$ there is a sequence $\{x_k\} \in l^p_X$ such that $\|\{x_k\}\|_{l^p_X} = 1$ and

$$\left(\sum_{k=1}^{n} \|c_k(g)\|_{X^*}^{p'}\right)^{1/p'} < \sum_{k=1}^{n} \langle x_k, c_k(g)\rangle + \varepsilon.$$

Since X has the type p, we have, by virtue of (5.3),

$$\begin{aligned}\sum_{k=1}^{n} \langle x_k, c_k(g)\rangle &= \int_0^1 \left\langle \sum_{k=1}^{n} r_k(t) x_k, g(t)\right\rangle dt \\ &\leqslant \left\|\sum_{k=1}^{n} r_k x_k\right\|_{L^{r'}_X} \|g\|_{L^r_{X^*}} \leqslant C\|g\|_{L^r_{X^*}}.\end{aligned}$$

This implies (5.9).

The converse can be proved similarly.

Proposition 5.18. *Let X be a Banach space, and let $1 < p \leqslant 2$. Suppose that there are numbers $C > 0$ and $1 < r < \infty$ such that for each function $g \in L^r_{X^*}[0,1]$ inequality* (5.9) *is valid. Then X has the type p.*

Remark 5.19. Propositions 5.17 and 5.18 remain valid if we interchange X and X^*.

The propositions we have proved lead to the following definition.

Definition 5.20. Let $2 \leqslant q < \infty$. We say that a Banach space X has the strong cotype q if for each r, $1 < r < \infty$, the space possesses the following property: there is a constant $A_r > 0$ such that for each function $f \in L^r_X[0,1]$ one has

$$\left(\sum_{k=1}^{\infty} \|c_k(f)\|^q\right)^{1/q} \leqslant A_r\|f\|_{L^r_X}, \tag{5.10}$$

where the $c_k(f)$ are the Fourier coefficients of f with respect to the Rademacher system.

Remark 5.21. If for fixed $q \in [2, \infty)$ the above-mentioned property holds for some $1 < r < \infty$, then it holds for any r, $1 < r < \infty$ (see Propositions 5.17 and 5.18 and Remark 5.19). In particular, by choosing $r = q'$, we obtain an analogue of the Hausdorff–Young–Riesz inequality (1.9).

Propositions 5.17 and 5.18 and Remark 5.19 imply the following result.

Proposition 5.22. *Let X be a Banach space and $1 \leqslant p \leqslant 2$. Then*

(i) *X has the type p if and only if X^* has the strong cotype p';*
(ii) *X^* has the type p if and only if X has the strong cotype p'.*

Taking account of Proposition 5.15, we obtain the following result.

Corollary 5.23. *A Banach space X has a non-trivial type if and only if X has a non-trivial strong cotype.*

It is obvious that if X has a strong cotype q, $2 \leqslant q < \infty$, then X has the cotype q. The converse is not true. It is also clear that the equivalence of the cotype and strong cotype is related to additional properties of the space, expressed by inequalities of the form (5.2).

Let $1 < r < \infty$. We say that a Banach space X is *K_r-convex* if there is a constant $A_r > 0$ such that for any function $f \in L^r_X[0,1]$ the partial sums of its Fourier series with respect to the Rademacher system satisfy the inequality

$$\biggl\| \sum_{k=1}^{n} c_k(f) r_k \biggr\|_{L^r_X} \leqslant A_r \|f\|_{L^r_X} \quad (n \in \mathbb{N}). \tag{5.11}$$

By virtue of the Kahane inequality (5.3), the exponent r on the left-hand side can be replaced by any $s > 0$. Therefore, for $r < s$, K_r-convexity implies K_s-convexity. Let us prove that this is also the case for $1 < s < r$.

Lemma 5.24. *Let X be a Banach space. Then the following assertions hold:*

(i) *if $1 < r < \infty$, then X is K_r-convex if and only if X^* is $K_{r'}$-convex;*
(ii) *if X is K_r-convex for some $1 < r < \infty$, then X is K_s-convex for any $1 < s < \infty$.*

Proof. Suppose that X is K_r-convex. Let $g \in L^{r'}_{X^*}$ and $n \in \mathbb{N}$. By Proposition 2.6, for any $\varepsilon > 0$ there is a function $f \in L^r_X$ with $\|f\|_{L^r_X} = 1$ such that

$$\biggl\| \sum_{k=1}^{n} c_k(g) r_k \biggr\|_{L^{r'}_{X^*}} < \int_0^1 \biggl\langle f(t), \sum_{k=1}^{n} c_k(g) r_k(t) \biggr\rangle \, dt + \varepsilon.$$

It is easy to see that the integral on the right-hand side is equal to

$$\int_0^1 \biggl\langle \sum_{k=1}^{n} c_k(f) r_k(\xi), g(\xi) \biggr\rangle \, d\xi.$$

By virtue of (5.11) the latter integral, in turn, does not exceed $A_r \|g\|_{L^{r'}_{X^*}}$. Thus, X^* is $K_{r'}$-convex. The converse can be proved in a similar way. Assertion (ii) follows from (i) and the Kahane inequality.

For $r = 2$, the definition of K_r-convexity was introduced in the paper [56]. Following this paper, we refer to K_r-convex spaces ($1 < r < \infty$) just as K-convex spaces.

It follows from the definition that if X is K-convex and has a cotype $q \in [2, \infty)$, then X has the strong cotype q.

The spaces $L^1[0,1]$ and l^1 are not K-convex (they have the cotype 2 but have no non-trivial strong cotype). The spaces $L^p(\Omega, \mu)$ $(1 < p < \infty)$ are K-convex (this easily follows from (5.2)).

Using Proposition 5.22 and Lemma 5.24, we obtain the following proposition [56].

Proposition 5.25. *Let X be a Banach space. Then the following assertions are valid.*

(i) *X is K-convex if and only if X^* is K-convex.*

(ii) *If X is K-convex, then X has a type p, $1 < p \leqslant 2$, if and only if X^* has the cotype p'. In particular,*

$$\frac{1}{p_X} + \frac{1}{q_{X^*}} = \frac{1}{p_{X^*}} + \frac{1}{q_X} = 1. \tag{5.12}$$

For $X = l^1$, equality (5.12) fails. In [56], the following conjecture was put forward: K-convex spaces are the only spaces that do not uniformly contain l^1_n. This conjecture was proved by Pisier [67].

Theorem 5.26. *A Banach space X is K-convex if and only if X does not uniformly contain l^1_n.*

By virtue of Theorem 5.14, we see that a space is K-convex if and only if it has a non-trivial type.

§6. The Fourier type of Banach spaces with respect to groups

In § 4 we introduced the concept of Fourier type of a Banach space X; it means that the Hausdorff–Young inequality holds for the Fourier transform of functions defined on $\mathbb{R}$ and taking values in X. If we consider functions defined on an arbitrary locally compact Abelian group, then we arrive at a more general concept of the Fourier type. Some results have already been obtained in this field. The present section deals with these results.

We start with a brief survey of necessary definitions and facts from the theory of topological groups (see [69], [36], [72]).

A topological group is a group G equipped with a topology such that the group operations in G are continuous with respect to the topology.

For the basic group operation, we use the sum or the product symbol. Definitions, as a rule, are given for additive groups.

In what follows we suppose that G is an Abelian group and a locally compact Hausdorff space. It is well known (see, for example, [72], [36]) that on any compact Abelian group G there is a unique probability Borel measure μ_G that has the following invariance properties:

$$\mu_G(E + h) = \mu_G(E), \qquad \mu_G(-E) = \mu_G(E), \tag{6.1}$$

for any Borel set $E \subset G$ and any $h \in G$. The measure μ_G is called the *Haar measure* on the group G.

For a locally compact Abelian group, the Haar measure exists and is defined uniquely up to a positive constant factor.

The following examples are most important for us.

1) The circle

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

is a multiplicative group with the topology induced from $\mathbb{C}$.

2) The real line $\mathbb{R}$ with the sum operation and the ordinary topology.

3) The group $\mathbb{Z}$ of integers with the discrete topology.

4) The Cantor group $\mathbb{D} = \{0,1\}^{\mathbb{N}}$ is the Cartesian product of a sequence of second-order groups $G_n = \{0,1\}$ $(n = 1, 2, \dots)$. The elements of $\mathbb{D}$ are all possible sequences $\{\delta_n\}$ $(\delta_n = 0, 1)$. The set $\mathbb{D}$ is equipped with coordinatewise addition modulo 2 and the direct product topology. According to the Tikhonov theorem, the group $\mathbb{D}$ is compact. Therefore, the mapping

$$\{\delta_n\} \to 2\sum_{n=1}^{\infty} \delta_n 3^{-n}$$

is a homeomorphism of $\mathbb{D}$ onto the classical Cantor set.

Let G be a locally compact Abelian group. A *character* of G is a continuous homomorphism of G into the group $\mathbb{T}$, that is, a continuous complex-valued function γ on G such that

$$|\gamma(t)| = 1 \quad \text{and} \quad \gamma(t+s) = \gamma(t)\gamma(s) \quad (t, s \in G).$$

The set $\widehat{G}$ of all characters of the group G is a multiplicative Abelian group (with pointwise multiplication). The identity element in $\widehat{G}$ is a function that is identically equal to 1; the inverse element is $\gamma^{-1} = \overline{\gamma}$. On $\widehat{G}$ we introduce the topology in which the convergence is equivalent to the uniform convergence on compact subsets of G. The group $\widehat{G}$ is called the dual group of G.

A group equipped with a discrete topology is called a discrete group. We note the following property: if a group G is compact, then the group $\widehat{G}$ is discrete; conversely, if the group G is discrete, then $\widehat{G}$ is compact.

Let us consider some examples.

If $G = \mathbb{Z}$ and γ is a character on $\mathbb{Z}$, then

$$\gamma(n) = \Big(\gamma(1)\Big)^n \quad \text{for any} \quad n \in \mathbb{Z}.$$

The value $\gamma(1)$ can be any complex number $z \in \mathbb{T}$. Thus, we can identify $\widehat{\mathbb{Z}}$ with the group $\mathbb{T}$, $\widehat{\mathbb{Z}} = \mathbb{T}$.

Now let $G = \mathbb{T}$. It is clear that for any $n \in \mathbb{Z}$ the function $\gamma_n(z) = z^n$ $(z \in \mathbb{T})$ is a character of $\mathbb{T}$. One can prove that $\mathbb{T}$ has no other characters, and the multiplicative group $\widehat{\mathbb{T}}$ is isomorphic to the additive group $\mathbb{Z}$.

The characters of the group $G = \mathbb{R}$ are defined by the formula

$$\gamma_x(t) = e^{2\pi ixt} \quad (t \in \mathbb{R}) \quad \text{for some} \quad x \in \mathbb{R}.$$

This formula defines all characters of $\mathbb{R}$, and the group $\widehat{\mathbb{R}}$ is isomorphic to $\mathbb{R}$ (with the ordinary topology).

For $G = \mathbb{D}$, each character has the form

$$\{\delta_j\}_{j=1}^{\infty} \to (-1)^{\sum_{\mu=1}^{k} \varepsilon_\mu \delta_\mu},$$

where $k \in \mathbb{N}$ and ε_μ is equal to 0 or 1 ($\mu = 1, \dots, k$). Thus, the character is uniquely determined by the number

$$n = \sum_{\mu=1}^{k} \varepsilon_\mu 2^{\mu-1}. \tag{6.2}$$

The dual group $\widehat{\mathbb{D}}$ is isomorphic to the group $\mathbb{N}_0$ of non-negative integers, where the sum operation is defined as follows: if

$$n = \sum_{\mu=1}^{k} \varepsilon_\mu(n) 2^{\mu-1}, \qquad m = \sum_{\nu=1}^{l} \varepsilon_\nu(m) 2^{\nu-1},$$

are the binary expansions of numbers $n, m \in \mathbb{N}_0$, then

$$m + n = \sum_{\mu=1}^{\max(k,l)} [\varepsilon_\mu(m) \oplus \varepsilon_\mu(n)] 2^{\mu-1},$$

where $\oplus$ stands for addition modulo 2. It is obvious that for this definition of the operation $+$, for each $m \in \mathbb{N}_0$ we have $m + m = 0$ and $-m = m$. It is convenient to identify the group $\mathbb{N}_0$ with a special sequence of functions defined on $[0, 1]$ and assuming values $+1$ and -1. Let

$$w_0(t) = 1 \quad \text{for} \quad t \in [0, 1].$$

Next, the formula

$$t = \sum_{j=1}^{\infty} \delta_j 2^{-j}$$

establishes a one-to-one correspondence between non-periodic elements of $\mathbb{D}$ and binary irrational points $t \in [0, 1]$. Starting from the binary expansion (6.2) of a positive integer n, we put

$$w_n(t) = (-1)^{\sum_{\mu=1}^{k} \varepsilon_\mu(n) \delta_\mu} \quad (n = 1, 2, \dots). \tag{6.3}$$

Since $r_\mu(t) = (-1)^{\delta_\mu}$, we have

$$w_n(t) = \prod_{\mu=1}^{k} \big(r_\mu(t)\big)^{\varepsilon_\mu(n)} \quad (n = 1, 2, \dots). \tag{6.4}$$

Here t is any binary irrational point. In the questions under consideration, the values of w_n on the countable set of binary rational points are inessential; so, at these points we agree to define the functions w_n $(n \geqslant 1)$ by formula (6.4) as well. In particular,

$$w_{2^j}(t) = r_j(t) \quad (j \in \mathbb{N},\ t \in [0,1]). \tag{6.5}$$

The sequence of functions $\{w_n\}_{n=0}^{\infty}$ is called the Walsh system.

We have seen that the groups $\mathbb{Z}$ and $\mathbb{T}$ are duals of each other. This fact is a special case of the following theorem.

The Pontryagin duality principle. *For any locally compact Abelian group G with a dual group $\Gamma = \widehat{G}$, the natural mapping of G into $\widehat{\Gamma}$ that takes each element $t \in G$ to the character t' on Γ defined by the formula*

$$t'(\gamma) = \gamma(t), \qquad \gamma \in \widehat{G}, \tag{6.6}$$

is an isomorphism of the topological groups G and $\widehat{\Gamma}$.

In what follows, we identify $t \in G$ and $t' \in \widehat{\Gamma}$.

Let G be a locally compact Abelian group. The Fourier transform of a function $f \in L^1(G)$ is defined by the formula

$$\mathcal{F}_G(f)(\gamma) = \int_\Gamma f(t)\overline{\gamma(t)}\, d\mu_G(t), \qquad \gamma \in \widehat{G}. \tag{6.7}$$

We sometimes write $\widehat{f}(\gamma)$ in place of $\mathcal{F}(f)(\gamma)$. An analogue of the Hausdorff–Young theorem holds for locally compact Abelian groups, which makes it possible to define the Fourier transform for functions $f \in L^p(G)$, $1 < p \leqslant 2$ (the Fourier transform is a bounded operator from $L^p(G)$ to $L^{p'}(\widehat{G})$ [36], Ch. 8).

If X is a Banach space, then the Fourier transform of a function $f \in L^1_X(G)$ is defined as in (6.7), where the integral is understood in the sense of Bochner.

Let $1 \leqslant p \leqslant 2$. For linear combinations of the form

$$\sum_{j=1}^{n} \varphi_j(t)x_j \quad (x_j \in X,\ \varphi_j \in L^p(G),\ j = 1,\dots,n), \tag{6.8}$$

the Fourier transform is defined by the formula

$$\mathcal{F}\biggl(\sum_{j=1}^{n} \varphi_j x_j\biggr)(\gamma) = \sum_{j=1}^{n} \widehat{\varphi}_j(\gamma)x_j. \tag{6.9}$$

The set of functions of the form (6.8) (the tensor product $L^p(G) \otimes X$) is everywhere dense in $L^p_X(G)$ (see § 2).

Definition 6.1. We say that a space X has a Fourier type p with respect to the group G if the operator $\mathcal{F}$ originally defined on $L^p(G) \otimes X$ by formula (6.9) can be extended to a bounded operator

$$\mathcal{F}\colon L^p_X(G) \to L^{p'}_X(\widehat{G}).$$

The norm of the operator $\mathcal{F}$ is denoted by $C_p(X, G)$.

Definition 6.1 was introduced by Milman [59].

Remark 6.2. Let G be a discrete countable group, $G = \{h_k\}_{k=1}^{\infty}$. It is easy to see that in this case Definition 6.1 is equivalent to the assertion that for any vectors $x_1, \ldots, x_n \in X$ one has

$$\left(\int_{\widehat{G}} \left\|\sum_{k=1}^{n} \gamma(h_k)x_k\right\|^{p'} d\gamma\right)^{1/p'} \leqslant C\left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p}. \tag{6.10}$$

We consider the concept of Fourier type in terms of duality. Let G be a locally compact Abelian group. According to the duality principle, the Fourier transform of a function $\psi \in L^1(\widehat{G})$ is defined by the formula

$$\widehat{\psi}(t) = \int_{\widehat{G}} \psi(\gamma)\overline{\gamma(t)}\, d\mu_{\widehat{G}}(\gamma), \qquad t \in G. \tag{6.11}$$

Theorem 6.3. *A Banach space X has a Fourier type p $(1 \leqslant p \leqslant 2)$ with respect to the group G if and only if the dual space X^* has the Fourier type p with respect to $\widehat{G}$. Here $C_p(X, G) = C_p(X^*, \widehat{G})$.*

Proof. Suppose that X has the Fourier type p with respect to G. Then for any function $f \in L^p_X(G)$ one has

$$\left(\int_{\widehat{G}} \|\widehat{f}(\gamma)\|_X^{p'}\, d\gamma\right)^{1/p'} \leqslant C\left(\int_G \|f(t)\|_X^p\, dt\right)^{1/p}. \tag{6.12}$$

Let us show that the Fourier transform can also be extended to a bounded operator

$$\mathcal{F}\colon L^p_{X^*}(\widehat{G}) \to L^{p'}_{X^*}(G).$$

Let

$$g(\gamma) = \sum_{k=1}^{m} \psi_k(\gamma)x_k^* \quad \big(\psi_k \in L^p(\widehat{G}),\ x_k^* \in X^*\big). \tag{6.13}$$

The Fourier transform $\widehat{g} = \sum_{k=1}^{m} \widehat{\psi}_k x_k^*$ belongs to $L^{p'}_{X^*}(G)$ (see (6.11)). By Proposition 2.6, for any $\varepsilon > 0$ there is a function $f \in L^p_X(G)$ with $\|f\|_{L^p_X(G)} = 1$ such that

$$\left(\int_G \|\widehat{g}(t)\|_{X^*}^{p'}\, d\mu_G(t)\right)^{1/p'} \leqslant (1+\varepsilon)\left|\int_G \langle f(t), \widehat{g}(t)\rangle\, d\mu_G(t)\right|.$$

Using the Fubini theorem and taking into account (6.11), (6.7) and (2.13), we obtain

$$\int_G \langle f(t), \widehat{g}(t)\rangle\, d\mu_G(t) = \int_{\widehat{G}} \langle \widehat{f}(\overline{\gamma}), g(\gamma)\rangle\, d\mu_{\widehat{G}}(\gamma).$$

Therefore, by virtue of (6.12) and the invariance of the measure $\mu_{\widehat{G}}$ with respect to the inverse element (see (6.1)), we have

$$\begin{aligned}\biggl(\int_G \|\widehat{g}(t)\|_{X^*}^{p'}\,d\mu_G(t)\biggr)^{1/p'}&\\ &\leqslant (1+\varepsilon)\int_{\widehat{G}} \|\widehat{f}(\overline{\gamma})\|_X\|g(\gamma)\|_{X^*}\,d\mu_{\widehat{G}}(\gamma)\\ &\leqslant (1+\varepsilon)\biggl(\int_{\widehat{G}} \|\widehat{f}(\overline{\gamma})\|_X^{p'}\,d\mu_{\widehat{G}}(\gamma)\biggr)^{1/p'}\biggl(\int_{\widehat{G}} \|g(\gamma)\|_{X^*}^{p}\,d\mu_{\widehat{G}}(\gamma)\biggr)^{1/p}\\ &\leqslant C(1+\varepsilon)\|g\|_{L^p_{X^*}(\widehat{G})}.\end{aligned}$$

Since the set of functions of the form (6.13) is everywhere dense in $L^p_{X^*}(\widehat{G})$, we see that X^* has the Fourier type p with respect to $\widehat{G}$, and $C_p(X^*,\widehat{G})\leqslant C_p(X,G)$.

The converse assertion can be obtained in a very similar way. The theorem is thereby proved.

In the case $G=\mathbb{R}$, we see that the Banach space X has the Fourier type p $(1\leqslant p\leqslant 2)$ if and only if X^* has the Fourier type p (this result was obtained by Peetre [62]).

Let us now establish that if a Banach space X has a Fourier type p with respect to one of the groups $\mathbb{T}$, $\mathbb{R}$ or $\mathbb{Z}$, then it has the Fourier type p with respect to each of the groups.

We use the following relations:

$$\sum_{k\in\mathbb{Z}}\biggl(\frac{\sin\theta}{\theta+k\pi}\biggr)^2=1 \quad \text{for any} \quad \theta\in\mathbb{R} \tag{6.14}$$

and

$$B_r\equiv\inf_{\theta\in\mathbb{R}}\biggl(\sum_{k\in\mathbb{Z}}\biggl|\frac{\sin\theta}{\theta+k\pi}\biggr|^r\biggr)^{1/r}>0 \quad (r>1). \tag{6.15}$$

The first relation follows readily from Parseval's equality (1.5) applied to the 1-periodic function $\varphi_\theta(x)=e^{-i\theta(x-[x])}$ (where $[x]$ is the integral part of a number $x\in\mathbb{R}$ and θ is fixed). The second inequality is obvious.

Theorem 6.4. *A Banach space X has a Fourier type p $(1\leqslant p\leqslant 2)$ with respect to $\mathbb{R}$ if and only if X has the Fourier type p with respect to $\mathbb{Z}$. Here*

$$C_p(X,\mathbb{R})\leqslant C_p(X,\mathbb{Z})\leqslant B_{p'}^{-1}C_p(X,\mathbb{R}).$$

Proof. We first suppose that X has the Fourier type p with respect to $\mathbb{R}$. Then for any function $f\in L^p_X(\mathbb{R})$ we have

$$\biggl(\int_{\mathbb{R}}\|\widehat{f}(\xi)\|^{p'}\,d\xi\biggr)^{1/p'}\leqslant C\biggl(\int_{\mathbb{R}}\|f(t)\|^p\,dt\biggr)^{1/p}. \tag{6.16}$$

We must show that the Fourier transform can be extended to a bounded operator $\mathcal{F}\colon L^p_X(\mathbb{Z}) \to L^{p'}_X(\mathbb{T})$. To this end, it suffices to prove the inequality

$$\left(\int_0^1 \left\| \sum_{n\in\mathbb{Z}} x_n e^{2\pi int} \right\|^{p'} dt\right)^{1/p'} \leqslant C' \left(\sum_{n\in\mathbb{Z}} \|x_n\|^p\right)^{1/p}, \tag{6.17}$$

where $\{x_n\}$ is an arbitrary finite sequence of vectors from X.

We consider the function

$$f(t) = \sum_{n\in\mathbb{Z}} x_n \chi_{[n,n+1)}(t), \qquad t \in \mathbb{R}.$$

We have

$$\|f(t)\|^p = \sum_{n\in\mathbb{Z}} \|x_n\|^p \chi_{[n,n+1)}(t), \qquad t \in \mathbb{R},$$

and consequently,

$$\left(\int_{\mathbb{R}} \|f(t)\|^p\,dt\right)^{1/p} = \left(\sum_{n\in\mathbb{Z}} \|x_n\|^p\right)^{1/p}. \tag{6.18}$$

Further,

$$\widehat{f}(\xi) = \sum_{n\in\mathbb{Z}} x_n e^{-2\pi in\xi} \widehat{\chi}_{[0,1)}(\xi) = \sum_{n\in\mathbb{Z}} x_n e^{-2\pi in\xi}\, \frac{1-e^{-2\pi i\xi}}{2\pi i\xi}$$

and

$$\|\widehat{f}(\xi)\| = \frac{1}{\pi}\left|\frac{\sin \pi\xi}{\xi}\right| \left\| \sum_{n\in\mathbb{Z}} x_n e^{-2\pi in\xi} \right\|.$$

Hence we obtain

$$\begin{aligned}\int_{\mathbb{R}} \|\widehat{f}(\xi)\|^{p'}\,d\xi &= \sum_{k\in\mathbb{Z}} \int_k^{k+1} \left|\frac{\sin \pi\xi}{\pi\xi}\right|^{p'} \left\| \sum_{n\in\mathbb{Z}} x_n e^{-2\pi in\xi}\right\|^{p'} d\xi \\ &= \int_0^1 \sum_{k\in\mathbb{Z}} \left|\frac{\sin \pi\xi}{\pi(\xi+k)}\right| \left\| \sum_{n\in\mathbb{Z}} x_n e^{-2\pi in\xi}\right\|^{p'} d\xi,\end{aligned}$$

and, by virtue of (6.15),

$$\left(\int_0^1 \left\| \sum_{n\in\mathbb{Z}} x_n e^{-2\pi in\xi} \right\|^{p'} d\xi\right)^{1/p'} \leqslant B_{p'}^{-1} \left(\int_{\mathbb{R}} \|\widehat{f}(\xi)\|^{p'}\,d\xi\right)^{1/p'}.$$

Using (6.16) and (6.18), we obtain (6.17) (with the constant $C' = B_{p'}^{-1}C$).

We now suppose that X has the Fourier type p with respect to $\mathbb{Z}$, so that (6.17) holds. From this we must derive (6.16), that is, the fact that X also has the Fourier type p with respect to $\mathbb{R}$. Obviously, it suffices to consider functions of the form

$$f(t) = \sum_{n\in\mathbb{Z}} x_n \chi_{[n\delta,(n+1)\delta)}(t),$$

where $\{x_n\}$ is a finite sequence of vectors from X and $\delta > 0$. For such functions we have

$$\int_{\mathbb{R}} \|f(t)\|^p \, dt = \delta \sum_{n\in\mathbb{Z}} \|x_n\|^p \tag{6.19}$$

and

$$\widehat{f}(\xi) = \sum_{n\in\mathbb{Z}} x_n e^{-2\pi i n\delta\xi} \, \frac{1 - e^{-2\pi i \delta\xi}}{2\pi i \xi}\,.$$

As above, using (6.14), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \|\widehat{f}(\xi)\|^{p'} \, d\xi &= \delta^{p'-1} \int_{\mathbb{R}} \left|\frac{\sin \pi\xi}{\pi\xi}\right|^{p'} \left\|\sum_{n\in\mathbb{Z}} x_n e^{-2\pi i n\xi}\right\|^{p'} d\xi \\ &\leqslant \delta^{p'-1} \int_0^1 \left\|\sum_{n\in\mathbb{Z}} x_n e^{-2\pi i n\xi}\right\|^{p'} d\xi. \end{aligned}$$

By virtue of (6.17) and (6.19), we obtain

$$\left(\int_{\mathbb{R}} \|\widehat{f}(\xi)\|^{p'} \, d\xi\right)^{1/p'} \leqslant C' \left(\delta \sum_{n\in\mathbb{Z}} \|x_n\|^p\right)^{1/p} = C' \|f\|_{L^p_X(\mathbb{R})},$$

as desired. The theorem is proved.

Since the group $\mathbb{R}$ is isomorphic to its dual group, we readily obtain the following results by applying Theorems 6.3 and 6.4.

Theorem 6.5. *A Banach space X has a Fourier type p with respect to $\mathbb{R}$ if and only if X has the Fourier type p with respect to $\mathbb{T}$. Here*

$$C_p(X, \mathbb{R}) \leqslant C_p(X, \mathbb{T}) \leqslant B_{p'}^{-1} C_p(X, \mathbb{R}).$$

Theorem 6.6. *For any Banach space X, the following properties are equivalent:*

(1) *X has a Fourier type p with respect to the circle $\mathbb{T}$;*
(2) *X has the Fourier type p with respect to the real line $\mathbb{R}$;*
(3) *X has the Fourier type p with respect to the group $\mathbb{Z}$ of integers.*

Bourgain [13] showed that if a Banach space X has a Fourier type p with respect to $\mathbb{T}$, then X also has the Fourier type p with respect to $\mathbb{R}$. Theorem 6.6 was completely proved in the papers [1], [28], [47] independently.

We consider some questions related to direct products of groups. Let $G_1, \dots, G_n$ be locally compact Abelian groups and $G = G_1 \times \cdots \times G_n$ their direct product. If $\gamma_j \in \widehat{G}_j$ $(j = 1, \dots, n)$, then we define a character γ on G by the formula

$$\gamma(t_1, \dots, t_n) = \prod_{j=1}^{n} \gamma_j(t_j), \qquad (t_1, \dots, t_n) \in G. \tag{6.20}$$

It is not hard to show (see [36], Ch. 6) that any character on G is of the form (6.20) and the mapping $(\gamma_1, \dots, \gamma_n) \mapsto \gamma$ given by (6.20) is a topological isomorphism of the direct product $\widehat{G}_1 \times \cdots \times \widehat{G}_n$ onto the character group $\widehat{G}$.

Proposition 6.7. *If a Banach space X has a Fourier type p $(1 \leqslant p \leqslant 2)$ with respect to each of the groups G_j $(j = 1, \dots, n)$, then X has the Fourier type p with respect to the direct product $G = G_1 \times \dots \times G_n$ and*

$$C_p(X, G) \leqslant \prod_{j=1}^{n} C_p(X, G_j).$$

Proof. Obviously, it suffices to consider the case $n = 2$. Let $f \in L^1_X(G) \cap L^p_X(G)$. Then

$$\widehat{f}(\gamma_1, \gamma_2) = \iint_G f(t_1, t_2)\overline{\gamma_1(t_1)\gamma_2(t_2)}\, dt_1 dt_2 = \mathcal{F}_{G_2}\{\mathcal{F}_{G_1}[f(\cdot, t_2)](\gamma_1)\}(\gamma_2).$$

Using the Minkowski inequality $(p'/p \geqslant 1)$, we obtain

$$\begin{aligned}
&\left(\iint_{\widehat{G}} \|\widehat{f}(\gamma_1, \gamma_2)\|_X^{p'}\, d\gamma_1 d\gamma_2\right)^{p/p'} \\
&\quad = \left(\int_{\widehat{G}_1} d\gamma_1 \left(\int_{\widehat{G}_2} \big\|\mathcal{F}_{G_2}\{\mathcal{F}_{G_1}[f(\,\cdot\,, t_2)](\gamma_1)\}(\gamma_2)\big\|_X^{p'}\, d\gamma_2\right) d\gamma_1\right)^{p/p'} \\
&\quad \leqslant C_p^p(B, G_2)\left(\int_{\widehat{G}_1} \left(\int_{G_2} \big\|\mathcal{F}_{G_1}[f(\,\cdot\,, t_2)](\gamma_1)\big\|_X^{p}\, dt_2\right)^{p'/p} d\gamma_1\right)^{p/p'} \\
&\quad \leqslant C_p^p(X, G_2) \int_{G_2} \left(\int_{\widehat{G}_1} \big\|\mathcal{F}_{G_1}[f(\,\cdot\,, t_2)](\gamma_1)\big\|_X^{p'}\, d\gamma_1\right)^{p/p'} dt_2 \\
&\quad \leqslant C_p^p(X, G_2) C_p^p(X, G_1) \int_{G_2}\int_{G_1} \big\|f(t_1, t_2)\big\|_X^{p}\, dt_1\, dt_2.
\end{aligned}$$

The proposition is thereby proved.

Theorem 6.4 can be generalized to higher dimensions (the proof is similar).

Theorem 6.8. *Let $n \in \mathbb{N}$. A Banach space X has a Fourier type p with respect to $\mathbb{R}^n$ if and only if X has the Fourier type p with respect to $\mathbb{Z}^n$. Here*

$$C_p(X, \mathbb{R}^n) \leqslant C_p(X, \mathbb{Z}^n) \leqslant B_{p'}^{-n} C_p(X, \mathbb{R}^n).$$

Further, by applying Theorem 6.3, we also obtain the following results.

Theorem 6.9. *Let $n \in \mathbb{N}$. A Banach space X has a Fourier type p with respect to $\mathbb{R}^n$ if and only if X has the Fourier type p with respect to $\mathbb{T}^n$. Here*

$$C_p(X, \mathbb{R}^n) \leqslant C_p(X, \mathbb{T}^n) \leqslant B_{p'}^{-n} C_p(X, \mathbb{R}^n).$$

Theorem 6.10. *Let $n \in \mathbb{N}$ and let X be a Banach space. Then the following three properties are equivalent:*

(1) *X has a Fourier type p with respect to $\mathbb{T}^n$;*
(2) *X has the Fourier type p with respect to $\mathbb{R}^n$;*
(3) *X has the Fourier type p with respect to $\mathbb{Z}^n$.*

Let us consider an interesting fact related to direct products. For some groups Γ, the following property holds: if a Banach space X has a Fourier type p with respect to Γ, then the constant $C_p(X, \Gamma^N)$ is the same for all N.[2] For the groups $\mathbb{T}$ and $\mathbb{Z}$,

[2]The group $\mathbb{R}$ does not possess such a property, as follows from the results of Babenko [3] and Beckner [5] (see § 1).

this fact was proved by Andersson [1]. We shall obtain a more general result, which implies that the above-mentioned property is also valid for the groups $\mathbb{D}$ and $\mathbb{N}_0$.

Let Γ be a discrete countable Abelian group, and let $G = \widehat{\Gamma}$ (we recall that in this case the group G is compact). We treat elements of Γ as characters of G. Therefore, it is natural to take multiplication as the basic group operation in Γ. The measure of each singleton in Γ is assumed to be equal to 1.

We say that sets $\Omega, \Omega' \subset \Gamma$ are independent if for any $\gamma, \delta \in \Omega$ and $\gamma', \delta' \in \Omega'$ the equality $\gamma\gamma' = \delta\delta'$ holds if and only if $\gamma = \delta$ and $\gamma' = \delta'$.

Definition 6.11. A discrete countable Abelian group Γ is said to be *dissipative* if for any bounded set $\Omega \subset \Gamma$ there is a set $\Omega' \subset \Gamma$ such that

1) the sets Ω and Ω' are independent;
2) there is a bijective mapping $\psi\colon \Omega \to \Omega'$ such that for any set $\{x_\gamma\}_{\gamma\in\Omega}$ of elements of an arbitrary Banach space X and any $y > 0$ one has

$$\mu_G\biggl(\biggl\{t \in G : \biggl\|\sum_{\gamma\in\Omega} \gamma(t)x_\gamma\biggr\| > y\biggr\}\biggr) = \mu_G\biggl(\biggl\{t \in G : \biggl\|\sum_{\gamma\in\Omega} \psi(\gamma)(t)x_\gamma\biggr\| > y\biggr\}\biggr). \tag{6.21}$$

This definition and the Fubini theorem readily imply the following assertion.

Proposition 6.12. *Let Γ and Λ be discrete countable Abelian groups. If Γ and Λ are dissipative, then their direct product $\Gamma \times \Lambda$ is also dissipative.*

Let us show that the groups $\mathbb{Z}$ and $\mathbb{N}_0$ are dissipative.

Let Ω be a bounded subset of $\mathbb{Z}$. We put

$$m_0 = 2\max_{n\in\Omega}|n| + 1, \quad \Omega' = \{m_0 n\}_{n\in\Omega} \quad \text{and} \quad \psi(n) = m_0 n \quad (n \in \Omega).$$

It is clear that Ω and Ω' are independent (as subsets of the additive group). To each element $n \in \mathbb{Z}$ we assign the function $e^{2\pi int}$ $(0 \leqslant t < 1)$; condition (6.21) follows easily from the periodicity of these functions.

Next, let Ω be a bounded subset in $\mathbb{N}_0$. Let

$$n = \sum_{\mu=1}^{k} \varepsilon_\mu(n)2^{\mu-1}$$

be the binary expansion of a number n, $1 \leqslant n < 2^k$. We choose a k_0 such that the inequality $n < 2^{k_0}$ holds for all $n \in \Omega$. We define a mapping ψ as follows: if $0 \in \Omega$, then we put $\psi(0) = 0$; for any $n \in \Omega$, $n \neq 0$, we put

$$\psi(n) = \sum_{\mu=1}^{k} \varepsilon_\mu(n)2^{\mu+k_0}.$$

Let $\Omega' = \bigl\{\psi(n)\bigr\}_{n\in\Omega}$. It is clear that ψ is a bijection and Ω, Ω' are independent. Furthermore, the group $\mathbb{N}_0$ can be identified with the Walsh sequence $\{w_n\}_{n=0}^\infty$ of functions. For any $n \in \Omega$ $(n \neq 0)$, we have (see (6.4))

$$w_n(t) = \prod_{\mu=1}^{k} [r_\mu(t)]^{\varepsilon_\mu(n)} \quad \text{and} \quad w_{\psi(n)}(t) = \prod_{\mu=1}^{k} [r_{\mu+k_0+1}(t)]^{\varepsilon_\mu(n)}.$$

This yields (6.21).

Theorem 6.13. *Let Γ be a discrete countable dissipative Abelian group, and let a Banach space X have a Fourier type p with respect to Γ. Then for any positive integer N one has*

$$C_p(X, \Gamma^N) = C_p(X, \Gamma).$$

Proof. The inequality $C_p(X, \Gamma^N) \geqslant C_p(X, \Gamma)$ is obvious, since Γ can be viewed as a subgroup of Γ^N. For the same reason, and by virtue of Proposition 6.12, one can see that to prove the opposite inequality it suffices to consider the case $N = 2$.

Let $\Gamma \times \Gamma = \{\eta_k\}_{k=1}^{\infty}$. By virtue of Remark 6.2, we must prove that for any $n \in \mathbb{N}$ and any vectors $x_1, \dots, x_n \in X$ the following inequality holds ($\mu \equiv \mu_G$):

$$\left(\int_G \int_G \| \sum_{k=1}^{n} \eta_k(t,s) x_k \|^{p'} \, d\mu(t)\, d\mu(s)\right)^{1/p'} \leqslant C_p(X, \Gamma) \left(\sum_{k=1}^{n} \|x_k\|^p\right)^{1/p}. \tag{6.22}$$

We have (see (6.20))

$$\eta_k(t,s) = \gamma_k(t)\widetilde{\gamma}_k(s) \quad (\gamma_k, \widetilde{\gamma}_k \in \Gamma;\ t, s \in G). \tag{6.23}$$

Let Ω be the set of all elements γ_k, $\widetilde{\gamma}_k$ $(k = 1, \dots, n)$. It follows that there is a set $\Omega' \subset \Gamma$ and a bijection $\psi \colon \Omega \to \Omega'$ such that Ω and Ω' are independent and for any $x_1, \dots, x_n \in X$ one has

$$\mu\left(\left\{s \in G : \left\|\sum_{k=1}^{n} x_k \widetilde{\gamma}_k(s)\right\| > y\right\}\right) = \mu\left(\left\{s \in G : \left\|\sum_{k=1}^{n} x_k \psi(\widetilde{\gamma}_k)(s)\right\| > y\right\}\right) \ (y > 0). \tag{6.24}$$

We put $\beta_k = \psi(\widetilde{\gamma}_k)$. Using the Fubini theorem and (6.24), we have

$$\begin{aligned}
\int_G \int_G \left\|\sum_{k=1}^{n} \eta_k(t,s) x_k\right\|^{p'} d\mu(t)\, d\mu(s) &= \int_G \int_G \left\|\sum_{k=1}^{n} \gamma_k(t)\widetilde{\gamma}_k(s) x_k\right\|^{p'} d\mu(t)\, d\mu(s) \\
&= \int_G \int_G \left\|\sum_{k=1}^{n} \gamma_k(t)\beta_k(s) x_k\right\|^{p'} d\mu(t)\, d\mu(s) \\
&= \int_G \int_G \left\|\sum_{k=1}^{n} \gamma_k(t)\beta_k(s+t) x_k\right\|^{p'} d\mu(t)\, d\mu(s) \\
&= \int_G \int_G \left\|\sum_{k=1}^{n} \gamma_k(t)\beta_k(t)\beta_k(s) x_k\right\|^{p'} d\mu(t)\, d\mu(s)
\end{aligned}$$

for any $x_1, \dots, x_n \in X$. For a given $s \in G$, we put $x_k'(s) = \beta_k(s) x_k$; then $\|x_k'(s)\| = \|x_k\|$. Consider the integral

$$J(s) = \int_G \left\|\sum_{k=1}^{n} \gamma_k(t)\beta_k(t) x_k'(s)\right\|^{p'} dt.$$

The critical point is that the elements $\gamma_k\beta_k$ are distinct for different k: this follows directly from the independence of the sets Ω, Ω' and (6.23). Hence we have

$$J(s) \leqslant C_p^{p'}(X,\Gamma)\biggl(\sum_{k=1}^{n} \|x_k\|^p\biggr)^{p'/p}.$$

for any $s \in G$. This implies (6.22). The theorem is thereby proved.

We now can supplement the above-cited results[3].

Theorem 6.14. *Let a Banach space X have a Fourier type p with respect to one of the groups $\mathbb{T}$, $\mathbb{Z}$, or $\mathbb{R}$. Then for any positive integer n the following properties hold:*

(1) *X has the Fourier type p with respect to $\mathbb{T}^\infty$ and*

$$C_p(X,\mathbb{T}) = C_p(X,\mathbb{T}^n) = C_p(X,\mathbb{T}^\infty);$$

(2) *X has the Fourier type p with respect to $\mathbb{R}^\infty$ and*

$$C_p(X,\mathbb{R}^\infty) \leqslant C_p(X,\mathbb{T}), \qquad C_p(X,\mathbb{R}^\infty) \leqslant C_p(X,\mathbb{Z});$$

(3) *X has the Fourier type p with respect to $\mathbb{Z}^\infty$ and*

$$C_p(X,\mathbb{Z}) = C_p(X,\mathbb{Z}^n) = C_p(X,\mathbb{Z}^\infty).$$

By applying Theorems 6.3 and 6.13 and also Theorem 7.15 of § 7 of the present paper, we obtain the following result.

Theorem 6.15. *If a Banach space X has a Fourier type p with respect to one of the groups $\mathbb{D}$ and $\mathbb{N}_0$, then it also has the Fourier type p with respect to the other group. Moreover, for any positive integer n*

1) *X has the Fourier type p with respect to $\mathbb{D}^\infty$ and*

$$C_p(X,\mathbb{D}) = C_p(X,\mathbb{D}^n) = C_p(X,\mathbb{D}^\infty);$$

2) *X has the Fourier type p with respect to $\mathbb{N}_0^\infty$ and*

$$C_p(X,\mathbb{N}_0) = C_p(X,\mathbb{N}_0^n) = C_p(X,\mathbb{N}_0^\infty).$$

To conclude this section we dwell on the results of Bourgain [12] devoted to inequalities of Hausdorff–Young type for characters of compact Abelian groups. In the paper [12], the following theorem is proved.

[3]For definitions related to direct products of sequences of locally compact Abelian groups see [36], Ch. 2, [34], § 38, [37], Ch. 6.

Theorem 6.16. *Let G be a compact Abelian group. If a Banach space X has a non-trivial Rademacher type, then there are numbers $p > 1$, $q < \infty$, and C, $0 < C < \infty$, such that for any vectors $x_1, \dots, x_n \in X$ and any characters $\gamma_1, \dots, \gamma_n \in \widehat{G}$ one has*

$$\left(\int_G \left\| \sum_{j=1}^n \gamma_j(t) x_j \right\|^2 dt\right)^{1/2} \leqslant C\left(\sum_{j=1}^n \|x_j\|^p\right)^{1/p}$$

and

$$\left(\sum_{j=1}^n \|x_j\|^q\right)^{1/q} \leqslant C\left(\int_G \left\| \sum_{j=1}^n \gamma_j(t) x_j \right\|^2 dt\right)^{1/2}.$$

The proof of this theorem is quite difficult. Later, Bourgain [13] obtained stronger results for the Cantor group $\mathbb{D}$ and the circle $\mathbb{T}$. In § 8 we give a complete proof of the Bourgain theorem for the group $\mathbb{D}$.

§7. Types and cotypes of Banach spaces with respect to general orthonormal systems

7.1. Previously, we have studied the concepts of the Rademacher type and cotype and the concept of the Fourier type and its generalization to arbitrary locally compact Abelian groups. The latter generalization includes also classical orthonormal systems such as the trigonometric system and the Walsh system. It is natural to proceed further and to reveal the general laws that are not related to specific features of particular systems but are inherent in each uniformly bounded orthonormal system. The present section is devoted to this problem. The starting point of our definitions is the Riesz theorem (Theorem 1.2).

Let (I, μ) be a measure space, and let $\mu(I) = 1$. Further, let $\Phi = \{\varphi_n\}_{n=1}^\infty$ be an orthonormal system of scalar (real or complex-valued) functions on I. We suppose that the system Φ is uniformly bounded,

$$|\varphi_n(t)| \leqslant M \text{ for all } t \in I \text{ and all } n \in \mathbb{N}. \tag{7.1}$$

Let X be a Banach space. For each function $f \in L^1_X(I)$, we denote its Fourier coefficients by

$$c_n(f) = \int_I f(t)\overline{\varphi_n(t)}\, d\mu, \qquad n \in \mathbb{N}, \tag{7.2}$$

where the integral is interpreted in the sense of Bochner. An analogue of Mercer's theorem holds for complex-valued functions:

If the system $\{\varphi_n\}$ is uniformly bounded, then for any function $f \in L^1_X$ one has

$$\lim_{n\to\infty} c_n(f) = 0.$$

The proof is similar to that for the scalar case (see [4], Russian p. 74).

Definition 7.1. Let $1 \leqslant p \leqslant 2$. We say that a Banach space X has a Φ-type p if there is a constant C such that for any bounded set $\{x_k\}_{k=1}^n$ of elements $x_k \in X$ one has

$$\left(\int_I \left\| \sum_{k=1}^n \varphi_k(t) x_k \right\|^{p'} d\mu\right)^{1/p'} \leqslant C\left(\sum_{k=1}^n \|x_k\|^p\right)^{1/p}. \tag{7.3}$$

Remark 7.2. Let X have a Φ-type p. Then, as is easy to see, for any sequence $\{x_k\} \in l_X^p$ the series $\sum_{k=1}^\infty \varphi_k x_k$ converges in the norm of $L_X^{p'}(I)$, and its sum f satisfies the inequality

$$\|f\|_{L_X^{p'}} \leqslant C\|\{x_k\}\|_{l_X^p}. \tag{7.4}$$

Moreover, for any rearrangement of its elements, this series converges in $L_X^{p'}$ to the same sum f.

Definition 7.3. Let $1 \leqslant p \leqslant 2$. We say that a Banach space X has a Φ-cotype p' if there is a constant C such that for any polynomial $\sum_{k=1}^n x_k\varphi_k$ with coefficients $x_k \in X$ one has

$$\left(\sum_{k=1}^n \|x_k\|^{p'}\right)^{1/p'} \leqslant C\left(\int_I \left\| \sum_{k=1}^n x_k \varphi_k(t) \right\|^{p} d\mu\right)^{1/p}. \tag{7.5}$$

Definition 7.4. Let $1 \leqslant p \leqslant 2$. We say that a Banach space X has a strong Φ-cotype p' if there is a constant C such that for any function $f \in L_X^p(I)$ the sequence $\{c_n(f)\}$ of its Fourier coefficients satisfies the inequality

$$\left(\sum_{n=1}^\infty \|c_n(f)\|^{p'}\right)^{1/p'} \leqslant C\left(\int_I \|f(t)\|^p \, d\mu\right)^{1/p}. \tag{7.6}$$

Remark 7.5. The Rademacher system is denoted by R. In the case $\Phi = R$, our definitions are equivalent to the definitions of § 5. Further, let G be a compact Abelian group with countable dual group $\Gamma = \{\gamma_n\}_{n=1}^\infty$. It is easy to see that Γ is an orthonormal system of functions on G.[4] The Fourier type p ($1 \leqslant p \leqslant 2$) of the Banach space X with respect to Γ is none other than the Γ-type p in the sense of Definition 7.1 (see Remark 6.2). On the other hand, the Fourier type p of the space X with respect to G means the same as the strong Γ-cotype p' (see Definitions 6.1 and 7.4). For example, if $G = \mathbb{T}$, then the system of characters is the trigonometric system $T = \{e^{2\pi int}\}_{n\in\mathbb{Z}}$ ($t \in [0,1]$). The strong T-cotype p' ($1 \leqslant p \leqslant 2$) of the Banach space X coincides with a Fourier type p with respect to $\mathbb{T}$. Furthermore, by Theorem 6.6, the Fourier type p with respect to $\mathbb{T}$ is equivalent to the Fourier type p with respect to $\mathbb{Z}$. But the group $\widehat{\mathbb{T}}$ is isomorphic to $\mathbb{Z}$; therefore, by virtue of the preceding, the Fourier type p with respect to $\mathbb{Z}$ coincides

[4]By virtue of the Stone–Weierstrass theorem, ([36], Vol. 1, p. 151), the system Γ is complete in $L^1(G)$ (for the definition see 7.2).

with a T-type p. Thus, the T-type p and the strong T-cotype p' $(1 \leqslant p \leqslant 2)$ of the Banach space X are equivalent to each other. A similar assertion holds for the Walsh system; later on we shall return to these questions.

We recall that the concept of the Fourier type (with respect to $\mathbb{R}$) was for the first time introduced by Peetre [62]. Blasco and Pełczyński [9] introduced another definition of the Fourier type p (in terms of coefficients with respect to the trigonometric system T). In our terminology, this definition implies the strong T-cotype p' and is equivalent to the Fourier type with respect to the group $\mathbb{T}$. Theorem 6.5 implies the equivalence of the "continual" and "discrete" Fourier types studied in the cited papers.

7.2. In the general case, the Φ-cotype p' does not imply the strong Φ-cotype p' (this question was discussed in § 5 for the Rademacher system). However, these concepts are equivalent for any complete system Φ.

We say that Φ is a *complete* system in a space $L^p(I)$ $(1 \leqslant p < \infty)$ if the set of all polynomials $\sum_{k=1}^{n} \alpha_k \varphi_k$ with scalar coefficients is everywhere dense in $L^p(I)$.

Lemma 7.6. *Let X be a Banach space, and Φ an orthonormal system on I complete in the space $L^p(I)$ $(1 \leqslant p < \infty)$. Then for any function $f \in L^p_X(I)$ and any $\varepsilon > 0$ there is a polynomial $\psi = \sum_{k=1}^{n} x_k \varphi_k$ $(x_k \in X)$ such that $\|f - \psi\|_{L^p_X} < \varepsilon$.*

This assertion is obvious if f is a simple function, $f = \sum_{j=1}^{N} c_j \chi_{E_j}$ $(c_j \in X)$. It remains to apply Lemma 2.4.

Proposition 7.7. *Let Φ be a complete uniformly bounded system in $L^p(I)$ $(1 \leqslant p \leqslant 2)$. If the Banach space X has a Φ-cotype p', then X also has the strong Φ-cotype p'.*

Proof. Let $f \in L^p_X(I)$. For given positive integer N, we write

$$\sigma_N = \left(\sum_{k=1}^{N} \|c_k(f)\|^{p'} \right)^{1/p'}.$$

Let $\varepsilon > 0$. By Lemma 7.6, there is a polynomial $\psi = \sum_{k=1}^{n} x_k \varphi_k$ $(x_k \in X)$ such that

$$\|f - \psi\| < \frac{\varepsilon}{MN^{1/p'}}$$

(M is the constant in (7.1)). Then

$$\sigma_N \leqslant \left(\sum_{k=1}^{N} \|c_k(\psi)\|^{p'} \right)^{1/p'} + \varepsilon \leqslant \left(\sum_{k=1}^{n} \|x_k\|^{p'} \right)^{1/p'} + \varepsilon.$$

Since X has the Φ-cotype p', we have

$$\left(\sum_{k=1}^{n} \|x_k\|^{p'} \right)^{1/p'} \leqslant C\|\psi\|_{L^p_X} \leqslant C\big(\|f\|_{L^p_X} + \varepsilon\big).$$

Thus, for any $N \in \mathbb{N}$ and any $\varepsilon > 0$,

$$\sigma_N \leqslant C\|f\|_{L^p_X} + (C+1)\varepsilon.$$

This implies (7.6). The proof of the proposition is complete.

7.3. In Proposition 5.22 we have established a relation between the concepts of the Rademacher type and strong cotype in terms of duality of Banach spaces. A similar assertion for locally compact Abelian groups is contained in Theorem 6.3. We now cite the corresponding result for general orthonormal systems. As above, Φ is an orthonormal system on I satisfying the condition (7.1).

Theorem 7.8. *Let X be a Banach space, and let $1 \leqslant p \leqslant 2$. Then*

(i) *X has a Φ-type p if and only if X^* has the strong Φ-cotype p';*
(ii) *X^* has a Φ-type p if and only if X has the strong Φ-cotype p'.*

Proof. (i) Suppose that X has the Φ-type p. Let $g \in L^{p'}_{X^*}(I)$. We put

$$\sigma_N = \biggl(\sum_{k=1}^{N} \|c_k(g)\|^{p'}_{X^*}\biggr)^{1/p'}.$$

By virtue of Proposition 2.6, for any $\varepsilon > 0$ there are vectors $x_1, \dots, x_N \in X$ such that $\bigl(\sum_{k=1}^{N} \|x_k\|^p\bigr)^{1/p} = 1$ and

$$\sigma_N < (1+\varepsilon) \sum_{k=1}^{N} \langle x_k, c_k(g)\rangle.$$

Next, by (2.13) and (7.3) we have

$$\begin{aligned} \sum_{k=1}^{N} \langle x_k, c_k(g)\rangle &= \sum_{k=1}^{N} \biggl\langle x_k, \int_I g(t)\overline{\varphi_k(t)}\,d\mu \biggr\rangle = \int_I \biggl\langle \sum_{k=1}^{N} x_k\,\varphi_k(t), g(t) \biggr\rangle\,d\mu \\ &\leqslant \int_I \biggl\| \sum_{k=1}^{N} x_k\varphi_k(t) \biggr\|_X \|g(t)\|_{X^*}\,d\mu \leqslant \biggl\| \sum_{k=1}^{N} x_k\varphi_k \biggr\|_{L^{p'}_X} \|g\|_{L^p_{X^*}} \leqslant C\|g\|_{L^p_{X^*}}. \end{aligned}$$

It follows from the resulting estimates that X has the strong Φ-type p. The converse of (i) can be proved in a similar way. We obtain assertion (ii) in the same way. The theorem is thereby proved.

Remark. Let X be a complex space. If multiplication by complex numbers in X^* is defined by the formula

$$(\alpha g)(a) = \alpha g(a) \quad (g \in X^*,\ a \in X),$$

then the Φ-type p of the space X is equivalent to the strong $\overline{\Phi}$-cotype p' of the space X^*, where $\overline{\Phi} = \{\overline{\varphi}_n\}$ is a system of complex conjugate functions. This is the case indeed if Φ is the system of all characters of a compact Abelian group G. This is also true if Φ is an arbitrary uniformly bounded orthonormal system and X is an involution space (see [44], p. 475). We point out that no example of a system Φ and a space X such that X has the Φ-type p, but does not have the $\overline{\Phi}$-type p, is available to the author.

In connection with Theorem 7.8, we point out that, generally speaking, the Φ-cotype p' of the space X^* does not imply the Φ-type p of the space X; for example, this is the case for $\Phi = R$ (see § 5).

7.4. It is obvious that each Banach space has the Φ-type 1 and the strong Φ-cotype ∞ for any uniformly bounded orthonormal system Φ. Using interpolation, we see that the properties of the Φ-type p and the strong Φ-cotype p' become stronger with p increasing ($1 \leqslant p \leqslant 2$).

Theorem 7.9. *Let $1 < p \leqslant 2$, and let a Banach space X have the Φ-type p. Then X has the Φ-type r for any $1 \leqslant r < p$.*

Proof. Using Remark 7.2, on l^p_X we define a linear operator J by putting

$$J\colon \{x_n\} \mapsto \sum_{n=1}^{\infty} x_n \varphi_n \quad (\text{convergence in } L^{p'}_X).$$

By virtue of (7.4), the operator J acts boundedly from l^p_X into $L^{p'}_X(I)$. Let $\overset{\circ}{L}{}^{\infty}_X(I)$ stand for the closure of the set of simple functions from $L^{\infty}_X(I)$ with respect to the norm of this space. It readily follows from (7.1) that J acts boundedly from l^1_X into $\overset{\circ}{L}{}^{\infty}_X(I)$. By Theorem 3.5, we have

$$[l^1_X, l^p_X]_\theta = l^r_X, \quad [L^{p'}_X, \overset{\circ}{L}{}^{\infty}_X]_\theta = L^{r'}_X \quad (\theta = p'/r').$$

Applying Theorem 3.4, we see that the operator J acts boundedly from l^r_X into $L^{r'}_X(I)$. Hence, X has the Φ-type r. The proof of the theorem is complete.

We note that in the case of the Rademacher system, Theorem 7.9 readily follows from the Jensen inequality (see Proposition 5.7). Of course, the Khinchine-Kahane inequalities are of the first importance here; in the general case, the use of the Jensen inequality on the right-hand side of (7.3) gives nothing.

Theorems 7.8 and 7.9 imply the following assertion.

Theorem 7.10. *Let $1 < p \leqslant 2$, and let a Banach space X have the strong Φ-cotype p'. Then for any $1 \leqslant r < p$, X has the strong Φ-cotype r'.*

Naturally, to prove this theorem one can directly apply interpolation. Let $\mathcal{F}$ be the operator that makes each function $f \in L^1_X(I)$ correspond to the sequence of its Fourier coefficients with respect to the system Φ. We denote by $\overset{\circ}{l}{}^{\infty}_X$ the set of all sequences $\{x_n\} \in l^\infty_X$ with $\lim_{n\to\infty} \|x_n\| = 0$ (this is a closed subset of l^∞_X). By Mercer's theorem (see 7.1), the operator $\mathcal{F}$ acts boundedly from $L^1_X(I)$ to $\overset{\circ}{l}{}^{\infty}_X$. Moreover, under our hypothesis, $\mathcal{F}$ is a bounded operator from $L^p_X(I)$ to $l^{p'}_X$. By virtue of Theorem 3.5, we have

$$[L^1_X, L^p_X]_\theta = L^r_X, \qquad [l^{p'}_X, \overset{\circ}{l}{}^{\infty}_X]_\theta = l^{r'}_X,$$

for $\theta = p'/r'$. It remains to apply Theorem 3.4.

Let us note that in the case of the Rademacher system the similar assertion for the cotype is valid and, moreover, directly follows from the Jensen inequality.

Theorems 7.9 and 7.10 can also be proved by means of real interpolation. In this case, even stronger results can be obtained.

Theorem 7.11. *Let $1 < p \leqslant 2$, and let X be a Banach space. Then*

(1) *if X has the Ψ-type p, then for any $1 < r < p$ and $1 \leqslant s < \infty$ there is a constant K such that for each finite sequence $\{x_k\}$ $(x_k \in X)$ one has*

$$\|f\|_{L_X^{r',s}} \leqslant K \,\|\{x_k\}\|_{l_X^{r,s}} \qquad \biggl(f = \sum_{k=1}^{\infty} x_k \varphi_k\biggr); \tag{7.7}$$

(2) *if X has the strong Ψ-cotype p', then for any $1 < r < p$ and $1 \leqslant s < \infty$ there is a constant K such that for each function $f \in L_X^{r,s}(I)$ one has*

$$\|\{c_n(f)\}\|_{l_X^{r',s}} \leqslant K \,\|f\|_{L_X^{r,s}}. \tag{7.8}$$

To prove the theorem, it suffices to use the same reasoning as above and apply Corollary 3.13 (and the remark following the corollary) and then Theorem 3.11.

By virtue of inequality (2.6), for $r = s$ the assertions of Theorem 7.11 are stronger than those of Theorems 7.9 and 7.10. We note that in this case inequalities (7.7) and (7.8) acquire the form

$$\biggl(\int_0^1 \xi^{r-2} f^*(\xi)^r \, d\xi\biggr)^{1/r} \leqslant K\biggl(\sum_{n=1}^{\infty} \|x_n\|^r\biggr)^{1/r} \tag{7.9}$$

and, respectively,

$$\biggl(\sum_{n=1}^{\infty} n^{r-2} c_n^*(f)^r\biggr)^{1/r} \leqslant K\biggl(\int_I \|f(t)\|^r \, d\mu(t)\biggr)^{1/r}. \tag{7.10}$$

These are vector-valued analogues of the Paley inequalities (see [82], Ch. 12).

It is not hard to prove Theorem 7.11 straightforwardly by using the corresponding estimates for rearrangements. For example, assertion (1) of this theorem readily follows from the inequality

$$f^*\biggl(\frac{1}{n}\biggr) \leqslant M \sum_{k=1}^{n} x_k^* + C n^{1/p'} \biggl(\sum_{k=n+1}^{\infty} (x_k^*)^p\biggr)^{1/p}, \qquad n \in \mathbb{N}, \tag{7.11}$$

and from Lemma 2.3. For the scalar case, inequality (7.11) is proved in the paper [60]; for the vector-valued case, the proof is similar.

The best possible value of type and cotype is 2 (see § 5). If H is a Hilbert space, then for any orthonormal system Φ, H has the Φ-type 2 and Φ-cotype 2. Indeed, by virtue of the orthogonality of the functions φ_k, we have

$$\int_I \biggl\|\sum_{k=1}^{n} \varphi_k(t) x_k\biggr\|_H^2 dt = \sum_{k=1}^{n} \|x_k\|_H^2.$$

In § 5 we cited the Kwapień theorem [50], by which a Banach space X that has the R-type 2 and R-cotype 2 is isomorphic to a Hilbert space. In § 8 we shall show that in the hypotheses of the theorem the Rademacher system can be replaced by any uniformly bounded system Φ. In the paper [50] a similar assertion was proved for complete systems. Namely, the following result was obtained in [50].

Theorem 7.12. *Let X be a Banach space and let $\{f_n\}$ be a complete orthonormal system of functions in $L^2([0,1])$. The space X is isomorphic to a Hilbert space if and only if there is a constant $C > 0$ such that for any $n \in \mathbb{N}$ and any vectors $x_1, \dots, x_n \in X$ one has*

$$C^{-1} \sum_{k=1}^{n} \|x_k\|^2 \leqslant \int_0^1 \left\| \sum_{k=1}^{n} f_k(t) x_k \right\|^2 dt \leqslant C \sum_{k=1}^{n} \|x_k\|^2.$$

For an arbitrary orthonormal system, this assertion does not hold (it suffices to take any sequence $\{f_n\}$ of functions on $[0,1]$ with pairwise disjoint supports and with $\|f_n\|_2 = 1$).

7.5. In the case of the Rademacher system there is no relation between the R-type and the R-cotype of a given space X (see Proposition 5.10). However, as was already mentioned above, for the trigonometric system

$$T = \left\{e^{2\pi i n t}\right\}_{n \in \mathbb{Z}}, \qquad t \in [0,1],$$

the following theorem holds.

Theorem 7.13. *A Banach space X has a T-type p $(1 \leqslant p \leqslant 2)$ if and only if X has the strong T-cotype p'.*

This theorem follows from Theorem 6.6 (see Remark 7.5). Of course, we can give a straightforward proof of Theorem 7.13 by applying the scheme of the proof of Theorem 6.4.

Let us show that a similar assertion is valid for the Walsh system $W = \{w_n\}_{n=0}^{\infty}$. The definition of the system is given in § 6. We write

$$\Delta_j^{(n)} = \left(\frac{j}{2^n}, \frac{j+1}{2^n}\right), \qquad 0 \leqslant j \leqslant 2^n - 1, \quad n = 0, 1, \dots .$$

For $0 \leqslant k < 2^n$, the function $w_k(t)$ assumes a constant value equal to 1 or -1 on each of the intervals $\Delta_j^{(n)}$, $0 \leqslant j < 2^n$; we denote this value by $\alpha_{kj}^{(n)}$. The matrix $(\alpha_{kj}^{(n)})$ of order $2^n \times 2^n$ is symmetric, and its columns are pairwise orthogonal (see [32], Ch. 1). Thus, for all $t \neq i2^{-n}$ $(i = 0, 1, \dots, 2^n)$ the following equalities are valid:

$$w_k(t) = \sum_{j=0}^{2^n - 1} \alpha_{kj}^{(n)} \chi_{\Delta_j^{(n)}}(t), \qquad 0 \leqslant k < 2^n, \tag{7.12}$$

$$\chi_{\Delta_j^{(n)}}(t) = 2^{-n} \sum_{k=0}^{2^n - 1} \alpha_{kj}^{(n)} w_k(t), \qquad 0 \leqslant j < 2^n. \tag{7.13}$$

Theorem 7.14. *A Banach space X has a W-type p $(1 \leqslant p \leqslant 2)$ if and only if X has the strong W-cotype p'.*

Proof. Let $x_j \in X$ $(j = 0, 1, \dots, 2^n - 1)$. For any $1 < r < \infty$, by virtue of (7.13) we have

$$\Big(\sum_{j=0}^{2^n-1} \|x_j\|^{r'}\Big)^{1/r'} = 2^{n/r'} \Big\| \sum_{j=0}^{2^n-1} x_j \chi_{\Delta_j^{(n)}} \Big\|_{L_X^{r'}} = 2^{-n/r} \Big\| \sum_{k=0}^{2^n-1} y_k w_k \Big\|_{L_X^{r'}}, \tag{7.14}$$

where $y_k = \sum_{j=0}^{2^n-1} \alpha_{kj}^{(n)} x_j$. Next, taking into account the symmetry of the matrix $(\alpha_{kj}^{(n)})$ and (7.12), we obtain

$$\begin{aligned}\Big(\sum_{k=0}^{2^n-1} \|y_k\|^{r}\Big)^{1/r} &= 2^{n/r} \Big\| \sum_{k=0}^{2^n-1} y_k \chi_{\Delta_k^{(n)}} \Big\|_{L_X^{r}} \\ &= 2^{n/r} \Big\| \sum_{k=0}^{2^n-1} \sum_{j=0}^{2^n-1} \alpha_{kj}^{(n)} x_j \chi_{\Delta_k^{(n)}} \Big\|_{L_X^{r}} = 2^{n/r} \Big\| \sum_{j=0}^{2^n-1} x_j w_j \Big\|_{L_X^{r}}.\end{aligned} \tag{7.15}$$

Suppose that X has the W-type p. Then

$$\Big\| \sum_{k=0}^{2^n-1} y_k w_k \Big\|_{L_X^{p'}} \leqslant C \Big(\sum_{k=0}^{2^n-1} \|y_k\|^{p}\Big)^{1/p}. \tag{7.16}$$

By putting $r = p$, we obtain (see (7.14)–(7.16))

$$\Big(\sum_{j=0}^{2^n-1} \|x_j\|^{p'}\Big)^{1/p'} \leqslant C \Big\| \sum_{j=0}^{2^n-1} x_j w_j \Big\|_{L_X^{p}}.$$

Thus, X has the W-cotype p', whence, by Proposition 7.7, it follows that X has the strong W-cotype p'.

We now suppose that X has the W-cotype p'. Then

$$\Big(\sum_{k=0}^{2^n-1} \|y_k\|^{p'}\Big)^{1/p'} \leqslant C' \Big\| \sum_{k=0}^{2^n-1} y_k w_k \Big\|_{L_X^{p}}. \tag{7.17}$$

By putting $r = p'$, by virtue of (7.14), (7.15), and (7.17), we have

$$\Big\| \sum_{j=0}^{2^n-1} x_j w_j \Big\|_{L_X^{p'}} \leqslant C' \Big(\sum_{j=0}^{2^n-1} \|x_j\|^{p}\Big)^{1/p}.$$

Therefore, X has the W-type p. The theorem is thereby proved.

Let us consider the Cantor group $\mathbb{D}$. Its dual group $\mathbb{N}_0$ can be identified with the Walsh system W of functions. For any Banach space X, the Fourier type p with respect to the group $\mathbb{N}_0$ coincides with the W-type p, and the Fourier type p with respect to $\mathbb{D}$ coincides with the strong W-cotype p' (see Remark 7.5). Thus, we obtain the following result (see [29]).

Theorem 7.15. *Let $1 \leqslant p \leqslant 2$. A Banach space X has the Fourier type p with respect to the group $\mathbb{D}$ if and only if X has the Fourier type p with respect to the dual group $\mathbb{N}_0$.*

It seems interesting to obtain the description of complete uniformly bounded orthonormal systems (compact Abelian groups), for which analogues of Theorems 7.13–7.15 hold.

7.6. Let $\{X_0, X_1\}$ be an interpolation pair of Banach spaces. Peetre [62] demonstrated that if the space X_0 has a Fourier type p_0 and the space X_1 a Fourier type p_1 $(1 \leqslant p_0, p_1 \leqslant 2)$, then for any $0 < \theta < 1$ the space $[X_0, X_1]_\theta$ has the Fourier type p, where $1/p = (1-\theta)/p_0 + \theta/p_1$. A similar theorem is also valid for general orthonormal systems Φ satisfying condition (7.1).

Theorem 7.16. *Suppose that $\{X_0, X_1\}$ is an interpolation pair of Banach spaces, $1 \leqslant p_0, p_1 \leqslant 2$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, and $X = [X_0, X_1]_\theta$. Then the following assertions hold:*

(i) *if X_j has a Φ-type p_j $(j = 0, 1)$, then X has the Φ-type p;*
(ii) *if X_j has a strong Φ-cotype p_j' $(j = 0, 1)$, then X has the strong Φ-cotype p'.*

Proof. (i) First, let $p_0, p_1 > 1$. We put $A_j = l^{p_j}_{X_j}$, $B_j = L^{p_j'}_{X_j}(I)$ $(j = 0, 1)$. It follows from Remark 7.2 that for any sequence $\{x_n\} \in A_0 + A_1$ the series

$$\sum_{n=1}^{\infty} x_n \varphi_n \tag{7.18}$$

converges in the norm of the space $B_0 + B_1$. Let J be the operator that takes each sequence $\{x_n\} \in A_0 + A_1$ to the sum of the series (7.18). By virtue of (7.4), the restriction of J to A_j is a bounded operator from A_j to B_j. By Theorem 3.4, the operator J acts boundedly from $[A_0, A_1]_\theta$ to $[B_0, B_1]_\theta$. But by Theorem 3.5,

$$[l^{p_0}_{X_0}, l^{p_1}_{X_1}]_\theta = l^p_X, \qquad [L^{p_0'}_{X_0}, L^{p_1'}_{X_1}]_\theta = L^{p'}_X.$$

This proves assertion (i) (for the case $p_0, p_1 > 1$). If, for example, $p_1 = 1$, then we put $B_1 = \overset{\circ}{L}{}^\infty_X$ and take into account the fact that J acts boundedly from A_1 to B_1 and $[L^{p_0'}_{X_0}, \overset{\circ}{L}{}^\infty_X] = L^{p'}_X$ (see Theorem 3.5).

Assertion (ii) can be proved in a similar way.

It is clear that Theorems 7.9 and 7.10 are contained in Theorem 7.16 (for the case $X_0 = X_1$).

We now consider the real interpolation of spaces with given types (strong cotypes). Here things become more complicated because of the fact that, generally speaking, the equality

$$(L^{p_0}_{X_0}, L^{p_1}_{X_1})_{\theta,r} = L^p_{(X_0,X_1)_{\theta,r}}$$

holds only for $r = p$ (see § 3). However, the following lemma is valid [47] (where spaces of vector-valued functions defined on an arbitrary measure space (S, ν) are considered).

Lemma 7.17. *Let* $1 < p_0, p_1 \leqslant 2$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $p \leqslant r \leqslant p'$, *and* $X = (X_0, X_1)_{\theta,r}$. *Then for any interpolation pair* $\{X_0, X_1\}$ *of Banach spaces, the continuous embeddings*

$$L^p_X \hookrightarrow \Big(L^{p_0}_{X_0}, L^{p_1}_{X_1}\Big)_{\theta,r}, \tag{7.19}$$

$$\big(L^{p_0'}_{X_0}, L^{p_1'}_{X_1}\big)_{\theta,r} \hookrightarrow L^{p'}_X \tag{7.20}$$

hold.

Proof. We consider the embedding (7.19). Let us use the L-method. We put $\eta = \theta p/p_1$. Let $f \in L^p_X$. By Theorem 3.8, for any $t \in S$ there is a representation of an element $f(t) \in X$

$$f(t) = f^0_\xi(t) + f^1_\xi(t), \qquad 0 < \xi < \infty,$$

such that

$$\|f(t)\|_X \geqslant C\bigg(\int_0^\infty \big[\xi^{-\eta}\big(\|f^0_\xi(t)\|^{p_0}_{X_0} + \xi\|f^1_\xi(t)\|^{p_1}_{X_1}\big)\big]^{r/p}\,\frac{d\xi}{\xi}\bigg)^{1/r},$$

where C is a positive constant. Hence, by virtue of the condition $p \leqslant r$ and the Minkowski inequality, we have

$$\begin{aligned}
\|f\|^p_{L^p_X} &\geqslant C^p \int_S \bigg(\int_0^\infty \big[\xi^{-\eta}\big(\|f^0_\xi(t)\|^{p_0}_{X_0} + \xi\|f^1_\xi(t)\|^{p_1}_{X_1}\big)\big]^{r/p}\frac{d\xi}{\xi}\bigg)^{p/r} d\nu(t)\\
&\geqslant C^p \bigg(\int_0^\infty \xi^{-\eta r/p}\bigg[\int_S \big(\|f^0_\xi(t)\|^{p_0}_{X_0} + \xi\|f^1_\xi(t)\|^{p_1}_{X_1}\big)\,d\nu(t)\bigg]^{r/p}\frac{d\xi}{\xi}\bigg)^{p/r}\\
&= C^p \bigg(\int_0^\infty \big[\xi^{-\eta}\big(\|f^0_\xi\|^{p_0}_{L^{p_0}_{X_0}} + \xi\|f^1_\xi\|^{p_1}_{L^{p_1}_{X_1}}\big)\big]^{r/p}\frac{d\xi}{\xi}\bigg)^{p/r}.
\end{aligned}$$

Applying Theorem 3.8 again, we obtain

$$\|f\|_{L^p_X} \geqslant C'\|f\|_{(L^{p_0}_{X_0}, L^{p_1}_{X_1})_{\theta,r}} \quad (C' > 0).$$

This implies the embedding (7.19). The embedding (7.20) is obtained by analogy. The lemma is proved.

In the discrete case $S = \mathbb{N}$, the embeddings (7.19) and (7.20) are of the form

$$l^p_X \hookrightarrow (l^{p_0}_{X_0}, l^{p_1}_{X_1})_{\theta,r}, \tag{7.21}$$

$$(l^{p_0'}_{X_0}, l^{p_1'}_{X_1})_{\theta,r} \hookrightarrow l^{p'}_X. \tag{7.22}$$

Theorem 7.18. *Let $\{X_0, X_1\}$ be an interpolation pair of Banach spaces. Suppose that $1 < p_0, p_1 \leqslant 2$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, $p \leqslant r \leqslant p'$, and $X = (X_0, X_1)_{\theta,r}$. Then the following assertions hold:*

(i) *if X_j has a Φ-type p_j $(j = 0, 1)$, then X has the Φ-type p;*
(ii) *if X_j has a strong Φ-cotype p_j' $(j = 0, 1)$, then X has the strong Φ-cotype p'.*

Proof. (i) As in the proof of Theorem 7.15, we put $A_j = l^{p_j}_{X_j}$, $B_j = L^{p_j'}_{X_j}(I)$ $(j = 0, 1)$ and define the operator $J\colon A_0 + A_1 \to B_0 + B_1$ that takes each sequence $\{x_n\} \in A_0 + A_1$ to the sum of the series (7.18) (convergence in $B_0 + B_1$). By virtue of (7.4), the restriction of J to A_j is a bounded operator from A_j to B_j $(j = 0, 1)$. By Theorem 3.11, the operator J acts boundedly from $(A_0, A_1)_{\theta,r}$ to $(B_0, B_1)_{\theta,r}$. But then, by virtue of (7.20) and (7.21), J is a bounded operator from l^p_X to $L^{p'}_X(I)$, and so X has the Φ-type p.

Assertion (ii) is obtained similarly. The theorem is proved.

In the case of the trigonometric system, the T-type p is equivalent to the strong T-cotype p'; these properties, in turn, are equivalent to the strong Fourier type p in the sense of Peetre (see Remark 7.5). For $\Phi = T$, Theorem 7.18 was obtained in the paper [47] (in terms of the Fourier type).

7.7. Let (Ω, ν) be an arbitrary measure space. In § 4 we showed that (see Proposition 4.4) the space $L^p(\Omega, \nu)$ $(1 < p < \infty)$ has the Fourier type $\min(p, p')$. By Remark 7.5, this implies that L^p has the T-type $\min(p, p')$ and the strong T-cotype $\max(p, p')$. A similar result holds for general orthonormal systems.

Proposition 7.19. *A space $L^p(\Omega, \nu)$ $(1 < p < \infty)$ has the Φ-type $\min(p, p')$ and the strong Φ-cotype $\max(p, p')$ for any uniformly bounded orthonormal system Φ.*

Proof. As was noted in 7.4, the space $L^2(\Omega, \nu)$ has the Φ-type 2. Moreover, $L^1(\Omega, \nu)$ has the Φ-type 1. Let $1 < p < 2$. We apply the interpolation, putting $\theta = 2/p'$. By Theorem 3.5, $L^p = [L^1, L^2]_\theta$. By Theorem 7.15, this implies that $L^p(\Omega, \nu)$ has the Φ-type p. By analogy, we see that for $p > 2$ the space $L^p(\Omega, \nu)$ has the Φ-type p'. By virtue of the duality and Theorem 7.8, the assertion on the strong Φ-cotype follows from the assertion on the Φ-type. The proof of the proposition is complete.

Let us note that Proposition 7.19 can readily be derived from Theorem 1.2 by using the generalized Minkowski inequality (see the proof of Proposition 4.4).

In the general case, Proposition 7.19 cannot be strengthened. Namely, the following proposition holds.

Proposition 7.20. *Suppose that a space Ω is not the union of finitely many ν-atoms. Let Φ be a uniformly bounded orthonormal system on I, and let $1 \leqslant p < s \leqslant 2$. Then*

(1) *the space $L^p(\Omega, \nu)$ does not have the Φ-type s and does not have the T- and W-cotype s';*
(2) *the space $L^{p'}(\Omega, \nu)$ does not have the T- and W-type s and does not have the Φ-cotype s'.*

Proof. The fact that $L^p(\Omega,\nu)$ does not have the Φ-type s follows from Proposition 5.10 and Theorem 8.2. Furthermore, if $L^p(\Omega,\nu)$ has the T-cotype s' or, respectively, W-cotype s', then by Proposition 7.7 and Theorems 7.8, 7.13, 7.14 the space $L^{p'}(\Omega,\nu)$ has the T-cotype s'. But then, by Theorem 8.2, $L^{p'}(\Omega,\nu)$ has the R-cotype s', which contradicts Proposition 5.10 (see also Remark 5.11). We obtain assertion (1). Assertion (2) can be proved similarly.

In Proposition 7.20, the trigonometric system T or the Walsh system W cannot be replaced by an arbitrary orthonormal system Φ (we recall that by Proposition 5.9, a space $L^r(\Omega,\nu)$, $1<r<\infty$, has the R-cotype $\max(r,2)$ and R-type $\min(r,2)$).

In the papers [47], [28], [29] the authors show that the space $L^{p,r}(\Omega,\nu)$ $(1<p,r<\infty)$ has the Fourier type $\min(p,p',r,r')$. A similar result is valid for general uniformly bounded orthonormal systems.

Proposition 7.21. *Let $1<p,r<\infty$. Then the space $L^{p,r}(\Omega)$ has the Φ-type $\min(p,p',r,r')$ and the strong Φ-cotype $\max(p,p',r,r')$ for any uniformly bounded orthonormal system Φ.*

Proof. For example, let $1<p\leqslant 2$. We first suppose that $p\leqslant r\leqslant p'$. If $p=2$, then there is nothing to prove; therefore, we suppose that $p<2$. We take $1<p_0<p$ and choose θ from the condition

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{2}\,.$$

By virtue of Corollary 3.13, we have $(L^{p_0},L^2)_{\theta,r}=L^{p,r}$. Next, L^{p_0} has the Φ-type p_0, and L^2 the Φ-type 2. Therefore, by Theorem 7.18, $L^{p,r}$ has the Φ-type p. By analogy, we see that $L^{p,r}$ has the strong Φ-cotype p'.

Now let $1<r<p\leqslant 2$. We take numbers $r<p_0<p$ and $2<p_1<r'$. We choose θ from the condition

$$\frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\,.$$

Then $(L^{p_0},L^{p_1})_{\theta,r}=L^{p,r}$ (see Corollary 3.13). Here L^{p_0} has the Φ-type p_0, and L^{p_1} has the Φ-type p_1'. Since $p_0,p_1'\geqslant r$, we see that both of these spaces have the Φ-type r. By applying Theorem 7.18, we see that $L^{p,r}$ has the Φ-type r. Similarly, $L^{p,r}$ has the strong Φ-cotype r'. Just as in the case $1<p\leqslant 2$, $p'<r<\infty$, we see that $L^{p,r}$ has the Φ-type r' and the strong Φ-cotype r. This proves Proposition 7.21 for $1<p\leqslant 2$.

In the case $2<p<\infty$ the proof is similar.

We note that, generally speaking, the values of type and cotype given in Proposition 7.21 for Lorentz spaces cannot be improved (see [20], [28], [29], [47]).

§8. Relations between types (cotypes) with respect to the Rademacher system and other orthonormal systems

The Rademacher system is of great importance in the questions considered in the present paper. This is partly due to the extremal properties of this system, which we now consider.

We use the following theorem, known as the contraction principle ([43], Ch. 2).

Theorem 8.1. *Let X be a Banach space, and let $f = \sum_{k=1}^{n} x_k r_k$ $(x_k \in X)$. If $\lambda = \{\lambda_k\}_{k=1}^{n}$ is a set of numbers with $|\lambda_k| \leqslant 1$ and $f_\lambda = \sum_{k=1}^{n} \lambda_k x_k r_k$, then for any $1 \leqslant p < \infty$ one has*

$$\|f_\lambda\|_{L^p_X} \leqslant \|f\|_{L^p_X}.$$

8.1. An extremal property of the Rademacher system. As above, suppose that (I, μ) is a measure space, $\mu(I) = 1$, and $\Phi = \{\varphi_n\}$ is an orthonormal system on I satisfying condition (7.1).

Theorem 8.2. *Let X be a Banach space, and let $1 < p \leqslant 2$. Then the following assertions hold:*

(i) *if X has the Φ-type p, then X has the R-type p;*
(ii) *if X has the Φ-cotype p', then X has the R-cotype p';*
(iii) *if X has the strong Φ-cotype p', then X has the strong R-cotype p'.*

Proof. (i) Let us show that for any $x_k \in X$ $(k = 1, \dots, n)$ and any $r > 0$ (see (7.1)) one has

$$\int_0^1 \left\| \sum_{k=1}^{n} r_k(\xi) x_k \right\|^r d\xi \leqslant (2M)^r \int_0^1 d\xi \int_I \left\| \sum_{k=1}^{n} r_k(\xi) \varphi_k(t) x_k \right\|^r d\mu(t). \tag{8.1}$$

It is easy to see that there is a sequence $\{\varepsilon_k(t)\}$ of measurable functions on I that assume the values ± 1 and are such that

$$\left| \int_I \varepsilon_k(t) \varphi_k(t)\, d\mu(t) \right| \geqslant \frac{1}{2M} \quad (k = 1, 2, \dots) \tag{8.2}$$

(we recall that the functions φ_k can assume complex values). It follows from the symmetry of the Rademacher functions that

$$\int_0^1 \left\| \sum_{k=1}^{n} r_k(\xi) \varepsilon_k(t) \varphi_k(t) x_k \right\|^r d\xi = \int_0^1 \left\| \sum_{k=1}^{n} r_k(\xi) \varphi_k(t) x_k \right\|^r d\xi \tag{8.3}$$

for any $t \in I$. Let $\lambda_k = \left(\int_I \varepsilon_k(t) \varphi_k(t)\, d\mu \right)^{-1}$; by virtue of (8.2), $|\lambda_k| \leqslant 2M$. Using the contraction principle, we obtain

$$\begin{aligned} \int_0^1 \left\| \sum_{k=1}^{n} x_k r_k(\xi) \right\|^r d\xi &\leqslant (2M)^r \int_0^1 \left\| \sum_{k=1}^{n} \frac{1}{\lambda_k} x_k r_k(\xi) \right\|^r d\xi \\ &\leqslant (2M)^r \int_0^1 d\xi \int_I \left\| \sum_{k=1}^{n} r_k(\xi) \varepsilon_k(t) \varphi_k(t) x_k \right\|^r d\mu(t). \end{aligned}$$

Applying the Fubini theorem and (8.3), we obtain (8.1). Assertion (i) follows from (8.1).

(ii) Let $x_k \in X$ $(k=1,\dots,n)$. By virtue of (7.1) and the contraction principle, we have

$$\int_0^1 \bigg\|\sum_{k=1}^n \varphi_k(t) r_k(\xi) x_k\bigg\|^p \,d\xi \leqslant M^p \int_0^1 \bigg\|\sum_{k=1}^n r_k(\xi) x_k\bigg\|^p \,d\xi \tag{8.4}$$

for any $t \in I$. On the other hand, for any binary irrational $\xi \in [0,1]$, by virtue of our condition (see (7.5)), we obtain

$$\bigg(\sum_{k=1}^n \|x_k\|^{p'}\bigg)^{p/p'} \leqslant C^p \int_I \bigg\|\sum_{k=1}^n \varphi_k(t) r_k(\xi) x_k\bigg\|^p \,d\mu(t).$$

If we integrate this inequality with respect to ξ and the inequality (8.4) with respect to t and apply the Fubini theorem, then we obtain

$$\bigg(\sum_{k=1}^n \|x_k\|^{p'}\bigg)^{1/p'} \leqslant C' \bigg(\int_0^1 \bigg\|\sum_{k=1}^n r_k(\xi) x_k\bigg\|^p \,d\xi\bigg)^{1/p}.$$

This proves assertion (ii).

(iii) This assertion follows from (i) by virtue of duality (see Theorem 7.8).

The theorem is thereby proved.

For the trigonometric system $(\Phi = T)$, Theorem 8.2 is proved in the papers [50] (the case $p = 2$), [47], [28].

As a result, we see that the Kwapién theorem (Theorem 5.6) assumes a more general statement.

Theorem 8.3. *Let Φ be a uniformly bounded orthonormal system. A Banach space X is isomorphic to a Hilbert space if and only if X has the Φ-type 2 and the Φ-cotype 2.*

8.2. Systems with the Sidon property. Pisier [66] proved that the Rademacher functions in the definitions of type and cotype (see § 5) can be replaced by an arbitrary sequence of compact Abelian group characters belonging to the Sidon set. In particular, this is valid for any lacuna subsystem $\{e^{2\pi i n_k t}\}$ $(n_k \in \mathbb{N},\ n_{k+1}/n_k \geqslant q > 1)$.

Later on, a similar assertion was proved by Pelczyński [64] for topological Sidon sets.

An increasing sequence $\{\lambda_k\}_{k=1}^\infty$ of positive real numbers is called a topological Sidon set if there is a compact set $K \subset \mathbb{R}$ and a constant $C > 0$ such that for any complex numbers $\alpha_1, \dots, \alpha_n$ one has

$$\sum_{k=1}^n |\alpha_k| \leqslant C \sup_{t \in K} \bigg|\sum_{k=1}^n \alpha_k e^{i\lambda_k t}\bigg|.$$

An important example of a topological Sidon set is any positive sequence $\{\lambda_k\}$ satisfying the Hadamard condition

$$\lambda_{k+1}/\lambda_k \geqslant q > 1 \quad (k = 1, 2, \dots)$$

(see [57], p. 185).

The following theorem was proved by Pelczyński [64].

Theorem 8.4. *Let $\{\lambda_k\} \subset \mathbb{R}$ be a topological Sidon set, and let $[a,b] \subset \mathbb{R}$. Then there is a constant $C > 0$ depending only on λ and $b-a$ such that for any $1 \leqslant p < \infty$ and each sequence $\{x_j\}_{j=1}^n$ of elements of an arbitrary Banach space X one has*

$$C^{-p} \int_0^1 \left\| \sum_{k=1}^n r_k(\xi) x_k \right\|^p d\xi \leqslant \int_a^b \left\| \sum_{k=1}^n e^{i\lambda_k t} x_k \right\|^p dt \leqslant C^p \int_0^1 \left\| \sum_{k=1}^n r_k(\xi) x_k \right\|^p d\xi. \tag{8.5}$$

Proof. We use the following assertion on topological Sidon sets: for any $\delta > 0$ there is a constant A (depending only on λ and δ) such that for any complex numbers $\alpha_1, \dots, \alpha_n$ one has

$$\sum_{k=1}^n |\alpha_k| \leqslant A \sup_{|t| \leqslant \delta} \left| \sum_{k=1}^n \alpha_k e^{i\lambda_k t} \right| \tag{8.6}$$

(see [57], p. 194). We put $\delta = (b-a)/4$.

The key idea is as follows. We fix an $n \in \mathbb{N}$. Let $\xi \in [0,1]$ be a binary irrational point. We consider a linear functional defined on the linear span of the system $\{e^{i\lambda_k t}\}_{k=1}^n$ of functions. The functional assumes the value $r_k(\xi)$ on the element $e^{i\lambda_k t}$ $(k = 1, \dots, n)$. By virtue of (8.6) and the Hahn–Banach theorem, this functional can be extended to a bounded linear functional on $C[-\delta, \delta]$, with a norm not greater than A. Therefore, by the Riesz theorem, there is a complex Borel measure μ_ξ on $[-\delta, \delta]$ such that $\|\mu_\xi\| \leqslant A$ and

$$\int_{-\delta}^{\delta} e^{i\lambda_k s} \, d\mu_\xi(s) = r_k(\xi) \quad (k = 1, \dots, n). \tag{8.7}$$

We now choose a sequence $\{x_k\}_{k=1}^n \subset X$ and put

$$f(t) = \sum_{k=1}^n e^{i\lambda_k t} x_k, \quad f_\xi(t) = \sum_{k=1}^n r_k(\xi) e^{i\lambda_k t} x_k \quad (t \in \mathbb{R}).$$

Then, by virtue of (8.7),

$$f(t) = \sum_{k=1}^n r_k(\xi) e^{i\lambda_k t} x_k \int_{-\delta}^{\delta} e^{i\lambda_k s} \, d\mu_\xi(s) = \int_{-\delta}^{\delta} f_\xi(t+s) \, d\mu_\xi(s).$$

Using the Hölder inequality and the estimate $\|\mu_\xi\| \leqslant A$, we obtain

$$\|f(t)\|^p \leqslant A^{p-1} \int_{-\delta}^{\delta} \|f_\xi(t+s)\|^p \, d\nu_\xi(s),$$

where ν_ξ is the variation of the measure μ_ξ. If we integrate with respect to t and apply the Fubini theorem, then we obtain

$$\int_a^b \|f(t)\|^p \, dt \leqslant A^p \int_{a-\delta}^{b+\delta} \|f_\xi(u)\|^p \, du. \tag{8.8}$$

Finally, integrating with respect to ξ and using the contraction principle, we obtain the second inequality in (8.5).

To prove the first inequality in (8.5), we replace $[a,b]$ by $[a+\delta, b-\delta]$ and f by f_ξ. As above (see (8.8)), we obtain

$$\int_{a+\delta}^{b-\delta} \|f_\xi(t)\|^p\,dt \leqslant A^p \int_a^b \|f(u)\|^p\,du.$$

It remains to apply the inequality

$$\int_0^1 \biggl\|\sum_{k=1}^n r_k(\xi)x_k\biggr\|^p d\xi \leqslant \biggl(\frac{2}{|J|}\biggr)^p \int_J \|f_\xi(t)\|^p\,dt$$

($J = [a+\delta, b-\delta]$), which we obtain in the same way as (8.1). The theorem is thereby proved.

Corollary 8.5. *Let $\{n_k\}$ be a sequence of positive integers satisfying the condition*

$$\frac{n_{k+1}}{n_k} \geqslant \lambda > 1 \quad (k = 1, 2, \dots). \tag{8.9}$$

Next, suppose that $\Phi = \{e^{2\pi i n_k t}\}$, $t \in [0,1]$, X is a Banach space, and $1 < p \leqslant 2$. Then

(i) *X has the R-type p if and only if X has the Φ-type p;*
(ii) *X has the R-cotype p' if and only if X has the Φ-cotype p';*
(iii) *X has the strong R-cotype p' if and only if X has the strong Φ-cotype p'.*

We note that (iii) follows from (i) and Theorem 7.8.

Corollary 8.5 is a special case of the Pisier theorem [66]. Suppose that G is a compact Abelian group with a dual group Γ and μ is a Haar measure on G.

A subset $E \subset \Gamma$ is called a Sidon set if there is a constant C depending only on E such that for any continuous complex-valued function f on G for which $\widehat{f}(\gamma) = 0$ for all $\gamma \in \Gamma \setminus E$, the following inequality holds:

$$\sum_{\gamma\in\Gamma} |\widehat{f}(\gamma)| \leqslant C\,\|f\|_\infty. \tag{8.10}$$

The least constant in (8.10) is called the Sidon constant of the set E.

The following theorem was proved by Pisier [66].

Theorem 8.6. *Suppose that $E = \{\gamma_n\} \subset \Gamma$ is a Sidon set. Then there is a constant C_0 depending only on the Sidon constant $S(E)$ such that for any Banach space X, any elements $x_1, \dots, x_N \in X$, and any $p \in [1, \infty)$ one has*

$$\begin{aligned} C_0^{-1}\biggl(\int_0^1 \biggl\|\sum_{n=1}^N x_n r_n(\xi)\biggr\|^p d\xi\biggr)^{1/p} &\leqslant \biggl(\int_G \biggl\|\sum_{n=1}^N x_n\gamma_n\biggr\|^p d\mu\biggr)^{1/p} \\ &\leqslant C_0\biggl(\int_0^1 \biggl\|\sum_{n=1}^N x_n r_n(\xi)\biggr\|^p d\xi\biggr)^{1/p}. \end{aligned}$$

In the paper [2], a more general assertion is proved: there is a constant C (depending only on $S(E)$) such that for all $\alpha > 0$

$$C^{-1}\left|\left\{\left\|\sum_{n=1}^{N} x_n r_n\right\| \geqslant C\alpha\right\}\right| \leqslant \mu\left(\left\{\left\|\sum_{n=1}^{N} x_n \gamma_n\right\| \geqslant \alpha\right\}\right)$$
$$\leqslant C\left|\left\{\left\|\sum_{n=1}^{N} x_n r_n\right\| \geqslant C^{-1}\alpha\right\}\right|.$$

The Pisier theorem was generalized in another sense by Pelczyński [64], who proved that if $\{\gamma_j\}$ and $\{\sigma_j\}$ are, respectively, sequences of characters of compact Abelian groups G and S such that for each sequence $\{a_j\}$ of scalars we have

$$\left\|\sum_{j=1}^{\infty} a_j \gamma_j\right\|_{\infty} \asymp \left\|\sum_{j=1}^{\infty} a_j \sigma_j\right\|_{\infty},$$

then for any $1 \leqslant p < \infty$ and any sequence $\{x_j\}$ of elements of an arbitrary Banach space X we have

$$\int_G \left\|\sum_{j=1}^{\infty} x_j \gamma_j\right\|^p d\mu \asymp \int_S \left\|\sum_{j=1}^{\infty} x_j \sigma_j\right\|^p d\nu.$$

Remark 8.7. In the above-mentioned cases, we consider character sequences that are Sidon sets. These special sets cannot be replaced by an arbitrary orthonormal Sidon system.

An orthonormal uniformly bounded on $[0,1]$ system $\Phi = \{\varphi_n\}$ of functions is called a Sidon system if there is a constant $C > 0$ such that for any scalars $\alpha_1, \dots, \alpha_n$ one has

$$\sum_{k=1}^{n} |\alpha_k| \leqslant C\left\|\sum_{k=1}^{n} \alpha_k \varphi_k\right\|_{\infty} \tag{8.11}$$

(see [45], Russian p. 327).

Important examples of Sidon systems are the Rademacher system and the system $\{e^{2\pi i n_k t}\}$ ($n_k \in \mathbb{N}$, $n_{k+1}/n_k \geqslant q > 1$).

Generally speaking, from the fact that a Banach space X has the R-type p, it does not follow that X has the Φ-type p for any Sidon system Φ. The same is true for the cotype.

Indeed, let $\Psi = \{\psi_n\}$ be a uniformly bounded orthonormal system on $[0,1]$. We put

$$\varphi_n(t) = \begin{cases} \psi_n(2t), & 0 \leqslant t < 1/2, \\ r_n(2t-1), & 1/2 \leqslant t \leqslant 1. \end{cases}$$

It is clear that $\Phi = \{\varphi_n\}$ is a uniformly bounded orthonormal system on $[0,1]$. Since the Rademacher system is a Sidon system, it follows that Φ is a Sidon system (see (8.11)). At the same time, it is easy to see (taking Theorem 8.2 into account) that for any space X the Φ-type and the Ψ-type coincide. If we take, say, $X = L^q[0,1]$ $(2 < q < \infty)$ and $\Psi = T$, then, by Proposition 5.9, the R-type of X is equal to 2, and the Φ-type is precisely q', and not better (see 7.7).

8.3. The Bourgain theorem. In § 6 we have already cited the Bourgain theorem [12] for characters of compact Abelian groups. Later, in the paper [13] Bourgain obtained stronger results (complete analogues of the Hausdorff–Young inequalities) for the Cantor group and the circle.

As above, we denote by T and W the trigonometric system and the Walsh system, respectively.

Bourgain [13] proved the following theorem[5].

Theorem 8.8. *Let a Banach space X have a non-trivial R-type. Then X has a non-trivial T-type and a non-trivial W-type.*

This theorem implies that a partial converse of Theorem 8.3 is valid for $\Phi = T$ and $\Phi = W$. It is not known whether a similar assertion is true for an arbitrary complete uniformly bounded orthonormal system (or for an abstract compact Abelian group).

Theorems 8.1 and 8.8 (also see Remark 7.5) yield the following assertion.

Corollary 8.9. *A Banach space X has a non-trivial Rademacher type if and only if X has a non-trivial Fourier type.*

The proof of Theorem 8.8 is based on Theorem 6.16 and is long and complicated. However, here we give the complete proof (with some simplifications) of the theorem for a Walsh system.

Proof of Theorem 8.8 (for a Walsh system). By Corollary 5.23, if X has a non-trivial R-type, then X has the strong R-cotype p' for some $1 < p \leqslant 2$. This means that there is a constant C_0 such that for any function $f \in L^2_X[0,1]$ its Fourier coefficients with respect to the Rademacher system satisfy the inequality

$$\biggl(\sum_{n=1}^{\infty} \|c_n(f)\|^{p'}\biggr)^{1/p'} \leqslant C_0 \|f\|_{L^2_X} \tag{8.12}$$

(see Definition 5.20 and Remark 5.21).

We put

$$\varphi(n) = \sup \biggl(\int_0^1 \biggl\| \sum_{k\in\Lambda} x_k w_k(u) \biggr\|^2 du \biggr)^{1/2},$$

where the upper bound is taken over all subsets $\Lambda \subset \mathbb{N}$ with cardinality $\Lambda^{\#} = n$ and over all $x_k \in X$ with $\|x_k\| \leqslant 1$.

Let us prove that

$$\varphi(n) \leqslant C_0 d^{-1/p'} 2^d \varphi(2^{-d+1} n) + \frac{2^d}{\sqrt{n}} \varphi(n) \tag{8.13}$$

for any $d \in \mathbb{N}$. Using the group property of the Walsh system, we have

$$\begin{aligned} J &\equiv \biggl(\int_0^1 \biggl\| \sum_{k\in\Lambda} x_k w_k(u) \biggr\|^2 du \biggr)^{1/2} \\ &= \biggl(\int_0^1 \cdots \int_0^1 \biggl(\frac{1}{d} \sum_{i=1}^{d} \int_0^1 \biggl\| \sum_{k\in\Lambda} w_k(t_i) w_k(u) x_k \biggr\|^2 du \biggr) dt_1 \cdots dt_d \biggr)^{1/2}. \end{aligned}$$

[5]We say that a Banach space X has a non-trivial Φ-type if there is a p, $1 < p \leqslant 2$, such that X has the Φ-type p.

For each point $t \in I^d$ ($I \equiv [0,1]$) and any $k \in \Lambda$, there is a point $\xi_{t,k} \in [0,1]$ such that

$$w_k(t_i) = r_i(\xi_{t,k}), \qquad i = 1, \ldots, d. \tag{8.14}$$

We denote by $\Lambda_\nu(t)$ ($t \in I^d$, $\nu = 0, 1, \ldots, 2^d - 1$) the set of all $k \in \Lambda$ such that $\xi_{t,k} \in \Delta_\nu^{(d)} \equiv (\nu 2^{-d}, (\nu+1)2^{-d})$. Then for $i = 1, \ldots, d$ (see (8.14)) we have

$$\sum_{k \in \Lambda} w_k(t_i) w_k(u) x_k = 2^d \sum_{\nu=0}^{2^d-1} \int_{\Delta_\nu^{(d)}} r_i(\xi)\, d\xi \sum_{k \in \Lambda_\nu(t)} x_k w_k(u). \tag{8.15}$$

Let $f(\xi) = \sum_{k \in \Lambda_\nu(t)} x_k w_k(u)$ for $\xi \in \Delta_\nu^{(d)}$ (u and t are fixed). Then f is a step function with values in X. Here the right-hand side of (8.15) is equal to[6]

$$2^d \int_0^1 f(\xi) r_i(\xi)\, d\xi,$$

where the integral is interpreted in the sense of Bochner. Therefore, by virtue of (8.12),

$$\left(\sum_{i=1}^d \left\| \sum_{k \in \Lambda} w_k(t_i) w_k(u) x_k \right\|^{p'} \right)^{2/p'} \leqslant C_0^2 2^d \sum_{\nu=0}^{2^d-1} \left\| \sum_{k \in \Lambda_\nu(t)} x_k w_k(u) \right\|^2. \tag{8.16}$$

Using the Hölder inequality and integrating with respect to u, we obtain

$$\begin{aligned} J_d(t) &\equiv \frac{1}{d} \sum_{i=1}^d \int_0^1 \left\| \sum_{k \in \Lambda} w_k(t_i) w_k(u) x_k \right\|^2 du \\ &\leqslant C_0^2 d^{-2/p'} 2^d \sum_{\nu=0}^{2^d-1} \int_0^1 \left\| \sum_{k \in \Lambda_\nu(t)} w_k(u) x_k \right\|^2 du \\ &\leqslant C_0^2 d^{-2/p'} 2^{2d} \max_\nu \big(\varphi(\Lambda_\nu(t)^\#) \big)^2. \end{aligned} \tag{8.17}$$

We put

$$\Omega_\nu = \{ t \in I^d : \Lambda_\nu(t)^\# > 2^{-d+1} n \}, \qquad \Omega = \bigcup_{\nu=0}^{2^d-1} \Omega_\nu.$$

Integrating the inequality (8.16) with respect to t and using the estimate (8.17) for $t \in I^d \setminus \Omega$, we obtain

$$J \leqslant C_0 d^{-1/p'} 2^d \varphi(2^{-d+1} n) + \sqrt{\operatorname{mes}_d \Omega}\, \varphi(n). \tag{8.18}$$

[6]This representation is one of the key points of the proof.

It remains to estimate $\operatorname{mes}_d \Omega$. Note that, by virtue of (8.14), for any fixed $\nu = 0, 1, \dots, 2^d - 1$ one has

$$2^d \Lambda_\nu(t)^\# = \sum_{k\in\Lambda} \prod_{i=1}^{d} \bigl(1 + \varepsilon_i w_k(t_i)\bigr) \equiv f_d(t_1, \dots, t_d),$$

where ε_i is the value of the function r_i in the interval $\Delta_\nu^{(d)}$. We obtain an estimate of $\operatorname{mes}_d \Omega$ by using the inequality

$$\|f_d - n\|_{L^2(I^d)} \leqslant \sqrt{2^d n}. \tag{8.19}$$

Indeed, it readily follows from (8.19) and the Chebyshev inequality that

$$\operatorname{mes}_d \Omega_\nu \leqslant \operatorname{mes}_d \bigl\{t \in I^d : |2^d \Lambda_\nu(t)^\# - n| \geqslant n\bigr\} \leqslant \frac{2^d}{n},$$

and hence, $\operatorname{mes}_d \Omega \leqslant 2^{2d}/n$. By (8.18), this implies (8.13). To prove the inequality (8.19), it suffices to use Parseval's equality and known properties of binomial coefficients; it is easy to see that

$$\|f_d - n\|^2_{L^2(I^d)} = (2^d - 1)n.$$

Now from (8.13) we derive the existence of a constant K such that

$$\varphi(n) \leqslant K n^{1-\varepsilon} \text{ for all } n \in \mathbb{N}, \ \ \varepsilon = (8C_0)^{-p'}. \tag{8.20}$$

We put $d = [(8C_0)^{p'}] + 1$ and choose K so that the following inequality holds:

$$\varphi(n) \leqslant K n^{1-\varepsilon} \quad \text{for} \quad n \leqslant 4^{d+1}.$$

Let $n > 4^{d+1}$. Suppose that (8.20) is valid for all numbers less than n. Then, by virtue of (8.13),

$$\varphi(n) \leqslant C_0 d^{-1/p'} 2^d K (2^{-d+1} n)^{1-\varepsilon} + \frac{1}{2}\varphi(n)$$

and $\varphi(n) \leqslant K n^{1-\varepsilon}$ according to the choice of d.

We now take $1 < s < \frac{1}{1-\varepsilon}$. Then there is a constant $A \equiv A_s$ such that for any finite sequence $\{x_k\} \subset X$ one has

$$\Bigl\|\sum_k x_k w_k\Bigr\|_{L^2_X} \leqslant A\Bigl(\sum_k \|x_k\|^s\Bigr)^{1/s}. \tag{8.21}$$

Indeed, without loss of generality we can assume that the sum on the right-hand side of (8.21) is equal to 1. Let $D_j = \{k : 2^{-j-1} < \|x_k\| \leqslant 2^{-j}\}$; then $D_j^\# \leqslant 2^{(j+1)s}$. Consequently, by virtue of (8.20), we have

$$\Bigl\|\sum_k x_k w_k\Bigr\|_{L^2_X} \leqslant \sum_{j=0}^{\infty} \Bigl\|\sum_{k\in D_j} x_k w_k\Bigr\|_{L^2_X} \leqslant A.$$

In the same way as in Theorem 7.14, it follows from (8.21) that for any sequence $\{y_k\}_{k=0}^N \subset X$ one has

$$\left(\sum_{k=0}^N \|y_k\|^q\right)^{1/q} \leqslant A\left\|\sum_{k=0}^N y_k w_k\right\|_{L_X^2} \qquad (q = s'). \tag{8.22}$$

We now proceed to the second part of the proof. First, we show that for any bounded set Λ of non-negative integers and any set $\{x_n\}_{n\in\Lambda} \subset X$ of vectors with $\|x_n\| \leqslant 1$

$$Q \equiv \left(\int_0^1 \left\|\sum_{n\in\Lambda} x_n w_n(t)\right\|^q dt\right)^{1/q} \leqslant C(\Lambda^\#)^{1/q'}. \tag{8.23}$$

We can suppose that $0 \notin \Lambda$ and $\Lambda^\# = 2^k$ ($k \geqslant 2$). Starting from (6.3), for each $j = 1, \dots, 2^k - 1$ we denote by P_j the set of ν, $1 \leqslant \nu \leqslant k$, such that

$$w_j(t) = \prod_{\nu\in P_j} r_\nu(t), \qquad t \in [0,1]. \tag{8.24}$$

We also define $P_0 = \varnothing$. Using the group structure of Walsh functions (see § 6), for any fixed vector $\xi = (\xi_1, \dots, \xi_k) \in I^k$ we make 2^k shifts in the integral (8.23) with steps $t_j = \sum_{\nu\in P_j} \xi_\nu$ ($j = 0, 1, \dots, 2^k - 1$). Thus we obtain[7]

$$\begin{aligned} Q &= 2^{-k/q}\left(\sum_{j=0}^{2^k-1} \int_0^1 \left\|\sum_{n\in\Lambda} x_n w_n\left(t + \sum_{\nu\in P_j} \xi_\nu\right)\right\|^q dt\right)^{1/q} \\ &\leqslant 2^{-k/q}\left(\sum_{j=0}^{2^k-1} \left\|\sum_{n\in\Lambda} \tilde{x}_n \prod_{\nu\in P_j} w_n(\xi_\nu)\right\|^q\right)^{1/q}, \end{aligned}$$

where $\tilde{x}_n \equiv \tilde{x}_n(\xi) = x_n w_n(t_0(\xi))$ ($t_0(\xi)$ is a point of the interval $[0,1]$).

We consider a polynomial in the Walsh system:

$$\sum_{j=0}^{2^k-1} y_j w_j(t) \equiv \sum_{j=0}^{2^k-1} \sum_{n\in\Lambda} \tilde{x}_n \prod_{\nu\in P_j} w_n(\xi_\nu) r_\nu(t).$$

Using (8.22) and (8.24), we have

$$Q \leqslant 2^{-k/q}\left(\sum_{j=0}^{2^k-1} \|y_j\|^q\right)^{1/q} \leqslant A\,2^{-k/q}\left(\int_0^1 \left(\sum_{n\in\Lambda}\left|\sum_{j=1}^{2^k-1} \prod_{\nu\in P_j} r_\nu(t) w_n(\xi_\nu)\right|\right)^2 dt\right)^{1/2}.$$

It is easy to prove by induction that

$$\sum_{j=0}^{2^k-1} \prod_{\nu\in P_j} a_\nu = \prod_{\nu=1}^k (1 + a_\nu) \quad (a_\nu \in \mathbb{R}).$$

[7] Let us explain that, taking into account a one-to one correspondence between the Cantor group $\mathbb{D}$ and the interval $[0,1]$, here addition is interpreted as the addition of elements of the group $\mathbb{D}$.

Therefore,

$$Q \leqslant A\, 2^{-k/q} \biggl(\int_0^1 \biggl(\sum_{n\in\Lambda} \biggl| \prod_{\nu\in P_j} (1 + r_\nu(t) w_n(\xi_\nu)) \biggr| \biggr)^2 dt \biggr)^{1/2}.$$

Squaring both sides of the inequality, integrating with respect to ξ in I^k, and taking into account the orthogonality of the functions w_n, we obtain:

$$Q^2 \leqslant A^2 2^{-2k/q} 2^{2k} = A^2 2^{2k/q'}.$$

Hence, inequality (8.23) is proved.

At the last stage, we use a well-known technique. Let $f = \sum_{k=0}^\infty x_k w_k$, where $\{x_k\} \subset X$ is a finite sequence. We put

$$B_j = \bigl\{ k : x^*_{2^{j+1}} < \|x_k\| \leqslant x^*_{2^j} \bigr\}, \qquad j = 0, 1, \dots .$$

Then $B_j^\# \leqslant 2^{j-1}$. Let $f_j(t) = \sum_{k\in B_j} x_k w_k(t)$. By (8.23), we obtain

$$\|f_j\|_{L^q_X} \leqslant C\, 2^{j/q'} x^*_{2^j}.$$

Therefore, for any $n \in \mathbb{N}$ we have

$$f^*(2^{-n}) \leqslant \biggl(2^n \int_0^{2^{-n}} f^*(u)^q \, du \biggr)^{1/q} \leqslant C \biggl(\sum_{j=0}^n 2^j x^*_{2^j} + 2^{n/q} \sum_{j=n}^\infty 2^{j/q'} x^*_{2^j} \biggr).$$

It readily follows from this estimate that for any $1 < r < q'$ one has

$$\|f\|_{L^{r',r}_X} \leqslant C_r \|\{x_k\}\|_{l^r_X}. \tag{8.25}$$

By virtue of (2.6), this implies that the space X has the W-type r. The proof of the theorem is complete.

Remark 8.10. The inequality (8.21) derived at the first stage plays an important role in the proof. We note that by means of the same method one can prove a similar inequality with the exponent 2 on the left-hand side of (8.21) replaced by any exponent $\alpha > 2$ (see [12]). Generally speaking, this yields the decrease of the exponent s on the right-hand side. But the method mentioned does not allow one to take the value $s = \alpha'$ and thus complete the proof of Theorem 8.8 (for the Walsh system).

Inequality (8.22) contains the special case of Theorem 6.16 (for the group $\mathbb{D}$). For an arbitrary compact Abelian group, Bourgain in addition uses a theorem due to Edgar [25]. Otherwise, the proof is almost the same as in the case of a Walsh system.

Remark 8.11. The number r in (8.27) (the W-type of the space X) is defined by the condition

$$1 < r < \frac{1}{1-\varepsilon}, \qquad \varepsilon = (8C_0)^{-p'}.$$

Here p' is the strong R-cotype of X and C_0 is the corresponding constant in (8.12). We recall that if X has non-trivial type and cotype $q \geqslant 2$, then X has the strong cotype q (see § 5). Here we must note that the W-type of the space X is not uniquely determined by its Rademacher type and cotype. Indeed, let us consider the following example (based on a scheme due to Bourgain [13]).

Let $1 < p < 2$. We say that a sequence $b = \{b_k\}_{k=0}^{\infty}$ of real numbers belongs to the space $\mathcal{F}(L^p)$ if there is a function $f \in L^p[0,1]$ such that $\{b_k\}$ is the sequence of its Fourier coefficients with respect to the Walsh system. We put $\|b\|_{\mathcal{F}(L^p)} = \|f\|_{L^p}$. By the Hausdorff–Young theorem, $\|b\|_{l^{p'}} \leqslant \|b\|_{\mathcal{F}(L^p)}$. Further, for any $\theta \in (0,1)$ we write

$$X_\theta = [l^{p'}, \mathcal{F}(L^p)]_\theta .$$

In § 9 we shall show that the spaces $l^{p'}$ and L^p have the W-type p. Since the space $\mathcal{F}(L^p)$ is isomorphic to L^p, we see that this space also has the W-type p. Consequently, by Theorem 7.16, the space X_θ has the W-type p. Let us show that X_θ does not have the W-type r for $r > p$. Assume the opposite. For each $n = 0, 1, \dots$ we denote by $b^{(n)}$ the sequence that has only one non-zero term $b_n^{(n)} = 1$. Let $f_N = \sum_{n=0}^{2^N-1} b^{(n)} w_n$ so that $f_N(t) = \{w_n(t)\}_{n=0}^{2^N-1}$. Using the inequality (see [79], 1.9.3)

$$\|f_N(t)\|_{X_\theta} \leqslant C_\theta \|f_N(t)\|_{l^{p'}}^{1-\theta} \|f_N(t)\|_{\mathcal{F}(L^p)}^{\theta}$$

and applying the Hölder inequality, we obtain

$$\|f_N\|_{L^r_{X_\theta}} \leqslant C\, 2^{N/p'} .$$

On the other hand, $\|b^{(n)}\|_{X_\theta} = 1$. Using our assumption, by which X_θ has the W-cotype r' and

$$\left(\sum_{n=0}^{2^N-1} \|b^{(n)}\|_{X_\theta}^{r'} \right)^{1/r'} \leqslant C \|f_N\|_{L^r_{X_\theta}},$$

we obtain $2^{N/r'} \leqslant C\, 2^{N/p'}$. This implies that $r \leqslant p$.

Furthermore, with the interpolation and reasoning similar to the above one can obtain

$$p_{X_\theta} = \left(\frac{1-\theta}{2} + \frac{\theta}{p} \right)^{-1}, \qquad q_{X_\theta} = \left(\frac{1-\theta}{p'} + \frac{\theta}{2} \right)^{-1}.$$

As θ changes from 0 to 1, the R-type of X_θ runs from 2 to p, and the R-cotype runs from p' to 2. At the same time, the W-type of X_θ is p for any θ. For $\theta = 1/2$ we obtain the optimal values $4p/(p+2)$ of R-type and $4p'/(p'+2)$ of R-cotype; they are self-conjugate numbers.

Let us note that a similar example for the trigonometric system is considered in [28].

§9. Theorems of Hardy and Paley for vector-valued functions

Let $H^{1,at}$ be the atomic Hardy space of 2π-periodic functions on $\mathbb{R}$. The following classical inequalities hold: *if* $f \in H^{1,at}$, *then*

$$\sum_{n=-\infty}^{+\infty} \frac{|\widehat{f}(n)|}{|n|+1} \leqslant c\|f\|_{1,at} \quad (\textit{the Hardy inequality}), \tag{9.1}$$

$$\left(\sum_{k=0}^{\infty} |\widehat{f}(\pm 2^k)|^2\right)^{1/2} \leqslant c\|f\|_{1,at} \quad (\textit{the Paley inequality}), \tag{9.2}$$

where the $\widehat{f}(n)$ *are the Fourier coefficients of the function* f.

We consider conditions on a Banach space X under which these inequalities remain valid for X-valued functions.

Let X be a complex Banach space. The Fourier coefficients of a 2π-periodic function $f \in L^1_X(-\pi,\pi)$ are defined by the formula

$$\widehat{f}(n) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt, \qquad n \in \mathbb{Z}.$$

Let $\Delta \subset \mathbb{R}$ be an interval of length $|\Delta| \leqslant 2\pi$. Each 2π-periodic function $a \in L^\infty_X(\mathbb{R})$ with values in X such that

(1) $a(t) = 0$ for all $t \notin \Delta + 2k\pi \quad (k \in \mathbb{Z})$;

(2) $\|a\|_{L^\infty_X} \leqslant |\Delta|^{-1}$;

(3) $\displaystyle\int_{-\pi}^{\pi} a(t)\,dt = 0$

is called an X-atom concentrated in Δ.

Each continuous function $a(t) \equiv x_0$, where $x_0 \in X$ and $\|x_0\| \leqslant (2\pi)^{-1}$, is also called an X-atom.

It is obvious that for any X-atom a one has

$$\int_{-\pi}^{\pi} \|a(t)\|\,dt \leqslant 1.$$

If a is an X-atom concentrated in the interval Δ, then, as is easy to see,

$$\|\hat{a}(n)\| \leqslant \frac{1}{8\pi}|n|\,|\Delta| \quad (n \in \mathbb{Z}). \tag{9.3}$$

We say that a 2π-periodic function $f \in L^1_X(-\pi,\pi)$ belongs to the atomic Hardy space $H^{1,at}_X$ if there are X-atoms a_k and coefficients $\lambda_k \in \mathbb{C}$ $(k = 1, 2, \dots)$ such that

$$\lim_{n\to\infty}\left\| f(t) - \sum_{k=1}^{n} \lambda_k a_k(t)\right\| = 0 \text{ for almost all } t \in \mathbb{R} \text{ and } \sum_{k=1}^{\infty} |\lambda_k| < \infty. \tag{9.4}$$

The norm in the space $H^{1,at}_X$ is defined by

$$\|f\|_{H^{1,at}_X} \equiv \|f\|_{1,at} = \inf \sum_{k=1}^{\infty} |\lambda_k|,$$

where the lower bound is taken over all such representations of f. It is clear that

$$\|f\|_{L^1_X} \leqslant \|f\|_{H^{1,at}_X} \text{ for all } f \in H^{1,at}_X. \tag{9.5}$$

On the other hand, if $f \in L^p_X(-\pi,\pi)$ $(1 < p < \infty)$, then $f \in H^{1,at}_X$ and

$$\|f\|_{H^{1,at}_X} \leqslant C_p \|f\|_{L^p_X} \tag{9.6}$$

(this follows from the boundedness of the maximum Hardy–Littlewood operator for vector-valued functions (see [70]).

Fefferman proved the following theorem describing the $(H^{1,at} - l^1)$-multipliers (see [73]).

Theorem 9.1. *Let $m = \{m_j\}_{j\in\mathbb{Z}}$ be a sequence of complex numbers and*

$$\mu(m) \equiv \sup\Big\{ \sum_{j=-\infty}^{\infty} |m_j \widehat{f}(j)| : \|f\|_{1,at} \leqslant 1 \Big\}, \tag{9.7}$$

where the upper bound is taken over all functions $f \in H^{1,at}$ with $\|f\|_{1,at} \leqslant 1$. Then $\mu(m) < \infty$ if and only if

$$\sigma(m) \equiv |m_0| + \sup_{N\geqslant 1} \Bigg(\sum_{k=1}^{\infty} \Bigg(\sum_{|j|=kN}^{(k+1)N-1} |m_j| \Bigg)^2 \Bigg)^{1/2} < \infty. \tag{9.8}$$

Moreover, there is a constant $C > 0$ such that

$$C^{-1}\sigma(m) \leqslant \mu(m) \leqslant C\sigma(m).$$

Note that this theorem implies inequalities (9.1) and (9.2).

It is obvious that each sequence satisfying (9.8) belongs to l^2. Using the Fefferman duality theorem [26], we obtain the following result

Corollary 9.2. *Let $\{m_j\}_{j\in\mathbb{Z}}$ be a sequence of complex numbers satisfying condition* (9.8). *Then the function*

$$f(t) = \sum_{j=-\infty}^{+\infty} m_j e^{ijt} \tag{9.9}$$

belongs to the space BMO *(bounded mean oscillation).*

Further, using another characterization of $(H^1 - l^1)$-multipliers ([24], p. 105), we see that if $m_j \geqslant 0$ $(j \in \mathbb{Z})$, then the condition (9.8) is also necessary for the function (9.9) to belong to BMO.

Now let $1 \leqslant p \leqslant 2$. Following the paper [9], we denote by FM_p the space of all sequences $m = \{m_j\}_{j=-\infty}^{+\infty}$ of complex numbers such that

$$\sigma_p(m) \equiv |m_0| + \sup_{N\geqslant 1} \Bigg(\sum_{k=1}^{\infty} \Bigg(\sum_{|j|=kN}^{(k+1)N-1} |m_j| \Bigg)^p \Bigg)^{1/p} < \infty. \tag{9.10}$$

It is clear that $FM_1 = l^1$ and

$$FM_p \subset FM_r \quad \text{for} \quad 1 \leqslant p < r \leqslant 2. \tag{9.11}$$

The next theorem is proved in the paper [9].

Theorem 9.3. *Let a Banach space X have a Fourier type $p \in [1,2]$. Then there is a constant $C>0$ such that for any sequence $m=\{m_j\} \in FM_p$ and any function $f \in H_X^{1,at}$*

$$\sum_{j=-\infty}^{+\infty} |m_j| \, \|\widehat{f}(j)\| \leqslant C\sigma_p(m)\|f\|_{H_X^{1,at}}. \tag{9.12}$$

Proof. It suffices to prove (9.12) in the case where f is a X-atom a concentrated in the interval $(-\delta,\delta)$ $(0<\delta<\pi)$. We can only consider $\hat{a}(j)$ for positive j (since $a(-t)$ is also an X-atom). Thus, we must prove that

$$\sum_{j=1}^{\infty} |m_j| \, \|\hat{a}(j)\| \leqslant C\sigma_p(m). \tag{9.13}$$

We choose an integer $s \geqslant 0$ from the condition $\pi\, 2^{-s-1} \leqslant \delta < \pi\, 2^{-s}$. Let $N=2^s$. Then

$$\sum_{j=1}^{N} j|m_j| \leqslant \sum_{k=0}^{s-1} 2^{k+1} \sum_{j=2^k}^{2^{k+1}-1} |m_j| \leqslant 2N\sigma_p(m).$$

Therefore, by (9.3), we obtain

$$\sum_{j=1}^{N-1} |m_j| \, \|\hat{a}(j)\| \leqslant \frac{1}{2}\,\sigma_p(m).$$

Next, for any $kN \leqslant j < (k+1)N$ $(k \in \mathbb{N})$ we have

$$\begin{aligned} \|\hat{a}(j)\| &\leqslant \frac{1}{N} \sum_{\nu=kN}^{(k+1)N-1} \|\hat{a}(\nu)\| + \sum_{\nu=kN}^{(k+1)N-1} \|\hat{a}(\nu) - \hat{a}(\nu+1)\| \\ &\leqslant N^{1/p-1} \left(\sum_{\nu=kN}^{(k+1)N-1} \|\hat{a}(\nu)\|^{p'} \right)^{1/p'} + N^{1/p} \left(\sum_{\nu=kN}^{(k+1)N-1} \|\hat{b}(\nu)\|^{p'} \right)^{1/p'}, \end{aligned}$$

where $b(t) = a(t)(1-e^{-it})$. Using the Hölder inequality and our assumption on X having the Fourier type[8] p, we readily obtain

$$\sum_{j=N}^{\infty} |m_j| \, \|\hat{a}(j)\| \leqslant CN^{1/p} \left(\frac{1}{N}\|a\|_p + \|b\|_p \right) \leqslant C'\sigma_p(m).$$

This completes the proof of (9.13).

Definition 9.4. We say that a Banach space X possesses the Hardy property if there is a constant $C>0$ such that

$$\sum_{n=-\infty}^{+\infty} \frac{|\widehat{f}(n)|}{|n|+1} \leqslant C\|f\|_{H_X^{1,at}} \tag{9.14}$$

for any function $f \in H_X^{1,at}$.

The following theorem holds.

[8] We recall that the Fourier type p is equivalent to the T-type p (see § 7).

Theorem 9.5. *For a Banach space X to possess the Hardy property, it is necessary and sufficient that X have a non-trivial Rademacher type.*

We recall that by Corollary 8.9 the space X has a non-trivial Rademacher type if and only if X has a non-trivial Fourier type. The sufficiency in Theorem 9.5 (due to Bourgain [13]) follows directly from Theorem 9.3. The proof of the necessity reduces to the fact that the space $L^1(\mathbb{T})$ does not possess the Hardy property (see [9], Proposition 2.6).

Remark 9.6. Let $1 < p \leqslant 2$. We say that a Banach space X possesses the property $(\mathcal{F}_p)$ if for any sequence $m \in FM_p$ and any function $f \in H_X^{1,at}$ we have $\{m_j \|\widehat{f}(j)\|\} \in l^1$. It would be interesting to give a full description of Banach spaces that have this property. By Theorem 9.3, if X has the Fourier type p, then X also has the property $(\mathcal{F}_p)$. On the other hand, in the paper [9] it is proved that if X has the property $(\mathcal{F}_p)$, then X^* has the Rademacher type p; by Proposition 5.22, this is equivalent to the fact that X has the strong R-cotype p'. At the same time, the Fourier type p is equivalent to the strong T-cotype p' (see § 7). Thus, there is a gap between necessary and sufficient conditions.

We now recall the definition of a BMO_X. Let $f \in L_X^1(-\pi, \pi)$ be a 2π-periodic function. We put

$$\|f\|_* = \sup \frac{1}{|\Delta|} \int_\Delta \|f(t) - f_\Delta\| \, dt,$$

where $f_\Delta = \dfrac{1}{|\Delta|} \displaystyle\int_\Delta f(u) \, du$ and the upper bound is taken over all intervals $\Delta \subset \mathbb{R}$ with $|\Delta| \leqslant 2\pi$. The space BMO_X consists of all functions $f \in L_X^1$ such that $\|f\|_* < \infty$.

The Fefferman theorem on the duality between H^1 and BMO holds in part for vector-valued functions. Namely, a space BMO_{X^*} is isomorphic to a subspace of a space $(H_X^{1,at})^*$ (see [9]).

However, with the use of Theorem 9.3 we easily derive an analogue of Corollary 9.2.

Theorem 9.7. *Let a Banach space X have a Fourier type $p \in (1, 2]$. Suppose that a sequence $\{x_n\}_{n \in \mathbb{Z}} \subset X$ is such that $\{\|x_n\|\} \in FM_p$. Then the series*

$$\sum_{n=-\infty}^{+\infty} x_n e^{int} \tag{9.15}$$

converges in $L_X^{p'}(-\pi, \pi)$ and its sum belongs to BMO_X.

Proof. The first assertion follows from the fact that $FM_p \subset l^p$ and X has the Fourier type p.

Next, we fix an interval Δ with $|\Delta| \leqslant 2\pi$; suppose that $f_\Delta = 0$. Using the Hahn–Banach theorem, it is easy to see that there is a strongly measurable function $g\colon [-\pi, \pi] \to X^*$ such that for any $t \in [-\pi, \pi]$

$$\langle f(t), g(t) \rangle = \|f(t)\|_X \quad \text{and} \quad \|g(t)\|_{X^*} = 1.$$

We put

$$a(t) = \frac{1}{|\Delta|}\chi_\Delta(t)[g(t) - g_\Delta].$$

Then a is a X^*-atom. Here (see (2.13))

$$\begin{aligned}\frac{1}{|\Delta|}\int_\Delta \|f(t)\|\,dt &= \frac{1}{|\Delta|}\int_\Delta \langle f(t), g(t) - g_\Delta\rangle\,dt \\ &= \int_{-\pi}^{\pi} \langle f(t), a(t)\rangle\,dt = 2\pi \sum_{n=-\infty}^{+\infty} \langle x_n, \hat{a}(n)\rangle \\ &\leqslant 2\pi \sum_{n=-\infty}^{+\infty} \|x_n\|_X \|\hat{a}(n)\|_{X^*}.\end{aligned}$$

Since X has the Fourier type p, we see that, by Theorem 6.3, the space X^* also has the Fourier type p. Therefore, by applying the inequality (9.12) we complete the proof of the theorem.

Corollary 9.8. *Let X have a non-trivial Fourier type. If $x_n \in X$ $(n \in \mathbb{Z})$ and*

$$\|x_n\| = O\left(\frac{1}{|n|+1}\right), \tag{9.16}$$

then the sum of the series (9.15) *belongs to* BMO_X.

Indeed, it follows from (9.16) that $\{\|x_n\|\} \in FM_p$ for any $p \in (1, 2]$.

Below we consider classes of Banach spaces for which inequalities of Paley type hold (see (9.2)).

We note that in inequalities of the form (9.1) or (9.2) it suffices to take only coefficients $\widehat{f}(n)$ for $n \geqslant 0$ (since if a function $f \in H^{1,at}$, then also $f(-t) \in H^{1,at}$, and the $H^{1,at}$-norms of these functions are equal).

We recall that a sequence $\{n_k\}_{k\geqslant 1}$ of positive integers is called a λ-lacuna in the sense of Hadamard $(\lambda > 1)$ if

$$n_1 = 1, \qquad \frac{n_{k+1}}{n_k} \geqslant \lambda \quad (k = 1, 2, \dots). \tag{9.17}$$

Definition 9.9. We say that a Banach space X possesses the $\mathcal{P}_q$-property $(2 \leqslant q < \infty)$ if for any $\lambda > 1$ there is a constant $C > 0$ such that for each λ-lacuna sequence $\{n_k\}$ and each function $f \in H^{1,at}_X$ one has

$$\left(\sum_{k=1}^{\infty} \|\widehat{f}(n_k)\|^q\right)^{1/q} \leqslant C\|f\|_{H^{1,at}_{H_X}}. \tag{9.18}$$

This concept is studied in the paper [9] for $q = 2$. In particular, the following theorem is proved in [9] for $q = 2$; in the general case $q \geqslant 2$, the proof is similar.

Theorem 9.10. *A Banach space X possesses the $\mathcal{P}_q$-property $(2 \leqslant q < \infty)$ if and only if X has the strong R-cotype q.*

Proof. First we suppose that X has the strong R-cotype q. By Proposition 5.22, this implies that X^* has the R-type $p = q'$.

Let $\lambda > 1$. It suffices to prove that there is a constant C_λ such that for any λ-lacuna sequence $\{n_k\}$ and any atom a concentrated in the interval $(-\delta, \delta)$ $(0 < \delta \leqslant \pi)$, the inequality

$$\left(\sum_{k=1}^{\infty} \|\hat{a}(n_k)\|^q\right)^{1/q} \leqslant C_\lambda \tag{9.19}$$

is valid. Let $\pi/n_{\nu+1} < \delta \leqslant \pi/n_\nu$. Then, by virtue of (9.3) and (9.17),

$$\left(\sum_{k=1}^{\nu} \|\hat{a}(n_k)\|^q\right)^{1/q} \leqslant \frac{1}{4n_\nu} \sum_{k=1}^{\nu} n_k \leqslant \frac{\lambda}{4(\lambda - 1)}. \tag{9.20}$$

Next, we fix a positive integer $N > \nu$ and choose vectors $x_k^* \in X^*$ $(\nu < k \leqslant N)$ so that the relations $\langle \hat{a}(n_k), x_k^* \rangle \geqslant 0$,

$$\sum_{k=\nu+1}^{N} \|x_k^*\|^p = 1 \quad \text{and} \quad \left(\sum_{k=\nu+1}^{N} \|\hat{a}(n_k)\|^q\right)^{1/q} = \sum_{k=\nu+1}^{N} \langle \hat{a}(n_k), x_k^* \rangle \tag{9.21}$$

hold. Let $g(t) = \sum_{k=\nu+1}^{N} x_k^* e^{in_k t}$. We have

$$\begin{aligned} \sum_{k=\nu+1}^{N} \langle \hat{a}(n_k), x_k^* \rangle &= \frac{1}{2\pi} \int_{-\delta}^{\delta} \langle a(t), g(t) \rangle \, dt \\ &\leqslant \frac{1}{4\pi\delta} \int_{-\delta}^{\delta} \|g(t)\|_{X^*} \, dt = \frac{1}{4\pi} \int_{-\pi}^{\pi} \|g(\delta s)\|_{X^*} \, ds. \end{aligned}$$

By Theorem 8.4, there is a constant C such that the latter integral does not exceed

$$C \int_0^1 \left\| \sum_{j=1}^{N-\nu+1} x_{\nu+j}^* r_j(\tau) \right\|_{X^*} d\tau.$$

Since X^* has the R-type p, using (9.21) we have

$$\left(\sum_{k=\nu+1}^{N} \|\hat{a}(n_k)\|^q\right)^{1/q} \leqslant C'.$$

Combining this inequality with (9.20), we obtain (9.19).

Now suppose that X has the $\mathcal{P}_q$-property. Let

$$g(t) = \sum_{k=1}^{N} x_k^* \, e^{i2^k t}, \qquad x_k^* \in X^*.$$

By virtue of Proposition 2.6,

$$\|g\|_{L^2_{X^*}} = \sup\left|\int_{-\pi}^{\pi} \langle f(t), g(t)\rangle\, dt\right|,$$

where the upper bound is taken over all functions $f \in L^2_X$ with unit norm. For any such function f, by (9.18) and (9.6) we have

$$\left|\int_{-\pi}^{\pi} \langle f(t), g(t)\rangle\, dt\right| = \left|\sum_{k=1}^{N} \langle \widehat{f}(2^k), x_k^*\rangle\right|$$
$$\leqslant \left(\sum_{k=1}^{N} \|\widehat{f}(2^k)\|^q\right)^{1/q} \left(\sum_{k=1}^{N} \|x_k^*\|^p\right)^{1/p} \leqslant C\left(\sum_{k=1}^{N} \|x_k^*\|^p\right)^{1/p}.$$

It follows that X^* has the R-type p, and hence, X has the strong R-cotype q (see Proposition 5.22). The theorem is proved.

In particular, Theorem 9.10 contains a description of Banach spaces X such that the Paley inequality

$$\left(\sum_{k=0}^{\infty} \|\widehat{f}(2^k)\|^2\right)^{1/2} \leqslant C\|f\|_{1,at}, \qquad f \in H_X^{1,at},$$

holds. These are the only spaces with the strong R-cotype 2.

In connection with the Paley inequalities (9.2), Hardy and Littlewood considered a more general question on the description of $(H^1 - l^q)$-multipliers $(2 \leqslant q < \infty)$. They proved the following theorem (see [24], p. 103).

Theorem 9.11. *Let $m = \{m_j\}_{j\geqslant 0}$ be a sequence of complex numbers and $2 \leqslant q < \infty$. For a sequence $\{m_j\widehat{f}(j)\}$ to belong to l^q for each function $f \in H^1$, it is necessary and sufficient to have*

$$\sum_{k=1}^{N} k^q\, |m_k|^q = O(N^q). \tag{9.22}$$

We note that the $\mathcal{P}_q$-property can be described equivalently in terms of multipliers. Namely, it is easy to see that the following proposition holds.

Proposition 9.12. *Let X be a Banach space and $2 \leqslant q < \infty$. Then the following conditions are equivalent:*

(i) *X has the $\mathcal{P}_q$-property;*
(ii) *for any function $f \in H_X^{1,at}$ and any sequence $\{m_j\}_{j\geqslant 0}$ satisfying (9.22), the sequence $\{m_j\|\widehat{f}(j)\|\}$ belongs to l^q.*

§10. Estimates for Fourier transforms of smooth functions

It is well known that additional conditions on the smoothness of a periodic function (in an integral or uniform metric) ensure a certain degree of convergence to zero of its Fourier coefficients. The classical result is the Bernstein theorem (see [4], Russian p. 608):

If $f \in \operatorname{Lip}\alpha$ $(\alpha > \frac{1}{2})$, then the Fourier series of the function f converges absolutely.

In this section we consider similar results for vector-valued functions. The results are given in terms of Fourier transforms; it is clear that the corresponding assertions also hold for Fourier coefficients of periodic functions.

For vector-valued functions on $\mathbb{R}^n$ the concepts of a modulus of continuity, generalized derivative, Sobolev and Besov spaces are defined as in the scalar case (see § 4).

The Fourier transform of a function f is denoted by $\mathcal{F}(f)$.

We start with the estimate for a rearrangement of the Fourier transform.

Lemma 10.1. *Let a Banach space X have a Fourier type $p \in [1,2]$. Then for any function $f \in L^p_X(\mathbb{R}^n)$ and any $r \in \mathbb{N}$ one has*

$$\mathcal{F}(f)^*(t) \leqslant C\, t^{1/p-1} \omega_r(f; t^{-1/n})_p \quad (t > 0). \tag{10.1}$$

Proof. We write

$$\varphi_h^{(k)}(x) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + jhe_k) \quad (h > 0;\ k = 1, \dots, n),$$

where the e_k are the unit coordinate vectors in $\mathbb{R}^n$. Let $E \subset \mathbb{R}^n$ be a set of measure t such that

$$|\mathcal{F}(f)(\xi)| \geqslant \mathcal{F}(f)^*(t) \quad \text{for any} \quad \xi \in E.$$

There is a subset $Q \subset E$ with $|Q| \geqslant t/(2n)$ and an index $k = k(t)$, $1 \leqslant k \leqslant n$, such that $|\xi_k| \geqslant t^{1/n}/2$ for all $\xi \in Q$. We have

$$\mathcal{F}(\varphi_h^{(k)})(\xi) = \mathcal{F}(f)(\xi)\sigma(h\xi_k), \quad \text{where} \quad \sigma(u) = (e^{2\pi i u} - 1)^r.$$

We put $\delta = rt^{-1/n}$. It is easy to see that

$$\frac{1}{\delta} \int_0^\delta |\sigma(h\xi_k)|\, dh > \frac{1}{2} \qquad (\xi \in Q)$$

and

$$\mathcal{F}(f)^*(t) \leqslant \frac{4n}{t} \frac{1}{\delta} \int_0^\delta dh \int_Q |\mathcal{F}(\varphi_h^{(k)})(\xi)|\, d\xi. \tag{10.2}$$

Since X has the Fourier type p, we have

$$\int_Q |\mathcal{F}(\varphi_h^{(k)})(\xi)|\, d\xi \leqslant |Q|^{1/p} \left(\int_Q |\mathcal{F}(\varphi_h^{(k)})(\xi)|^{p'}\, d\xi \right)^{1/p'}$$
$$\leqslant C|Q|^{1/p} \|\varphi_h^{(k)}\|_p \leqslant Ct^{1/p} \|\varphi_h^{(k)}\|_p.$$

By virtue of (10.2), this yields (10.1). For $p = 1$, the estimate (10.1) follows directly from (10.2). The lemma is proved.

Using Lemma 10.1, we obtain a new proof of the following result [28], [47].

Theorem 10.2. *Suppose that a Banach space X has a Fourier type $p \in [1,2]$. Let $\alpha > 0$, $1 \leqslant \theta < \infty$ and $1/q = 1/p' + \alpha/n$. Then for any function $f \in B^{\alpha}_{p,\theta}(\mathbb{R}^n, X)$ the Fourier transform $\mathcal{F}(f)$ belongs to $L^{q,\theta}_X(\mathbb{R}^n)$; moreover*

$$\|\mathcal{F}(f)\|_{L^{q,\theta}_X} \leqslant C\biggl(\int_0^\infty [t^{-\alpha}\omega_r(f;t)_p]^\theta \,\frac{dt}{t}\biggr)^{1/\theta} \qquad (r > \alpha,\ r \in \mathbb{N}). \tag{10.3}$$

In particular, an analogue of the Bernstein–Szasz theorem holds [75].

Corollary 10.3. *If X has a Fourier type $p \in [1,2]$, then for any function $f \in B^{n/p}_{p,1}(\mathbb{R}^n, X)$ one has*

$$\int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|\,d\xi < \infty.$$

In the paper of König [47], for the periodic case examples are given showing that Theorem 10.2 cannot be improved. In particular, even in the case $X = \mathbb{C}$ the assertion of this theorem is optimal for Lorentz spaces. On the other hand, the condition on the modulus of continuity cannot be weakened. This holds even for Corollary 10.3: as was shown for $n = 1$ by Ul'yanov [80], for any modulus of continuity $\omega(\delta)$ such that

$$\int_0^1 \frac{\omega(\delta)}{\delta^{1+1/p}}\,d\delta = \infty \quad (1 < p < \infty)$$

there is an essentially unbounded function $f \in L^p$ with $\omega_p(f;\delta) = O(\omega(\delta))$.

Further, we give the following example.

Example 10.4. Let $1 < p \leqslant 2$ and $1 < r < p$. Then there is a Banach space X such that X has the Fourier type r and there is a function $f \in B^{n/p}_{p,1}(\mathbb{R}^n, X)$ such that

$$\int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|\,d\xi = \infty. \tag{10.4}$$

Indeed, for simplicity we take $n = 1$ and set $X = L^{r'}(\mathbb{R})$. Then X has the Fourier type r (see Proposition 4.4). We define a function $f\colon \mathbb{R} \to X$ by the formula

$$f(t)(y) = \frac{1}{1+|y|}\,e^{2\pi i t y}\,\varphi(t),$$

where $\varphi \in C_0^\infty(\mathbb{R})$, $\varphi \geqslant 0$, and $\displaystyle\int_0^1 |\mathcal{F}(\varphi)(\xi)|^{r'}\,d\xi = 1$. It is easy to see that for any $t \in \mathbb{R}$ and $h \in (0,1]$ one has

$$\|f(t) - f(t+h)\|_X \leqslant Ch^{1-1/r'}\varphi(t) + |\varphi(t) - \varphi(t+h)|.$$

Therefore, $f \in B^{1/p}_{p,1}(\mathbb{R}, X)$. On the other hand,

$$\mathcal{F}(f)(\xi)(y) = \frac{1}{1+|y|}\,\mathcal{F}(\varphi)(\xi - y),$$

whence

$$\|\mathcal{F}(f)(\xi)\|_X \geqslant \frac{1}{\xi} \quad \text{for} \quad \xi \geqslant 1,$$

and we arrive at (10.4).

Let us next cite another useful estimate for the rearrangement of the Fourier transform.

Lemma 10.5. *Let a Banach space X have a Fourier type $1 < p \leqslant 2$. Then for any function $f \in L^1_X(\mathbb{R}^n)$ one has*

$$\left(\int_0^t (\mathcal{F}(f)^*(u))^{p'}\,du\right)^{1/p'} \leqslant t^{1/p'} \int_0^\tau f^*(s)\,ds + C\left(\int_\tau^\infty f^*(x)^p\,ds\right)^{1/p} \tag{10.5}$$

for any $t, \tau > 0$.

In the scalar case ($p = 2$), this lemma is proved in the papers [42], [60]. In the vector case, the proof remains the same. Namely, we choose a set $Q \subset \mathbb{R}^n$ of measure τ such that

$$\{t : \|f(t)\| > f^*(\tau)\} \subset Q \subset \{t : \|f(t)\| \geqslant f^*(\tau)\}.$$

Let $g = f\chi_Q$, $h = f - g$. Then

$$\|g\|_1 = \int_0^\tau f^*(s)\,ds, \qquad \|h\|_p = \left(\int_\tau^\infty f^*(s)^p\,ds\right)^{1/p}.$$

We have

$$\mathcal{F}(f)^*(u) \leqslant \int_0^\tau f^*(s)\,ds + \mathcal{F}(h)^*(u) \quad (u > 0).$$

Since X has the Fourier type p, we have (10.5).

By means of Lemma 10.5 we easily derive the following theorem, proved in another way in [47].

Theorem 10.6. *Suppose that a Banach space X has a Fourier type $p_0 \in (1, 2]$. Let $r \in \mathbb{N}$, $1 < p < p_0$, and $1/q = 1/p' + r/n$. Then for any function $f \in W^r_p(\mathbb{R}^n, X)$ the Fourier transform $\mathcal{F}(f)$ belongs to $L^{q,p}_X(\mathbb{R}^n)$, where*

$$\|\mathcal{F}(f)\|_{L^{q,p}_X} \leqslant c \sum_{j=1}^n \|D^r_j f\|_{L^p_X}. \tag{10.6}$$

Proof. Note that

$$|\mathcal{F}(f)(\xi)| = (2\pi)^{-r} |\xi|^{-r} \left(\sum_{j=1}^n |\mathcal{F}(g_j)(\xi)|^{2/r}\right)^{r/2},$$

where $g_j(x) = D^r_j f(x)$. Using inequalities (2.2) and (2.3), we obtain

$$\mathcal{F}(f)^*(t) \leqslant C\, t^{-r/n} \sum_{j=1}^n \mathcal{F}(g_j)^*\left(\frac{t}{2n}\right).$$

Applying Lemma 10.5 to the functions g_j, we obtain

$$\left(\int_0^t [\mathcal{F}(f)^*(u) u^{r/n}]^{p_0'}\,du\right)^{1/p'}$$
$$\leqslant C \sum_{j=1}^n \left(t^{1/p_0'} \int_0^\tau g_j^*(u)\,du + \left(\int_\tau^\infty g_j^{*\,p_0}(u)\,du\right)^{1/p_0}\right) \tag{10.7}$$

for any $t, \tau > 0$. Using this estimate (for $\tau = 1/t$) and Lemma 2.2, we obtain inequality (10.6). The theorem is proved.

Remark 10.7. For $p = p_0$, inequality (10.6) does not hold (see [47]). In this case the estimate

$$\|\mathcal{F}(f)\|_{L_X^{q,p'}} \leqslant C \sum_{j=1}^{n} \|D_j^r f\|_{L_X^p} \tag{10.8}$$

is valid (it readily follows from (10.7) for $\tau = 0$). This estimate cannot be improved by substituting a smaller number for the index p' on the left-hand side [47].

We now consider the case $p = 1$ in Theorem 10.6. This case is of special interest because it is related to integrability conditions for the Fourier transform on $\mathbb{R}^n$. We note that this case is included in the statement of the corresponding theorem in [47] (Theorem 4, p. 223); however, the proof cited in [47] does not hold for $p = 1$. Moreover, Theorem 10.6 is not valid for $p = n = 1$. For example, it is well known that the Fourier series of an absolutely continuous function does not have to converge absolutely: an example is given by the function

$$f(x) = \sum_{n=2}^{\infty} \frac{\sin nx}{n \ln n}.$$

Nevertheless, for $p = 1$, $n \geqslant 2$ Theorem 10.6 is true; however, even for scalar functions this case is much more complex (in particular, it cannot be exhausted by applying Lemma 10.5).

The following theorem (an analogue of the Hardy inequality) holds for scalar functions. It was proved by Bourgain [14] for the periodic case, and by Pelczyński and Wojciechowski [65] for the Fourier transform.

Theorem 10.8. *Let $f \in W_1^r(\mathbb{R}^n)$ ($n \geqslant 2$, $r \in \mathbb{N}$). Then*

$$\int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|\, |\xi|^{r-n}\, d\xi \leqslant C \sum_{|s|=r} \|D^s f\|_1. \tag{10.9}$$

In [46] it was proved that a more precise inequality is valid:[9]

$$\|\mathcal{F}(f)\|_{n/r,1} \leqslant C \sum_{j=1}^{n} \|D_j^r f\|_1 \quad (1 \leqslant r \leqslant n,\ n \geqslant 2) \tag{10.10}$$

(thus, (10.6) also holds for $p = 1$, $X = \mathbb{C}$). In [46] the author notes that the inequality (10.10) also remains valid for functions assuming values in a Banach space X if X has a non-trivial Fourier type $1 < p \leqslant 2$. We note that this is not true without additional conditions on X, as is seen from the following example.

[9] In comparison with (10.9), there are only unmixed derivatives on the right-hand side of (10.10).

Example 10.9. Let $X = L^\infty(\mathbb{R}^n)$, $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\varphi \geqslant 0$, and $\mathcal{F}(\varphi)(0) = 1$. We put

$$f(t)(y) = \frac{1}{1+|y|^n} e^{2\pi i t\cdot y}\varphi(t), \qquad f\colon \mathbb{R}^n \to X.$$

It is easy to see that $f \in W_1^n(\mathbb{R}^n; X)$. Then

$$\mathcal{F}(f)(\xi)(y) = \frac{\mathcal{F}(\varphi)(\xi - y)}{1+|y|^n}.$$

Therefore,

$$\|\mathcal{F}(f)(\xi)\|_X \geqslant \frac{\mathcal{F}(\varphi)(0)}{1+|\xi|^n} = \frac{1}{1+|\xi|^n}$$

and $\mathcal{F}(f) \notin L^1(\mathbb{R}^n)$.

We now note that an analogue of Theorem 10.6 also holds for Sobolev–Liouville fractional spaces. The space $\mathcal{L}_p^\alpha(\mathbb{R}^n, X)$ is defined as the set of all functions $f\colon \mathbb{R}^n \to X$ representable in the form of the convolution

$$f(x) = \int_{\mathbb{R}^n} G_\alpha(x-y)\, g(y)\, dy,$$

where $g \in L_X^p(\mathbb{R}^n)$ and G_α is a Bessel kernel (see [74], Ch. 5). Here, by definition,

$$\|f\|_{\mathcal{L}_p^\alpha(\mathbb{R}^n, X)} = \|g\|_{L_X^p}.$$

Taking into account the fact that

$$\mathcal{F}(G_\alpha)(\xi) = (1 + 4\pi^2|\xi|^2)^{-\alpha/2}$$

and using Lemma 10.5, we readily obtain the following result.

Theorem 10.10. *Suppose that a Banach space X has a Fourier type $p_0 \in (1, 2]$. Let $\alpha > 0$, $1 < p < p_0$, and $1/q = 1/p' + \alpha/n$. Then for any function $f \in \mathcal{L}_p^\alpha(\mathbb{R}^n, X)$ we have $\mathcal{F}(f) \in L_X^{q,p}(\mathbb{R}^n)$; moreover,*

$$\|\mathcal{F}(f)\|_{L_X^{q,p}} \leqslant C\,\|f\|_{\mathcal{L}_p^\alpha(\mathbb{R}^n, X)}.$$

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